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Affine surfaces which admit several affine immersions in $\mathbb{R}^3$

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Abstract. Let $\mathcal{F} : \Sigma \longrightarrow \mathbb{R}^3$ be a Blaschke immersion of an affine surface (Σ, ∇) with a positive definite affine fundamental form such that $\dim \operatorname{Im} R = 1$ where R is the curvature tensor. Suppose that there exists another immersion of the same surface with the same induced affine connection ∇ which is not affine equivalent to the first one. Then we give explicitely $\mathcal{F}$. Therefore all immersions which admit another immersion which is not affine equivalent to the original one are classified.

Key words: Blaschke immersion, affine surface.

1. Introduction

For a long time, mathematicians have been interested in the study of immersions of hypersurfaces and in particular in the problem of rigidity of such immersions. Around 1980, K. Nomizu posed the following global problem in affine differential geometry:

Problem a *Assume that $f : M \longrightarrow \mathbb{R}^{n+1}$ and $g : M \longrightarrow \mathbb{R}^{n+1}$ are two ovaloids (i.e. connected, compact non degenerate hypersurfaces) with the same induced affine Blaschke connection. Are both immersions affine equivalent?*

In the case that $n = 2$, K. Nomizu and B. Opozda (see [4]) proved the following theorems; given a positive answer to Problem a, under some additional assumptions:

Theorem a *Let M be a connected, compact orientable 2-manifold and let f, $\overline{f} : M \longrightarrow \mathbb{R}^3$ be two nondegenerate embeddings with equiaffine transversal vector fields ξ, $\overline{\xi}$. Assume that $\det S$ (where S is the shape operator) is nowhere 0. If the induced connections coincide and if $\det S = \det \overline{S}$ at every point, then f and $\overline{f}$ are affine equivalent.*

Theorem b *Let M be a connected, compact orientable 2-manifold and let*

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f, $\overline{f} : M \longrightarrow \mathbb{R}^3$ be a nondegenerate embedding and a nondegenerate immersion, respectively, with equiaffine transversal vector fields ξ, $\overline{\xi}$. Assume that they have the same induced connection ∇ with nondegenerate Ricci tensor. If $\operatorname{tr} S = \operatorname{tr} \overline{S}$ and if $\det S \leq \det \overline{S}$, then f and $\overline{f}$ are affine equivalent.

And U. Simon (see [7]) showed a theorem which for Blaschke immersions reduces to:

Theorem c *Let $x : M \longrightarrow \mathbb{R}^3$ and $x' : M \longrightarrow \mathbb{R}^3$ be two ovaloids in $\mathbb{R}^3$. Assume that the Blaschke connections induced by x and x' coincide. Then $x : M \longrightarrow \mathbb{R}^3$ and $x' : M \longrightarrow \mathbb{R}^3$ are affine equivalent.*

Note that as in Blaschke geometry the volume form is always parallel, the previous theorem solves the case $n = 2$ for ovaloids and up to now, the case $n \geq 3$ remains open.

In this paper we are interested in a local analog of this previous problem.

Problem b *Assume that $f : M \longrightarrow \mathbb{R}^{n+1}$ and $g : M \longrightarrow \mathbb{R}^{n+1}$ are two positive definite affine immersions with the same induced affine Blaschke connection. Is it possible to find an affine transformation A of $\mathbb{R}^{n+1}$ such that $f = A \circ g$?*

K. Nomizu and L. Vrancken (see [6]) answered positively in case $n \geq 3$ and the dimension of the image of the curvature tensor is at least equal to 2 ($\dim \operatorname{Im} R \geq 2$). In 2003, L. Vrancken solved the case $n \geq 3$ and $\dim \operatorname{Im} R = 1$ (see [8]). In this case, such immersions are generically locally rigid and he gave a complete description of non locally rigid immersions in terms of differential equations of Monge-Ampere type.

We study the case $n = 2$ and $\dim \operatorname{Im} R = 1$. We consider an affine surface (Σ, ∇) with a positive definite affine fundamental form and an immersion $\mathcal{F} : \Sigma \longrightarrow \mathbb{R}^3$. Since $\dim \operatorname{Im} R = 1$, we can choose X_1 and X_2 orthonormally, with respect to the affine metric introduced by $\mathcal{F}$, such that we have $R(X_1, X_2)X_1 = 0$ and $R(X_1, X_2)X_2 = \lambda_1 X_1$, where λ_1 is a non vanishing function defined on the surface. This is equivalent to have $S_1 X_2 = 0$ and $S_1 X_1 = \lambda_1 X_1$ by using the Gauss equation. The aim of this paper is to give a description of affine surfaces which admit several immersions in $\mathbb{R}^3$ with the same induced connection, by giving the position vector for the immersion $\mathcal{F}$. More precisely we have the following theorems:

Theorem 1 *Let $\mathcal{F} : \Sigma \longrightarrow \mathbb{R}^3$ be a Blaschke immersion with induced connection ∇, positive definite affine fundamental form h_1 such that the image of the curvature tensor has dimension 1. Suppose that there exists another Blaschke immersion with the same induced connection ∇, positive definite affine fundamental form h_2 such that $h_2(X_1, X_2) \neq 0$, then under a suitable choice of coordinates (u, v) on Σ, the immersion $\mathcal{F}$ is given by the formulas*:

if $\lambda_1 > 0$, we have $\Delta\left(\frac{1}{\lambda_1}\right) = -\frac{1}{\lambda_1}$ and

$$\mathcal{F} = \begin{pmatrix} \displaystyle\int_0^u \lambda_1(\widetilde{u}, v)^{-1} \cos(\widetilde{u}) d\widetilde{u} - f_1(v) \\ \displaystyle\int_0^u \lambda_1(\widetilde{u}, v)^{-1} \sin(\widetilde{u}) d\widetilde{u} + f_2(v) \\ v \end{pmatrix};$$

if $\lambda_1 < 0$, we have $\Delta\left(\frac{1}{\lambda_1}\right) = \frac{1}{\lambda_1}$ and

$$\mathcal{F} = \begin{pmatrix} \displaystyle\int_0^u -\lambda_1(\widetilde{u}, v)^{-1} \cosh(\widetilde{u}) d\widetilde{u} + f_1(v) \\ \displaystyle\int_0^u -\lambda_1(\widetilde{u}, v)^{-1} \sinh(\widetilde{u}) d\widetilde{u} - f_2(v) \\ v \end{pmatrix},$$

where f_1, f_2 are functions satisfying $f_1''(v) = \frac{\partial \lambda_1^{-1}}{\partial u}(0, v)$, $f_2''(v) = \lambda_1^{-1}(0, v)$.

Theorem 2 *Let $\mathcal{F} : \Sigma \longrightarrow \mathbb{R}^3$ be a Blaschke immersion with induced connection ∇, positive definite affine fundamental form h_1 such that the image of the curvature tensor has dimension 1. Suppose that there exists another Blaschke immersion with the same induced connection ∇, positive definite affine fundamental form h_2 such that $h_2(X_1, X_2) = 0$. Then under a suitable choice of coordinates (u, v) on Σ, either there exist a non degenerate equiaffine curve γ in $\mathbb{R}^2$, constant vectors k_1, k_2, e and constants d_1, d_2 such that the immersion $\mathcal{F}$ is given by one of the following expressions*:

$$\mathcal{F}(u, v) = \gamma(u) + \frac{e}{2} v^2 + k_1 v,$$

$$\mathcal{F}(u,v) = (d_1 v + d_2)\gamma(u) + \frac{e}{6d_1^2}(d_1 v + d_2)^3 + k_2 v,$$

or there exist a non degenerate centroaffine curve $\widetilde{\gamma} = (\gamma_1, \gamma_2, 0)$*, constants* d_1*,* d_2*,* d *and* e *such that* $\mathcal{F}$ *is given by one of the following expressions*:

$$\begin{aligned}\mathcal{F}(u,v) = \big(&\big(d_1 \exp(\sqrt{d}\, v) + d_2 \exp(-\sqrt{d}\, v)\big)\gamma_1(u),\\ &\big(d_1 \exp(\sqrt{d}\, v) + d_2 \exp(-\sqrt{d}\, v)\big)\gamma_2(u), ev\big),\\ \mathcal{F}(u,v) = \big(&\big(d_1 \cos(\sqrt{-d}\, v) + d_2 \sin(\sqrt{-d}\, v)\big)\gamma_1(u),\\ &\big(d_1 \cos(\sqrt{-d}\, v) + d_2 \sin(\sqrt{-d}\, v)\big)\gamma_2(u), ev\big).\end{aligned}$$

2. Preliminaries

In this section, we will introduce all the material we need. For more details see [5]. Let $f : M \longrightarrow (\mathbb{R}^{n+1}, D)$ be an immersion of an n dimensional manifold in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat affine connection D.

For each point of M we can choose locally a transversal vector field ξ. Then we have a torsion free induced connection ∇ satisfying:

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi \quad \text{(Gauss formula)}, \tag{1}$$

where h is a symmetric bilinear function on the space $\mathcal{X}(M)$ of vector fields on M. To simplify, we will write $D_X Y = \nabla_X Y + h(X,Y)\xi$.

Definition 2.1 This symmetric bilinear form h is called the affine fundamental form of f with respect to ξ.

Remark 2.1 Each choice of a transversal ξ gives us such form h.

For all $X \in \mathcal{X}(M)$ we have:

$$D_X \xi = -f_*(SX) + \tau(X)\xi \quad \text{(Weingarten formula)}, \tag{2}$$

where S is a tensor of type $(1,1)$, called the affine shape operator and τ is a 1-form called the transversal connection form. To simplify, we will write $D_X\xi = -SX + \tau(X)\xi$.

On $\mathbb{R}^{n+1}$ we have a parallel volume element Ω given by the determinant. Parallel means that $D_X\Omega = 0$ for all vector fields on $\mathbb{R}^{n+1}$. Now we suppose that h is nondegenerate. This condition is independent of the choice of ξ.

In this case, we say that the immersion is nondegenerate. In fact h is a pseudo-riemannian metric called the affine metric. We define a volume element ω on M by setting $\omega = i_\xi \Omega$, where i is the interior product.

Proposition 2.1 (see [5]) *We have $\nabla_X \omega = \tau(X)\omega$ for all X in $\mathcal{X}(M)$. Consequently, the following conditions are equivalent*:

(a) $\nabla\omega = 0$,
(b) $\tau = 0$*, that is $D_X\xi$ is tangential for all X in $\mathcal{X}(M)$.*

Definition 2.2 We say that f is an equiaffine immersion if condition (b) is verified. In this case ξ is said to be equiaffine.

Remark 2.2 Every hypersurface immersion admits locally an equiaffine transversal vector field ξ.

In this case, we have the following equations:

Gauss equation

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{3}$$

Codazzi equations for h and S

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{4}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \tag{5}$$

Ricci equation

$$h(X,SY) - h(SX,Y) = 0. \tag{6}$$

Relative to a coordinate system $(x^1, x^2, \ldots, x^n)$, we can express the components of h as follows: $h_{ij} = h\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big)$. If the immersion is nondegenerate, we define the volume element by $\omega_h\big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\big) = \sqrt{|\det(h_{ij})|}$. Let (c): $\omega = \omega_h$.

There exists a unique choice of ξ, such that conditions (b) and (c) hold. In this case, ξ is called the affine normal and f is a Blaschke immersion.

Definition 2.3 Let $f : M \longrightarrow \mathbb{R}^{n+1}$ and $g : M \longrightarrow \mathbb{R}^{n+1}$ be two immersions. We say that f and g are affine equivalent or affine congruent if there exists an affine transformation A of $\mathbb{R}^{n+1}$ such that $f = A \circ g$.

3. Computations

Assume now that we have two Blaschke immersions of the same surface $\mathcal{F} : \Sigma \longrightarrow \mathbb{R}^3$ and $\mathcal{G} : \Sigma \longrightarrow \mathbb{R}^3$, which are not affine congruent, but which have the same induced connection ∇. We also assume that the dimension of the image of the curvature tensor R is 1. We denote by ξ the affine normal, h_1 the nondegenerate positive definite affine metric, S_1 the affine shape operator and ω_1 the volume form for the first immersion $\mathcal{F}$. We use the notations $\widetilde{\xi}$, h_2, S_2 and ω_2 for the second one $\mathcal{G}$. We denote by X_1 and X_2 a basis of vector fields on Σ.

The volume forms ω_1 and ω_2 are parallel with respect to ∇; that is $\nabla_X \omega_i(X_1, X_2) = 0$ $(i = 1, 2)$. Hence ω_2 is a constant multiple of ω_1. By using a homothety, we can suppose that $\omega_1 = \omega_2$. So, we will write the volume form ω, without indices.

We pose:

$$\nabla_{X_1} X_1 = a_1 X_1 + a_2 X_2,$$
$$\nabla_{X_1} X_2 = a_3 X_1 + a_4 X_2,$$
$$\nabla_{X_2} X_1 = a_5 X_1 + a_6 X_2,$$
$$\nabla_{X_2} X_2 = a_7 X_1 + a_8 X_2,$$

where the a_i are functions defined on the surface.

As $\dim \operatorname{Im} R = 1$, choosing X_1 and X_2 as indicated before, we have $h_1(X_1, X_1) = h_1(X_2, X_2) = 1$, $h_1(X_1, X_2) = 0$, $S_1 X_2 = 0$ and $S_1 X_1 = \lambda_1 X_1$.

By using the Codazzi equations for h_1 and S_1 ((4) and (5)) and $\nabla\omega = 0$, a straightforward computation shows:

$$\nabla_{X_1} X_1 = a_1 X_1 + a_2 X_2,$$
$$\nabla_{X_1} X_2 = a_3 X_1 - a_1 X_2,$$
$$\nabla_{X_2} X_1 = \frac{1}{2}(a_2 + a_3) X_1,$$
$$\nabla_{X_2} X_2 = -2a_1 X_1 - \frac{1}{2}(a_2 + a_3) X_2,$$

with $X_2(\lambda_1) = -a_3\,\lambda_1$.

Now, we look at the second immersion in order to deduce more informations. We pose $b_{11} = h_2(X_1, X_1)$, $b_{22} = h_2(X_2, X_2)$ and $b_{12} = h_2(X_1, X_2)$. In this case, in general, $b_{12} \neq 0$ and we have the equation $b_{11}b_{22} - b_{12}^2 = 1$. Since the two immersions have the same connection, they have the same curvature tensor R. Using (3) for the second immersion, we find $S_2X_1 = \lambda_1 b_{11} X_1$ and $S_2X_2 = \lambda_1 b_{12} X_1$.

Writing (5) for the second immersion in the direction of X_1, we get:

$$X_1(\lambda_1 b_{12}) + \lambda_1 a_1 b_{12} - a_3\lambda_1 b_{11} - a_4 \lambda_1 b_{12} = X_2(\lambda_1 b_{11}),$$

so $X_1(\lambda_1)b_{12} + \lambda_1 a_1 b_{12} - a_3\lambda_1 b_{11} + a_1\lambda_1 b_{12} + \lambda_1 X_1(b_{12}) - \lambda_1 X_2(b_{11}) = -\lambda_1 a_3 b_{11}$.

Using (4), we have $X_1(b_{12}) + a_2(b_{11} - b_{22}) - X_2(b_{11}) = 0$.

Then

$$X_1(\lambda_1)b_{12} + 2\lambda_1 a_1 b_{12} + \lambda_1 a_2(b_{22} - b_{11}) = 0. \tag{7}$$

Using (5) in the direction of X_2, we obtain $\lambda_1 b_{12} a_2 = 0$.

Thus we get $a_2 b_{12} = 0$.

Lemma 3.1 *We always have $a_2 = 0$.*

Proof. The case $b_{12} \neq 0$ is obvious. Now suppose that $b_{12} = 0$. We have $X_1(\lambda_1)b_{12} + 2\lambda_1 a_1 b_{12} + \lambda_1 a_2(b_{22} - b_{11}) = 0$. So $a_2(b_{11} - b_{22}) = 0$.

If $b_{11} = b_{22}$, since $b_{12} = 0$, we have $b_{11} = b_{22} = 1$ and the second immersion equals to the first one (see [3]). Hence $b_{11} \neq b_{22}$ and we find $a_2 = 0$. □

To further simplify the problem, we now introduce special isothermal coordinates. It is well known that general isothermal coordinates exist for 2 dimensional regular surfaces (see [2]). However we want to find isothermal coordinate vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ such that $\frac{\partial}{\partial u}$ is a multiple of X_1 and $\frac{\partial}{\partial v}$ is a multiple of X_2. The existence of such coordinates is equivalent to the existence of a positive function ρ such that $[\rho X_1, \rho X_2] = 0$. As

$$\begin{aligned}[\rho X_1, \rho X_2] &= \nabla_{\rho X_1}\rho X_2 - \nabla_{\rho X_2}\rho X_1 \\ &= \rho\Big[(X_1(\rho) - \rho a_1)X_2 + \Big(\rho\frac{1}{2}a_3 - X_2(\rho)\Big)X_1\Big],\end{aligned}$$

such isothermal coordinates exist if and only if we can find a function ρ satisfying the following differential equations:

$$\begin{cases} X_1(\ln(\rho)) = a_1, \\ X_2(\ln(\rho)) = \frac{1}{2}a_3. \end{cases} \tag{8}$$

Lemma 3.2 *The above equations have solutions and hence isothermal coordinate vector fields* $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ *exist.*

Proof. After straightforward computations, we find that these equations have solutions if and only if the following integrability equation holds: $0 = X_1\left(\frac{1}{2}a_3\right) - X_2(a_1)$.

By using the following Gauss equation with $X = Z = X_1$ and $Y = X_2$:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = h_1(Y,Z)S_1 X - h_1(X,Z)S_1 Y,$$

we find that $X_1\left(\frac{1}{2}a_3\right) - X_2(a_1) = 0$. $\square$

Using these isothermal coordinates, we have:

$$\begin{aligned} \mathcal{F}_{uu} &= 2\rho a_1 \mathcal{F}_u + \rho^2 \xi, \\ \mathcal{F}_{uv} &= \mathcal{F}_{vv} = \rho a_3 \mathcal{F}_u, \\ \mathcal{F}_{vv} &= -2\rho a_1 \mathcal{F}_u + \rho^2 \xi, \\ \xi_u &= -\lambda_1 \mathcal{F}_u, \\ \xi_v &= 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathcal{G}_{uu} &= 2\rho a_1 \mathcal{G}_u + \rho^2 b_{11} \widetilde{\xi}, \\ \mathcal{G}_{uv} &= \mathcal{G}_{vv} = \rho a_3 \mathcal{G}_u + \rho^2 b_{12} \widetilde{\xi}, \\ \mathcal{G}_{vv} &= -2\rho a_1 \mathcal{G}_u + \rho^2 b_{22} \widetilde{\xi}, \\ \widetilde{\xi}_u &= -\lambda_1 b_{11} \mathcal{G}_u, \\ \widetilde{\xi}_v &= -\lambda_1 b_{12} \mathcal{G}_u, \end{aligned} \tag{10}$$

where the b_{ij}, $i, j = 1, 2$ satisfy:

$$\begin{cases} b_{11}b_{22} - b_{12}^2 = 1, \\ \dfrac{\partial b_{12}}{\partial u} - \dfrac{\partial b_{11}}{\partial v} = 0 \text{ because } a_2 = 0 \text{ (Cf. (4))}, \\ -\dfrac{\partial b_{22}}{\partial u} + \dfrac{\partial b_{12}}{\partial v} + 2\rho a_1 b_{11} + 2\rho a_3 b_{12} - 2\rho a_1 b_{22} = 0 \text{ (Cf. (4))}, \end{cases}$$

and

$$\lambda_1 = -2X_1(a_1) - X_2(a_3) - \frac{3}{2}a_3^2 - 6a_1^2 \text{ (Cf. (3))}. \tag{11}$$

From (9), after integration, we get $\mathcal{F}_u = C(u)\exp(\int \rho a_3 dv)$. Moreover $X_2(\rho) = \rho\frac{1}{2}a_3$.

Then

$$\begin{aligned} \mathcal{F}_u &= C(u)\exp\left(\int 2X_2(\rho)dv\right) \\ &= C(u)\exp\left(\int 2\frac{1}{\rho}\frac{\partial\rho}{\partial v}dv\right) \\ &= C(u)\exp\left(\int 2\frac{\partial(\ln(\rho))}{\partial v}dv\right) \\ &= C(u)\exp(\ln(\rho^2)) \\ &= C(u)\rho^2. \end{aligned}$$

Therefore we have $\mathcal{F}_{uu} = C'(u)\rho^2 + 2\rho C(u)\frac{\partial\rho}{\partial u}$.

Moreover $\mathcal{F}_{uu} = 2\rho a_1\mathcal{F}_u + \rho^2\xi = 2\rho^3 a_1 C(u) + \rho^2 C'(u)$ then $\xi = C'(u)$ and $\xi_u = C''(u) = -\lambda_1\rho^2 C(u)$. We deduce from this that $\lambda_1\rho^2$ doesn't depend on v.

So we have to solve the following differential equation:

$$C''(u) + \lambda_1\rho^2 C(u) = 0. \tag{12}$$

Remark 3.1 As the surface is nondegenerate, $(C(u), C'(u))$ is linearly independent.

We have the following well known result (see [1, p. 243]):

Lemma 3.3 *The solutions of the previous equation* (12) *are of the form* $C(u) = C_1\alpha_1(u) + C_2\alpha_2(u)$ *where* $C_1, C_2 \in \mathbb{R}^3$ *and* (α_1, α_2) *is a fundamental system of solutions.*

By using an equiaffine transformation, we can suppose that $C_1 = (k, 0, 0)$ and $C_2 = (0, k', 0)$, where $k, k' \in \mathbb{R}^*$. Applying the matrix $M = \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k'} & 0 \\ 0 & 0 & kk' \end{pmatrix}$ on the vector space, we find that $C_1 = (1, 0, 0)$ and $C_2 = (0, 1, 0)$. Therefore, we have $C(u) = (\alpha_1(u), \alpha_2(u), 0)$ and $\mathcal{F}_u = (\rho^2\alpha_1(u), \rho^2\alpha_2(u), 0)$.

Finally we have:

$$\mathcal{F} = \begin{pmatrix} \displaystyle\int^u \rho^2(\widetilde{u}, v)\alpha_1(\widetilde{u})d\widetilde{u} + f_1(v) \\ \displaystyle\int^u \rho^2(\widetilde{u}, v)\alpha_2(\widetilde{u})d\widetilde{u} + f_2(v) \\ f_3(v) \end{pmatrix},$$

where the f_i, $i = 1, 2, 3$ are integration functions with conditions on their second derivative given by $\mathcal{F}_{uu} + \mathcal{F}_{vv} = 2\rho^2\ (\alpha_1', \alpha_2', 0)$.

4. Case $h_2(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) \neq 0$.

Since $b_{12} \neq 0$, by using Lemma 3.1 and (7), we get:

$$\begin{cases} X_1(\lambda_1) = -2\lambda_1 a_1, \\ X_2(\lambda_1) = -a_3\lambda_1. \end{cases} \tag{13}$$

By (8) and (13), we have $\rho^2|\lambda_1| = c$, where c is a constant. As ρ is determined up to a constant, we can take $c = 1$.

By (11) and (13), we find that:

$$\lambda_1^3 = (X_1X_1(\lambda_1) + X_2X_2(\lambda_1))\lambda_1 - \frac{5}{2}(X_2(\lambda_1))^2 - \frac{5}{2}(X_1(\lambda_1))^2. \tag{14}$$

In isothermal coordinates, (14) reduces to

$$\rho^2\lambda_1^3 = \lambda_1\frac{\partial^2\lambda_1}{\partial u^2} + \lambda_1\frac{\partial^2\lambda_1}{\partial v^2} - 2\left(\frac{\partial\lambda_1}{\partial u}\right)^2 - 2\left(\frac{\partial\lambda_1}{\partial v}\right)^2. \tag{15}$$

If $\rho^2\lambda_1 = 1$, the equation (12) becomes $C''(u) + C(u) = 0$ and we can choose initial condition such that:

$$\mathcal{F} = \begin{pmatrix} \displaystyle\int_0^u \frac{1}{\lambda_1(\widetilde{u}, v)}\cos(\widetilde{u})d\widetilde{u} + f_1(v) \\ \displaystyle\int_0^u \frac{1}{\lambda_1(\widetilde{u}, v)}\sin(\widetilde{u})d\widetilde{u} + f_2(v) \\ f_3(v) \end{pmatrix},$$

$$\begin{aligned} |C(u), C'(u)| &= \begin{vmatrix} \cos(u) & -\sin(u) \\ \sin(u) & \cos(u) \end{vmatrix} \\ &= \cos^2(u) + \sin^2(u) \\ &= 1 \end{aligned}$$

and $\frac{\partial}{\partial u}|C(u), C'(u)| = |C(u), C''(u)| = 0$.

The equation (15) gives:

$$\lambda_1^2 = \lambda_1\frac{\partial^2\lambda_1}{\partial u^2} + \lambda_1\frac{\partial^2\lambda_1}{\partial v^2} - 2\left(\frac{\partial\lambda_1}{\partial u}\right)^2 - 2\left(\frac{\partial\lambda_1}{\partial v}\right)^2.$$

So

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_1^2}\frac{\partial^2\lambda_1}{\partial u^2} + \frac{1}{\lambda_1^2}\frac{\partial^2\lambda_1}{\partial v^2} - \frac{2}{\lambda_1^3}\left(\frac{\partial\lambda_1}{\partial u}\right)^2 - \frac{2}{\lambda_1^3}\left(\frac{\partial\lambda_1}{\partial v}\right)^2.$$

Then λ_1 verifies the following differential equation:

$$\Delta\left(\frac{1}{\lambda_1}\right) = -\frac{1}{\lambda_1}.$$

We have:

$$\mathcal{F}_{vv} = \begin{pmatrix} \displaystyle\int_0^u \frac{\partial^2}{\partial v^2}\left(\frac{1}{\lambda_1}\right)\cos(\widetilde{u})d\widetilde{u} + f_1''(v) \\ \displaystyle\int_0^u \frac{\partial^2}{\partial v^2}\left(\frac{1}{\lambda_1}\right)\sin(\widetilde{u})d\widetilde{u} + f_2''(v) \\ f_3''(v) \end{pmatrix}$$

$$= \begin{pmatrix} \displaystyle\int_0^u \left(-\frac{1}{\lambda_1} - \frac{\partial^2}{\partial \widetilde{u}^2}\left(\frac{1}{\lambda_1}\right)\right)\cos(\widetilde{u})d\widetilde{u} + f_1''(v) \\ \displaystyle\int_0^u \left(-\frac{1}{\lambda_1} - \frac{\partial^2}{\partial \widetilde{u}^2}\left(\frac{1}{\lambda_1}\right)\right)\sin(\widetilde{u})d\widetilde{u} + f_2''(v) \\ f_3''(v) \end{pmatrix}.$$

By making two integrations by parts we get:

$$-\int_0^u \frac{\partial^2}{\partial \widetilde{u}^2}\left(\frac{1}{\lambda_1}\right)\cos(\widetilde{u})d\widetilde{u}$$
$$= -\frac{\partial}{\partial u}\left(\frac{1}{\lambda_1}\right)\cos(u) + \left(\frac{\partial}{\partial u}\left(\frac{1}{\lambda_1}\right)\right)_{|u=0} - \frac{1}{\lambda_1}\sin(u) + \int_0^u \frac{1}{\lambda_1}\cos(\widetilde{u})d\widetilde{u}$$

and

$$-\int_0^u \frac{\partial^2}{\partial \widetilde{u}^2}\left(\frac{1}{\lambda_1}\right)\sin(\widetilde{u})d\widetilde{u}$$
$$= -\frac{\partial}{\partial u}\left(\frac{1}{\lambda_1}\right)\sin(u) + \frac{1}{\lambda_1}\cos(u) - \left(\frac{1}{\lambda_1}\right)_{|u=0} + \int_0^u \frac{1}{\lambda_1}\sin(\widetilde{u})d\widetilde{u}.$$

So

$$\mathcal{F}_{vv} = \begin{pmatrix} -\dfrac{\partial}{\partial u}\left(\dfrac{1}{\lambda_1}\right)\cos(u) - \dfrac{1}{\lambda_1}\sin(u) + \left(\dfrac{\partial}{\partial u}\left(\dfrac{1}{\lambda_1}\right)\right)_{|u=0} + f_1''(v) \\ -\dfrac{\partial}{\partial u}\left(\dfrac{1}{\lambda_1}\right)\sin(u) + \dfrac{1}{\lambda_1}\cos(u) - \left(\dfrac{1}{\lambda_1}\right)_{|u=0} + f_2''(v) \\ f_3''(v) \end{pmatrix}.$$

Using $\mathcal{F}_{uu} - 2\rho a_1 \mathcal{F}_u = \rho^2\xi$, we obtain $\rho^2\xi$. Then

$$-2\rho a_1\mathcal{F}_u+\rho^2\xi=\begin{pmatrix} -4\rho a_1\dfrac{1}{\lambda_1}\cos(u)-\dfrac{1}{\lambda_1}\sin(u)-\dfrac{\partial}{\partial u}(\lambda_1)\times\dfrac{1}{\lambda_1^2}\cos(u)\\ -4\rho a_1\dfrac{1}{\lambda_1}\sin(u)+\dfrac{1}{\lambda_1}\cos(u)-\dfrac{\partial}{\partial u}(\lambda_1)\times\dfrac{1}{\lambda_1^2}\sin(u)\\ 0\end{pmatrix}.$$

Since $\mathcal{F}_{vv}=-2\rho a_1\mathcal{F}_u+\rho^2\xi$, finally we find that $\big(\frac{\partial}{\partial u}\big(\frac{1}{\lambda_1}\big)\big)_{|u=0}+f_1''(v)=0$, $-\big(\frac{1}{\lambda_1}\big)_{|u=0}+f_2''(v)=0$ and $f_3''(v)=0$.

Therefore we can assume that $f_1''(v)=-\big(\frac{\partial}{\partial u}\big(\frac{1}{\lambda_1}\big)\big)_{|u=0}$, $f_2''(v)=\big(\frac{1}{\lambda_1}\big)_{|u=0}$ and $f_3(v)=v$.

This completes the proof of the first case of Theorem 1.

If $\rho^2\lambda_1=-1$, similar computations give the second case.

Remark 4.1 Conversely, given a strictly positive function f on an open domain of $\mathbb{R}^2$ satisfying $\Delta(f)=\mp f$ and putting $\lambda_1=\pm\frac{1}{f}$ in formulas of Theorem 1, we can construct an immersion $\mathcal{F}$. It is straightforward to check that this immersion admits several immersions with the same induced connection. An example of this is given in the final section.

5. Case $h_2\big(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\big)=0$

In this case $b_{12}=0$. We have $b_{11}b_{22}=1$. We also recall that the function $\lambda_1\rho^2$ depends only on u. In this case, we follow the approach of section 3 of Vrancken ([8]). More precisely we have:

Lemma 5.1 *For any Y in the second direction, we have*:

$$X_1\big(Y(\ln(|\lambda_1|))\big)-(\nabla_{X_1}Y)^*\big(\ln(|\lambda_1|)\big)=0$$

where Z^* is the component in the second direction; *i.e.* $Z=Z^*+h_1(Z,X_1)X_1$.

Proof. By using the Codazzi equation for h_2 ((4)), we obtain:

$$\begin{cases} X_1(b_{22})=2a_1(b_{11}-b_{22})=2a_1\left(\dfrac{1}{b_{22}}-b_{22}\right),\\ X_2(b_{11})=X_2(b_{22})=0.\end{cases}$$

These equations have solutions if and only if

$$\begin{aligned}
\mathcal{B}(b_{22}) &= [X_1, X_2](b_{22}) - (\nabla_{X_1}X_2 - \nabla_{X_2}X_1)(b_{22}) = 0.\\
\mathcal{B}(b_{22}) &= X_1(X_2(b_{22})) - X_2(X_1(b_{22})) - \nabla_{X_1}X_2(b_{22}) + \nabla_{X_2}X_1(b_{22})\\
&= 0 - X_2\left(2a_1\left(\frac{1}{b_{22}} - b_{22}\right)\right) - a_3X_1(b_{22}) + \frac{1}{2}a_3X_1(b_{22})\\
&= -2X_2(a_1)\left(\frac{1}{b_{22}} - b_{22}\right) - \frac{1}{2}a_3\left(2a_1\left(\frac{1}{b_{22}} - b_{22}\right)\right)\\
&= -\left(\frac{1}{b_{22}} - b_{22}\right)(2X_2(a_1) + a_1a_3).
\end{aligned}$$

Since $\left(\frac{1}{b_{22}} - b_{22}\right) \neq 0$, we find that:

$$2X_2(a_1) + a_1a_3 = 0. \tag{16}$$

Since Y belongs to the second direction, there exists some function f such that $Y = f\ X_2$. We have:

$$\begin{aligned}
X_1(Y(\ln(|\lambda_1|))) &= X_1(f \times (-a_3))\\
&= -X_1(f)a_3 - fX_1(a_3)
\end{aligned}$$

and $(\nabla_{X_1} fX_2) = X_1(f)X_2 + f(a_3X_1 - a_1X_2)$. So

$$\begin{aligned}
(\nabla_{X_1}Y)^*(\ln(|\lambda_1|)) &= X_1(f)X_2(\ln(|\lambda_1|)) - fa_1 \times (-a_3)\\
&= -X_1(f)a_3 + fa_1a_3.
\end{aligned}$$

Then using (16), we find:

$$\begin{aligned}
&X_1\big(Y(\ln(|\lambda_1|))\big) - (\nabla_{X_1}Y)^*(\ln(|\lambda_1|))\\
&\quad = -fX_1(a_3) - fa_1a_3\\
&\quad = f[-2X_2(a_1) - a_1a_3]\\
&\quad = f\big[X_1\big(X_2(\ln(|\lambda_1|))\big) - (\nabla_{X_1}X_2)^*(\ln(|\lambda_1|))\big]\\
&\quad = 0.
\end{aligned}$$

□

Lemma 5.2 *We have* $\frac{\partial}{\partial u}\frac{\partial}{\partial v}\ln(|\lambda_1|)=0$.

Proof. By using the previous lemma we get:

$$X_1\left(\frac{\partial}{\partial v}(\ln(|\lambda_1|))\right)-\left(\nabla_{X_1}\frac{\partial}{\partial v}\right)^*(\ln(|\lambda_1|))=0.$$

So $\rho X_1\big(\frac{\partial}{\partial v}(\ln(|\lambda_1|))\big)-\big(\rho\nabla_{X_1}\frac{\partial}{\partial v}\big)^*(\ln(|\lambda_1|))=0$ and then $\frac{\partial}{\partial u}\big(\frac{\partial}{\partial v}(\ln(|\lambda_1|))\big)-\big(\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v}\big)^*(\ln(|\lambda_1|))=0$.

$$\begin{aligned}\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v}&=\nabla_{\rho X_1}\rho X_2\\&=\rho X_1(\rho)X_2+\rho^2\nabla_{X_1}X_2\\&=\rho^2a_1X_2+\rho^2(a_3X_1-a_1X_2)\\&=\rho a_3\frac{\partial}{\partial u}.\end{aligned}$$

Therefore $\big(\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v}\big)^*=0$ and $\frac{\partial}{\partial u}\big(\frac{\partial}{\partial v}(\ln(|\lambda_1|))\big)=0$. □

This previous lemma implies that there exist a function λ (depending only on the variable u) and a function μ (depending only on the variable v) such that:

$$|\lambda_1|=\frac{\lambda(u)}{\mu(v)}.$$

As $|\lambda_1|$ is positive, we may assume that λ and μ are positive functions.

Lemma 5.3 *There exists a curve γ depending only on the variable u such that $\mathcal{F}_u=\mu(v)\gamma'(u)$.*

Proof. As $\frac{\partial(|\lambda_1|)}{\partial v}=-\rho a_3|\lambda_1|$, we find that:

$$\rho a_3=-\frac{\frac{\partial(|\lambda_1|)}{\partial v}}{|\lambda_1|}=-\mu\frac{\partial\left(\frac{1}{\mu}\right)}{\partial v}=\frac{\mu'}{\mu}.$$

Since $\frac{\partial\mathcal{F}_u}{\partial v}=\rho a_3\mathcal{F}_u=\frac{\mu'}{\mu}\mathcal{F}_u$, we have:

$$\mu \frac{\partial}{\partial v}(\mathcal{F}_u) - \mu' \mathcal{F}_u = 0,$$

so

$$\frac{\partial}{\partial v}\left(\frac{\mathcal{F}_u}{\mu}\right) = 0. \qquad \square$$

Lemma 5.4 *If $\mu'' \neq 0$, there exists a curve $\widetilde{\gamma}$ depending only on the variable u such that $\mathcal{F}_{vv} = \mu''(v)\widetilde{\gamma}(u)$.*

Proof. From the previous lemma, we deduce that $\mathcal{F}_{vv} = \mu''\gamma(u) + c(v)$ and $\rho a_3 = \frac{\mu'}{\mu}$.

We know that $\frac{\partial}{\partial v}(\rho) = \frac{1}{2}\rho^2 a_3$, so $\frac{\mu'}{\mu} = \frac{2\frac{\partial}{\partial v}(\rho)}{\rho}$.

Therefore there exists a function g depending only on the variable u such that $\rho = \sqrt{\mu} g$.

Then

$$\begin{aligned}
\mathcal{F}_{vv} &= -2\rho a_1 \mathcal{F}_u + \rho^2 \xi \\
&= -2\rho a_1 \mu \gamma' + \mu g^2 \xi \\
&= -2\sqrt{\mu} g a_1 \mu \gamma' + \mu g^2 \xi \\
&= \mu\big(-2g(\sqrt{\mu} a_1)\gamma' + g^2 \xi\big).
\end{aligned}$$

Using (16) we find that:

$$\begin{aligned}
\frac{\partial}{\partial v}(\rho a_1) &= \frac{1}{2}\rho^2 a_1 a_3 + \rho^2 X_2(a_1) \\
&= 0.
\end{aligned}$$

So $\frac{\partial}{\partial v}(\sqrt{\mu}\, a_1) = 0$.

We deduce that the function $(-2g(\sqrt{\mu} a_1)\gamma' + g^2\xi)$ depends only on the variable u, so

$$\frac{\partial}{\partial v}\left(\frac{1}{\mu}\mathcal{F}_{vv}\right) = 0,$$

i.e.

$$\frac{\partial}{\partial v}\left(\frac{\mu''}{\mu}\right)\gamma(u) + \frac{\partial}{\partial v}\left(\frac{c(v)}{\mu}\right) = 0.$$

Since $\mathcal{F}_u = \mu\gamma' \neq 0$, we have $\gamma'(u) \neq 0$. Then $\frac{\partial}{\partial v}\left(\frac{\mu''}{\mu}\right) = 0$ and $\frac{\partial}{\partial v}\left(\frac{c(v)}{\mu}\right) = 0$.

Therefore there are non zero constant d and constant vector e such that $\mu'' = d\mu$ and $c = e\mu$.

So

$$\begin{aligned}\mathcal{F}_{vv} &= d\mu\gamma(u) + e\mu \\ &= d\mu\left(\gamma(u) + \frac{e}{d}\right).\end{aligned}$$

Taking $\widetilde{\gamma}(u) = \gamma(u) + \frac{e}{d}$ completes the proof. □

5.1. Case $\mu'' = 0$

We have $\mathcal{F}_{vv} = e\mu$ and $\mu(v) = d_1 v + d_2$ where d_1, d_2 are constants. So $\mathcal{F}_u = (d_1 v + d_2)\, \gamma'(u)$ where γ is a non degenerate equiaffine curve.

If $d_1 = 0$, then $\mathcal{F}(u, v) = \gamma(u) + k(v)$ such that $k''(v) = e$.

Therefore there exists constant vector k_1 such that $k(v) = \frac{e}{2}v^2 + k_1 v$.

We find that $\mathcal{F}(u, v) = \gamma(u) + \frac{e}{2}v^2 + k_1 v$.

If $d_1 \neq 0$, then $\mathcal{F}(u, v) = (d_1 v + d_2)\, \gamma(u) + k(v)$ such that

$$k''(v) = e(d_1 v + d_2).$$

Therefore there exists constant vector k_2 such that

$$k(v) = e\frac{1}{6d_1^2}(d_1 v + d_2)^3 + k_2 v.$$

We get $\mathcal{F}(u, v) = (d_1 v + d_2)\gamma(u) + \frac{e}{6d_1^2}(d_1 v + d_2)^3 + k_2 v$.

In isothermal coordinates, we obtain that $\gamma''(u) = e + 4\rho a_1 \gamma'(u)$, by using the equality $2\rho a_1\, \mathcal{F}_u = \mathcal{F}_{uu} - \frac{1}{2}(\mathcal{F}_{uu} + \mathcal{F}_{vv})$.

5.2. Case $\mu'' \neq 0$

We have $\mathcal{F}(u, v) = \mu(v)\widetilde{\gamma}(u) + k(v)$ such that the function k verifies $k''(v) = 0$. The curve $\widetilde{\gamma}$ verifies $\widetilde{\gamma}'(u) = \gamma'(u)$.

Since $\mu'' = d\mu$, there are two cases.

If $d > 0$, we have $\mu(v) = d_1 \exp(\sqrt{d}\ v) + d_2 \exp(-\sqrt{d}\ v)$.

And if $d < 0$, we have $\mu(v) = d_1 \sin(\sqrt{-d}\, v) + d_2 \cos(\sqrt{-d}\, v)$ where d_1 and d_2 are constants.

Since $\widetilde{\gamma}$ is a non degenerate equiaffine curve in $\mathbb{R}^2$, there exist functions γ_1 and γ_2 which verify $\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'' \neq 0$, constants d_1, d_2, d and e such that $\mathcal{F}$ is given by one of the following expressions:

$$\begin{aligned}\mathcal{F}(u,v) = \big(&\big(d_1 \exp(\sqrt{d}\, v) + d_2 \exp(-\sqrt{d}\, v)\big)\gamma_1(u),\\ &\big(d_1 \exp(\sqrt{d}\, v) + d_2 \exp(-\sqrt{d}\, v)\big)\gamma_2(u), ev\big),\\ \mathcal{F}(u,v) = \big(&\big(d_1 \cos(\sqrt{-d}\, v) + d_2 \sin(\sqrt{-d}\, v)\big)\gamma_1(u),\\ &\big(d_1 \cos(\sqrt{-d}\, v) + d_2 \sin(\sqrt{-d}\, v)\big)\gamma_2(u), ev\big).\end{aligned}$$

In each case, we calculate $\mathcal{F}_{uu} + \mathcal{F}_{vv}$.

We know that $2\rho a_1\ \mathcal{F}_u = \mathcal{F}_{uu} - \frac{1}{2}(\mathcal{F}_{uu} + \mathcal{F}_{vv})$. Then we obtain that $\gamma_1''(u) = 4\rho a_1 \gamma_1'(u) + d\gamma_1(u)$ and $\gamma_2''(u) = 4\rho a_1 \gamma_2'(u) + d\gamma_2(u)$.

So $\gamma_1'(u)\gamma_2''(u) - \gamma_2'(u)\gamma_1''(u) = d(\gamma_1'(u)\gamma_2(u) - \gamma_2'(u)\gamma_1(u))$.

Therefore $\gamma_1'(u)\gamma_2(u) - \gamma_2'(u)\gamma_1(u) \neq 0$, i.e. $\widetilde{\gamma} = (\gamma_1, \gamma_2, 0)$ is a non degenerate centroaffine curve.

In isothermal coordinates, we notice that we have:

$$\big(\gamma_1'', \gamma_2'', 0\big) = d(\gamma_1, \gamma_2, 0) + 4\rho a_1 \big(\gamma_1', \gamma_2', 0\big).$$

This completes the proof of Theorem 2.

6. Examples

In this section, we will construct some explicit examples using Theorem 1.

6.1. Case $h_2\big(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\big) \neq 0$ and $\rho^2\lambda_1 = 1$

We define $\mathcal{F}$ and $\mathcal{G}$ on $\big\{(u,v) \in \mathbb{R}^2 / -\frac{\pi}{2} < u < \frac{\pi}{2}\big\}$ by

$$\mathcal{F} = \begin{pmatrix} \frac{1}{4}\sin(2u) + \frac{1}{2}u \\ -\frac{1}{4}\cos(2u) + \frac{1}{2}u + \frac{1}{2}v^2 \\ v \end{pmatrix},$$

$$\mathcal{G} = \big(\exp(f(u))\cos(g(u) - b_{12}v), -\exp(f(u))\sin(g(u) - b_{12}v), v\big),$$

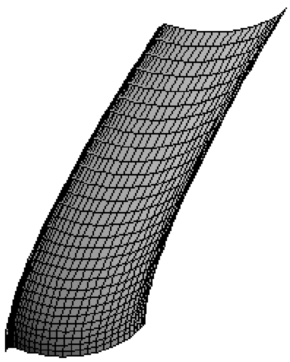
where b_{12} is a non-zero constant, and f, g are functions satisfying:

$$g'(u) = \frac{-\sqrt{2 + \tan^2(u) + b_{12}^2} \times b_{12}^2}{2 + 2\tan^2(u) + b_{12}^2}$$

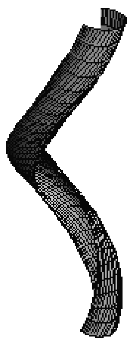
and

$$\exp(f(u)) = \sqrt{4 + b_{12}^2 + b_{12}^2\cos(2u)} - \sqrt{4 + b_{12}^2 + b_{12}^2} + 1.$$

Here is a picture of $\mathcal{F}$:



Here is a picture of $\mathcal{G}$, with $b_{12} = 1$:



These immersions have the same induced connection ∇ with $\dim \operatorname{Im} R = 1$ and they are not affine equivalent. In fact, the induced connection ∇ is given by:

$$\nabla_{X_u} X_u = -\tan(u) X_u,$$
$$\nabla_{X_u} X_v = 0,$$
$$\nabla_{X_v} X_u = 0,$$
$$\nabla_{X_v} X_v = \tan(u) X_u.$$

The affine metric h_1 of $\mathcal{F}$ is given by $h_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The affine h_2 of $\mathcal{G}$ is given by $h_2 = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{smallmatrix}\right)$, with $b_{11} = \frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}}$ and $b_{22} = \sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}$.

To construct $\mathcal{F}$, we set $\lambda_1 = \frac{1}{\cos(u)}$ in Theorem 1. And we construct $\mathcal{G}$ as follows.

We take b_{ij} depending only on the variable u. So we find that b_{12} is a constant, $b_{11}b_{22} = 1 + b_{12}^2$ and $-\frac{\partial b_{22}}{\partial u} - \tan(u)(b_{11} - b_{22}) = 0$. Then after integration, we get $|1 + b_{12}^2 - b_{22}^2| = \frac{1}{\cos^2(u)}$.

We take b_{22} such that $b_{22}^2 = 1 + b_{12}^2 + \frac{1}{\cos^2(u)}$. We find $b_{22} = \sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}$ and $b_{11} = \frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}}$.

We have:

$$\begin{aligned} \mathcal{G}_{uvv} &= \rho^2 b_{12} \widetilde{\xi}_v \\ &= \rho^2 b_{12}(-\lambda_1 b_{12} \mathcal{G}_u) \\ &= -b_{12}^2 \mathcal{G}_u. \end{aligned}$$

So there exist differentiable vectors D_1, D_2 and D_3 such that $\mathcal{G} = D_1(u)\cos(b_{12}v) + D_2(u)\sin(b_{12}v) + D_3(v)$.

From $\mathcal{G}_{uv} = \rho^2 b_{12} \widetilde{\xi}$, we deduce that:

$$\widetilde{\xi} = \frac{1}{\cos(u)}\big(- D_1'(u)\sin(b_{12}v) + D_2'(u)\cos(b_{12}v)\big).$$

From $\mathcal{G}_{uu} = -\tan(u)\mathcal{G}_u + \rho^2 b_{11}\widetilde{\xi}$, we obtain that:

$$D_1''(u) = -\tan(u)D_1'(u) + D_2'(u)\frac{1 + b_{12}^2}{\sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}},$$

$$D_2''(u) = -\tan(u)D_2'(u) - D_1'(u)\frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}}.$$

From $\mathcal{G}_{vv} = \tan(u)\mathcal{G}_u + \rho^2 b_{22}\widetilde{\xi}$, we find that:

$$\begin{aligned}
D_3''(v) &= b_{12}^2(D_1(u)\cos(b_{12}v) + D_2(u)\sin(b_{12}v)) \\
&\quad + \tan(u)\big(D_1'(u)\cos(b_{12}v) + D_2'(u)\sin(b_{12}v)\big) \\
&\quad + \sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}\,\big(-D_1'(u)\sin(b_{12}v) + D_2'(u)\cos(b_{12}v)\big).
\end{aligned}$$

Then

$$\begin{aligned}
D_3(v) &= -(D_1(u)\cos(b_{12}v) + D_2(u)\sin(b_{12}v)) \\
&\quad - \frac{\tan(u)}{b_{12}^2}\big(D_1'(u)\cos(b_{12}v) + D_2'(u)\sin(b_{12}v)\big) \\
&\quad - \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}\,\big(-D_1'(u)\sin(b_{12}v) + D_2'(u)\cos(b_{12}v)\big) \\
&\quad + E \times v,
\end{aligned}$$

where E is a constant vector.

Separating the variables u and v, we get that there exist constant vectors E_1 and E_2 such that:

$$D_1(u) + \frac{\tan(u)}{b_{12}^2}D_1'(u) + \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}\,D_2'(u) = E_1$$

$$\text{and}\quad D_2(u) + \frac{\tan(u)}{b_{12}^2}D_2'(u) - \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}\,D_1'(u) = E_2.$$

So we have:

$$\begin{aligned}
&(D_1 + iD_2 - (E_1 + iE_2)) \\
&\quad = \left(-\frac{\tan(u)}{b_{12}^2} + \frac{i}{b_{12}^2}\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}\right)(D_1 + iD_2 - (E_1 + iE_2))'.
\end{aligned}$$

Then there exist constants vectors A and B such that:

$$\begin{aligned}
&(D_1 + iD_2)\\
&\quad = (A + iB)\exp\left(\int_0^u \frac{b_{12}^2}{-\tan(u) + i\sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}}\,du\right) + (E_1 + iE_2).
\end{aligned}$$

We write $\int_0^u \frac{b_{12}^2}{-\tan(u)+i\sqrt{1+b_{12}^2+\frac{1}{\cos^2(u)}}}\,du = f(u) + ig(u)$, where f and g are functions depending on the variable u.

We have:

$$\begin{aligned}
f'(u) + ig'(u) &= \frac{b_{12}^2}{-\tan(u) + i\sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}}\\
&= \frac{b_{12}^2\Big(-\tan(u) - i\sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}\Big)}{\tan^2(u) + 1 + b_{12}^2 + \frac{1}{\cos^2(u)}}\\
&= \frac{b_{12}^2\Big(-\tan(u) - i\sqrt{1 + b_{12}^2 + \frac{1}{\cos^2(u)}}\Big)}{2 + 2\tan^2(u) + b_{12}^2}.
\end{aligned}$$

So $f'(u) = \frac{-\tan(u)\times b_{12}^2}{2+2\tan^2(u)+b_{12}^2}$ and $g'(u) = \frac{-\sqrt{2+\tan^2(u)+b_{12}^2}\times b_{12}^2}{2+2\tan^2(u)+b_{12}^2}$.

We obtain that:

$$\begin{cases}
D_1(u) = A\exp(f(u)) \times \cos(g(u)) - B\exp(f(u)) \times \sin(g(u)) + E_1,\\
D_2(u) = A\exp(f(u)) \times \sin(g(u)) + B\exp(f(u)) \times \cos(g(u)) + E_2,\\
D_3(v) = -E_1\cos(b_{12}) - E_2\sin(b_{12}) + E \times v.
\end{cases}$$

Finally we get:

$$\begin{aligned}
\mathcal{G} &= \exp(f(u))\big(\cos(g(u))\cos(b_{12}v) + \sin(g(u))\sin(b_{12}v)\big)A\\
&\quad + \exp(f(u))\big(\cos(g(u))\sin(b_{12}v) - \sin(g(u))\cos(b_{12}v)\big)B + E \times v.
\end{aligned}$$

Since $\mathcal{G}$ is non degenerate, (A, B, E) are linearly independent. So by an affine transformation, we can assume that:

$$\mathcal{G} = \big(\exp(f(u))\big(\cos(g(u))\cos(b_{12}v) + \sin(g(u))\sin(b_{12}v)\big),$$
$$\exp(f(u))\big(\cos(g(u))\sin(b_{12}v) - \sin(g(u))\cos(b_{12}v)\big), v\big).$$

So $\mathcal{G} = (\exp(f(u))\cos(g(u) - b_{12}v), -\exp(f(u))\sin(g(u) - b_{12}v), v)$ and

$$\widetilde{\xi} = \frac{\exp(f(u))}{\cos(u)}\big(\cos(g(u) - b_{12}v) + f'(u)\sin(g(u) - b_{12}v),$$
$$-\sin(g(u) - b_{12}v) + f'(u)\cos(g(u) - b_{12}v), 0\big),$$

with $f'(u) = \frac{-\tan(u)\times b_{12}^2}{2+2\tan^2(u)+b_{12}^2}$, $g'(u) = \frac{-\sqrt{2+\tan^2(u)+b_{12}^2}\times b_{12}^2}{2+2\tan^2(u)+b_{12}^2}$ and $\exp(f(u)) = \sqrt{4 + b_{12}^2 + b_{12}^2\cos(2u)} - c$, where c is a constant.

Since $f(0) = 0$, we find that $c = \sqrt{4 + 2b_{12}^2} - 1$.

Therefore $\exp(f(u)) = \sqrt{4 + b_{12}^2 + b_{12}^2\cos(2u)} - \sqrt{4 + 2b_{12}^2} + 1$.

6.2. Case $h_2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) \neq 0$ and $\rho^2\lambda_1 = -1$

Like before, we define $\mathcal{F}$ and $\mathcal{G}$ on R^2 by

$$\mathcal{F} = \begin{pmatrix} \frac{1}{2}u - \frac{1}{4}\exp(-2u) + \frac{v^2}{2} \\ \frac{1}{2}u + \frac{1}{4}\exp(-2u) + \frac{v^2}{2} \\ v \end{pmatrix},$$

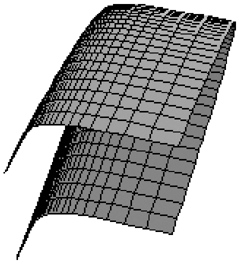
$$\mathcal{G} = \left(\frac{1}{2}\exp(f(u) + b_{12}v), \frac{1}{2}\exp(g(u) - b_{12}v), v\right),$$

where b_{12} is a non-zero constant, and f, g are functions satisfying:

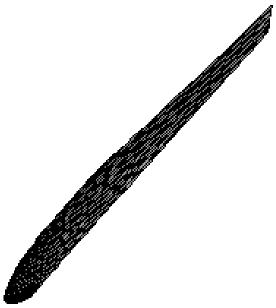
$$f'(u) = \frac{b_{12}^2}{1 + \sqrt{1 + b_{12}^2 + \exp(2u)}}$$

$$\text{and} \quad g'(u) = \frac{b_{12}^2}{1 - \sqrt{1 + b_{12}^2 + \exp(2u)}}.$$

Here is a picture of $\mathcal{F}$:



Here is a picture of $\mathcal{G}$, with $b_{12} = 1$:



These immersions have the same induced connection ∇ with $\dim \operatorname{Im} R = 1$ and are not affine equivalent. The induced connection ∇ is given by:

$$\begin{aligned}
\nabla_{X_u} X_u &= -X_u,\\
\nabla_{X_u} X_v &= 0,\\
\nabla_{X_v} X_u &= 0,\\
\nabla_{X_v} X_v &= X_u.
\end{aligned}$$

The affine metric h_1 of $\mathcal{F}$ is given by $h_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The affine h_2 of $\mathcal{G}$ is given by $h_2 = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{smallmatrix}\right)$, with $b_{11} = \frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\exp(2u)}}$ and $b_{22} = \sqrt{1 + b_{12}^2 + \exp(2u)}$.

To construct $\mathcal{F}$, we set $\lambda_1 = -\exp(u)$ in Theorem 1 and we construct $\mathcal{G}$ as follows.

We take b_{ij} depending only on the variable u. We find that b_{12} is a constant and we choose b_{22} such that:

$$b_{22} = \sqrt{1 + b_{12}^2 + \exp(2u)} \quad \text{and} \quad b_{11} = \frac{1 + b_{12}^2}{\sqrt{1 + b_{12}^2 + \exp(2u)}}.$$

We make similar computations than before and we obtain that there exist differentiable vectors D_1, D_2 and D_3 such that:

$$\mathcal{G} = D_1(u)\cosh(b_{12}v) + D_2(u)\sinh(b_{12}v) + D_3(v),$$

with

$$D_1''(u) = -D_1'(u) + D_2'(u)\frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\exp(2u)}},$$

$$D_2''(u) = -D_2'(u) + D_1'(u)\frac{1+b_{12}^2}{\sqrt{1+b_{12}^2+\exp(2u)}}$$

and $$\begin{aligned} D_3(v) = &-(D_1(u)\cosh(b_{12}v) + D_2(u)\sinh(b_{12}v)) \\ &+\frac{1}{b_{12}^2}\big(D_1'(u)\cosh(b_{12}v) + D_2'(u)\sinh(b_{12}v)\big) \\ &+\frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\exp(2u)}\ \big(D_1'(u)\sinh(b_{12}v) + D_2'(u)\cosh(b_{12}v)\big) \\ &+E\times v, \end{aligned}$$

where E is a constant vector.

Separating the variables u and v, we get that there exist constant vectors E_1 and E_2 such that:

$$-D_1(u) + \frac{1}{b_{12}^2}D_1'(u) + \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\exp(2u)}\ D_2'(u) = E_1$$

$$\text{and} \quad -D_2(u) + \frac{1}{b_{12}^2}D_2'(u) + \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\exp(2u)}\ D_1'(u) = E_2.$$

Then

$$\begin{aligned} &(D_1(u) + D_2(u) + E_1 + E_2) \\ &\quad= \left(\frac{1}{b_{12}^2} + \frac{1}{b_{12}^2}\sqrt{1+b_{12}^2+\exp(2u)}\right)(D_1(u) + D_2(u) + E_1 + E_2)' \end{aligned}$$

and

$$
\begin{aligned}
&(D_1(u) + E_1 - D_2(u) - E_2) \\
&\quad = \left(\frac{1}{b_{12}^2} - \frac{1}{b_{12}^2}\sqrt{1 + b_{12}^2 + \exp(2u)}\right)(D_1(u) + E_1 - D_2(u) - E_2)'.
\end{aligned}
$$

We obtain that there exist constant vectors V_1 and V_2 such that:

$$
\begin{cases}
(D_1(u) + D_2(u) + E_1 + E_2) = V_1 \exp\left(\displaystyle\int_0^u \frac{b_{12}^2}{1 + \sqrt{1 + b_{12}^2 + \exp(2u)}}\, du\right), \\[2ex]
(D_1(u) - D_2(u) + E_1 - E_2) = V_2 \exp\left(\displaystyle\int_0^u \frac{b_{12}^2}{1 - \sqrt{1 + b_{12}^2 + \exp(2u)}}\, du\right).
\end{cases}
$$

If we write:

$$
\int_0^u \frac{b_{12}^2}{1 + \sqrt{1 + b_{12}^2 + \exp(2u)}}\, du = f(u)
$$

$$
\text{and} \quad \int_0^u \frac{b_{12}^2}{1 - \sqrt{1 + b_{12}^2 + \exp(2u)}}\, du = g(u),
$$

we get:

$$
\begin{cases}
2D_1(u) + 2E_1 = \big(V_1 \exp(f(u)) + V_2 \exp(g(u))\big), \\
2D_2(u) + 2E_2 = \big(V_1 \exp(f(u)) - V_2 \exp(g(u))\big).
\end{cases}
$$

Then

$$
\begin{cases}
D_1(u) = \dfrac{1}{2}\big(V_1 \exp(f(u)) + V_2 \exp(g(u))\big) - E_1, \\[2ex]
D_2(u) = \dfrac{1}{2}\big(V_1 \exp(f(u)) - V_2 \exp(g(u))\big) - E_2, \\[2ex]
D_3(v) = E_1 \cosh(b_{12}) + E_2 \sinh(b_{12}) + E \times v.
\end{cases}
$$

Moreover $\widetilde{\xi} = \exp(u)(D_1'(u)\sinh(b_{12}v) + D_2'(u)\cosh(b_{12}v))$.
Finally we get:

$$\mathcal{G} = \frac{1}{2}\big(V_1 \exp(f(u)) + V_2 \exp(g(u))\big) \cosh(b_{12}v) + \frac{1}{2}\big(V_1 \exp(f(u)) - V_2 \exp(g(u))\big) \sinh(b_{12}v) + E \times v.$$

Since $\mathcal{G}$ is non degenerate, (V_1, V_2, E) are linearly independent. So by an affine transformation, we can assume that:

$$\mathcal{G} = \bigg(\frac{1}{2}\exp(f(u))(\cosh(b_{12}v) + \sinh(b_{12}v)), \frac{1}{2}\exp(g(u))(\cosh(b_{12}v) - \sinh(b_{12}v)), v\bigg).$$

So $\mathcal{G} = \big(\frac{1}{2}\exp(f(u) + b_{12}v), \frac{1}{2}\exp(g(u) - b_{12}v), v\big)$ and $\widetilde{\xi} = \exp(u)\big(\frac{1}{2}f'(u)\exp(f(u) + b_{12}v), -\frac{1}{2}g'(u)\exp(g(u) - b_{12}v), 0\big)$, with $f'(u) = \frac{b_{12}^2}{1+\sqrt{1+b_{12}^2+\exp(2u)}}$ and $g'(u) = \frac{b_{12}^2}{1-\sqrt{1+b_{12}^2+\exp(2u)}}$.

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