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AN EARLIER FRACTAL GRAPH

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ is *additive* if $f(x + y) = f(x) + f(y)$ for all real numbers x and y. We give examples of an additive function whose graph is fractal.

1 Introduction

A fractal subset X of Euclidean space $\mathbb{R}^n$ can be *self-similar* in the sense that X be of a shape similar to arbitrarily tiny portions of itself. Sometimes the fractal similarity dimension k of an object can be obtained by scaling it up with a zoom-up factor of n, to create a similar larger object comprised of m congruent copies of the original; in this case, k, n, and m satisfy the equation $n^k = m$. By definition, X is *fractal* if its topological dimension, $\dim_T X$, is less than its Hausdorff dimension, $\dim_H X$.

The continuous nowhere differentiable real function,

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$ and b is an odd integer with $ab > 1 + \frac{3\pi}{2}$, presented by Weierstrass in 1872, turns out to have a fractal graph; see [1, 4]. According to [2], half a century earlier and prior to 1821, Cauchy found that a real additive function is either continuous or totally discontinuous. Later, Hamel developed a method for constructing such a discontinuous additive function. A *Hamel basis* for $\mathbb{R}$ is a subset $\{x_\alpha : \alpha \in A\}$ of real numbers such that each real number x can be expressed uniquely as $x = \sum_{\alpha \in A} r_\alpha x_\alpha$, where all but finitely many

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of the rational numbers r_α are zero. Hamel defined an additive function f by first defining it any way whatsoever on this Hamel basis and then extending it to any $x = \sum_{\alpha \in A} r_\alpha x_\alpha \in \mathbb{R}$ by defining $f(x) = \sum_{\alpha \in A} r_\alpha f(x_\alpha)$. In 1942, F. B. Jones used a Hamel basis for $\mathbb{R}$ to construct both connected graphs and disconnected graphs of discontinuous additive functions on $\mathbb{R}$; see [2]. We show that for some additive function, its connected graph is fractal.

2 Preliminary results

The topological dimension of a continuous real function is 1 by [4, Exercise 1.8]. In fact, $\dim_T f \leq 1$ for the graph of an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. This follows because for each $(x, f(x)) \in f$, there are arbitrarily small open squares S centered at $(x, f(x))$ whose horizontal boundary edges contain no open interval subset lying in the graph of f and whose vertical boundary edges each contain at most one point of the graph of f. Otherwise, some $(x, f(x))$ would be the center of an open square S such that each concentric open square inside S would have a horizontal boundary edge containing an open interval subset lying in the graph of f. Then the projection of all those open intervals into the x-axis would give uncountably many disjoint open intervals in the separable space $\mathbb{R}$, which is impossible. Since $\dim_T(f \cap bd(S)) \leq 0$, $\dim_T f \leq 1$.

If three vertices of a parallelogram belong to the graph of an additive function $f : \mathbb{R} \to \mathbb{R}$, then it follows that its fourth vertex also belongs. Moreover, $f(rx + sy) = rf(x) + sf(y)$ for all rational numbers r and s; see [2]. In Theorem 3 of [2], Jones gives an example of a discontinuous additive real function f with totally disconnected graph for which $\dim_T f = 0$ and $\dim_H f = 1$. Hence its graph is fractal. In Theorem 4 of [2], he constructs an example of a discontinuous additive real function f which has a connected graph because its graph intersects each perfect set in $\mathbb{R}^2$ not lying in the union of countably many vertical lines and therefore intersects each continuum in $\mathbb{R}^2$ not lying wholly in a vertical line. His same construction remains valid if we simply replace "perfect set" with "closed set," and upon doing so, we can now show that $\dim_H f = 2$.

3 Main result

Theorem 1. *Jones's additive function $f : \mathbb{R} \to \mathbb{R}$ whose graph intersects each closed set in $\mathbb{R}^2$ not lying in the union of countably many vertical lines has a fractal graph.*

Proof. For completeness, we first provide details for Jones' argument found in [2]. Since the collection of all closed subsets of $\mathbb{R}^2$ whose x-projection is uncountable has cardinality c, this collection has a well ordering $\Gamma = \{F_\gamma : \gamma < c\}$ where each element F_γ is preceded by less than c-many elements of Γ. Let (x_1, y_1) denote a point of F_1 such that $x_1 \neq 0$. Define $f(x_1) = y_1$ and $f(r_1 x_1) = r_1 f(x_1)$ for all rational numbers r_1. Assume that γ is a fixed ordinal less than c and that $f(x_\alpha)$ is defined for all $\alpha < \gamma$ such that $x_\alpha \neq 0$, $(x_\alpha, f(x_\alpha)) \in F_\alpha$, and if $x = \sum_{\beta \leq \alpha} r_\beta x_\beta$ where all but finitely many of the rational numbers r_β are 0, then $f(x) = \sum_{\beta \leq \alpha} r_\beta f(x_\beta)$. Each element of Γ must contain points of c distinct vertical lines because the element's x-projection is an uncountable F_σ-set which contains a Cantor set, and $f(x)$ is so far defined for less than c values of x. Therefore, let (x_γ, y_γ) denote a point of F_γ such that $x_\gamma \neq 0$ and $x_\gamma \notin LIN(\{x_\alpha : \alpha < \gamma\})$, with $LIN(C)$ denoting the linear subspace of $\mathbb{R}$ over the rationals set $\mathbb{Q}$ generated by a subset C of $\mathbb{R}$. Define $f(x_\gamma) = y_\gamma$, and if $x = \sum_{\alpha \leq \gamma} r_\alpha x_\alpha$ where not more than a finite number of the rational numbers r_α are different from 0, define $f(x) = \sum_{\alpha \leq \gamma} r_\alpha f(x_\alpha)$. By transfinite induction, the resulting function f is single-valued on the set $\{x_\alpha : \alpha < c\}$, which is linearly independent over $\mathbb{Q}$, and therefore single-valued on $LIN(\{x_\alpha : \alpha < c\})$, and the graph of f intersects every closed set in $\mathbb{R}^2$ not contained in the union of a countable collection of vertical lines. By Theorem 4.2.1 of [3], this linearly independent set $\{x_\alpha : \alpha < c\}$ is a subset of a Hamel basis H_o of $\mathbb{R}$. Define f arbitrarily on $H_o \setminus \{x_\alpha : \alpha < c\}$. By Theorem 5.2.2 of [3], the restriction $f|H_o$ has a unique additive extension $f : \mathbb{R} \to \mathbb{R}$. Finally, according to Theorem 2 of [2], the graph of f is connected because it contains a point of each continuum in $\mathbb{R}^2$ not contained in a vertical line. Therefore $\dim_T f = 1$.

Now, suppose P is the closed parallelogram region with vertices $v_1(0,0)$, $v_2(x, f(x))$, $v_3(y, f(y))$ lying on the graph of f . By additivity, its fourth vertex $v_4(x+y, f(x)+f(y))$ also belongs to the graph of f . Subdivide P into four congruent closed parallelogram regions P_1 , P_2, P_3, and P_4 similar to P such that each P_i has v_i as a vertex along with the centroid of P as a vertex. For i=1,2,3,4, let $\phi_i = \frac{1}{2}I + \frac{1}{2}v_i$, which is a translation of the radial $\frac{1}{2}$-contraction, $\frac{1}{2}I$, by the amount $\frac{1}{2}v_i$, where I is the identity transformation. Each ϕ_i maps $P \cap f$ onto $P_i \cap f$, and so $P \cap f$ is a fixed set of the transformation $\Phi = \phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4$, i.e., $\Phi(P \cap f) = P \cap f$. From $\frac{m}{n^k} = 1$ with $m = 4$ and $n = 2$, we get that the similarity dimension is $k = 2$.

To show that $\dim_H(P \cap f) = 2$, suppose $\delta > 0$ and $P \cap f \subset \bigcup_{i=1}^\infty U_i$, where each U_i is a relative open subset of P and has diameter $|U_i| < \delta$. Since $B = P \setminus \bigcup_{i=1}^\infty U_i$ is a closed set in $\mathbb{R}^2$ and $B \cap f = \emptyset$, B is contained in the union of countably many vertical lines. Therefore, for each positive number $\varepsilon < \delta$, there exist relative open subsets $V_1, V_2, V_3, \ldots$ of P such that

1. $B \subset \bigcup_{i=1}^\infty V_i$,

2. $|V_i| < \delta$, and
3. $\sum_{i=1}^{\infty} |V_i|^2 < \varepsilon$.

Since $P = (\bigcup_{i=1}^{\infty} U_i) \cup B = (\bigcup_{i=1}^{\infty} U_i) \cup (\bigcup_{i=1}^{\infty} V_i)$, its 2-dimensional Hausdorff outer measure obeys

$$\begin{aligned} H^2(P) &\leq \lim_{\delta\to 0} \inf_{U_i, V_i} \sum_{i=1}^{\infty} (|U_i|^2 + |V_i|^2) \\ &\leq \lim_{\delta\to 0} \inf_{U_i} (\sum_{i=1}^{\infty} |U_i|^2 + \varepsilon) \\ &= \lim_{\delta\to 0} \inf_{U_i} \sum_{i=1}^{\infty} |U_i|^2 = H^2(P \cap f). \end{aligned}$$

So $H^2(P) = H^2(P \cap f)$ and $\dim_H f = \dim_H (P \cap f) = 2$. Therefore the graph of f is fractal. □

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