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CHOQUET INTEGRAL IN CAPACITY

Abstract

In this paper we introduce and study the new concept of Choquet integral in capacity, which generalizes the Riemann integral in probability and the classical Choquet integral. Properties of this new integral are proved and some applications are presented.

1 Introduction

Let (E, B, P) be a field of probability, where E is a nonempty set, B is a field of subsets of E and P is a composite probability on B. Let us denote by $L(E, B, P)$, the space of all real random variables (also called the space of all real stochastic processes) defined on E and a.e. P-finite.

The stochastic modelling of various processes, in, e.g., nature, economy and finance, naturally imposes the study of functions with values in the space $L(E, B, P)$.

It is well-known the fact that the classical Choquet integral defined for real-valued functions, has many applications in statistics, economic decisions, finance, cooperative game theory, image processing and computer vision, pattern recognition (see, e.g., Chapter 15 in [9] and Preface in [3]) or in potential theory (see, e.g., [1]). In this context, the extension of the classical Choquet integral to functions with values of real random variables, can present an interest for possible stochastic approaches of these applications (for example, a stochastic potential theory).

Let us recall the following concept of integral which generalizes the concept of classical Riemann integral, introduced and studied in [8], p. 50.

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Definition 1. *Let $a, b \in \mathbb{R}, a < b$. The random function $f : [a, b] \to L(E, B, P)$ is called Riemann integrable in probability on $[a, b]$, if there exists a random variable $I = I(\omega) \in L(E, B, P)$ with the following property: for all $\varepsilon > 0, \eta > 0$, there exists $\delta = \delta(\varepsilon, \eta) > 0$, such that for all divisions $d : a = x_0 < x_1 < ... < x_n = b$ with the norm $\nu(d) < \delta$ and all $\xi_i \in [x_i, x_{i+1}]$, $i \in \{0, ..., n-1\}$, we have*

$$P(\{\omega \in E; |S(f; d, \xi_i)(\omega) - I(\omega)| \geq \varepsilon\}) < \eta,$$

where $\nu(d) = \max\{x_{i+1} - x_i; i = 0, 1, ..., n-1\}$ and

$$S(f; d, \xi_i)(\omega) = \sum_{i=0}^{n-1} f(\xi_i, \omega)(x_{i+1} - x_i).$$

In this case, $I(\omega)$ is called the Riemann integral in probability of f on $[a, b]$ and it is denoted by $I(\omega) = (P)\int_a^b f(t, \omega)dt$.

Remark 1. *By using the method in, e.g., [5], [7], in [4] we have introduced and studied the concept of Kurzweil-Henstock integral in probability, which generalizes both the classical Kurzweil-Henstock integral and the above Riemann integral in probability.*

The main purpose of the present paper is to generalize the concept in Definition 1 to the case of Choquet integral in capacity. Some properties are proved. The new integral generalizes simultaneously the integral in probability and the classical Choquet integral.

Section 2 contains preliminaries on capacities and on the classical Choquet integral. In Section 3 we introduce the concept of the new integral and in Section 4 we obtain some of its basic properties. At the end, we present some applications of this new integral.

2 Classical Choquet integral

The aim of this section is to present known concepts and results on the Choquet integral which are used in the next sections.

Definition 2. *Let $(\Omega, \mathcal{C})$ be a measurable space, i.e. Ω is a nonempty set and $\mathcal{C}$ is a σ-algebra of subsets in Ω.*

(i) (see e.g. [9], p. 63) The set function $\nu : \mathcal{C} \to [0, +\infty)$ is called a monotone set function, or a capacity, if $\nu(\emptyset) = 0$ and $A, B \in \mathcal{C}$, with $A \subset B$, implies $\nu(A) \leq \nu(B)$. If $\nu(\Omega) = 1$, then ν is called normalized.

(ii) *(see [2], or [9], p. 233) Let ν be a normalized, monotone set function defined on $\mathcal{C}$. Recall that $f : \Omega \to \mathbb{R}$ is called $\mathcal{C}$-measurable if for any B, Borelian subset in $\mathbb{R}$, we have $f^{-1}(B) \in \mathcal{C}$.*

If $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable, then the Choquet integral of f on $A \in \mathcal{C}$, with respect to the capacity ν is defined by

$$(C)\int_A f d\nu = \int_0^{+\infty} \nu(F_\alpha(f)\bigcap A)d\alpha + \int_{-\infty}^0 [\nu(F_\alpha(f)\bigcap A) - \nu(A)]d\alpha,$$

where $F_\alpha(f) = \{\omega \in \Omega; f(\omega) \geq \alpha\}$. If $(C)\int_A f d\nu < +\infty$ then f is called Choquet integrable on A. Note that if $f \geq 0$ on A, then the integral $\int_{-\infty}^0$ in the above formula becomes equal to zero.

When ν is countably additive, then the Choquet integral $(C)\int_A X d\nu$ reduces to the Lebesgue type integral with respect to ν.

(iii) *Given $(\Omega, \mathcal{C})$ a measurable space and $\nu : \mathcal{C} \to \mathbb{R}_+$ a capacity on $\mathcal{C}$, we denote by $C(\Omega, \mathcal{C}, \nu)$ the space of all $\mathcal{C}$-measurable functions $g : \Omega \to \overline{\mathbb{R}}$, ν-a.e. finite, that is $\nu(\{\omega \in \Omega; |g(\omega)| = +\infty\}) = 0$.*

(iv) *$f, g : \Omega \to \mathbb{R}$ are called comonotonic, if*

$$[f(\omega_1) - f(\omega_2)] \cdot [g(\omega_1) - g(\omega_2)] \geq 0, \text{ for all } \omega_1, \omega_2 \in \Omega.$$

Let, e.g., $\Omega \subset \mathbb{R}$. It easily follows that if both f, g are nondecreasing functions or both nonincreasing functions, then f and g are comonotonic functions.

In what follows, we list some known properties.

Remark 2. *Let us suppose that ν is a monotone set function. Then, we have:*

(i) *For any $a \geq 0$, we have $(C)\int_A afd\nu = a \cdot (C)\int_A fd\nu$ (see, e.g., [3], p. 64, Proposition 5.1, (ii)).*

(ii) *If $f \leq g$ on A then $(C)\int_A fd\nu \leq (C)\int_A gd\nu$ (see, e.g., [9]).*

(iii) *If $f \geq 0$ and ν is subadditive, then by definition it is immediate that*

$$(C)\int_{A\bigcup B} fd\nu \leq (C)\int_A fd\nu + (C)\int_B fd\nu.$$

(iv) *If $f, g : \Omega \to \mathbb{R}_+$ are comonotonic, then*

$$(C)\int_\Omega [f(\omega) + g(\omega)]d\nu(\omega) = (C)\int_\Omega f(\omega)d\nu(\omega) + (C)\int_\Omega g(\omega)d\nu(\omega),$$

(see e.g. [3], p. 82, Corollary 6.7).

(v) Simple concrete examples of monotone and subadditive set functions ν, can be obtained from a probability measure M, by the formula $\nu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ (see, e.g., [3], pp. 16-17, Example 2.1).

3 Choquet integral in capacity

Analysing Definition 2, (ii), it is clear that the Choquet integral is an improper Riemann-kind integral and could be equivalently expressed as follows.

Definition 3. *Let $(\Lambda, \mathcal{B}, \nu)$ be with $(\Lambda, \mathcal{B})$ a measurable space and $\nu : \mathcal{B} \to \mathbb{R}_+$ a capacity.*

For $\varphi : \Lambda \to \mathbb{R}$, $\mathcal{B}$-measurable and $A \in \mathcal{B}$, denote

$$F_{A,\nu}(\varphi)(t) = \nu(\{\lambda \in \Lambda \bigcap A; \varphi(\lambda) \geq t\}),$$

which is a nonincreasing function of $t \in \mathbb{R}$.

We say that φ is Choquet ν-integrable on A, if there exist finite the limits

$$L_1 = \lim_{k \to \infty} (R) \int_0^k F_{A,\nu}(\varphi)(t)dt,$$
$$L_2 = \lim_{k \to \infty} (R) \int_{-k}^0 [F_{A,\nu}(\varphi)(t) - \nu(A)]dt.$$

In this case, by definition, we take $(C) \int_A \varphi(t)d\nu(t) = L_1 + L_2$.

Let us observe that if the Riemann integrals above are replaced by Kurzweil-Henstock integrals, then we don't get a more general concept. Indeed, as function of t, $F_{A,\nu}(\varphi)(t)$ is nonincreasing and therefore is Riemann integrable on any compact subinterval of $\mathbb{R}$.

The Choquet integral in Definition 3 however can be generalized in the spirit of Definition 1, as follows :

Definition 4. *Let $(\Lambda, \mathcal{B})$, $(\Omega, \mathcal{C})$ be two measurable spaces and $\nu : \mathcal{B} \to \mathbb{R}_+$, $\mu : \mathcal{C} \to \mathbb{R}_+$ two capacities with the additional properties that ν is normalized and μ is countably subadditive.*

Given a property $\mathcal{P}$ on Ω, everywhere in this paper we will say that $\mathcal{P}$ holds μ-a.e. on Ω if $\mu(\{\omega \in \Omega; \mathcal{P}(\omega)$ does not hold $\}) = 0$. Note that the subadditivity of μ implies that $\mu(\{\omega \in \Omega; \mathcal{P}(\omega)$ holds $\}) = \mu(\Omega)$.

Now, for $f : \Lambda \to C(\Omega, \mathcal{C}, \mu)$ and $A \in \mathcal{B}$, let us define

$$F_{A,\nu}(f)(t, \omega) = \nu(\{\lambda \in \Lambda \bigcap A; f(\lambda, \omega) \geq t\}),$$

for all $t \in \mathbb{R}$ *and* $\omega \in \Omega$.

We say that f *is Choquet* ν*-integrable on* $A \in \mathcal{B}$ *in the capacity* μ, *if for any* $k > 0$, *there exist finite the* (μ) *integrals of the type in Definition 1 (with* P *replaced by* μ*), i.e.,* $I_k^{(1)}(\omega) = (\mu)\int_0^k F_{A,\nu}(f)(t,\omega)dt$ *and* $I_k^{(2)}(\omega) = (\mu)\int_{-k}^0 [F_{A,\nu}(f)(t,\omega) - \nu(A)]dt$, *and the finite limits* μ*-a.e.* $\omega \in \Omega$

$$I_1(\omega) = \lim_{k\to\infty} I_k^{(1)}(\omega),$$
$$I_2(\omega) = \lim_{k\to\infty} I_k^{(2)}(\omega).$$

In this case, $I(\omega) = I_1(\omega) + I_2(\omega)$ *is called the Choquet* ν*-integral of* f *on* A *in the capacity* μ *and it is denoted by*

$$I(\omega) = (C,\mu)\int_A f(t,\omega)d\nu(t).$$

Remark 3.

(i) If the above (μ) *integrals on* $[0,k]$ *and* $[-k,0]$ *are well defined, then the* $(C,\mu)\int_A f(t,\omega)d\nu(t)$ *integral is well defined. Indeed, by Theorem 2.2 in [4], if* P *is a probability (i.e. countably additive), then the* (P) *integral of* f *on* $[a,b]$ *is well defined. Therefore, if* $I_1(\omega), I_2(\omega)$ *are integrals in probability* P *of* f *on* $[a,b]$, *then* $P(\{\omega \in E; I_1(\omega) \neq I_2(\omega)\}) = 0$. *But analysing the proof of Theorem 2.2 mentioned above, it is easily seen that the monotonicity and the countably subadditivity of* P *are enough for its validity (see the last part of the proof, where with the notations used there, in fact we have* $P(A_n) = 0$ *for all* $n \in \mathbb{N}$.*) Therefore, we conclude that the* $(C,\mu)\int_A f(t,\omega)d\nu(t)$ *integral in Definition 4 is well defined.*

(ii) The Choquet ν*-integral on* $A \in \mathcal{B}$ *in the capacity* μ *in Definition 4, generalizes both integrals in Definition 1 and in Definition 3. Indeed, if* $f(t,\omega) = f(t)$, *for all* $t \in \Lambda$ *and* $\omega \in \Omega$, *then the* $(C,\mu)\int_A f(t,\omega)d\nu(t)$ *integral one reduces to the classical Choquet integral* $(C)\int_A f(t)d\nu(t)$ *in Definition 3. Also, if* μ *and* ν *are both normalized and countably additive, then the* $(C,\mu)\int_A f(t,\omega)d\nu(t)$ *integral becomes, in essence, the integral in probability in Definition 1.*

(iii) Let us consider that ν *is normalized and* μ *is countably subadditive and continuous by decreasing sequences of sets, that is* $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_n \supseteq \ldots.$ *and* $\mu(E_1) < \infty$, *implies* $\lim_{n\to\infty}\mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$. *If* f *is Choquet* ν *integrable in the capacity* μ, *then for all* $\omega \in \Omega$ *we have*

$$(C,\mu)\int_A f(t,\omega)d\nu(t) = (C)\int_A f(t,\omega)d\nu(t).$$

Indeed, let $k > 0$ be fixed, arbitrary. Since $F_{A,\nu}(f)(t,\omega)$ is bounded and nondecreasing as function of $t \in [-k,0]$ and $t \in [0,k]$, for all $\omega \in \Omega$, there exists finite the Riemann integrals

$$\int_{-k}^{0} [F_{A,\nu}(f)(t,\omega) - \nu(A)]dt \text{ and } \int_{0}^{k} F_{A,\nu}(f)(t,\omega)dt.$$

Let us consider a sequence $(d_n)_n$ of divisions of $[0,k]$ with the norm tending to zero. It follows that if $n \to \infty$, then for the integral sums we have $\lim_{n\to\infty} S(F_{A,\nu}(f); d_n, \xi_i)(\omega) \to \int_0^k F_{A,\nu}(f)(t,\omega)dt$, pointwise for all $\omega \in \Omega$. A similar relationship holds if we make the above considerations on the interval $[-k,0]$. Now, since μ is continuous from above, by Theorem 7.9, p. 159 in [9], it follows that the corresponding integral sums converge in the "measure" μ to $\int_{-k}^{0}[F_{A,\nu}(f)(t,\omega) - \nu(A)]dt$ and $\int_0^k F_{A,\nu}(f)(t,\omega)dt$, respectively. This immediately implies that for all $\omega \in \Omega$ we have

$$\begin{aligned}(C,\mu)&\int_A f(t,\omega)d\nu(t)\\ &= \int_{-\infty}^{0}[F_{A,\nu}(f)(t,\omega) - \nu(A)]dt + \int_0^{+\infty} F_{A,\nu}(f)(t,\omega)dt\\ &= (C,\nu)\int_A f(t,\omega)dt.\end{aligned}$$

4 Properties of the (C, μ)-integral

In this section, we will prove some properties of the (C,μ) integral in capacity. The first result is the following.

Theorem 1. *Let $(\Lambda,\mathcal{B})$, $(\Omega,\mathcal{C})$ be two measurable spaces, $\nu : \mathcal{B} \to \mathbb{R}_+$ a normalized capacity, $\mu : \mathcal{C} \to \mathbb{R}_+$ a countably subadditive capacity and let $f, g : \Lambda \to C(\Omega,\mathcal{C},\mu)$ be Choquet ν-integrable on $A \in \mathcal{B}$ in the capacity μ.*

(i) If $f(\lambda,\omega) \le g(\lambda,\omega)$ for all $\lambda \in \Lambda$ and μ-a.e. $\omega \in \Omega$, then

$$(C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda) \le (C,\mu)\int_A g(\lambda,\omega)d\nu(\lambda),\ \mu-\ a.e.\ \omega \in \Omega.$$

(ii) If $a \ge 0$, then μ - a.e. $\omega \in \Omega$, we have

$$(C,\mu)\int_A a\cdot f(\lambda,\omega)d\nu(\lambda) = a\cdot (C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda).$$

(iii) Suppose, in addition, that ν is finitely subadditive. Then μ-a.e. $\omega \in \Omega$

$$(C,\mu)\int_A [f(\lambda,\omega)+g(\lambda,\omega)]d\nu(\lambda) \\ \leq 2\cdot\left\{(C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda)+(C,\mu)\int_A g(\lambda,\omega)d\nu(\lambda)\right\}.$$

Moreover, if in addition, $f(\lambda,\omega), g(\lambda,\omega) \geq 0$, for all $\lambda \in \Lambda$, μ-a.e. $\omega \in \Omega$ and $\{\lambda \in \Lambda; f(\lambda,\omega) > 0\}\bigcap\{\lambda \in \Lambda; g(\lambda,\omega) > 0\} = \emptyset$, μ-a.e. $\omega \in \Omega$, then

$$(C,\mu)\int_A [f(\lambda,\omega)+g(\lambda,\omega)]d\nu(\lambda) \\ \leq (C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda)+(C,\mu)\int_A g(\lambda,\omega)d\nu(\lambda),$$

μ-a.e. $\omega \in \Omega$.

(iv) For any constant $c > 0$ such that $f(\lambda,\omega)+c \geq 0$, for all $\lambda \in \Lambda$, μ-a.e. $\omega \in \Omega$, we have

$$(C,\mu)\int_A [f(\lambda,\omega)+c]d\nu(\lambda) = (C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda)+c\cdot\nu(A),$$

μ-a.e. $\omega \in \Omega$.

Proof. (i) With the notations in Definition 4, the monotonicity of ν immediately implies $F_{A,\nu}(f)(t,\omega) \leq F_{A,\nu}(g)(t,\omega)$, for all $t \in \Lambda$, μ-a.e. $\omega \in \Omega$. Then, with the notations for the integral sum and for the division in Definition 1, we immediately get

$$S_1(F_{A,\nu}(f); d^{(1)}, \xi_i)(\omega) \leq S_1(F_{A,\nu}(g); d^{(1)}, \xi_i)(\omega),$$

and

$$S_2(F_{A,\nu}(f)-\nu(A); d^{(2)}, \zeta_i)(\omega) \leq S_2(F_{A,\nu}(g)-\nu(A); d^{(2)}, \zeta_i)(\omega),$$

μ-a.e. $\omega \in \Omega$. Here recall that $d^{(1)}$ represents a division of $[0,k]$ and $d^{(2)}$ represents a division of $[-k,0]$.

Let us fix $k \in \mathbb{N}$, arbitrary and let us consider the divisions $d^{(1)}(n)$ (of $[0,k]$) and $d^{(2)}(n)$ (of $[-k,0]$), $n \in \mathbb{N}$, with the norms tending to zero and their corresponding intermediary points denoted by $\xi_{i,n}$ and $\zeta_{i,n}$, respectively.

According to Definition 4 (and Definition 1), we have the following convergences (for $n \to \infty$) in the monotone and countable subadditive "measure", μ:

$$
\begin{aligned}
S_1(F_{A,\nu}(f); d^{(1)}(n), \xi_{i,n})(\omega) &\xrightarrow{\mu} I_k^{(1)}(\omega)(f),\\
S_1(F_{A,\nu}(g); d^{(1)}(n), \xi_{i,n})(\omega) &\xrightarrow{\mu} I_k^{(1)}(\omega)(g),\\
S_2(F_{A,\nu}(f) - \nu(A); d^{(2)}(n), \zeta_{i,n})(\omega) &\xrightarrow{\mu} I_k^{(2)}(\omega)(f),\\
S_2(F_{A,\nu}(g) - \nu(A); d^{(2)}(n), \zeta_{i,n})(\omega) &\xrightarrow{\mu} I_k^{(2)}(\omega)(g).
\end{aligned}
$$

But since the Borel-Cantelli Lemma holds for μ monotone and countably subadditive (see, e.g., the proof in [6], p. 111) and reasoning exactly as in the proof of Theorem, page 112 in [6], it follows the Riesz's result for μ too, that is $S_1(F_{A,\nu}(f); d^{(1)}(n), \xi_{i,n})(\omega)$, $n \in \mathbb{N}$, contains a subsequence, pointwise convergent to $I_k^{(1)}(\omega)(f)$, μ-a.e. $\omega \in \Omega$. Then, since for the sequence of divisions corresponding to this subsequence, we also have

$$
S_1(F_{A,\nu}(g); \cdot, \cdot)(\omega) \xrightarrow{\mu} I_k^{(1)}(\omega)(g),
$$

this immediately implies that for each $k \in \mathbb{N}$, there is a set $A_k^{(1)}$ with $\mu(A_k^{(1)}) = 0$, such that $I_k^{(1)}(\omega)(f) \leq I_k^{(1)}(\omega)(g)$, for all $\omega \in \Omega \setminus A_k^{(1)}$. Then, for $A^{(1)} = \bigcup_{k=1}^{\infty} A_k^{(1)}$, we get $0 \leq \mu(A^{(1)}) \leq \sum_{k=1}^{\infty} \mu(A_k^{(1)}) = 0$ and for all $\omega \in \Omega \setminus A^{(1)}$ we have $I_1(\omega)(f) \leq I_1(\omega)(g)$.

Since for $S_2(F_{A,\nu}(f) - \nu(A); d^{(2)}, \zeta_i)(\omega)$, $S_2(F_{A,\nu}(g) - \nu(A); d^{(2)}, \zeta_i)(\omega)$, similar results hold, this finally leads to the first inequality in the statement.

(ii) For $a = 0$, the equality in the statement is trivial. Let us suppose that $a > 0$. We immediately can write

$$
S_1(F_{A,\nu}(a\cdot f); d^{(1)}, \xi_i)(\omega) = a\cdot\sum_{i=0}^{n-1} F_{A,\nu}(f)(d^{(1)}/a, \xi_i/a)(\omega)(x_{i+1}/a - x_i/a),
$$

where $d^{(1)}/a$ denotes a corresponding division of $[0, k/a]$. Since $k > 0$ is arbitrary, by applying Definition 4 we easily get that for any sequence of divisions $(d^{(1)}(n))_n$ with the norms tending to zero, we have that

$$
\sum_{i=0}^{n-1} F_{A,\nu}(f)(d^{(1)}/a, \xi_i/a)(\omega)(x_{i+1}/a - x_i/a)
$$

converges in the "measure" μ to $I_{k/a}^{(1)}(\omega)$. Since $\lim_{k\to\infty} I_{k/a}^{(1)} = I_1(\omega)$, applying similar reasonings for $I_2(\omega)$, we easily arrive at the desired formula.

(iii) For the first part of (iii), it easily follows

$$\{\lambda \in \Lambda \cap A; f(\lambda,\omega) + g(\lambda,\omega) \geq t\}$$
$$\subset \{\lambda \in \Lambda \cap A; f(\lambda,\omega) \geq t/2\} \bigcup \{\lambda \in \Lambda \cap A; g(\lambda,\omega) \geq t/2\}.$$

Applying ν, we get $F_{A,\nu}(f+g)(t,\omega) \leq F_{A,\nu}(f)(t/2,\omega) + F_{A,\nu}(g)(t/2,\omega)$. Now, for a division $d^{(1)}$ of $[0,k]$, of norm $< \delta_{\varepsilon,\eta,k}$, we obtain

$$\begin{aligned} S_1(F_{A,\nu}(f+g); d^{(1)}, \xi_i)(\omega) &\leq S_1(F_{A,\nu}(f); d^{(1)}, \xi_i/2)(\omega) \\ &\quad + S_1(F_{A,\nu}(g); d^{(1)}, \xi_i/2)(\omega) \\ &= 2[S_1(F_{A,\nu}(f); d^{(1)}/2, \xi_i/2)(\omega) \\ &\quad + S_1(F_{A,\nu}(g); d^{(1)}/2, \xi_i/2)(\omega)], \end{aligned}$$

where $d^{(1)}/2$ denotes a division of the interval $[0,k/2]$, of norm value of $< \delta_{\varepsilon,\eta,k}(\xi/2)$. Therefore, the hypothesis in Definition 4 for f and g are fulfilled, with $I_k^{(1)}$ replaced there by $I_{k/2}^{(1)}$. Since $k > 0$ is arbitrary and $I_{k/2}^{(1)}(\omega)$ converges, as $k \to \infty$, to $I_1(\omega)$, μ-a.e. $\omega \in \Omega$, reasoning exactly as at the above point (i), we immediately obtain that $I_1(\omega)(f+g) \leq 2[I_1(\omega)(f) + I_1(\omega)(g)]$.

A similar inequality holds for $I_2(\omega)$ too, which leads to the required inequality.

For the second part of (iii), from the hypothesis on f and g and on ν, we easily get

$$\{\lambda \in \Lambda \cap A; f(\lambda,\omega) + g(\lambda,\omega) \geq t\}$$
$$\subset \{\lambda \in \Lambda \cap A; f(\lambda,\omega) \geq t\} \bigcup \{\lambda \in \Lambda \cap A; g(\lambda,\omega) \geq t\}.$$

Consequently,

$$\begin{aligned} F_{A,\nu}(f+g)(t,\omega) &= \nu(\{\lambda \in \Lambda \cap A; f(\lambda,\omega) + g(\lambda,\omega) \geq t\}) \\ &\leq F_{A,\nu}(f)(t,\omega) + F_{A,\nu}(g)(t,\omega), \end{aligned}$$

$$\begin{aligned} &S_1(F_{A,\nu}(f+g); d^{(1)}, \xi_i)(\omega) \\ &\qquad \leq S_1(F_{A,\nu}(f); d^{(1)}, \xi_i)(\omega) + S_1(F_{A,\nu}(g); d^{(1)}, \xi_i)(\omega). \end{aligned}$$

Since a similar inequality holds in the case of $S_2(F_{A,\nu}(f+g)-\nu(A); d^{(2)},\zeta_i)(\omega)$, we immediately get the required conclusion.

(iv) For $k>0$, let us consider $d^{(1)}$ a division of $[0,k]$ such that c is an interior point of that division, that is there is $i_0\in\{1,...,n-2\}$ such that $x_{i_0}=c$. We can write

$$\begin{aligned}
&S_1(F_{A,\nu}(f+c);d^{(1)},\xi_i)(\omega)\\
&=\sum_{i=0}^{i_0-1}\nu(\{\lambda\in\Lambda\bigcap A; f(\lambda,\omega)\geq\xi_i-c\})(x_{i+1}-x_i)\\
&+\sum_{i=i_0}^{n-1}\nu(\{\lambda\in\Lambda\bigcap A; f(\lambda,\omega)\geq\xi_i-c\})[(x_{i+1}-c)-(x_i-c)]\\
&=\nu(A)\cdot c+S_1(F_{A,\nu}(f);\tilde{d}^{(1)},\xi_i)(\omega),
\end{aligned}$$

where $\tilde{d}^{(1)}$ is a division of $[0,k-c]$, consisting in $n-i_0$ points and having the same norm with $d^{(1)}$. Since a similar formula holds for $S_2(F_{A,\nu}(f+c)-\nu(A);d^{(2)},\zeta_i)(\omega)$, taking into account Definition 4, from this point the required equality is immediate.

□

Also, we have:

Theorem 2. *Let $(\Lambda,\mathcal{B})$, $(\Omega,\mathcal{C})$ be two measurable spaces, $\nu:\mathcal{B}\to\mathbb{R}_+$ a normalized capacity, $\mu:\mathcal{C}\to\mathbb{R}_+$ a countably subadditive capacity, continuous by decreasing sequences of sets (that is, if $A_{m+1}\subset A_m$ for all $m\in\mathbb{N}$, then $\lim_{m\to\infty}\mu(A_m)=\mu(M)$, where $M=\bigcap_{m=1}^{\infty}A_m$).*

If $\alpha\in C(\Omega,\mathcal{C},\mu)$ satisfies $\alpha(\omega)>0$, μ-a.e. on Ω and $g:\Lambda\to\mathbb{R}$ is Choquet ν- integrable on $A\in\mathcal{B}$, then $f(\lambda,\omega)=g(\lambda)\cdot\alpha(\omega)$ is Choquet ν-integrable on A in the capacity μ and we have

$$(C,\mu)\int_A f(t,\omega)d\nu(t)=\alpha(\omega)\cdot(C)\int_A g(t)d\nu(t).$$

Proof. Denoting $H=\{\omega\in\Omega;\alpha(\omega)>0\}$, by hypothesis and by the monotonicity and subadditivity of μ we easily get that $\mu(H)=\mu(\Omega)$.

Let us keep the notations in Definitions 1 and 4. If $d^{(1)}$ is the division $0=x_0<...<x_n=k$ with $\xi_i\in[x_i,x_{i+1}]$, $x_{i+1}-x_i<\delta^{(1)}_{\varepsilon,\eta,k}$, $i=0,...,n-1$, and if $d^{(2)}$ is the division $-k=y_0<...<y_n=0$ with $\zeta_i\in[y_i,y_{i+1}]$, $y_{i+1}-y_i<\delta^{(2)}_{\varepsilon,\eta,k}$, $i=0,...,n-1$, then it is easy to see that for any $\omega\in H$ we have:

$$\begin{aligned}
&S_1(F_{A,\nu}(f);d^{(1)},\xi_i)(\omega)\\
&=\alpha(\omega)\cdot\sum_{i=0}^{n-1}\nu\left(\left\{\lambda\in\Lambda\bigcap A;g(\lambda)>\frac{\xi_i}{\alpha(\omega)}\right\}\right)\cdot\left(\frac{x_{i+1}}{\alpha(\omega)}-\frac{x_i}{\alpha(\omega)}\right)\\
&=\alpha(\omega)\cdot S_1(F_{A,\nu}(g);d^{(1)}/\alpha(\omega),\xi_i/\alpha(\omega)),
\end{aligned}$$

where

$$d^{(1)}/\alpha(\omega):0=x_0/\alpha(\omega)<...<x_n/\alpha(\omega)=k/\alpha(\omega),$$

$$\xi_i/\alpha(\omega)\in[x_i/\alpha(\omega),x_{i+1}/\alpha(\omega)]$$

with $x_{i+1}/\alpha(\omega)-x_i/\alpha(\omega)<\delta^{(1)}_{\varepsilon,\eta,k}/\alpha(\omega)$, $i=0,...,n-1$.

Analogously, for any $\omega\in H$ we have

$$\begin{aligned}
&S_2(F_{A,\nu}(f)-\nu(A);d^{(2)},\zeta_i)(\omega)\\
&=\alpha(\omega)\cdot\sum_{i=0}^{n-1}\left[\nu\left(\left\{\lambda\in\Lambda\bigcap A;g(\lambda)>\frac{\zeta_i}{\alpha(\omega)}\right\}\right)-\nu(A)\right]\cdot\left(\frac{y_{i+1}}{\alpha(\omega)}-\frac{y_i}{\alpha(\omega)}\right)\\
&=\alpha(\omega)\cdot S_2(F_{A,\nu}(g)-\nu(A);d^{(2)}/\alpha(\omega),\zeta_i/\alpha(\omega)),
\end{aligned}$$

where

$$d^{(2)}/\alpha(\omega):-k/\alpha(\omega)=y_0/\alpha(\omega)<...<y_n/\alpha(\omega)=0/\alpha(\omega),$$

$$\zeta_i/\alpha(\omega)\in[y_i/\alpha(\omega),y_{i+1}/\alpha(\omega)],$$

with $y_{i+1}/\alpha(\omega)-y_i/\alpha(\omega)<\delta^{(2)}_{\varepsilon,\eta,k}(\zeta_i)/\alpha(\omega)$, $i=0,...,n-1$.

Let us denote $I=(C)\int_A g(t)d\nu(t)=I_1+I_2$, where $I_1=\lim_{k\to+\infty}I_k^{(1)}$, $I_2=\lim_{k\to+\infty}I_k^{(2)}$ and $A_m=\{\omega\in\Omega;|\alpha(\omega)|\geq m\}$. Obviously, $A_{m+1}\subset A_m$, $m\in\mathbb{N}$. Denoting $M=\bigcap_{m=1}^{\infty}A_m$, since $\alpha\in C(\Omega,\mathcal{C},\mu)$ we get $\mu(M)=0$ and by μ continuous by decreasing sequences of sets, it follows $\lim_{m\to\infty}\mu(A_m)=\mu(M)=0$.

Consequently, if $\eta>0$, there exists $N(\eta)\in\mathbb{N}$, such tat for all $m\in\mathbb{N},m\geq N(\eta)$, we have

$$\mu(\{\omega\in H;|\alpha(\omega)|\geq m\})<\eta.$$

For fixed $m\geq N(\eta)$, $\omega\in H$, let us consider $\varepsilon>0$, such that $1/\varepsilon\geq m$.

Consider $\varepsilon^2>0$. Because g is Choquet ν-integrable on A (see Definitions 1 and 3), there exist $\delta^{(1)}_{\varepsilon^2,k}/\alpha(\omega)>0$, $\delta^{(2)}_{\varepsilon^2,k}/\alpha(\omega)>0$, such that for any division

$d^{(1)}/\alpha(\omega) : 0 = x_0/\alpha(\omega) < ... < x_n/\alpha(\omega) = k/\alpha(\omega)$ of $[0, k/\alpha(\omega)]$, $\xi_i/\alpha(\omega) \in [x_i/\alpha(\omega), x_{i+1}/\alpha(\omega)]$, $x_{i+1}/\alpha(\omega) - x_i/\alpha(\omega) < \delta^{(1)}_{\varepsilon^2,k}/\alpha(\omega)$, $i = 0, ..., n-1$, and any division $d^{(2)}/\alpha(\omega) : -k/\alpha(\omega) = y_0/\alpha(\omega) < ... < y_n/\alpha(\omega) = 0$, $\zeta_i/\omega(\omega) \in [y_i/\alpha(\omega), y_{i+1}/\alpha(\omega)]$ and

$$y_{i+1}/\alpha(\omega) - y_i/\alpha(\omega) < \delta^{(2)}_{\varepsilon^2,k}/\alpha(\omega),\ i = 0, ..., n-1,$$

we have

$$|S_1(F_{A,\nu}(g); d^{(1)}/\alpha(\omega), \xi_i/\alpha(\omega)) - I^{(1)}_k| < \varepsilon^2,$$

$$|S_2(F_{A,\nu}(g) - \nu(A); d^{(2)}/\alpha(\omega), \zeta_i/\alpha(\omega)) - I^{(2)}_k| < \varepsilon^2,$$

where $F_{A,\nu}(g)(t) = \nu(\{\lambda \in \Lambda \bigcap A; g(\lambda) \geq t\})$.

We have

$$\begin{aligned}
&\{\omega \in \Omega; |S_1(F_{A,\nu}(f); d^{(1)}, \xi_i)(\omega) - \alpha(\omega) \cdot I^{(1)}_k| \geq \varepsilon\} \\
&= \{\omega \in \Omega; |\alpha(\omega)| \cdot |S_1(F_{A,\nu}(g); d^{(1)}/\alpha(\omega), \xi/\alpha(\omega)) - I^{(1)}_k| \geq \varepsilon\} \\
&\subset \{\omega \in \Omega; |\alpha(\omega)| \geq 1/\varepsilon\} \subset \{\omega \in \Omega; |\alpha(\omega)| \geq m\},
\end{aligned}$$

i.e. $\mu(\{\omega \in \Omega; |S_1(F_{A,\nu}(f); d^{(1)}, \xi_i)(\omega) - \alpha(\omega) \cdot I^{(1)}_k| \geq \varepsilon\}) < \eta$, for any division $d^{(1)} : 0 = x_0 < ... < x_n = k$, $\xi_i \in [x_i, x_{i+1}]$ with $x_{i+1} - x_i < \delta^{(1)}_{\varepsilon,\eta,k}$, $i = 0, ..., n-1$ (in fact, ε depends on m, which depends on η, therefore δ_{ε^2} depends on η too).

Analogously we get

$$\mu(\{\omega \in \Omega; |S_2(F_{A,\nu}(f) - \nu(A); d^{(2)}, \zeta_i)(\omega) - \alpha(\omega) \cdot I^{(2)}_k| \geq \varepsilon\}) < \eta,$$

for any division $d^{(2)} : -k = y_0 < ... < y_n = 0$, $\zeta_i \in [y_i, y_{i+1}]$ with $y_{i+1} - y_i < \delta^{(2)}_{\varepsilon,\eta,k}$, $i = 0, ..., n-1$.

Then, by Definition 4, f is Choquet ν-integrable on A in the capacity μ and we have

$$(C, \mu) \int_A f(t, \omega) d\nu(t) = \alpha(\omega) \cdot (C) \int_A g(t) d\nu(t),$$

which proves the theorem. □

Remark 4. *Since the Choquet integrability of a function $g : [a, b] \to \mathbb{R}$ is more general than the Riemann integrability, Theorem 2 furnishes simple examples of random functions which are integrable in the sense of Defintion 4, but which are not integrable in the senses of Definitions 1 and 3.*

The following Fubini-type result holds.

Theorem 3. *Let $(\Lambda, \mathcal{B})$, $(\Omega, \mathcal{C})$ be two measurable spaces, $\nu : \mathcal{B} \to \mathbb{R}_+$ a normalized capacity, $\mu : \mathcal{C} \to \mathbb{R}_+$ a countably subadditive capacity and let $f : \Lambda \to C([a,b], \mathcal{C}, \mu)$ be Choquet ν-integrable on $A \in \mathcal{B}$ in the capacity μ, with the property*

$$0 \leq f(\lambda, \omega) \leq M, \quad \text{for all } \lambda \in \Lambda, \omega \in [a,b]. \tag{1}$$

Suppose also that $f(\lambda, \cdot)$ is nondecreasing on $[a,b]$ for any $\lambda \in \Lambda$ fixed (or $f(\lambda, \cdot)$ is nonincreasing on $[a,b]$ for any $\lambda \in \Lambda$ fixed). Then, with the notation for $F_{A,\nu}(f)(t, \omega)$ in Definition 4, we have

$$(C) \int_{[a,b]} \left[(C, \mu) \int_A f(\lambda, \omega) d\nu(\lambda) \right] d\mu(\omega)$$
$$= (R) \int_0^\infty \left[(C) \int_{[a,b]} F_{A,\nu}(f)(t, \omega) d\mu(\omega) \right] dt.$$

Proof. Let us denote $I_1(\omega) = (C, \mu) \int_A f(\lambda, \omega) d\nu(\lambda)$, $\omega \in [a,b]$. Since f is Choquet ν-integrable on $A \in \mathcal{B}$ in the capacity μ, by (1) we get $0 \leq I_1(\omega) \leq M \cdot \nu(A)$ for all $\omega \in [a,b]$ and according to Definitions 1 and 4, there exists a generalized sequence $I_k^{(1)} \in C([a,b], \mathcal{C}, \mu)$, $I_k^{(1)}(\omega) \in \mathbb{R}$, $\omega \in [a,b]$, $k > 0$, with $\lim_{k \to +\infty} I_k^{(1)} = I_1 \in C([a,b], \mathcal{C}, \mu)$, pointwise limit on the whole $[a,b]$, in addition satisfying :

for $\varepsilon > 0$, $k > 0$ and $\eta = 1/m$, $m \in \mathbb{N}$, there exist $\delta_{\varepsilon,m,k}^{(1)} > 0$, such that for any division $d_m^{(1)} : 0 = x_0^{(m)} < ... < x_{n_m}^{(m)} = k$ of $[0,k]$, any $\xi_i^{(m)} \in [x_i^{(m)}, x_{i+1}^{(m)}]$ with $x_{i+1}^{(m)} - x_i^{(m)} < \delta_{\varepsilon,m,k}^{(1)}$, $i = 0, ..., n_m - 1$, we have

$$\mu(\{\omega \in [a,b]; |S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega) - I_k^{(1)}(\omega)| \geq \varepsilon\}) < 1/m, \ m \in \mathbb{N}.$$

This means that the divisions can be chosen such that for $m \to \infty$, we have

$$S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega) \to I_k^{(1)}(\omega),$$

in the "measure" μ on $[a,b]$. Moreover, reasoning exactly as in the proof of Theorem 1, (i), the divisions can be chosen such that the above convergence holds pointwise, μ-a.e. $\omega \in [a,b]$. Since $S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega) \geq 0$ for all $\omega \in [a,b]$, we get $I_k^{(1)}(\omega) \geq 0$, μ-a.e. $\omega \in [a,b]$. Then, since by the monotonicity hypothesis on f, it easily follows that as functions of ω, all $F_{A,\nu}(f)(\xi_i^{(m)}, \omega)$ are nondecreasing functions (or all $F_{A,\nu}(f)(\xi_i^{(m)}, \omega)$ are nonincreasing functions), we easily get that $S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega)$ are monotonous functions of ω, of the same monotonicity. This implies that all $I_k^{(1)}(\omega)$ are of the same monotonicity as functions of ω and consequently, $I_1(\omega)$ is of the same monotonicity μ-a.e. $\omega \in [a, b]$.

On the other hand, for all $m \in \mathbb{N}$ and $\omega \in [a, b]$ we have

$$|S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega)| = \sum_{i=0}^{n_m-1} F_{A,\nu}(f)(\xi_i^{(m)}, \omega)(x_{i+1}^{(m)} - x_i^{(m)}) \leq k.$$

Now, since μ is countably subadditive and monotone, it easily follows that it is null-additive (that is, for E, F with $E \bigcap F = \emptyset$ and $\mu(F) = 0$ we get $\mu(E \bigcup F) = \mu(E)$) and by Theorem 11.10, p. 236 in [9], we immediately obtain

$$(C)\int_{[a,b]} I_k^{(1)}(\omega)d\mu(\omega) = \lim_{m\to\infty}(C)\int_{[a,b]} S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega)d\mu(\omega).$$

But, since as function of ω, all $F_{A,\nu}(f)(\xi_i^{(m)}, \omega)$ are nondecreasing functions (or all $F_{A,\nu}(f)(\xi_i^{(m)}, \omega)$ are nonincreasing functions), by Remark 2, (iv) and (i), we immediately obtain

$$\begin{aligned}
&(C)\int_{[a,b]} I_k^{(1)}(\omega)d\mu(\omega) \\
&\qquad = \lim_{m\to\infty}\sum_{i=0}^{n_m-1}\left[(C)\int_{[a,b]} F_{A,\nu}(f)(\xi_i^{(m)}, \omega)d\mu(\omega)\right]\cdot(x_{i+1}^{(m)} - x_i^{(m)}) \\
&\qquad = (R)\int_0^k \left[(C)\int_{[a,b]} F_{A,\nu}(f)(t, \omega)d\mu(\omega\right]dt.
\end{aligned}$$

By (1), it follows $F_{A,\nu}(f)(t, \omega) = 0$, for all $t > M$, a.e. $\omega \in [a, b]$, which implies that for a fixed $N_0 > M$, we have

$$\begin{aligned}
S_1(F_{A,\nu}(f); d_m^{(1)}, \xi_i^{(m)})(\omega) &\leq \sum_{i=0, x_{i+1}^{(m)}<N_0}^{n_m-1} F_{A,\nu}(f)(\xi_i^{(m)}, \omega)(x_{i+1}^{(m)} - x_i^{(m)}) \\
&\leq \sum_{i=0, x_{i+1}^{(m)}<N_0}^{n_m-1} (x_{i+1}^{(m)} - x_i^{(m)}) \leq N_0.
\end{aligned}$$

Since at the beginning we proved that for $m \to \infty$, $S_1(F(f);)(\omega)$ converges pointwise to $I_k^{(1)}(\omega)$, this implies that for $N_0 > M$ and all $k \in \mathbb{N}$, we get

$$|I_k^{(1)}(\omega)| = I_k^{(1)}(\omega) \leq N_0, \text{ a.e. } \omega \in [a,b].$$

Now, since μ is countably subadditive and monotone, it is null-additive too (see the previous lines in the proof) and by Theorem 11.10, p. 236, we immediately obtain

$$\lim_{k\to\infty}(C)\int_{[a,b]} I_k^{(1)}(\omega)d\mu(\omega) = (C)\int_{[a,b]} I_1(\omega)d\mu(\omega).$$

This implies

$$\begin{aligned}
&(R)\int_0^\infty \left[(C)\int_{[a,b]} F_{A,\nu}(f)(t,\omega)d\mu(\omega\right] dt \\
&= \lim_{k\to\infty}(C)\int_{[a,b]} I_k^{(1)}(\omega)d\mu(\omega) \\
&= (C)\int_{[a,b]} I_1(\omega)d\mu(\omega) \\
&= (C)\int_{[a,b]} \left[(C,\mu)\int_A f(\lambda,\omega)d\nu(\lambda)\right] d\mu(\omega),
\end{aligned}$$

which proves the theorem. □

Remark 5. *It is clear that the concept of Choquet ν-integral in capacity μ and its properties remain valid in the case when μ is countably subadditive and ν is any normalized Choquet capacity of order $k \geq 2$ defined as in [2].*

Remark 6. *According to Subsection 15.8, pp. 335-337 in [9], the classical Choquet integral has applications in classification theory. Very briefly (see the mentioned subsection for all the details), the classifying boundary is identified by the equation*

$$(C)\int_A [a + bf(x)]d\nu(x) = c,$$

where a, b are vectors (of real constants), c is a real constant and f represents a (deterministic) matrix of observation data.

But it is clear that we can similarly consider for study the more general stochastic (non-deterministic) model for classification, when a, b are vector random variables, c is a random variable and f is a matrix of nondeterministic observation data, that is $a = a(\omega)$, $b = b(\omega)$, $c = c(\omega)$ and $f = f(t,\omega)$, with $\omega \in \Omega$, $t \in [\alpha,\beta]$. In this case, choosing another capacity μ supposed to be *countably subadditive, the stochastic classifying boundary will be identified by the equation*

$$(C,\mu)\int_A [a(\omega)+b(\omega)f(t,\omega)]d\nu(t)=c(\omega), \mu-a.e.\ \ \omega\in\Omega.$$

Remark 7. *According to Subsection 15.7, pp. 333-335 in [9], in a similar way with classification theory, the classical Choquet integral has also applications in multiregression. Very briefly, with the notations in Remark 6, the multiregression (deterministic) model can be written as*

$$y=c+(C,\mu)\int_A [a+bf(x)]d\nu(x)+N(0,\sigma^2),$$

where $N(0,\sigma^2)$ *is a normally distributed random variable with mean* 0 *and variance* σ^2.

As in the above Remark 6, we can consider for study the more general nondeterministic (stochastic) regression model

$$y(\omega)=c(\omega)+(C)\int_A [a(\omega)+b(\omega)f(t,\omega))]d\nu(t)+N(0,\sigma^2).$$

Remark 8. *In the hypothesis and with the notations in Definition 4 and as a generalization of the classical Fredholm integral equation, we can introduce and study the following stochastic Fredholm-Choquet integral equation*

$$\varphi(x,\omega)=f(x,\omega)+\lambda\cdot(C,\mu)\int_\Lambda K(x,s)\varphi(s,\omega)d\nu(s), x\in\Lambda, \mu-a.e.\ \ \omega\in\Omega. \quad (2)$$

with the given data $\lambda\in\mathbb{R}$, $f:\Lambda\to C(\Omega,\mathcal{C},\nu)$, $K:\Lambda\times\Lambda\to\mathbb{R}$ *and the unknown function* $\varphi:\Lambda\to C(\Omega,\mathcal{C},\nu)$.

Without to enter here into details, we mention that under the additional hypothesis on ν *to be submodular and continuous by increasing sequences and on* μ *to be continuous by decreasing sequences (for these concepts see, e.g., [3], p. 16), by using the classical fixed point theorem of Banach, we can deduce the existence and uniqueness of the solution of equation (2) in various spaces and under some suitable hypothesis on* λ *and* K. *This equation will be studied in details elsewhere.*

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References

[1] D. R. Adams, *Choquet integrals in potential theory*, Publ. Mat., **42** (1998), 3–66.

[2] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), **5**(1953–1954), 131–292.

[3] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer Academic Publisher, Dordrecht, Boston, London, 2010.

[4] S. G. Gal, *On the Kurzweil-Henstock integral in probability*, Math. Rep. (Bucur.), **47(3-4)**(1995), 263–269.

[5] R. Henstock, *Theory of Integration*, Butterworths, London, 1963.

[6] B. Makarov and A. Podkorytov, *Real Analysis : Measures, Integrals and Applications*, Universitext, Springer, New York, 2013.

[7] R. M. MacLeod, *The Generalized Riemann Integral*, Carus Mathematical Monographs, The Mathematical Association of America, 1980.

[8] O. Onicescu, Gh. Cenusa and I. Sacuiu, *Random Function Almost Periodic in Probability (Romanian)*, Academic Press, Bucharest, 1983.

[9] S. Wang and G. J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.