Real Analysis Exchange
Vol. 43(1), 20127, pp. 239–242
DOI: 10.14321/realanalexch.41.1.0239

Hajrudin Fejzić,* Department of Mathematics, California State University, San Bernardino, CA 92407, U.S.A. email: hfejzic@csusb.edu

A NOTE ON THE LUZIN-MENCHOFF THEOREM

The Luzin-Menchoff theorem asserts that if E is a measurable set on the real line, and $K \subset E$ is a closed subset such that every $x \in K$ is a density point of E, then there is a perfect set, F, such that $K \subset F \subset E$ and every $x \in K$ is a density point of F.

Although it is been attributed to Luzin and Menchoff, the two never published a proof. The earliest published proofs of this result are by Luzin's student Bogomolova [1] and by Zahorski [10]. In [9], Zahorski used the Luzin-Menchoff theorem to show that if E is an F_σ set such that every point of E is its density point, then there is an approximately continuous function f with the property that $0 < f \leq 1$ on E and $f = 0$ elsewhere. In this paper, Zahorski points to a paper by Maximoff [4] for a proof of the Luzin-Menchoff theorem. Another proof is given in [2] and one in [3]. See [5] for an interesting take on the Luzin-Menchoff theorem and its relationship to the well-known Urysohn lemma.

In [6], O'Malley used the Luzin-Menchoff theorem to establish the so-called O'Malley property for F_σ sets, which is used to supply proofs to a number of monotonicity theorems for real valued functions. A Luzin-Menchoff type result where the density one is replaced by weaker density conditions would strengthen these monotonicity theorems from [6]. It turns out that the proof given in [2] can be modified (and greatly simplified) to achieve this objective. Moreover, the Luzin-Menchoff Theorem presented in this note is also generalized to $\mathbb{R}^m$ for $m \geq 1$.

First we will review necessary definitions and basic properties of Lebesgue measure and perfect sets.

Mathematical Reviews subject classification: Primary: 28A05
Key words: measurable sets, perfect sets, density
Received by the editors September 15, 2017
Communicated by: Luisa Di Piazza
*This research was presented at XLI Summer Symposium in Real Analysis, Wooster, OH 2017

By $\lambda(E)$ we denote Lebesgue measure in $\mathbb{R}^m$, and by $B_r(c)$ we denote the open ball centered at c and with the radius r. The volume of m dimensional ball is equal to Br^m where the constant B can be expressed in terms of Gamma function as $B = \frac{\pi^{m/2}}{\Gamma\left(m/2+1\right)}$. Lower, $d^-(E,u)$, and upper, $d^+(E,u)$, densities of E at u are defined as $\liminf_{r\to 0^+} \frac{\lambda(E\cap B_r(u))}{Br^m}$ and $\limsup_{r\to 0^+} \frac{\lambda(E\cap B_r(u))}{Br^m}$ respectively. When the two are equal, we use $d(E,u)$ to denote the common value. When $d(E,u) = 1$ we say that u is a density point of E. It is a remarkable fact that almost every $u \in E$ is a density point of E. (See [8] page 141.) Hence if E is measurable, and $E' = \{u \in E : u \text{ is a density point of } E\}$ then E' is measurable and $\lambda(E') = \lambda(E)$.

We will need the following fact about Lebesgue measure. For every measurable set E and for every $\epsilon > 0$, there is a closed set $F \subset E$ such that $\lambda(E \setminus F) < \epsilon$. (See [8], Theorem 2.20 (b) page 50.) Here F closed can be replaced with F perfect (i.e. a closed set with no isolated points) and we will do so. This replacement is justified by the property that closed sets can be decomposed (uniquely) as $P \cup C$, where P is perfect and C is countable. (See [7], Exercise 28 page 45.)

Finally if $|a-b|$ denotes the Euclidean distance, then $dist(x,C)$ is the distance from a point x to a set C, that is $dist(x,C) = \inf\{|x-c| : c \in C\}$. In the proof of Theorem 2 below we will use the simple fact that the distance, $dist(x,C)$, is a continuous function of x. (In fact for every x, y, $|\mathrm{dist}(x,C) - \mathrm{dist}(y,C)| \le |x-y|$.)

Luzin-Menchoff Theorem. *Let E be a measurable set in $\mathbb{R}^m$ and K a closed subset of E. Then there is a closed set, F, such that $K \subset F \subset E$ and for all $u \in K$, $d^-(F,u) = d^-(E,u)$ and $d^+(F,u) = d^+(E,u)$. Moreover $d(E,x) = 1$ for every $x \in F \setminus K$, and if $d^+(E,u) > 0$ for every $u \in K$, then F is perfect.*

Proof. Let $E' = \{u \in E : u \text{ is a density point of } E\}$. Let $S_n = E' \cap \{x : \frac{1}{n+1} < dist(x,K) \le \frac{1}{n}\}$. As an intersection of two measurable sets, S_n is measurable. Let $K_n \subset S_n$ be a perfect set such that $\lambda(S_n \setminus K_n) < \frac{1}{2^n}$ and define $F = \bigcup_{n=1}^{\infty} K_n \cup K$. Since $K_n \subset E'$, $d(E,x) = 1$ for every $x \in F \setminus K$.

To show that F is a closed set, let x_n be a sequence of points from F that converges to x. If $x \notin K$, since K is closed there exist a positive integer p such that $\frac{1}{p+1} < dist(x,K)$. The continuity of the distance implies that for all sufficiently large k, $\frac{1}{p+1} < dist(x_k,K)$. Thus for all sufficiently large k, $x_k \in \cup_{j=1}^{p} K_j$ and since $\cup_{j=1}^{p} K_j$ is closed it must contain x. Hence F is a closed set.

Fix $u \in K$, and let $B_r(u)$ be a ball of radius $r < 1$. Let N be the unique integer such that $\frac{1}{N+1} \le r < \frac{1}{N}$. If $B_r(u) \cap S_n \ne \emptyset$, then for $e \in B_r(u) \cap S_n$

we have $\frac{1}{N} > r > |e-u| \geq dist(e,K) > \frac{1}{n+1}$. Thus if $B_r(u) \cap S_n \neq \emptyset$, then $n \geq N$. This observation justifies the second and the third equalities below:

$$\begin{aligned}\lambda(F \cap B_r(u)) &= \lambda(K \cap B_r(u)) + \sum_{n=1}^{\infty} \lambda(K_n \cap B_r(u)) \\ &= \lambda(K \cap B_r(u)) + \sum_{n=N}^{\infty} \lambda(K_n \cap B_r(u)) \\ &\geq \lambda(K \cap B_r(u)) + \sum_{n=N}^{\infty} \lambda(S_n \cap B_r(u)) - \sum_{n=N}^{\infty} \frac{1}{2^n} \\ &= \lambda(E' \cap B_r(u)) - \frac{1}{2^{N-1}} = \lambda(E \cap B_r(u)) - \frac{1}{2^{N-1}}.\end{aligned}$$

Hence

$$\lambda(E \cap B_r(u)) \geq \lambda(F \cap B_r(u)) \geq \lambda(E \cap B_r(u)) - \frac{1}{2^{N-1}}. \tag{1}$$

Since $r \geq \frac{1}{N+1}$, it follows that $\frac{1}{Br^m} \leq \frac{(N+1)^m}{B}$. From (1) we get

$$\frac{\lambda(E \cap B_r(u))}{Br^m} \geq \frac{\lambda(F \cap B_r(u))}{Br^m} \geq \frac{\lambda(E \cap B_r(u))}{Br^m} - \frac{(N+1)^m}{B2^{N-1}}. \tag{2}$$

The result about equalities of the densities follows from (2) and the observation that as $r \to 0$, $N \to \infty$ so that $\frac{(N+1)^m}{B2^{N-1}} \to 0$.

Finally if E has positive upper density at every $x \in K$, then $d^+(F,x) = d^+(E,x) > 0$. Hence x can't be an isolated point of F. Thus F is perfect. □

When we are working on the real line it is common to consider one-sided left and right upper and lower densities. The theorem remains true if densities under consideration are one-sided.

References

[1] W. Bogomolova, *Ob odnom klassie funkcij wsudu assimptoticeski neprerywnych,* Mat. Sb., **32** (1924), 152–171.

[2] A. Bruckner, *Differentiation of Real Functions,* CRM Monograph Series, American Mathematical Society, (1994).

[3] L. I. Kaplan and S. G. Slobodnik, *Monotone transformations and differential properties of functions,* Mat. Zametki, **22(6)** (1977), 859–871.

[4] I. Maximoff, *On density points and approximetly continuous functions,* Tôhoku Math. J., **47** (1940), 237–250.

[5] J. Mioduszewski, *Urysohn lemma or Luzin-Menshov theorem?,* Monograph on the occasion of 100th birthday anniversary of Zygmunt Zahorski, Wydaw. Politech. l., Gliwice, (2015), 191–199.

[6] R. O'Malley, *A density property and applications,* Trans. Amer. Math. Soc., **199** (1974) 75–87.

[7] W. Rudin, *Principles of Mathematical Analysis,* International Series in Pure and Applied Mathematics, (1976).

[8] W. Rudin, *Real and Complex Analysis,* McGraw-Hill Series in Higher Mathematics, (1987).

[9] Z. Zahorski, *Sur la premiére dèrivèe*, Trans. Amer. Math. Soc., **69** (1950), 1–54.

[10] Z. Zahorski, *Über die Menge der Punkte in welchen die ableitung unendlich ist,* Tôhoku Math. J., **48** (1941), 321–330.