PLENARY LECTURE

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MEASURING ANISOTROPY IN PLANAR SETS

Abstract

We define and discuss a pure mathematics formulation of an approach proposed in the physics literature to analysing anistropy of fractal sets.

1 Introduction.

There are numerous real world phenomena that display fractal characteristics and many notions of dimension have been introduced to try to measure the complexity of such objects. However objects with the same dimension may appear very different in structure, such as displaying differing amounts of isotropy. For example, the computer simulations displayed in Figure 1 have similar (correlation) dimension but visually appear very different. There have been sporadic attempts over the years to define appropriate measures of (an)isotropy (see for Example [1]), but most have suffered from being difficult to calculate either numerically or theoretically. In this note, we describe a numerical approach first given in [2] and show how to put it on sound theoretical footing.

1.1 The numerical physics approach taken in [2]

Suppose you are given a (finite) point set in the plane and fix some parameter $\alpha \in (0, 1)$.

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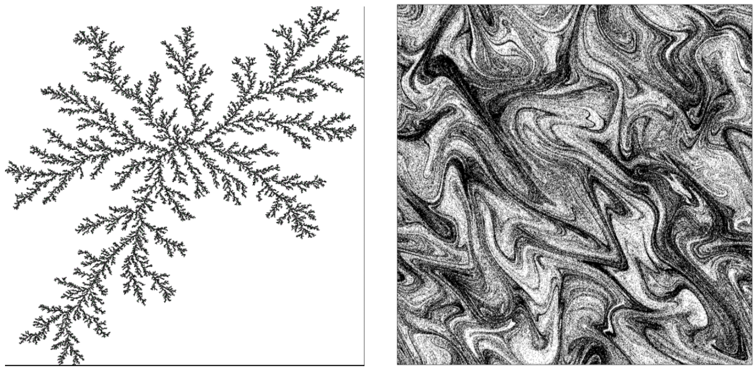

Figure 1: **left:** (Numerical) DLA with correlation dimension $\simeq 1.71$
right: Numerical simulation of inertial particles in an area preserving velocity field with correlation dimension $\simeq 1.76$. (Both images from [2])

For each point x in the set, determine the maximum number of points of the set that can be covered by an ellipse centred on x that has semi-minor axis of length ε and semi-major axis of length ε^α: denote the number of points covered by this ellipse by $N(\varepsilon, \alpha)$ (we are suppressing the dependence on x).

Do this for each point in the set and compute the *mean* value, $\langle N(\varepsilon, \alpha)\rangle$, of the maximum number of points covered by an appropriate ellipse. If the set has a regular enough structure (and enough points in the sample), then one should find that for a large range of $\varepsilon > 0$,

$$\langle N(\varepsilon, \alpha)\rangle \sim \varepsilon^{\beta(\alpha)},$$

where the exponent β depends solely upon α.

In the extreme case where $\alpha = 1$, the associated ellipse is a circle, and we should obtain the correlation dimension: $D_2 = \beta(1)$ (definition follows). When $\alpha = 0$, we are effectively maximising over 'infinite' strips of width ε.

Definition 1. *The upper and lower correlation dimension of a (finite Borel regular) measure μ on the plane are given by*

$$\overline{D}_2(\mu) = \limsup_{\varepsilon \searrow 0} \frac{\log \int \mu(B(x,\varepsilon))\, d\mu(x)}{\log \varepsilon}$$

and

$$\underline{D}_2(\mu) = \liminf_{\varepsilon \searrow 0} \frac{\log \int \mu(B(x,\varepsilon))\, d\mu(x)}{\log \varepsilon},$$

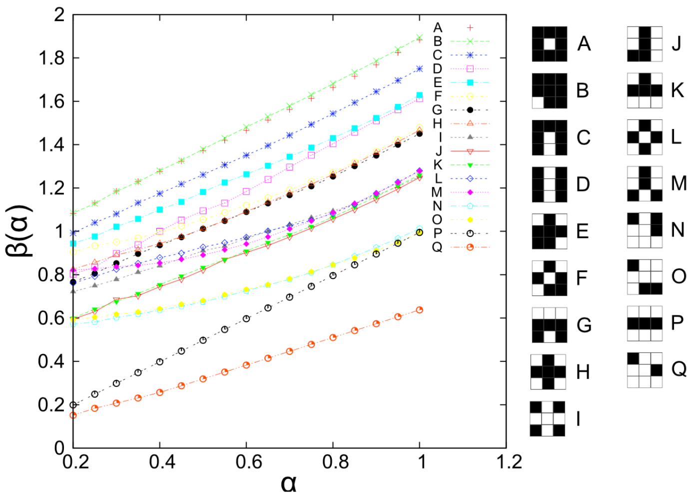

Figure 2: Estimates of $\beta(\alpha)$ for particular self-similar sets. (From [2]).

respectively. When these values agree, we obtain the correlation dimension of μ*,* $D_2(\mu)$*.*

It is easy to verify that for a point set based on a line segment, one obtains $\beta(\alpha) = \alpha$ and for the unit square, one obtains $\beta(\alpha) = 1 + \alpha$ and, arguing heuristically, the authors claimed [2] that for any locally isotropic point set,

$$\beta(\alpha) \leq 1 + \alpha(D_2 - 1), \tag{1}$$

where D_2 is the correlation dimension of the set (but see Theorem 2).

Using this approach, they numerically investigated $\beta(\alpha)$ for computer-generated point sets that had fractal-like structure. The main class of examples they looked at were self-similar sets constructed from dividing a unit square into nine similar subsquares, selecting a fixed subcollection for retention, and then iterating. Figure 2, taken from their paper, summarises their results.

2 Turning this approach into rigorous mathematics

A finite point set $\{x_1, \dots, x_n\}$ can be represented as a sum of dirac measures

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i},$$

which is a finite Borel measure. For $0 < \alpha \leq 1$ and ε small, we find

$$\langle N(\varepsilon, \alpha)\rangle = \int \max_{\theta \in [-\pi/2, \pi/2)} \mu(\mathcal{E}(x, \theta, \varepsilon^{\alpha}, \varepsilon))\, d\mu(x),$$

where $\mathcal{E}(x, \theta, \varepsilon^{\alpha}, \varepsilon)$ denotes the (closed) ellipse centered on x, with semi-minor axis of length ε, semi-major axis of length ε^{α} and inclined at an angle of θ to the positive x-axis.

Conversely, finite Borel regular measures can (in an appropriate sense) be well-approximated by sums of dirac measures. This leads us to the following definition of the anistropic dimension spectrum.

Definition 2. *For μ a finite Borel regular measure on the plane and $0 \leq \alpha \leq 1$, define the upper anistropic dimension spectrum by*

$$\overline{\beta}_{\mu}(\alpha) = \limsup_{\varepsilon \searrow 0} \frac{\log \mathbf{E}_{\mu}(\varepsilon, \alpha)}{\log \varepsilon}$$

and define the lower anisotropic dimension spectrum by

$$\underline{\beta}_{\mu}(\alpha) = \liminf_{\varepsilon \searrow 0} \frac{\log \mathbf{E}_{\mu}(\varepsilon, \alpha)}{\log \varepsilon},$$

where

$$\mathbf{E}_{\mu}(\varepsilon, \alpha) := \int \operatorname*{ess\,sup}_{\theta \in [-\pi/2, \pi/2)} \mu(\mathcal{E}(x, \theta, \varepsilon^{\alpha}, \varepsilon))\, d\mu(x).$$

(When $\alpha = 1$, we recover the upper and lower correlation) dimensions of μ.)

(The replacement of max by ess sup in the above is helpful theoretically and does not change the value of the integrand for finite sums of dirac measures.)

The following estimate allows us to turn an apparently hard estimate involving the mass of ellipses into a more symmetric one involving the mass of balls.

Lemma 1. *Suppose $0 < r \leq s \leq 1$. Then*

$$\int_{\theta} \mu(\mathcal{E}(x, \theta, \varepsilon^{\alpha}, \varepsilon))\, d(\theta/\pi) \asymp \frac{r}{s}\mu(B(x, s)) + r\int_{r}^{s} \sigma^{-2}\mu(B(x, \sigma))\, d\sigma.$$

(Here $f(r) \asymp g(r)$ means that there are positive constants c_1 and c_2 so that $c_1 g(r) \le f(r) \le c_2 g(r)$ for sufficiently small r.) It is also possible to obtain similar but (more complicated looking) expressions for higher powers of the mass of ellipses and we note that, since

$$\begin{aligned} &\operatorname*{ess\,sup}_{\theta\in[-\pi/2,\pi/2)} \mu(\mathcal{E}(x,\theta,\varepsilon^\alpha,\varepsilon)) \\ &\qquad = \lim_{p\to\infty} \left(\int_{|\theta|\le\pi/2} \mu(\mathcal{E}(x,\theta,\varepsilon^\alpha,\varepsilon))^p \, d\theta/\pi \right)^{1/p}, \end{aligned}$$

estimates of $\int \mu(\mathcal{E}(x,\theta,\varepsilon^\alpha,\varepsilon))^p d\theta/\pi$ (for large p) gives information about the anisotropy spectrum.

We can use Lemma 1 to derive the correct upper estimate of the anisotropic dimension spectrum for isotropic measures, compare with equation (1):

Theorem 2. *If μ is a (compact, finite Radon) measure in the plane and $0 < \alpha \le 1$, then*

$$\overline{\beta}_\mu(\alpha) \le \min\{\overline{D}_2(\mu),\, 1 + \alpha(\overline{D}_2(\mu) - 1)\}.$$

If μ is supported in a line segment with $\mu(B(x,r)) \asymp r^{D_2(\mu)}$ for almost every x, then we expect to find that for small ε,

$$\int \operatorname*{ess\,sup}_{\theta} \mu(\mathcal{R}(x,\theta,\varepsilon^\alpha,\varepsilon)) \, d\mu(x) \asymp \varepsilon^{\alpha D_2}$$

and then

$$\beta_\mu(\alpha) \asymp \alpha D_2.$$

2.1 Examples

In all the examples illustrated in Figure 2, we associate the natural uniform measure that is supported on the limiting self-similar set.

If the measure μ has support in a line, then it is easy to see that

$$\operatorname{ess\,sup} \mu(\mathcal{R}(x,\theta,\varepsilon^\alpha,\varepsilon)) = \mu(B(x,\varepsilon^\alpha))$$

and so

$$\overline{\beta}_\mu(\alpha) = \alpha\overline{D}_2(\mu) \text{ and } \underline{\beta}_\mu(\alpha) = \alpha\underline{D}_2(\mu).$$

This situation occurs for examples P and Q in Figure 2.

In case D of Figure 2, the resulting measure is the product of length measure on the unit interval together with the uniform measure on the 1/3-Cantor set

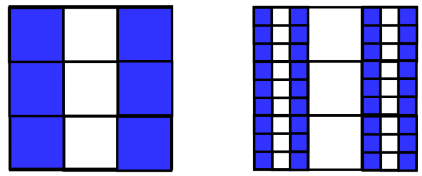

Figure 3: Construction of D in Figure 2.

For this example, it is not too hard to show that maximal ellipses are aligned vertically and we find that for a general point x,

$$\operatorname{ess\,sup} \mu(\mathcal{E}(x, \theta, \varepsilon^\alpha, \varepsilon)) \asymp \varepsilon^{\log 6/\log 3 + \alpha - 1},$$

giving

$$\beta_\mu(\alpha) = \log 6/\log 3 + \alpha - 1 = \alpha + \log 2/\log 3.$$

For more general cross-products such as denoted by Case I, there is a lot of symmetry and we find that for a general point x, if μ is the natural uniform measure defined on the limiting self-similar set, then

$$\operatorname{ess\,sup} \mu(\mathcal{R}(x, \theta, \varepsilon^\alpha, \varepsilon)) \asymp \varepsilon^{(1+\alpha)\log 2/\log 3},$$

giving

$$\beta_\mu(\alpha) = (1 + \alpha)\log 2/\log 3.$$

We are able to deal with some of the other examples given in Figure 2 but do not have a general approach that works in all cases and in particular, we do not have a technique that can cope with any reasonable self-similar set.

References

[1] P. Grassberger. Generalizations of the Hausdorff dimension of fractal measures. Phys. Lett., A107, 101-5, (1985)

[2] M. Wilkinson, H. R. Kennard, M. A. Morgan. Anisotropic covering of fractal sets. arXiV: 1204.3718v1