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ON SOME GAMIDOV INTEGRAL INEQUALITIES ON TIME SCALES AND APPLICATIONS

Abstract

In the present paper, we extend Gamidov's integral inequalities to time scales. The obtained results can be used as tools in the study of certain properties of dynamical equations on time scales.

1 Introduction

Integral inequalities play a fundamental role in development of the linear and nonlinear differential, and integral equations theory, they present an important tools in the study of the existence, uniqueness, boundedness, stability, and other qualitative properties of solutions. One of the best known and widely used is the so-called Gronwall-Bellman integral inequality. In view of its important applications, many and various generalizations, extensions, and variants have appeared in the literature, we can mention [1, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19] and references cited therein.

Recently Hilger [12] introduced the theory of time scales in order to unify continuous and discrete analysis. Many authors have extended some fundamental integral inequalities used in differential and integral equations theory on time scales; for more detail about time scales, we refer readers to [2, 3, 14].

Bainov et al.[4] discussed the following inequality, known as Gamidov inequality

$$u(t) \leq k + \int\limits_{0}^{t} g(s)u(s)ds + \int\limits_{0}^{T} h(s)u(s)ds, \tag{1}$$

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where k is a nonnegative constant and T is a positive real number.

Pachpatte [18] gave an extension of (1) as follows:

$$u(t) \leq k + \int_0^t g(t,s)u(s)ds + \int_0^T h(t,s)u(s)ds, \tag{2}$$

where k is a nonnegative constant and T is a positive real number.

Motivated by the above results, in the present paper, we extend Gamidov's integral inequalities to time scales. The obtained results can be used as tools in the study of certain properties of dynamical equations on time scales.

2 Preliminaries

In this section, we recall without proof some fundamental definitions and results from the calculus on time scales; for more details about time scales, we refer the reader to [7].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^\kappa$ which is derived from time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t) f(t) \neq 0$, $t \in \mathbb{T}^\kappa$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions and $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0\}$. We denote by $[a,b]_\mathbb{T}$ the interval in $\mathbb{T}$ which is defined by $\{t \in \mathbb{T} : a \leq t \leq b\}$ where a and b are points in $\mathbb{T}$ with $a < b$.

Definition 1. *[7, Definition 1.10] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| < \varepsilon \ |\sigma(t) - s| \quad \textit{for all } s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

Lemma 2. *[7, Theorem 1.117] Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$ and assume $g : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $g^\Delta(t,.)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$\left|g(\sigma(t),\tau) - g(s,\tau) - g^\Delta(t,\tau)(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s| \quad \textit{for all } s \in U, \tag{3}$$

where g^{Δ} denotes the derivative of g with respect to the first variable. Then

$$(i)\ g(t) := \int_a^t f(t,\tau)\Delta\tau \quad \textit{implies } g^{\Delta}(t) = \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t),t).$$

Lemma 3. *[7, Comparison Theorem] Let $u, b \in C_{rd}$ and $a \in \mathcal{R}^{+}$. If*

$$u^{\Delta}(t) \le a(t)\,u(t) + b(t), \quad \textit{for all } t \in \mathbb{T}$$

then

$$u(t) \le u(t_0)e_a(t,t_0) + \int_{t_0}^t b(\tau)e_a(t,\sigma(\tau))\Delta\tau, \quad \textit{for all } t \in \mathbb{T}.$$

Lemma 4. *[13, Lemma 2] Assume that $a \ge 0$, $p \ge q \ge 0$ and $p \ne 0$, for any $\xi > 0$, we have*

$$a^{\frac{q}{p}} \le \tfrac{q}{p}\xi^{\frac{q-p}{p}}a + \tfrac{p-q}{p}\xi^{\frac{q}{p}}.$$

3 Main results

Theorem 5. *Let $u(t), f(t), h(t), g(t), w(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^{+})$ such that $h(t), h(t)$ are positive functions and $g(t)$ is a nondecreasing function for all $t \in [a,b]_{\mathbb{T}}$. If the following inequality*

$$u(t) \le f(t) + h(t)\int_a^t g(s)u(s)\Delta s + \int_a^b w(s)u(s)\Delta s \tag{4}$$

holds then $u(t)$ has the following estimate

$$u(t) \le f(t) + h(t)\left(L(t) + e_{hg}(t,a)\frac{\int_a^b w(t)L(t)\Delta t}{1 - \int_a^b w(t)e_{hg}(t,a)\Delta t} \right), \tag{5}$$

where

$$L(t) = \int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau + e_{hg}(t,a)\int_a^b w(s)h^{-1}(s)f(s)\Delta s, \tag{6}$$

provided $\int_a^b w(t)e_{hg}(t,a)\Delta t < 1$.

Proof. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (4) as follows

$$u(t) \leq f(t) + h(t)\left[\int_a^t g(s)u(s)\Delta s \ + \int_a^b w(s)h^{-1}(s)u(s)\Delta s\right]. \tag{7}$$

Define a function $z(t)$ by

$$z(t) = \int_a^t g(s)u(s)\Delta s \ + \int_a^b w(s)h^{-1}(s)u(s)\Delta s. \tag{8}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq f(t) + h(t)z(t), \tag{9}$$

and

$$z(a) = \int_a^b w(s)h^{-1}(s)u(s)\Delta s. \tag{10}$$

Differentiating (8) with respect to t, we obtain

$$z^{\Delta}(t) = g(t)u(t). \tag{11}$$

Substituting (9) into (11), we get

$$z^{\Delta}(t) \leq g(t)f(t) + h(t)g(t)z(t). \tag{12}$$

According to Lemma 3, (12) gives

$$z(t) \leq z(a)e_{hg}(t,a) + \int_a^t g(\tau)f(\tau)e_{hg}(t,\sigma(\tau))\Delta\tau.$$

Using (10), and then (9) in the above inequality, we get

$$z(t) \leq L(t) + e_{hg}(t,a)\int_a^b w(s)z(s)\Delta s, \tag{13}$$

where $L(t)$ is given by (6).

Clearly $\int_a^b w(t)z(t)\Delta t$ is constant. Multiplying both sides of (13) by $w(t)$, then integrating the resultant inequality with respect to t from a to b, we obtain

$$\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)L(t)\Delta t + \left(\int_a^b w(s)z(s)\Delta s\right)\left(\int_a^b w(t)e_{hg}(t,a)\Delta t\right). \tag{14}$$

Since $\int_a^b w(t)e_{hg}(t,a)\Delta t < 1$, from (14) we have

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)L(t)\Delta t}{1 - \int_a^b w(t)e_{hg}(t,a)\Delta t}. \tag{15}$$

Combining (15), (13), and (9), we obtain the desired inequality. $\square$

Corollary 6. *Assume that all the assumptions of Theorem 5 are satisfied, and let $\mathbb{T} = \mathbb{Z}$. If the following inequality*

$$u(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(\tau)u(\tau) + \sum_{s=a}^{b-1} w(s)u(s)$$

holds then $u(t)$ has the following estimate

$$u(t) \leq f(t) + h(t)\left(L(t) + \frac{\prod_{\tau=a}^{\tau=t}[1 + h(\tau)g(\tau)]\sum_{s=a}^{b-1} w(s)L(s)}{1 - \sum_{s=a}^{b-1} w(s)\prod_{\tau=a}^{\tau=s}[1 + h(\tau)g(\tau)]}\right),$$

where

$$\begin{aligned} L(t) = &\sum_{s=a}^{t-1}\left(\prod_{\tau=s+1}^{\tau=t}[1 + h(\tau)g(\tau)]\right)g(s)f(s) \\ &+ \prod_{s=a}^{s=t}[1 + h(s)g(s)]\sum_{s=a}^{b-1} w(s)h^{-1}(s)f(s), \end{aligned}$$

provided $\sum_{s=a}^{b-1}\left(w(s)\prod_{\tau=a}^{\tau=s}[1 + h(\tau)g(\tau)]\right) < 1$, with $h(t)g(t) \neq -1$ for all $t \in [a,b]_{\mathbb{T}}$.

Theorem 7. *Let $p, q, r, \xi \in \mathbb{R}_0^+$ such that $p \geq q > 0$, $p \geq r > 0$. Assume that all the assumptions of Theorem 5 are satisfied, furthermore*

$$\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\Delta t < 1. \tag{16}$$

If the following inequality

$$u^p(t) \leq f(t) + h(t)\int_a^t g(s)u^q(s)\Delta s \ + \int_a^b w(s)u^r(s)\Delta s \tag{17}$$

holds then $u(t)$ has the following estimate

$$u(t) \leq \left\{ f(t) + h(t)\left(L(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}}e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\int_a^b w(t)L(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\Delta t} \right) \right\}^{\frac{1}{p}}, \tag{18}$$

where

$$\begin{aligned} L(t) = & \int_a^t \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}f(\tau) + \tfrac{p-q}{p}\xi^{\frac{q}{p}} \right) g(\tau)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,\sigma(\tau))\Delta\tau \\ & + e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\int_a^b w(s)h^{-1}(s)\left(\tfrac{r}{p}\xi^{\frac{r-p}{p}}f(s) + \tfrac{p-r}{p}\xi^{\frac{r}{p}} \right)\Delta s \end{aligned} \tag{19}$$

PROOF. Since $h(t)$ is positive and monotonic nondecreasing, we can restate (17) as follows

$$u^p(t) \leq f(t) + h(t)\left(\int_a^t g(s)u^q(s)\Delta s \ + \int_a^b w(s)h^{-1}(s)u^r(s)\Delta s \right).$$

Denoting by $z(t)$

$$z(t) = \int_a^t g(s)u^q(s)\Delta s \ + \int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \tag{20}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq \{f(t) + h(t)z(t)\}^{\frac{1}{p}}, \tag{21}$$

and

$$z(a) = \int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \tag{22}$$

Differentiating (20) with respect to t, then using (21) in the resultant equality, we obtain

$$\begin{aligned} z^{\Delta}(t) =& g(t)u^q(t) \\ \leq& g(t)\{f(t) + h(t)z(t)\}^{\frac{q}{p}}. \end{aligned} \tag{23}$$

Now, Applying Lemma 4 for (23) we get

$$z^{\Delta}(t) \leq \tfrac{q}{p}\xi^{\frac{q-p}{p}} g(t)h(t)z(t) + \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(t) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) g(t). \tag{24}$$

According to Lemma 3, we have

$$\begin{aligned} z(t) \leq& z(a)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a) \\ &+ \int_a^t \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(\tau) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) g(\tau)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,\sigma(\tau))\Delta\tau. \end{aligned} \tag{25}$$

Substituting (22) in (25), we obtain

$$\begin{aligned} z(t) \leq& \int_a^t \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(\tau) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) g(\tau)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,\sigma(\tau))\Delta\tau \\ &+ e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \end{aligned} \tag{26}$$

Using (21) in (26), and then applying Lemma 4 for the resultant inequality, we get

$$z(t) \leq L(t) + \tfrac{r}{p}\xi^{\frac{r-p}{p}} e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\int_a^b w(s)z(s)\Delta s, \tag{27}$$

where $L(t)$ is defined by (19).
Now, multiplying both sides of (27) by $w(t)$, then integrating the result with respect to t over $[a,b]$, we get

$$\int_a^b w(t)z(t)\Delta t \leq \frac{r}{p}\xi^{\frac{r-p}{p}}\left(\int_a^b w(t)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\Delta t\right)\int_a^b w(s)z(s)\Delta s + \int_a^b w(t)L(t)\Delta t. \tag{28}$$

From (16) and (28) we have

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)L(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_{\frac{q}{p}\xi^{\frac{q-p}{p}}gh}(t,a)\Delta t}. \tag{29}$$

Combining (29), (27), and (21), we get the desired result. □

Corollary 8. *Assume that all the assumptions of Theorem 7 are satisfied, and let* $\mathbb{T}=\mathbb{Z}$ *. If the following inequality*

$$u^p(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(\tau)u^q(\tau) + \sum_{s=a}^{b-1} w(s)u^r(s)$$

holds then $u(t)$ *has the following estimate*

$$u(t) \leq \left\{f(t)+h(t)\left(L(t)+\frac{\frac{r}{p}\xi^{\frac{r-p}{p}}\prod_{\tau=a}^{\tau=t}\left[1+\frac{q}{p}\xi^{\frac{q-p}{p}}h(\tau)g(\tau)\right]\sum_{s=a}^{b-1}w(s)L(s)}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\sum_{s=a}^{b-1}w(s)\prod_{\tau=a}^{\tau=s}\left[1+\frac{q}{p}\xi^{\frac{q-p}{p}}h(\tau)g(\tau)\right]}\right)\right\}^{\frac{1}{p}},$$

where

$$\begin{aligned} L(t) =& \sum_{s=a}^{t-1}\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}f(s)+\tfrac{p-q}{p}\xi^{\frac{q}{p}}\right)g(s)\left(\prod_{\tau=s+1}^{\tau=t}\left[1+\tfrac{q}{p}\xi^{\frac{q-p}{p}}h(\tau)g(\tau)\right]\right) \\ &+ \prod_{s=a}^{s=t}\left[1+\tfrac{q}{p}\xi^{\frac{q-p}{p}}h(s)g(s)\right]\sum_{s=a}^{t-1}w(s)h^{-1}(s)\left(\tfrac{r}{p}\xi^{\frac{r-p}{p}}f(s)+\tfrac{p-r}{p}\xi^{\frac{r}{p}}\right), \end{aligned}$$

provided $\frac{r}{p}\xi^{\frac{r-p}{p}}\sum_{s=a}^{b-1}w(s)\prod_{\tau=a}^{\tau=s}\left[1+\frac{q}{p}\xi^{\frac{q-p}{p}}h(\tau)g(\tau)\right]<1$
and $\frac{q}{p}\xi^{\frac{q-p}{p}}h(\tau)g(\tau)\neq -1$ *for all* $t\in[a,b]_{\mathbb{T}}$.

Theorem 9. *Let* $u(t), f(t), h(t), w(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ *with* $h(t)$ *a positive function, and let* $g(t,s)$ *be defined as in Lemma 2 such that* $g^{\Delta}(t,s) \geq 0$ *for all* $t \geq s$ *satisfying* (3). *Assume that*

$$\int_a^b w(t) e_L(t,a) \Delta t < 1. \tag{30}$$

If the following inequality

$$u(t) \leq f(t) + h(t) \int_a^t g(t,s) u(s) \Delta s + \int_a^b w(s) u(s) \Delta s \tag{31}$$

holds then $u(t)$ *has the following estimate*

$$u(t) \leq f(t) + h(t) \left(Q(t) + \frac{e_L(t,a) \int_a^b w(t) Q(t) \Delta t}{1 - \int_a^b w(t) e_L(t,a) \Delta t} \right), \tag{32}$$

where

$$L(t) = g(\sigma(t), t) h(t) + \int_a^t g^{\Delta}(t,s) h(s) \Delta s, \tag{33}$$

and

$$\begin{aligned} Q(t) = &\int_a^t \left(g(\sigma(\tau), \tau) f(\tau) + \int_a^\tau g^{\Delta_t}(\tau, s) f(s) \Delta s \right) e_L(t, \sigma(\tau)) \Delta \tau \\ &+ e_L(t,a) \int_a^b w(s) h^{-1}(s) f(s) \Delta s. \end{aligned} \tag{34}$$

PROOF. Since $h(t)$ is positive and monotonic nondecreasing we can restate (31) as follows

$$u(t) \leq f(t) + h(t) \left(\int_a^t g(t,s) u(s) \Delta s + \int_a^b w(s) h^{-1}(s) u(s) \Delta s \right).$$

Define a function $z(t)$ by

$$z(t)=\int_a^t g(t,s)u(s)\Delta s\ +\int_a^b w(s)h^{-1}(s)u(s)\Delta s. \tag{35}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t)\leq f(t)+h(t)z(t), \tag{36}$$

and

$$z(a)=\int_a^b w(s)h^{-1}(s)u(s)\Delta s. \tag{37}$$

Differentiating (35) with respect to t it yields

$$z^{\Delta}(t)=g(\sigma(t),t)u(t)+\int_a^t g^{\Delta_t}(t,s)u(s)\Delta s. \tag{38}$$

Substituting (36) into (38) we get

$$\begin{aligned} z^{\Delta}(t)\leq & g(\sigma(t),t)f(t)+\int_a^t g^{\Delta_t}(t,s)f(s)\Delta s \\ & +g(\sigma(t),t)h(t)z(t)+\int_a^t g^{\Delta_t}(t,s)h(s)z(s)\Delta s. \end{aligned} \tag{39}$$

Since $z(t)$ is monotonic nondecreasing, we can restate (38) as follows

$$z^{\Delta}(t)\leq g(\sigma(t),t)f(t)+\int_a^t g^{\Delta_t}(t,s)f(s)\Delta s+L(t)z(t). \tag{40}$$

where $L(t)$ is defined as in (33).
Applying Lemma 3 for (40), we obtain

$$\begin{aligned} z(t)\leq & z(a)e_L(t,a) \\ & +\int_a^t\left(g(\sigma(\tau),\tau)f(\tau)+\int_a^\tau g^{\Delta_t}(\tau,s)f(s)\Delta s\right)e_L(t,\sigma(\tau))\Delta\tau. \end{aligned} \tag{41}$$

Now, substituting (37) in (41), then using (36) in the resultant inequality, we get

$$z(t) \leq Q(t) + e_L(t,a)\int_a^b w(s)z(s)\Delta s, \tag{42}$$

where $Q(t)$ is defined by (34).
Now, multiplying both sides of (42) by $w(t)$ then integrating the resultant inequality with respect to t from a to b it yields

$$\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)Q(t)\Delta t + \left(\int_a^b w(t)e_L(t,a)\Delta t\right)\int_a^b w(s)z(s)\Delta s. \tag{43}$$

From (30), (43) gives

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)Q(t)\Delta t}{1-\int_a^b w(t)e_L(t,a)\Delta t}. \tag{44}$$

Combining (44), (42), and (36), we obtain the desired inequality. □

Corollary 10. *Assume that all the assumptions of Theorem 9 are satisfied, and let $\mathbb{T} = \mathbb{Z}$. If the following inequality*

$$u(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(t,\tau)u(\tau) + \sum_{s=a}^{b-1} w(s)u(s)$$

holds then $u\,(t)$ has the following estimate

$$u(t) \leq f(t) + h(t)\left(Q(t) + \frac{\prod_{\tau=a}^{\tau=t}[1+L(\tau)]\sum_{s=a}^{b-1} w(s)Q(s)}{1-\sum_{s=a}^{s=b-1}\left(w(s)\prod_{\tau=a}^{\tau=s}[1+L(\tau)]\right)}\right),$$

where

$$L(t) = g(t+1,t)h(t) + \sum_{s=a}^{t-1}\left(g(t+1,s)-g(t,s)\right)h(t),$$

and

$$Q(t) = \sum_{\tau=a}^{t-1}\left[\left(g(\tau+1,\tau)f(\tau)+\sum_{s=a}^{\tau-1}\left(g(\tau+1,s)-g(\tau,s)\right)f(s)\right)\right.$$
$$\left.\times\prod_{s=\tau+1}^{s=t}\left[1+L\left(s\right)\right]\right]+\prod_{s=a}^{s=t}\left[1+L\left(s\right)\right]\sum_{s=a}^{b-1}w(s)h^{-1}(s)f(s).$$

Provided $\sum\limits_{s=a}^{b-1}\left(w(s)\prod_{\tau=a}^{\tau=s}\left[1+L\left(\tau\right)\right]\right)<1,$ *and* $L\left(t\right)\neq -1$ *for all* $t\in\left[a,b\right]_{\mathbb{T}}$.

Theorem 11. *Under the assumptions of Theorem 9, and let* $p,q,r,\xi\in\mathbb{R}_0^+$ *such that* $p\geq q>0$, $p\geq r>0$. *Assume that*

$$\tfrac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t<1. \tag{45}$$

If the following inequality

$$u^p(t)\leq f(t)+h(t)\int_a^t g(t,s)u^q(s)\Delta s+\int_a^b w(s)u^r(s)\Delta s \tag{46}$$

holds then $u\left(t\right)$ *has the following estimate*

$$u(t)\leq\left\{f(t)+h(t)\left(K(t)+\frac{\frac{r}{p}\xi^{\frac{r-p}{p}}e_Q(t,a)\int_a^b w(t)K(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t}\right)\right\}^{\frac{1}{p}}, \tag{47}$$

where

$$L(t)=\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}f(t)+\tfrac{p-q}{p}\xi^{\frac{q}{p}}\right)g(\sigma\left(t\right),t)$$
$$+\int_a^t g^{\Delta t}(t,s)\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}f(s)+\tfrac{p-q}{p}\xi^{\frac{q}{p}}\right)\Delta s, \tag{48}$$

$$Q(t)=\tfrac{q}{p}\xi^{\frac{q-p}{p}}h(t)g(\sigma\left(t\right),t)+\tfrac{q}{p}\xi^{\frac{q-p}{p}}\int_a^t g^{\Delta t}(t,s)h(s)\Delta s, \tag{49}$$

and

$$K(t) = \int_a^t L(\tau) e_Q(t, \sigma(\tau)) \Delta\tau + e_Q(t, a) \int_a^b w(s) h^{-1}(s) \left(\tfrac{r}{p} \xi^{\frac{r-p}{p}} f(s) + \tfrac{p-r}{p} \xi^{\frac{r}{p}} \right) \Delta s. \tag{50}$$

PROOF. Since $h(t)$ is positive and monotonic nondecreasing we can restate (46) as follows

$$u^p(t) \le f(t) + h(t) \left(\int_a^t g(t, s) u^q(s) \Delta s + \int_a^b w(s) h^{-1}(s) u^r(s) \Delta s \right)$$

Denoting by $z(t)$

$$z(t) = \int_a^t g(t, s) u^q(s) \Delta s + \int_a^b w(s) h^{-1}(s) u^r(s) \Delta s \tag{51}$$

It is clear that $z(t)$ is nonnegative, nondecreasing,

$$u(t) \le \{ f(t) + h(t) z(t) \}^{\frac{1}{p}}, \tag{52}$$

and

$$z(a) = \int_a^b w(s) h^{-1}(s) u^r(s) \Delta s. \tag{53}$$

Differentiating (51) with respect to t, then using (52) we get

$$\begin{aligned} z^{\Delta}(t) =& g(\sigma(t), t) u^q(t) + \int_a^t g^{\Delta t}(t, s) u^q(s) \Delta s \\ \le& g(\sigma(t), t) \left(f(t) + h(t) z(t) \right)^{\frac{q}{p}} & (54) \\ &+ \int_a^t g^{\Delta t}(t, s) \left(f(s) + h(s) z(s) \right)^{\frac{q}{p}} \Delta s. & (55) \end{aligned}$$

Now, Applying Lemma 4 for (54) we obtain

$$z^{\Delta}(t) \leq \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(t) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) g(\sigma(t), t) + \int_a^t g^{\Delta t}(t,s)\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(s) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right)\Delta s + \tfrac{q}{p}\xi^{\frac{q-p}{p}} h(t)z(t)g(\sigma(t), t) + \tfrac{q}{p}\xi^{\frac{q-p}{p}}\int_a^t g^{\Delta t}(t,s)h(s)z(s)\Delta s. \tag{56}$$

Since $z(t)$ is monotonic nondecreasing, (55) can be restated

$$z^{\Delta}(t) \leq L(t) + Q(t)z(t), \tag{57}$$

where $L(t)$ and $Q(t)$ are defined by (48) and (49) respectively.
According to Lemma 3, (56) gives

$$z(t) \leq z(a)e_Q(t,a) + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{58}$$

Substituting (53) in (57), we obtain

$$z(t) \leq \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau + e_Q(t,a)\int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \tag{59}$$

Using (52) in (57), then applying Lemma 4 for the resultant inequality we get

$$z(t) \leq K(t) + \tfrac{r}{p}\xi^{\frac{r-p}{p}} e_Q(t,a)\int_a^b w(s)z(s)\Delta s, \tag{60}$$

where $K(t)$ is defined in (50).
Multiplying both sides of (59) by $w(t)$, then integrating the resultant inequality with respect to t over $[a,b]_{\mathbb{T}}$, we get

$$\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)K(t)\Delta t + \left(\tfrac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t\right)\int_a^b w(s)z(s)\Delta s. \tag{61}$$

From (45), (60) gives

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)K(t)\Delta t}{1 - \frac{r}{p}\xi^{\frac{r-p}{p}} \int_a^b w(t)e_Q(t,a)\Delta t}. \tag{62}$$

Combining (61), (59), and (52), we get the desired result. □

Corollary 12. *Assume that all the assumptions of Theorem 11 are satisfied, and let $\mathbb{T} = \mathbb{Z}$. If the following inequality*

$$u^p(t) \leq f(t) + h(t)\sum_{\tau=a}^{t-1} g(t,\tau)u^q(\tau) + \sum_{s=a}^{b-1} w(s)u^r(s)$$

holds then $u(t)$ has the following estimate

$$u(t) \leq \left\{ f(t) + h(t)\left(K(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}} \prod_{\tau=a}^{\tau=t}[1+Q(\tau)] \sum_{s=a}^{b-1} w(s)K(s)}{1-\frac{r}{p}\xi^{\frac{r-p}{p}} \sum_{s=a}^{b-1}\left(w(s)\prod_{\tau=a}^{\tau=s}[1+Q(\tau)]\right)} \right) \right\}^{\frac{1}{p}},$$

where

$$\begin{aligned} L(t) = &\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(t) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) g(t+1,t) \\ &+ \sum_{s=a}^{t-1} \left(g(t+1,s) - g(t,s)\right)\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}} f(s) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right), \end{aligned}$$

$$Q(t) = \tfrac{q}{p}\xi^{\frac{q-p}{p}} h(t)g(t+1,t) + \tfrac{q}{p}\xi^{\frac{q-p}{p}} \sum_{s=a}^{t-1} \left(g(t+1,s) - g(t,s)\right)h(s),$$

and

$$\begin{aligned} K(t) = &\sum_{s=a}^{t-1}\left(L(s) \prod_{\tau=s+1}^{\tau=t} [1+Q(\tau)] \right) \\ &+ \prod_{s=a}^{s=t}[1+Q(s)] \sum_{s=a}^{b-1} w(s)h^{-1}(s)\left(\tfrac{r}{p}\xi^{\frac{r-p}{p}} f(s) + \tfrac{p-r}{p}\xi^{\frac{r}{p}}\right). \end{aligned}$$

Provided $\frac{r}{p}\xi^{\frac{r-p}{p}} \sum_{s=a}^{b-1}\left(w(s)\prod_{\tau=a}^{\tau=s}[1+Q(\tau)]\right) < 1$, *and* $Q(t) \neq -1$ *for all* $t \in [a,b]_{\mathbb{T}}$.

4 Applications

In this section, we present two applications of our results.
Consider the following dynamic equation on time scales

$$u^p(t) = f(t) + h(t)\int_a^t \varphi\left(t, s, u(s)\right)\Delta s + \int_a^b \psi\left(s, u(s)\right)\Delta t, \tag{63}$$

where $p > 1$ is a constant, $f, h : \mathbb{T} \to \mathbb{R}$ are right-dense continuous functions on $\mathbb{T}$, $\varphi : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a right-dense continuous function on $\mathbb{T} \times \mathbb{T}$ and continuous on $\mathbb{R}$, and $\psi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a right-dense continuous function on $\mathbb{T}$ and continuous on $\mathbb{R}$.

Proposition 13. *Assume that*

$$\begin{aligned} \left|\varphi\left(t, s, u(s)\right)\right| \leq & g(s)\left|u(s)\right|^q \\ \left|\psi\left(s, u(s)\right)\right| \leq & w(s)\left|u(s)\right|^r, \end{aligned} \tag{64}$$

where f, g, h, w and q, r satisfy the hypotheses of Theorem 7. If $u(t)$ is any solution of (63)-(64)*, then $u(t)$ satisfies the following estimate*

$$|u(t)| \leq \left\{ f(t) + h(t)\left(L(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}} e_{\frac{q}{p}\xi^{\frac{q-p}{p}} gh}(t,a)\int_a^b w(t)L(t)\Delta t}{1 - \frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t) e_{\frac{q}{p}\xi^{\frac{q-p}{p}} gh}(t,a)\Delta t} \right) \right\}^{\frac{1}{p}}, \tag{65}$$

where $L(t)$ is defined by (19)*.*

Proof. Let $u(t)$ be a solution of (63), using the properties of modulus, we obtain

$$|u(t)|^p \leq |f(t)| + |h(t)|\int_a^t \left|\varphi\left(t, s, u(s)\right)\right|\Delta s + \int_a^b \left|\psi\left(s, u(s)\right)\right|\Delta s. \tag{66}$$

Using (64) and the assumptions imposed on f, g, h and w, we can restate (66) as follows

$$|u(t)|^p \leq f(t) + h(t)\int_a^t g(s)\left|u(s)\right|^q \Delta s + \int_a^b w(s)\left|u(s)\right|^r \Delta t. \tag{67}$$

Now, an application of Theorem 7 for (67) gives the estimate (65). $\square$

Proposition 14. *Assume that*

$$\begin{aligned} |\varphi(t,s,u(s))| &\leq g(t,s)\,|u(s)|^q \\ |\psi(s,u(s))| &\leq w(s)\,|u(s)|^r, \end{aligned} \tag{68}$$

where $f(t), g(t,s), h(t)$ *and* $w(t)$ *satisfy the hypotheses of Theorem 11. If* $u(t)$ *is any solution of* (63) *and* (68)*, then* $u(t)$ *satisfies the following estimate*

$$|u(t)| \leq \left\{ f(t) + h(t)\left(K(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}} e_Q(t,a)\int_a^b w(t)K(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t} \right) \right\}^{\frac{1}{p}}, \tag{69}$$

where $L(t), Q(t)$ *and* $K(t)$ *are defined by* (48)-(50) *respectively.*

Proof. Let $u(t)$ be a solution of (63), using the properties of modulus, we obtain

$$|u(t)|^p \leq |f(t)| + |h(t)| \int_a^t |\varphi(t,s,u(s))|\,\Delta s + \int_a^b |\psi(s,u(s))|\,\Delta s. \tag{70}$$

Using (68) and the assumptions imposed on f, g, h and w, we can restate (70) as follows

$$|u(t)|^p \leq f(t) + h(t)\int_a^t g(t,s)\,|u(s)|^q\,\Delta s + \int_a^b w(s)\,|u(s)|^r\,\Delta t. \tag{71}$$

Now, an application of Theorem 11 for (71) gives the estimate (69). □

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