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# Effective Divisors in $\overline{\mathcal{M}}_{g,n}$ from Abelian Differentials

Scott Mullane

Abstract. We compute many new classes of effective divisors in $\overline{\mathcal{M}}_{g,n}$ coming from the strata of Abelian differentials. Our method utilizes maps between moduli spaces and the degeneration of Abelian differentials.

## 1. Introduction

The moduli space of Abelian differentials $\mathcal{H}(\kappa)$ consists of pairs $(C,\omega)$ where $\omega$ is a holomorphic or meromorphic differential on a smooth genus $g$ curve $C$ and the multiplicity of the zeros and poles of $\omega$ is fixed of type $\kappa$, an integer partition of $2g-2$. The previous seminal work has exposed the fundamental algebraic attributes of these spaces [KZ; Mc; EM; EMM]. From the perspective of algebraic geometry, geometrically defined codimension one subvarieties, or divisors, have been used to study many aspects of moduli spaces of curves including the Kodaira dimension and the cone of effective divisors [HMu; EH1; F1]. In this paper, we compute the class of many effective divisors in $\overline{\mathcal{M}}_{g,n}$ defined by the strata of Abelian differentials.

The divisor $D^n_\kappa$ in $\overline{\mathcal{M}}_{g,n}$ for $\kappa=(k_1,\dots,k_m)$ and $n\geq 0$ with $\sum k_i=2g-2$ is defined as

$$D^n_\kappa=\overline{\left\{[C,p_1,\dots,p_n]\in\mathcal{M}_{g,n}\middle|\exists[C,p_1,\dots,p_m]\in\mathcal{M}_{g,m}\text{ with}\sum k_ip_i\sim K_C\right\}},$$

where $m=n+g-2$ or $n+g-1$ for holomorphic and meromorphic signature $\kappa$, respectively. When all $k_i$ are even, this divisor has two irreducible components based on spin structure, which we denote by the indices *odd* and *even*. We use this notation to denote both the divisor and the class[1] of the divisor in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})\otimes\mathbb{Q}$.

The results of this paper are the computation of the class of the divisor $D^n_\kappa$ and the even and odd spin structure components of this divisor in the cases given in Table 1.

For example, the divisor $D^{2,\text{even}}_{-2,2,2^{g-1}}$ in $\overline{\mathcal{M}}_{g,2}$ is the even spin structure component of the coupled partition divisor given by $g\geq 2$, $n=2$, and $\kappa=(-2,2,2^{g-1})$.

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[1] Our class expressions are given modulo the labeling of the unmarked points. See Section 2.8 for an explicit description.

**Table 1** Classes of effective divisors in $\overline{\mathcal{M}}_{g,n}$ from Abelian differentials

| $g$ and $n$ | Signature $\kappa$ | Reference |
|---|---|---|
| Divisors for $n=1$ | | |
| $g \geq 3, n=1$ | $\kappa = (g-k, k+1, 1^{g-3})$ for $k=1,\dots,g-1$ | Theorem 3 |
| $g \geq 2, n=1$ | $\kappa = (-h, g+h, 1^{g-2})$ for $h \geq 2$ | Theorem 3 |
| Coupled partition divisors | | |
| $g \geq 2, n=2$ | $\kappa = (1, 1, 2^{g-2})$ | Theorem 4.1 |
| $g \geq 2, n \geq 2$ | $\kappa = (d_1, \dots, d_n, 2^{g-1})$, some $d_i < 0$ and $\sum d_i = 0$, and even and odd spin structure components | Theorem 4.1 |
| Pinch partition divisors | | |
| $g \geq 3, n \geq 1$ | $\kappa = (d_1, \dots, d_n, 1^{g-3}, 2)$ for all $d_i \geq 0$ and $\sum d_i = g-1$ | Theorem 5.1 |
| $g \geq 2, n \geq 2$ | $\kappa = (d_1, \dots, d_n, 1^{g-2}, 2)$ for one $d_i \leq -2$ and $\sum d_i = g-2$ | Theorem 5.1 |

The class of this divisor is given in Theorem 4.1 as

$$2^{g-3}\Bigg((2^g+1)\lambda + 2(2^g+1)(\psi_1+\psi_2) - 2^{g-3}\delta_0 \\ - \sum_{i=0}^{g-1}(2^i-1)(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}(2^i+1)(2^{g-i}+1)\delta_{i:\{2\}}\Bigg).$$

In [M] the author used the method of test curves to obtain a closed formula for all divisors $D^0_\kappa$ in $\overline{\mathcal{M}}_g$ for all holomorphic signatures $\kappa$. The setting of $\overline{\mathcal{M}}_{g,n}$ allows us the opportunity to provide an exposition of a different method of calculating divisor classes. In this paper, we employ maps between moduli spaces to compute the classes of interest. Bainbridge, Chen, Gendron, Grushevsky, and Möller [BCGGM] have recently provided a full compactification of the space of Abelian differentials. With this understanding of the degeneration of differentials, we are able to explicitly describe the components of the pullback of a divisor coming from the strata of differentials under the maps described in Section 2.7 obtained by gluing in marked tails of different genus at marked points and gluing marked points together. Hence, knowing the class of the components of the pullback of an unknown divisor class or identifying an unknown divisor as a component of the pullback of a known class, we obtain many of the coefficients or the full class of the unknown divisor. In Section 2.9, we survey the many known results of divisor classes previously computed by classical methods that can be efficiently reproduced from this perspective.

The difficulties in the computation arise from obtaining sufficient relations to find all coefficients and from computing the multiplicity of each component of the pullback of a divisor in such a relation. Obtaining the multiplicity requires enumerating certain holomorphic and meromorphic sections of a line bundle on a general curve. We use a variant of the well-known de Jonquières formula in the holomorphic case. In the meromorphic case the Picard variety method realizes the unknown number as the degree of a map between $C^g$ and the Picard variety. The difficulty of calculating the multiplicity of different specific solutions that violate the global residue condition can be overcome by investigating the ramification locus of this map.

In Section 3, we provide an illustrative introduction to the techniques that will be developed in later sections by generalizing the Weierstrass divisor $W = D^1_{g,1^{g-2}}$, the closure in $\overline{\mathcal{M}}_{g,1}$ of the locus of Weierstrass points originally calculated by Cukierman [Cu]. The families of divisors computed in Sections 4 and 5 were chosen to best expose the utility of our methods and are relevant in our search for extremal effective divisor classes. The coupled partition divisors present the simplest case to provide an exposition of our method as it pertains to strata of Abelian differentials with multiple components. When all the marked points have even multiplicities, these divisors have two components based on the spin structure. The pinch partition divisors provide a useful exposition of our method in the case that the unmarked points have different multiplicities, and the first example of a pinch partition divisor $D^{g-1}_{1^{2g-4},2}$ was shown to be extremal in the effective cone by Farkas and Verra [FV1].

## 2. Preliminaries

### *2.1. Strata of Abelian Differentials*

A partition of $2g-2$ of the form $\kappa = (k_1, \dots, k_n)$ with all $k_i \in \mathbb{Z} \setminus \{0\}$ is known as a *signature.* We say that $\kappa$ is *holomorphic,* denoted $\kappa > 0$, if all entries $k_i > 0$, and $\kappa$ is *meromorphic,* denoted $\kappa \not> 0$, if some $k_i < 0$. We define the *stratum of Abelian differentials with signature* $\kappa$ as

$$\mathcal{H}(\kappa) := \{(C, \omega) \mid g(C) = g, (\omega) = k_1 p_1 + \cdots + k_n p_n \text{ for distinct } p_i\},$$

where $\omega$ is a meromorphic differential on $C$. Hence $\mathcal{H}(\kappa)$ is the space of Abelian differentials with prescribed multiplicities of zeros and poles given by $\kappa$. By relative period coordinates $\mathcal{H}(\kappa)$ has dimension $2g+n-1$ if $\kappa > 0$ and $2g+n-2$ if $\kappa \not> 0$ [K].

A related object of interest in our study of the birational geometry of $\mathcal{M}_{g,n}$ is the *stratum of canonical divisors with signature* $\kappa$*,* which we define as

$$\mathcal{P}(\kappa) := \{[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n} \mid k_1 p_1 + \cdots + k_n p_n \sim K_C\}.$$

Forgetting this ordering of the zeros or poles of the same multiplicity, we obtain the projectivization of $\mathcal{H}(\kappa)$. If all $k_i$ are distinct, then this finite cover becomes an isomorphism of $\mathcal{P}(\kappa)$ with the projectivization of $\mathcal{H}(\kappa)$ under the $\mathbb{C}^*$ action that scales the differential $\omega$. Hence we have that the dimension of $\mathcal{P}(\kappa)$ is $2g+n-2$ if $\kappa > 0$ and $2g+n-3$ if $\kappa \not> 0$.

A line bundle $\eta$ on a smooth curve $C$ such that $\eta^{\otimes 2} \sim K_C$ is known as a *theta characteristic.* The *spin structure* of $\eta$ is the parity of $h^0(C, \eta)$, which Mumford

[Mu] showed to be deformation invariant. Consider an Abelian differential $(C,\omega)$ where $\omega$ has signature $\kappa=(k_1,\dots,k_n)$. If all $k_i$ are even, then an Abelian differential of this type specifies a theta characteristic on the underlying curve

$$\eta\sim\sum_{i=1}^{n}\frac{k_i}{2}p_i.$$

As the parity of $h^0(C,\eta)$ is deformation invariant, the loci $\mathcal{H}(\kappa)$ and $\mathcal{P}(\kappa)$ are reducible and break up into disjoint components with even and odd parity of $h^0(C,\eta)$.

A signature $\kappa$ is of *hyperelliptic type* if all odd entries in the signature occur in pairs $\{j,j\}$ or $\{-j,-j\}$. A *hyperelliptic differential* of type $\kappa$ for such $\kappa$ is a differential on a hyperelliptic curve resulting from pulling back a degree $g-1$ rational function under the unique hyperelliptic cover of $\mathbb{P}^1$, with the minimum number of zeros occurring at ramification points of the hyperelliptic involution known as *Weierstrass points.* Hence the subvariety of hyperelliptic differentials in $\mathcal{H}(\kappa)$ has dimension $2g+(n-m)/2$, where $m$ is the number of zeros that occur at Weierstrass points in each hyperelliptic differential, and this is the subvariety of maximum dimension that can be built in $\mathcal{H}(\kappa)$ from the locus of hyperelliptic curves. Kontsevich and Zorich [KZ] showed that there can be at most three connected components in total of $\mathcal{H}(\kappa)$ for $\kappa>0$ and hence $\mathcal{P}(\kappa)$, corresponding to the case that the hyperelliptic differentials become a connected component of $\mathcal{H}(\kappa)$ distinct from the remaining differentials that provide two further connected components based on odd or even spin structure. Boissy [B] showed that this holds in the meromorphic case ($\kappa\not>0$) for $g\geq 2$ and completely classified the connected components of $\mathcal{H}(\kappa)$ when $g=1$.

### *2.2. Degeneration of Abelian Differentials*

The investigation of how Abelian differentials degenerate as the underlying curve becomes singular has recently attracted much attention. In calculating the Kodaira dimension of a number of the strata of Abelian differentials, Gendron [G] used analytic methods to investigate the degeneration of Abelian differentials. Chen [C] used algebraic methods to consider the limiting position of Weierstrass points on general curves of compact type. Farkas and Pandharipande [FP] extended these ideas to all nodal curves defining *the moduli space of twisted canonical divisors of type* $\kappa$ and showed that this space was, in general, reducible and contained extra boundary components. Janda, Pandharipande, Pixton, and Zvonkine in the Appendix to this paper provided a conjectural description of the cohomology classes of the strata. In [M], the author obtained a closed formula for the class of the strata closure that form a codimension one subvariety in $\overline{\mathcal{M}}_g$.

A twisted canonical divisor of type $\kappa=(k_1,\dots,k_n)$ is a collection of (possibly meromorphic) canonical divisors $D_j$ on each irreducible component $C_j$ of $C$ such that:

(a) The support of $D_j$ is contained in the set of marked points and the nodes lying in $C_j$; moreover, if $p_i\in C_j$, then $\operatorname{ord}_{p_i}(D_j)=k_i$.

(b) If $q$ is a node of $C$ and $q \in C_i \cap C_j$, then $\mathrm{ord}_q(D_i) + \mathrm{ord}_q(D_j) = -2$.
(c) If $q$ is a node of $C$ and $q \in C_i \cap C_j$ such that $\mathrm{ord}_q(D_i) = \mathrm{ord}_q(D_j) = -1$, then for any $q' \in C_i \cap C_j$, we have $\mathrm{ord}_{q'}(D_i) = \mathrm{ord}_{q'}(D_j) = -1$. We write $C_i \sim C_j$.
(d) If $q$ is a node of $C$ and $q \in C_i \cap C_j$ such that $\mathrm{ord}_q(D_i) > \mathrm{ord}_q(D_j)$, then for any $q' \in C_i \cap C_j$, we have $\mathrm{ord}_{q'}(D_i) > \mathrm{ord}_{q'}(D_j)$. We write $C_i \succ C_j$.
(e) There does not exist a directed loop $C_1 \succeq C_2 \succeq \cdots \succeq C_k \succeq C_1$ unless all $\succeq$ are $\sim$.

The natural question is what other conditions are required to distinguish the main component coming from twisted canonical divisors on smooth curves from the boundary components. Bainbridge, Chen, Gendron, Grushevsky, and Möller [BCGGM] have recently provided the global residue condition required to distinguish the main component from the boundary components giving a full compactification for the strata of Abelian differentials. Let $\Gamma$ be the dual graph of $C$. They show that a twisted canonical divisor of type $\kappa$ is the limit of twisted canonical divisors on smooth curves if there exists a collection of meromorphic differentials $\omega_i$ on $C_i$ with $\mathrm{Div}(\omega_i) = D_i$ that satisfy the following conditions:

(a) If $q$ is a node of $C$ and $q \in C_i \cap C_j$ such that $\mathrm{ord}_q(D_i) = \mathrm{ord}_q(D_j) = -1$, then $\mathrm{res}_q(\omega_i) + \mathrm{res}_q(\omega_j) = 0$.
(b) There exists a *level graph* $\overline{\Gamma} = (\Gamma, \mathrm{level})$, where $\mathrm{level} : \mathrm{vertices}(\Gamma) \longrightarrow \mathbb{R}$ satisfies

$$\begin{aligned} \mathrm{level}(C_i) &= \mathrm{level}(C_i) \quad \text{if } C_i \sim C_j, \text{ and} \\ \mathrm{level}(C_i) &> \mathrm{level}(C_i) \quad \text{if } C_i \succ C_j, \end{aligned}$$

such that, for any level $L \in \mathbb{R}$ and any connected component $Y$ of $\overline{\Gamma}_{>L}$ that does not contain any prescribed pole, we have

$$\sum_{\substack{\mathrm{level}(q)=L,\\ q\in C_i \in Y}} \mathrm{res}_q(\omega_i) = 0.$$

Part (b) is known as the *global residue condition.* Consider the following example.

Example 2.1. Consider the nodal genus $g = 4$ curve $C$ with three irreducible components $g(X) = g(Y) = 0$ and $g(Z) = 3$ and $X \cap Z = \{q_1, q_2\}$, $Y \cap Z = \{q_3\}$ and all other intersections zero. Figure 1 depicts a twisted canonical divisor of type $\kappa = (6, -1, 5, -4)$ on $C$ with

$$\begin{aligned} D_X &= 6p_1 - p_2 - 4q_1 - 3q_2, \\ D_Y &= 5p_3 - 4p_4 - 3q_3, \\ D_Z &= 2q_1 + q_2 + q_3, \end{aligned}$$

where $D_j \sim K_j$. To find the conditions on such a twisted canonical divisor being smoothable, we must consider the possible level graphs of the components and see what conditions these will place on the residues. Consider the meromorphic differential on $Y \cong \mathbb{P}^1$ with signature $(5, -4, -3)$. By the cross ratio we can set

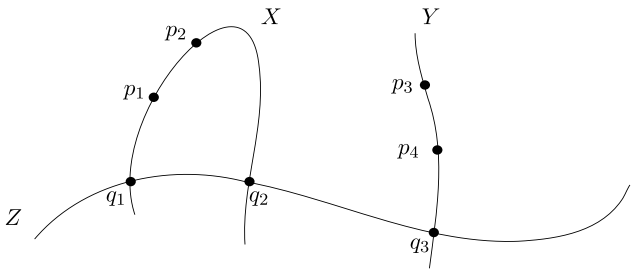


**Figure 1** A twisted canonical divisor of type $\kappa = (6, -1, 5, -4)$ on a nodal curve

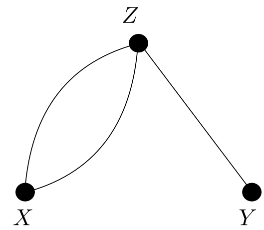


**Figure 2** The level graph giving the global residue condition

the poles to 0 and $\infty$ and the zero to 1. The resulting differential is given locally at 0 by

$$c\frac{(1-z)^5}{z^4}\,dz$$

for some constant $c \in \mathbb{C}^*$. Hence the residues at 0 and $\infty$ are nonzero. The flat geometric way of presenting this is that on a genus $g = 0$ surface with one conical singularity, the length of any saddle connection is obtained by integrating the differential along the saddle connection. Hence this length is equal to the sum of the residues the path encloses. Hence no flat surface of this signature can have zero residues at both poles, and we will have $\mathrm{res}_{q_3}(\omega_Y) \neq 0$. This shows that in the level graph the components $X$ and $Y$ must sit at the same level and the only possible level graph is shown in Figure 2. The global residue condition on this level graph becomes

$$\mathrm{res}_{q_1}(\omega_X) + \mathrm{res}_{q_2}(\omega_X) + \mathrm{res}_{q_3}(\omega_Y) = 0.$$

By the residue theorem we know that the sum of the residues on any $\omega_j$ is zero and hence our condition is equivalently

$$\mathrm{res}_{q_3}(\omega_Y) = \mathrm{res}_{p_2}(\omega_X).$$

We have seen that $\mathrm{res}_{q_3}(\omega_Y) \neq 0$ and as $p_2$ is a simple pole, $\mathrm{res}_{p_2}(\omega_X) \neq 0$. Hence, as there exist $\omega_i$ that satisfy $\mathrm{Div}(\omega_i) = D_i$, by scaling we can always satisfy this global residue condition, and we have shown that all twisted canonical divisors of this type are smoothable.

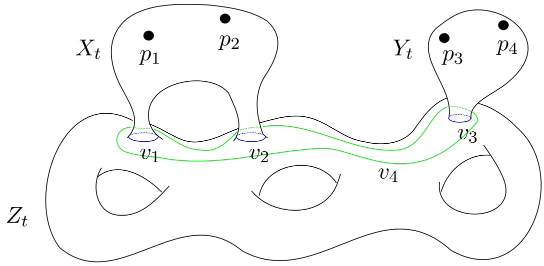


**Figure 3** A twisted canonical divisor of type $\kappa = (6, -1, 5, -4)$ on a smooth curve

By investigating topologically a family of twisted canonical divisors on smooth curves degenerating to a nodal curve we can see why this condition on the residues is necessary. Let $\chi$ be a family of meromorphic differentials $(C_t, \omega_t)$ with $\omega_t$ of type $\kappa = (6, -1, 5, -4)$ on smooth curves $C_t$ for $t \neq 0$, degenerating to the nodal curve $C_0 = C$ at $t = 0$. Figure 3 depicts topologically an element of this family for $t \neq 0$. Let $v_i$ for $i = 1, 2, 3$ be the vanishing cycles on $X_t \cup Y_t \cup Z_t$ that shrink to the nodes $q_i$ at $t = 0$ such that $X_t \cap Z_t = \{v_1, v_2\}$ and $Y_t \cap Z_t = v_3$ with $X_t \to X$, $Y_t \to Y$, and $Z_t \to Z$ as $t \to 0$. As there are no poles on the component $Z_t$, we observe by an application of Stokes formula to the cycle $v_4$ that

$$\int_{v_1+v_2+v_3} \omega_t = \int_{v_4} \omega_t = 0$$

for $t \neq 0$. Our residue condition is simply the limit of this condition as $t \to 0$. This shows that this residue condition is necessary. Complex-analytic plumbing techniques and flat geometry are used in [BCGGM] to show that in all cases the condition is sufficient.

### 2.3. *Degeneration of Theta Characteristics and Spin Structures*

Distinguishing how different components of the strata of canonical divisors extend to the boundary of $\overline{\mathcal{M}}_{g,n}$ will depend on understanding how theta characteristics and spin structures degenerate. Cornalba [Co] investigated how theta characteristics degenerate to nodal curves of pseudocompact type, which are curves with dual graph equal to a tree after removing any self-edges. We begin by considering a curve $C$ with a nonseparating node that is the only node in the curve. Let $\tilde{C}$ be the normalization of $C$, and let $x$ and $y$ be such that $C = \tilde{C}/\{x \sim y\}$. There are two types of theta characteristic on such a curve. Consider $\tilde{\eta}$ on $\tilde{C}$ such that $\tilde{\eta}^{\otimes 2} \sim K_{\tilde{C}} + x + y$. As $K_{\tilde{C}} + x$ has a base point $x$ for any $x$, a section of $H^0(\tilde{C}, \tilde{\eta})$ vanishes at $x$ if and only if it vanishes at $y$. Hence such sections are codimension one in $H^0(\tilde{C}, \tilde{\eta})$. For any $\tilde{\eta}$, there are two ways to glue sections $f \in H^0(\tilde{C}, \tilde{\eta})$ to agree at the node and hence descend to $C$. Sections can be glued as $f(x) = f(y)$

or $f(x) = -f(y)$. Hence $h^0(C, \eta)$ will differ by $+1$ for these two cases, representing even and odd spin structures. This obtains $2^{2g-2}$ even and $2^{2g-2}$ odd spin structures on the curve $C$.

By blowing up at the node and inserting a rational bridge between $x$ and $y$ we obtain the other type of theta characteristic on the curve

$$(\tilde{\eta}, \mathcal{O}(1)),$$

where $\tilde{\eta}^{\otimes 2} \sim K_{\tilde{C}}$, and the global sections are glued together at the nodes. However, as $h^0(\mathbb{P}^1, \mathcal{O}(1)) = 2$, the values at $x$ and $y$ completely determine the section on the rational bridge, and we obtain that the parity of these theta characteristics is thus $h^0(\tilde{C}, \tilde{\eta})$ mod 2. There are $2^{g-2}(2^{g-1}+1)$ even and $2^{g-2}(2^{g-1}-1)$ odd theta characteristics of this type. Each has multiplicity 2, which gives the expected $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta characteristics on $C$.

Consider now a curve $C$ of pseudocompact type with irreducible components $C_1, \dots, C_k$. Blowing up and inserting an exceptional component at every separating node, we obtain the theta characteristic on $C$ to be

$$(\eta_1, \dots, \eta_k, \{\mathcal{O}(1)\}_{i=1}^{k-1}),$$

where $\eta_i$ is a theta characteristic on $C_i$, and $\mathcal{O}(1)$ is a line bundle of degree one on the exceptional rational components. This gives the total degree $\sum_{i=1}^k (g_i - 1) + (k-1) = g-1$ as expected, and we observe the parity to be

$$\sum_{i=1}^k h^0(C_i, \eta_i) \bmod 2,$$

where if any component $C_i$ has self nodes, then the $\eta_i$ is of the types discussed earlier.

## *2.4. De Jonquières' Formula*

We will require some tools for enumerating the occurrence of meromorphic differentials of specified signatures in general curves in the moduli space. The first such tool is de Jonquières' formula, which enumerates the number of sections with a specified type of vanishing in a general $g^r_d$. The number of sections with ordered zeros of multiplicity $k_i$ for $i = 1, \dots, \rho$ with $\sum k_i = d$ and $\rho = d - r$ in a general $g^r_d$ on a general genus $g$ curve is

$$\begin{aligned}&\mathrm{dJ}[g; k_1, \dots, k_\rho]\\ &\quad = \frac{g!}{(g-\rho-1)!} \prod_{i=1}^{\rho} k_i \left( \sum_{j=0}^{\rho-1} \left( \frac{(-1)^j}{g-\rho+j} \sum_{|I|=j} \left( \prod_{i \notin I} k_i \right) \right) + \frac{(-1)^\rho}{g} \right),\end{aligned}$$

where $I$ is a subset of $\{1, \dots, \rho\}$, and $|I|$ denotes the number of elements in $I$. We define

$$\|I\| := \sum_{i \in I} k_i.$$

This formula can be found throughout the literature in many forms. This presentation is equivalent to that provided in [ACGH] on page 359 when all $k_i$ are distinct. We present this version for its relative computational ease. It is a variation of that developed in [Cool], p. 288. We will use the convention that $\mathrm{dJ}[1;\emptyset] = 1$.

Letting $k_1 = r + 1$ and $k_i = 1$ for $i = 2, \ldots, d - r$ recovers the well-known Plücker formula enumerating the number of simple ramification points in a general $g^r_d$ as

$$(r + 1)d + (r + 1)r(g - 1),$$

after allowing for the factor $(d - r - 1)!$ labeling the simple zeros.

### *2.5. The Picard Variety Method*

De Jonquières' formula enumerates the holomorphic sections of a general line bundle of a specified type. In some cases, however, we want to enumerate the meromorphic sections of a specified type. The Picard variety method enumerates the solutions to a particular equation in the Picard group. Here we provide a summary of this method as presented in [M].

For a specified line bundle $L$ of degree $d = \sum_{i=1}^{g} k_i$ on a general genus $g$ curve $C$, we want to enumerate $(p_1, \ldots, p_g) \in C^g$ so that

$$\sum_{i=1}^{g} k_i p_i \sim L.$$

We will follow the treatment in genus $g = 2$ of [CT], Section 2. Consider the map

$$\begin{aligned} f : C^g &\longrightarrow \mathrm{Pic}^d(C) \\ (p_1, \ldots, p_g) &\longmapsto \textstyle\sum_{i=1}^{g} k_i p_i. \end{aligned}$$

The fiber of this map above $L \in \mathrm{Pic}^d(C)$ gives us precisely the solutions of interest. We observe that the domain and range of $f$ are both of dimension $g$. Hence our answer will come from the degree of the map $f$ and an analysis of this fiber. Take a general point $e \in C$ and consider the isomorphism

$$\begin{aligned} h : \mathrm{Pic}^d(C) &\longrightarrow J(C) \\ L &\longmapsto L \otimes \mathcal{O}_C(-de). \end{aligned}$$

Now let $F = h \circ f$. Then we have $\deg F = \deg f$. We observe

$$F(p_1, \ldots, p_g) = \mathcal{O}_C\left(\sum_{i=1}^{g} k_i (p_i - e)\right).$$

Let $\Theta$ be the fundamental class of the theta divisor in $J(C)$. By [ACGH], Section 1.5, we have

$$\deg \Theta^g = g!,$$

and the dual of the locus of $\mathcal{O}_C(k(x-e))$ for varying $x \in C$ has class $k^2\Theta$ in $J(C)$. Hence

$$\begin{aligned}\deg F &= \deg F_* F^*([\mathcal{O}_C]) \\ &= \deg\left(\prod_{i=1}^{g} k_i^2 \Theta\right) \\ &= g!\left(\prod_{i=1}^{g} k_i^2\right).\end{aligned}$$

In practice, we may want to discount this number by any specific solutions that we may omit for some reason. For example, we will omit any solutions where $p_i = p_j$ for $i \neq j$. In this case, we need to know not only the existence of any specific solutions that we are discounting by, but also the multiplicity of these solutions. We calculate the multiplicity by investigating the branch locus of $F$. First, we look locally analytically at $F$ around each point. If $f_0\, d\omega, \dots, f_{g-1}\, d\omega$ is a basis for $H^0(C, K_C)$, then locally analytically the map becomes

$$(p_1, \dots, p_g) \longmapsto \left(\sum_{i=1}^{g} k_i \int_e^{p_i} f_0\, d\omega, \dots, \sum_{i=1}^{g} k_i \int_e^{p_i} f_{g-1}\, d\omega\right)$$

modulo $H_1(C, K_C)$. The map on tangent spaces at any fixed point $(p_1, \dots, p_g) \in C^g$ is the Jacobian of $F$ at the point, which is

$$\mathrm{DF}(p_1, \dots, p_g) = \mathrm{diag}(k_1, \dots, k_g)\begin{pmatrix} f_0(p_1) & \dots & f_0(p_g) \\ f_1(p_1) & \dots & f_1(p_g) \\ \dots & \dots & \dots \\ f_{g-1}(p_1) & \dots & f_{g-1}(p_g) \end{pmatrix}.$$

Ramification in the map $F$ occurs when the map on tangent spaces is not injective which takes place at the points where $\mathrm{rk(DF)} < g$. The ramification index at a point $(p_1, \dots, p_g) \in C^g$ is equal to the vanishing order of the determinant of $\mathrm{DF}(p_1, \dots, p_g)$ at the point.

We observe that there are two components to the branch locus of $F$:

$$\begin{aligned}\Delta &= \{(p_1, \dots, p_g) \in C^g \mid p_i = p_j \text{ for some } i \neq j\}, \\ \mathcal{K} &= \{(p_1, \dots, p_g) \in C^g \mid h^0(C, K_C - p_1 - \dots - p_g) > 0\},\end{aligned}$$

where $\mathcal{K}$ is irreducible, and $\Delta$ has $g(g-1)/2$ irreducible components defined by

$$\Delta_{i,j} = \{(p_1, \dots, p_g) \in C^g \mid p_i = p_j\}$$

for $i, j = 1, \dots, g$ and $i < j$. Hence finding the multiplicity of any point in the branch locus is simply a matter of investigating how these loci meet at the particular point.

### 2.6. Divisor Theory on $\overline{\mathcal{M}}_{g,n}$

Let $\lambda$ denote the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$, and let $\psi_i$ denote the first Chern class of the cotangent bundle on $\overline{\mathcal{M}}_{g,n}$ associated with the $i$th marked point where $1 \leq i \leq n$. These classes are extensions of classes defined on $\mathcal{M}_{g,n}$ that generate $\mathrm{Pic}(\mathcal{M}_{g,n}) \otimes \mathbb{Q}$; however, $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$ contains more classes.

The boundary $\Delta = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ parameterizing marked stable curves of genus $g$ with at least one node is codimension one. Let $\Delta_0$ be the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with a nonseparating node. Let $\Delta_{i:S}$ for $0 \leq i \leq g$, $S \subseteq \{1, \ldots, n\}$ be the locus of curves with a separating node that separates the curve into a genus $i$ component containing the marked points from $S$ and a genus $g-i$ component containing the marked points from $S^C$, the complement of $S$. We require $|S| \geq 2$ for $i = 0$ and $|S| \leq n-2$ for $i = g$. We observe that $\Delta_{i:S} = \Delta_{g-i:S^C}$. These boundary divisors are irreducible and can intersect each other and self-intersect. Denote the class of $\Delta_{i:S}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$ by $\delta_{i:S}$, and in the case that $n = 1$, we denote $\delta_{i:\{1\}}$ by $\delta_i$. See [AC; HMo] for more information.

For $g \geq 3$, these divisor classes freely generate $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$. For $g = 2$, the classes $\lambda$, $\delta_0$, and $\delta_1$ generate $\mathrm{Pic}(\overline{\mathcal{M}}_2) \otimes \mathbb{Q}$ with the relation

$$\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1.$$

Similarly, $\mathrm{Pic}(\overline{\mathcal{M}}_{2,n}) \otimes \mathbb{Q}$ is freely generated by $\lambda$, $\psi_i$, and $\delta_{i:S}$ with this relation pulled back under the map $\varphi : \overline{\mathcal{M}}_{2,n} \longrightarrow \overline{\mathcal{M}}_2$ that forgets the $n$ marked points.

### 2.7. Maps Between Moduli Spaces

There are a number of maps between moduli spaces that prove very useful in divisor class calculations. In this section, we present these maps and how the generators of the Picard group pullback under these maps. These results are produced in [AC], p. 161.

Let $(X, q, q_1 \ldots, q_{j+1})$ be a general genus $h$ curve marked at $j+2$ general points for $h \geq 0$ and $j \geq -1$ with $j \geq 1$ if $h = 0$. Define the map

$$\begin{aligned} \pi_{g,h}^{n,j} : \quad \overline{\mathcal{M}}_{g,n} \quad & \rightarrow \quad \overline{\mathcal{M}}_{g+h,n+j} \\ [C, p_1, \ldots, p_n] & \mapsto \left[ C \bigcup_{p_1 = q} X, q_1, p_2, \ldots, p_n, q_2, q_3, \ldots, q_{j+1} \right]. \end{aligned}$$

Letting $\pi = \pi_{g,h}^{n,j}$ for ease of notation, we have, for $j \geq 0$,

$$\pi^*\lambda = \lambda, \qquad \pi^*\delta_0 = \delta_0, \qquad \pi^*\delta_{h:\{1,n+1,\ldots,n+j\}} = -\psi_1,$$

and

$$\pi^*\psi_i = \begin{cases} 0 & \text{for } i = 1 \text{ and } n+1 \leq i \leq n+j, \\ \psi_i & \text{otherwise.} \end{cases}$$

Now let $T = \{1, n+1, \ldots, n+j\}$. For $i > h$,

$$\pi^*\delta_{i:S} = \begin{cases} \delta_{i:S} & \text{for } S \cap T = \emptyset, \\ \delta_{i-h:(S\setminus T)\cup\{1\}} & \text{for } T \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = h$ and $h < g$,

$$\pi^*\delta_{i:S} = \begin{cases} \delta_{i:S} & \text{for } S \cap T = \emptyset, \\ \delta_{0:(S\setminus T)\cup\{1\}} & \text{for } T \subset S \text{ and } T \neq S, \\ -\psi_1 & \text{for } T = S, \\ 0 & \text{otherwise.} \end{cases}$$

For $i < h$,

$$\pi^*\delta_{i:S} = \begin{cases} \delta_{i:S} & \text{for } S \cap T = \emptyset \text{ and } i < g, \\ 0 & \text{otherwise.} \end{cases}$$

When $j = -1$, let $[X, q]$ be a general genus $h$ curve marked at a general point. The map becomes

$$\begin{aligned} \pi^{n,-1}_{g,h} : \quad \overline{\mathcal{M}}_{g,n} \quad &\to \quad \overline{\mathcal{M}}_{g+h,n-1} \\ [C, p_1, \ldots, p_n] &\mapsto \Bigl[C \bigcup_{p_1=q} X, p_2, \ldots, p_n\Bigr]. \end{aligned}$$

Again, let $\pi = \pi^{n,-1}_{g,h}$ for ease of notation. Then

$$\pi^*\lambda = \lambda, \qquad \pi^*\delta_0 = \delta_0, \qquad \pi^*\psi_i = \psi_{i+1},$$
$$\pi^*\delta_{h:\emptyset} = -\psi_1 + \delta_{h:\emptyset}, \qquad \pi^*\delta_{i:S} = \delta_{i:S} + \delta_{i-h:S\cup\{1\}} \quad \text{for } i:S \neq h:\emptyset,$$

where $\delta_{i:S} = 0$ for $i > g$.

We can also create more complicated maps gluing in multiple tails of different genus with different numbers of marked points to suit our needs. We will describe these maps as needed.

The map $\vartheta$ identifies the first and second marked points

$$\begin{aligned} \vartheta : \quad \overline{\mathcal{M}}_{g,n} \quad &\to \quad \overline{\mathcal{M}}_{g+1,n-2} \\ [C, p_1, \ldots, p_n] &\mapsto [C/\{p_1 \sim p_2\}, p_3, \ldots, p_n]. \end{aligned}$$

We have

$$\vartheta^*\lambda = \lambda, \qquad \vartheta^*\delta_0 = \delta_0 + \sum_{p_1\in S,\, p_2\notin S} \delta_{i:S},$$
$$\vartheta^*\delta_{i:S} = \delta_{i:S} + \delta_{i-1:S\cup\{p_1,p_2\}}, \qquad \vartheta^*\psi_i = \psi_{i+2}.$$

The map $\varphi_j$ forgets the $j$th marked point:

$$\begin{aligned} \varphi_j : \quad \overline{\mathcal{M}}_{g,n} \quad &\to \quad \overline{\mathcal{M}}_{g,n-1} \\ [C, p_1, \ldots, p_n] &\mapsto [C, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n]. \end{aligned}$$

We have

$$\varphi_j^*\lambda = \lambda, \qquad \varphi_j^*\delta_0 = \delta_0, \qquad \varphi_j^*\delta_{h:S} = \delta_{h:S} + \delta_{h:S\cup\{j\}}, \qquad \varphi_j^*\psi_i = \psi_i - \delta_{0:\{i,j\}}.$$

In the case that $n = 1$, let $\varphi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_g$ be the map forgetting the marked point. In this case, for $g$ even, we have the one exception that $\varphi^*\delta_{g/2} = \delta_{g/2:\{1\}}$.

### 2.8. *Divisor Notation*

The divisor notation used in this paper differs based whether the signature $\kappa$ is meromorphic or holomorphic.

DEFINITION 2.2. For $|\kappa| = g - 2 + n$ if $\kappa > 0$ and $|\kappa| = g - 1 + n$ if $\kappa \not> 0$, write $\kappa$ in the form

$$\kappa = (k_1, \dots, k_n, d_1^{\alpha_1}, \dots, d_r^{\alpha_r}),$$

where $d_i \neq d_j$ for $i \neq j$. Then $D^n_\kappa$ for $n \geq 1$ is the divisor in $\overline{\mathcal{M}}_{g,n}$ defined by

$$D^n_\kappa := \frac{1}{\alpha_1! \cdots \alpha_r!} \varphi_* \overline{\mathcal{P}}(\kappa),$$

where $\varphi$ forgets the last $r = g - 2$ or $r = g - 1$ marked points for $\kappa > 0$ or $\kappa \not> 0$, respectively.

### 2.9. *Previous Computations*

Divisors in $\overline{\mathcal{M}}_{g,n}$ from the strata of Abelian differentials have been presented previously in various places, though often under different guises. Logan [L] investigated the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ through the use of pointed Brill–Noether divisors. For $\underline{d} = (d_1, \dots, d_n)$, $d_i \geq 0$ with $\sum d_i = g$, these divisors are the closure of $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ such that $|\sum d_i p_i|$ is $g^1_g$. From our perspective, these are the divisors $D^n_{\underline{d},1^{g-2}}$ coming from holomorphic strata where all unmarked points are simple zeros. Müller [Mü] and Grushevsky and Zakharov [GZ] computed the classes of the closure of the pullback of the theta divisor, that is, for $\underline{d} = (d_1, \dots, d_n)$ with $\sum d_i = g - 1$ and some $d_i < 0$, the closure of $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ such that $h^0(\sum d_i p_i) \geq 1$. From our perspective, these are the divisors $D^n_{\underline{d},1^{g-1}}$ coming from meromorphic strata where all unmarked points are simple zeros. From this perspective, the irreducibility of these divisors in the holomorphic case for $g \geq 3$ and the meromorphic case for $g \geq 2$ follows from the irreducibility results on the strata of Abelian differentials of Kontsevich and Zorich [KZ] and Boissy [B], respectively. Farkas and Verra [FV2] computed the class of the closure of the antiramification locus in $\overline{\mathcal{M}}_{g,g-1}$ for $g \geq 3$ defined as the closure of $[C, p_1, \dots, p_{g-1}] \in \mathcal{M}_{g,g-1}$ such that $h^0(K_C - p_1 - \cdots - p_{g-1} - 2q) \geq 1$ for some $q \in C$. From our perspective, this divisor is $D^{g-1}_{1^{2g-4},2}$. The class of strata with more than one irreducible connected component has also been considered in one isolated case. Teixidor and Bigas [T] computed the closure of the locus of curves in $\overline{\mathcal{M}}_g$ with a vanishing theta null or curves that admit a semicanonical pencil. Pulling this locus back under the map $\varphi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_g$, we obtain the locus of points on a curve with a vanishing theta null or the even component of the divisor $D^1_{2^{g-1}}$, which we denote $D^{1,\text{even}}_{2^{g-1}}$. Farkas and Verra [FV2] computed the odd spin structure component

$D^{1,\mathrm{odd}}_{2^{g-1}}$ as the closure of the loci of points in the support of an odd theta characteristic. All of these previously known results can be efficiently reproduced through the techniques used in this paper to compute new divisor classes.

## 3. Effective Divisors in $\overline{\mathcal{M}}_{g,1}$

This section provides a simple exposition of the techniques that will be developed in more complicated situations in the next sections. We calculate the classes of divisors in $\overline{\mathcal{M}}_{g,1}$ that are the closure of loci of points on smooth curves that form poles or zeros of holomorphic or meromorphic differentials of certain signatures. Unlike in [M], where the author used test curves to compute the divisor class, in this paper, we primarily use the method of pulling back divisor classes under maps between different moduli spaces of curves. Through our understanding of the degeneration of meromorphic differentials, we are able to explicitly describe the components of the pullback of a divisor coming from the strata of meromorphic differentials. The multiplicity of the components can be computed by an application of the Picard variety method or de Jonquières' formula. Hence, knowing the class of the components of the pullback of an unknown divisor class or identifying an unknown divisor as a component of the pullback of a known class, we obtain the class or many of the coefficients of the class of the unknown divisor.

We record the results of this section as the following theorem.

THEOREM 3.1 (Divisors in $\overline{\mathcal{M}}_{g,1}$). *The divisor $D^1_{g-k,k+1,1^{g-3}}$ for $g \geq 3$ and $k = 1, \dots, g-1$ is given by*[2]

$$D^1_{g-k,k+1,1^{g-3}} = c_\psi \psi + c_\lambda \lambda + c_0 \delta_0 + \sum_{i=1}^{g-1} c_i \delta_i,$$

*where*

$$c_\psi = \frac{(k+1)(g-k)((k+1)g^2 - (k^2+k+1)g - 2)}{2},$$

$$c_\lambda = \frac{(k+1)(4-2g+10k-2gk+11k^2+3k^3)}{2},$$

$$c_0 = \frac{(k+1)^2 - (k+1)^4}{6},$$

$$c_i = \begin{cases} -\frac{(k+1)(i(g-i+1)(g-i)(k+1)+(g-i-k)((k+1)(g-i)^2-(k^2+k+1)(g-i)-2))}{2} \\ \textit{for } 1 \leq i \leq g-k, \\ -\frac{(g-i)(k+1)(-3g+g^2+4i-gi+3k-4gk+g^2k+5ik-gik+3k^2-2gk^2+2ik^2+k^3)}{2} \\ \textit{for } g-k+1 \leq i \leq g-1. \end{cases}$$

[2]This formula is reproduced in the case that $k = g-1$ in the Appendix by the more labor intensive methods of Porteous' formula and test curves. This divisor is irreducible in all cases except $g = 3$ and $k = 1$, which is discussed in Remark 3.4.

*The divisor* $D^1_{-h,g+h,1^{g-2}}$ *for* $h \geq 2$ *is given by*[3]

$$D^1_{-h,g+h,1^{g-2}} = c_\psi \psi + c_\lambda \lambda + c_0\delta_0 + \sum_{i=1}^{g-1} c_i \delta_i,$$

*where*

$$c_\psi = \frac{g(g+h+1)(h-1)(h^2+gh+g+1)}{2},$$
$$c_\lambda = \frac{(1+g+h)(2-3g^2+3g^3-2h-4gh+9g^2h-h^2+9gh^2+3h^3)}{2},$$
$$c_0 = \frac{(g+h)^2-(g+h)^4}{6},$$
$$\begin{aligned} c_i = {} & \frac{(i-g)}{2}(g^3(2h+i)+g^2(5h^2+2ih+2h+i-1) \\ & + g(4h^3+h^2(i+4)+2h(i-1)+i) \\ & - 2h-i+1+h(h+2)(i+h^2)) \quad \textit{for } 1 \leq i \leq g-1. \end{aligned}$$

### 3.1. The Weierstrass Divisor

Setting $k = 0$ in the first formula, we recover $(g-2)W$ for $W = D^1_{g,1^{g-2}}$ the known class of the Weierstrass divisor computed by Cukierman [Cu],

$$W = \frac{g(g+1)}{2}\psi - \lambda - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2}\delta_i.$$

### 3.2. The Residual Divisor

The first generalization of the Weierstrass divisor in $\overline{\mathcal{M}}_{g,1}$ is the closure of the locus of points that are residual to Weierstrass points, that is,

$$R = \overline{\{[C,q] \in \mathcal{M}_{g,1} \mid h^0(K_C - gp - q) > 0 \text{ for some } p \in C\}}.$$

We call this the residual divisor, and in our notation, we have $R = D^1_{1,g,1^{g-3}}$. In this section, we calculate the class of $R$ through the use of maps between moduli spaces of curves and previously known classes. For $g \geq 4$, consider the map

$$\begin{aligned} \pi : \overline{\mathcal{M}}_{g-1,1} &\to \quad \overline{\mathcal{M}}_{g,1} \\ [C,y] &\mapsto \left[C \bigcup_{x=y} E, q\right], \end{aligned}$$

which for $[C,y] \in \overline{\mathcal{M}}_{g-1,1}$ identifies the point $y$ with the point $x$ of a general marked elliptic curve $[E,x,q]$ as described in Section 2.7. Consider how $R$ pulls back under this map. Restricting to the smooth locus of $\mathcal{M}_{g-1,1}$, there are two

[3] When $g = 2$ and $h$ is even, this divisor has two connected components based on spin structure. If $4 \mid h$, then $D^{1,\text{odd}}_{-h,h+2} = 5W$, and $D^{1,\text{even}}_{-h,h+2} = 5W$ otherwise.

cases possible. The limit of Weierstrass points, say $p$, residual to $q$ lies on $C$ or not on $C$ in the nodal curve $[C\bigcup_{x=y}E, q]$. There are only two configurations of zeros across the two components that give a codimension one condition on $[C, y]\in\mathcal{M}_{g-1,1}$. In the case that $p$ lies on $E$, in Section 2.2, we show that a twisted differential of the form required is differentials $\omega_E$ on $E$ and $\omega_C$ on $C$ such that

$$(\omega_E) = -(g+1)x + q + gp \sim \mathcal{O}_E,$$
$$(\omega_C) = (g-1)y + \sum_{j=1}^{g-3} q_i,$$

for some points $p\in E$ and $q_i\in C$. By the group law on $E$ there are $g^2$ such points $p$ in $E$. The point $y$ in $C$ is required to be a Weierstrass point. Further, in this case the global residue condition becomes

$$\mathrm{res}_x(\omega_E) = 0,$$

which is always satisfied, showing that these solutions are always smoothable and hence appear in the divisor $R$.

In the case that $p$ lies on $C$, in Section 2.2, we show that a twisted differential of the form required is differentials $\omega_E$ on $E$ and $\omega_C$ on $C$ such that

$$(\omega_E) = -2x + q + q_{g-3} \sim \mathcal{O}_E,$$
$$(\omega_C) = 0y + gp + \sum_{j=1}^{g-4} q_i,$$

for some points $q_{g-3}\in E$ and $p, q_i\in C$ for $i=1,\dots,g-4$. There is a unique point $q_{g-3}$ in $E$ satisfying this relation. To satisfy the second relation, $p$ must be an exceptional Weierstrass point, that is, $h^0(K_C-gp)>0$. Then any point $y$ on $C$ satisfies the second relation. Further, in this case the global residue condition becomes

$$\mathrm{res}_x(\omega_E) = 0,$$

which is always satisfied, showing that these solutions are always smoothable and hence appear in the divisor $R$. Any other configuration of points places a higher codimension condition on $[C, y]\in\mathcal{M}_{g-1}$. Such loci are contained in the closure of the two we have described.

The boundary of $\overline{\mathcal{M}}_{g,1}$ is codimension one. A simple check shows that a general point in any irreducible boundary component $\delta_i$ is not included in $\pi^*R$, and indeed we have found all components.

Let $\varphi:\overline{\mathcal{M}}_{g-1,1}\to\overline{\mathcal{M}}_{g-1}$ be the map that forgets the marked point. Then our analysis yields the relation

$$\pi^*R = g^2W + \varphi^*D,$$

where $W$ is closure of the locus of Weierstrass points calculated by Cukierman [Cu], and $D$ is the closure of the locus of curves containing an exceptional Weierstrass point calculated by Diaz [D], which agrees with the formula in [M]. In

$\overline{\mathcal{M}}_{g-1}$, this divisor is

$$D = \frac{g(g+1)(3g^2-3g+2)}{2}\lambda - \frac{g^2(g-1)(g+1)}{6}\delta_0 - \sum_{i=1}^{[(g-1)/2]} \frac{gi(g-i-1)(g+1)^2}{2}\delta_i.$$

Knowing the classes of $W$ and $D$ by the pullback relations given in Section 2.7, we obtain all coefficients of $R$ except the coefficient of $\psi$. A simple test curve created by allowing the marked point to vary in a fixed general curve provides the coefficient of $\psi$. This well-known test curve[4] has intersection $2g-2$ times the coefficient of $\psi$. We also know that any general curve has $(g+1)g(g-1)(g-2)$ residual points. Hence we have, for $g \geq 4$,

$$R = \frac{g(g+1)(g-2)}{2}\psi + \frac{g(3g^3-3g+2)}{2}\lambda + \frac{g^2-g^4}{6}\delta_0 + \sum_{i=1}^{g-1} \frac{g(i-g)(g^2i+gi-g+i-1)}{2}\delta_i.$$

In the Appendix, we laboriously reproduce this result by the different methods of Porteous' formula and test curves and show that this formula extends to the case $g=3$.

### 3.3. Divisors from Meromorphic Strata

The divisor $D^1_{-h,g+h,1^{g-2}}$ in $\overline{\mathcal{M}}_{g,1}$ for $h \geq 2$ is defined to be

$$D^1_{-h,g+h,1^{g-2}} = \overline{\{[C,p] \in \mathcal{M}_{g,1} \mid h^0(K_C + hp - (g+h)q) > 0 \text{ for some } q \in C \text{ with } p \neq q\}}.$$

To compute the class of this divisor, we consider the map

$$\begin{aligned} \pi : \overline{\mathcal{M}}_{g,1} &\to \overline{\mathcal{M}}_{g+h,1} \\ [C,y] &\mapsto [C \textstyle\bigcup_{x=y} X, q], \end{aligned}$$

which for $[C,y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general marked genus $h \geq 2$ curve $[X,x,q]$ as introduced in Section 2.7. Consider how $R$ pulls back under this map. Restricting to the smooth locus of $\mathcal{M}_{g,1}$, there are again two cases possible. The limit of Weierstrass points, say $p$, residual to $q$ lies on $C$ or not on $C$ in the nodal curve $[C \bigcup_{x=y} X, q]$. There are only two configurations of zeros across the two components that give a codimension one condition on $[C,y] \in \mathcal{M}_{g,1}$. In the case that $p$ lies on $X$, Section 2.2 shows that a twisted

[4] This test curve is presented in detail in the Appendix.

differential of the form required will be differentials $\omega_E$ on $E$ and $\omega_C$ on $C$ such that

$$(\omega_X) = -(g+2)x + q + (g+h)p + \sum_{j=1}^{h-1} q_j \quad \text{and}$$

$$(\omega_C) = gy + \sum_{j=h}^{g+h-3} q_j$$

for some points $p, q_j \in X$ for $j = 1, \dots, h-1$ and $q_i \in C$ for $j = h, \dots, g+h-3$. The point $y$ in $C$ is required to be a Weierstrass point. Further, in this case the global residue condition becomes

$$\mathrm{res}_x(\omega_E) = 0,$$

which is always satisfied, showing that these solutions are always smoothable and hence appear in the divisor $R$. To enumerate the solutions on $X$, we are looking for points $p, q_j \neq x$ on a general genus $h$ curve $X$ that satisfy

$$(g+h)p + \sum_{j=1}^{h-1} q_j \sim K_X + (g+2)x - q,$$

where $x$ is the node, and $q$ is the marked point, and we have placed them in general position. Using the Picard variety method, we consider the map

$$\begin{aligned} f : C^h &\longrightarrow \mathrm{Pic}^{g+2h-1}(C) \\ (p, q_1 \dots, p_{h-1}) &\longmapsto (g+h)p + \sum_{j=1}^{h-1} q_j, \end{aligned}$$

which by Section 2.5 has degree $(g+h)^2 h$. To enumerate the solutions of the type we are interested in, we simply need to discount for the unique solution $p = x$ of multiplicity $h-1$, where the multiplicity is $(h-2)x + \sum_{j=1}^{h-1} q_j \sim K_X - q$, so the determinant of the Brill–Noether matrix vanishes with multiplicity $h-2$ at this point, and hence the point has multiplicity $h-1$.

In the case that $p$ lies on $C$, Section 2.2 shows that a twisted differential of the form required is differentials $\omega_X$ on $X$ and $\omega_C$ on $C$ such that

$$(\omega_X) = (h-2)x + q + \sum_{j=1}^{h-1} q_j \quad \text{and}$$

$$(\omega_C) = -hy + (g+h)p + \sum_{j=h}^{g+h-3} q_j$$

for some points $q_j \in X$ for $j = 1, \dots, h-1$ and $p, q_i \in C$ for $i = h, \dots, g+h-3$. There is a unique set of points $q_j$ for $j = 1, \dots, h-1$ in $X$ satisfying this relation.

The second relation describes the divisor $D^1_{-h,g+h,1^{g-2}}$. Further, in this case the global residue condition becomes

$$\mathrm{res}_y(\omega_C) = 0,$$

which is always satisfied, showing that these solutions are always smoothable and hence appear in the divisor $R$. Any other configuration of points will place a higher codimension condition on $[C, y] \in \mathcal{M}_g$. Such loci will be contained in the closure of the two we have described.

The boundary of $\overline{\mathcal{M}}_{g,1}$ is codimension one. Again, a simple check shows that a general point in any irreducible boundary component $\delta_i$ is not included in $\pi^* R$, and indeed we have found all components.

Hence we have

$$\pi^* R = ((g+h)^2 h - (h-1))W + D^1_{-h,g+h,1^{g-2}}.$$

By the pullback formula provided in Section 2.7 we have

$$\begin{aligned}\pi^* R = {} & \frac{g(g+h)((g+h)^2 h + (g+h)h - g - 1)}{2}\psi \\ & + \frac{(g+h)(3(g+h)^3 - 3(g+h) + 2)}{2}\lambda + \frac{(g+h)^2 - (g+h)^4}{6}\delta_0 \\ & + \sum_{i=1}^{g-1} \frac{(g+h)(i-g)((g+h)^2(i+h) + (g+h)(i+h) - g + i - 1)}{2}\delta_i,\end{aligned}$$

which gives the result.

REMARK 3.2. As discussed in Section 2.1, this divisor is irreducible for $g \geq 3$ and $g = 2$ for $h$ odd. When $g = 2$ and $h$ is even, this divisor has two connected components based on spin structure, $D^{1,\mathrm{odd}}_{-h,h+2} = 5W$ for $4 \mid h$ and $D^{1,\mathrm{even}}_{-h,h+2} = 5W$ otherwise.

### 3.4. The Remaining Residual Divisors

The classes $D^1_{(g-k,k+1,1^{g-3})}$ for $k = 1, \ldots, g-2$ complete the calculation of divisors on $\overline{\mathcal{M}}_{g,1}$ coming from strata of differentials with only zeros away from the marked point, all but one of which are simple. For $g \geq 3$ and $k = 1, \ldots, g-2$, let

$$D^1_{(g-k,k+1,1^{g-3})} = c_\psi \psi + c_\lambda \lambda + c_0 \delta_0 + \sum_{i=1}^{g-1} c_i \delta_i$$

in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1}) \otimes \mathbb{Q}$. The coefficient $c_\psi$ can be computed by a simple test curve. Consider the test curve $B$ in $\overline{\mathcal{M}}_{g,1}$ created by fixing a general genus $g$ curve $C$ and allowing the marked point to vary in the curve. It is well known that $B \cdot \psi = 2g - 2$, and hence

$$B \cdot D^1_{(g-k,k+1,1^{g-3})} = (2g-2)c_\psi.$$

Further, we can compute the intersection of this test curve with the divisor directly by de Jonquières' formula introduced in Section 2.4, which shows that there are

$$(k+1)(g-k)((k+1)g^2-(k^2+k+1)g-2)(g-1)$$

sections of the type required in the canonical series on the general genus $g$ curve $C$. Hence

$$c_\psi = \frac{(k+1)(g-k)((k+1)g^2-(k^2+k+1)g-2)}{2}.$$

To compute all other coefficients, we again consider the map

$$\begin{aligned} \pi : \overline{\mathcal{M}}_{g-i,1} &\to \quad \overline{\mathcal{M}}_{g,1} \\ [C, y] &\mapsto \left[ C \bigcup_{x=y} X, q \right], \end{aligned}$$

which for $[C, y] \in \overline{\mathcal{M}}_{g-i,1}$ identifies the point $y$ with the point $x$ of a general marked genus $i \geq 2$ curve $[X, x, q]$ as introduced in Section 2.7. Consider the pullback of the divisor $D^1_{g-k,k+1,1^{g-3}}$. Restricting to the smooth locus of $\mathcal{M}_{g-i,1}$, we again observe that there will be two components based on the position of the point of multiplicity $(k+1)$ in the twisted canonical divisor and there are only two configurations of zeros across the two components that give a codimension one condition on $[C, y] \in \mathcal{M}_{g-i,1}$. In the case that the point of multiplicity $(k+1)$, say $p$, lies on $X$, Section 2.2 shows that a twisted differential of the form required is differentials $\omega_X$ on $X$ and $\omega_C$ on $C$ such that

$$(\omega_X) = (-g+i-2)x + (k+1)p + (g-k)q + \sum_{j=1}^{i-1} q_j \quad \text{and}$$

$$(\omega_C) = (g-i)y + \sum_{j=i}^{g-3} q_i$$

for some points $p, q_j \in X$ for $j = 1, \ldots, i-1$ and $q_j \in C$ for $j = i, \ldots, g-3$. Hence $y$ is required to be a Weierstrass point. The global residue condition of such a twisted canonical divisor is

$$\mathrm{res}_x(\omega_X) = 0,$$

which is always satisfied. To enumerate such solutions on $X$, we are looking for points $p, q_j$ distinct from $x, q$ satisfying

$$(-g+i-2)x + (k+1)p + (g-k)q + \sum_{j=1}^{i-1} q_j \sim K_X$$

for a general curve $X$ of genus $i$ and fixed general points $x$ and $q$. Hence the $p$ are the ramification points of $|K_X + (g-i+2)x - (g-k)q|$, which is a $g^k_{k+i}$ and hence has

$$(k+1)(k+i) + (k+1)k(i-1) = (k+1)^2 i$$

ramification points (alternatively, this can be found by the Picard variety method). There are no solutions with $p = q$ as this would require $q$ to be the ramification points of $|K_X + (g - i + 2)x|$ and not a general point. There are no solutions with $p = x$ if $i \le g - k$ as this would contradict the assumption that $x$ and $q$ are general. If $i > g - k$, then we have the unique solution

$$(i - (g - k) - 1)x + \sum_{j=1}^{i-1} q_j \sim K_X - (g - k)q,$$

which by the Picard variety method in Section 2.5 has multiplicity $i - (g - k)$.

In the case that $p$ lies on $C$, Section 2.2 shows that a twisted differential of the form required is differentials $\omega_X$ on $X$ and $\omega_C$ on $C$ such that

$$(\omega_X) = (-g + i + k - 2)x + (g - k)q + \sum_{j=1}^{i} q_j \quad \text{and}$$

$$(\omega_C) = (g - i - k)y + (k + 1)p + \sum_{j=i+1}^{g-3} q_i$$

for some points $q_j \in X$ for $j = 1, \ldots, i$ and $p, q_j \in C$ for $j = i + 1, \ldots, g - 3$. The global residue condition of such a twisted canonical divisor is

$$\mathrm{res}_x(\omega_X) = 0,$$

which is always satisfied. Further, there is a unique solution to the points $q_j \in X$ as $h^0(K_X - (-g + i - k - 2)x - (g - k)q) = 1$ for general $x$ and $q$.

Finally, we again see that a general point of any boundary divisor $\delta_i$ is not included in $\pi^* D^1_{g-k,k+1,1^{g-3}}$, and hence there are no extra boundary components. For any $i \le g - k$, we have

$$\pi^* D^1_{g-k,k+1,1^{g-3}} = (k + 1)^2 i W^{g-i} + D^1_{g-i-k,k+1,1^{g-i-3}},$$

and for $i > g - k$, we have

$$\pi^* D^1_{g-k,k+1,1^{g-3}} = ((k + 1)^2 i - (k - g + i))W^{g-i} + D^1_{g-i-(k+1),k+1,1^{g-i-2}},$$

where, for $g - i - (k + 1) \le -2$, we calculated the class of $D^1_{g-i-(k+1),k+1,1^{g-i-2}}$ in Section 3.3.

The first relation yields

$$\begin{aligned}
c_i &= -\text{Coefficient of } \psi \text{ in } \pi^* D^1_{g-k,k+1,1^{g-3}} \\
&= -\frac{(k + 1)^2 i(g - i + 1)(g - i)}{2} \\
&\quad - \frac{(k + 1)(g - i - k)((k + 1)(g - i)^2 - (k^2 + k + 1)(g - i) - 2)}{2} \\
&= -(k + 1)(i(g - i + 1)(g - i)(k + 1) + (g - i - k) \\
&\quad \times ((k + 1)(g - i)^2 - (k^2 + k + 1)(g - i) - 2))/2
\end{aligned}$$

for $i \leq g-k$. The second relation yields

$$\begin{aligned}
c_\lambda &= ((k+1)^2 i - (k-g+i))(-1)\\
&\quad + \frac{1}{2}(1+(k+1))(2-3(g-i)^2+3(g-i)^3-2(k+i+1-g)\\
&\quad - 4(g-i)(k+i+1-g)+9(g-i)^2(k+i+1-g)\\
&\quad - (k+i+1-g)^2+9(g-i)(k+i+1-g)^2\\
&\quad + 3(k+i+1-g)^3)\\
&= \frac{(k+1)(4-2g+10k-2gk+11k^2+3k^3)}{2},\\
c_0 &= \frac{(k+1)^2-(k+1)^4}{6},\\
c_i &= -\text{Coefficient of } \psi \text{ in } \pi^* D^1_{g-k,k+1,1^{g-3}}\\
&= -\frac{((k+1)^2 i - (k-g+i))(g-i+1)(g-i)}{2}\\
&\quad - (g-i)(k+2)(i+k-g)\\
&\quad \times ((i+k+1-g)^2+(g-i)(i+k+1-g)+(g-i)+1)/2\\
&= -(g-i)(k+1)(-3g+g^2+4i-gi+3k\\
&\quad - 4gk+g^2k+5ik-gik+3k^2-2gk^2+2ik^2+k^3)/2
\end{aligned}$$

for $i > g-k$. This completes the computation of the classes presented in Theorem 3.1.

REMARK 3.3. Setting $k = g-1$, the coefficients match those computed for the residual divisor $R = D^1_{1^{g-2},g}$ in Section 3.2. Setting $k=0$, we obtain $(g-2)W = (g-2)D^1_{g,1^{g-2}}$.

REMARK 3.4. This divisor is defined for $g \geq 3$. As discussed in Section 2.1, by Kontsevich and Zorich [KZ] this divisor is irreducible in all cases except $g = 3$, $k = 1$. In this case, we have that $D^1_{2,2}$ contains two irreducible components distinguished by spin structure:

$$D^1_{2,2} = \overline{\Theta}_{3,1} + \varphi^* \overline{H},$$

where $\overline{\Theta}_{3,1}$ is the closure of the locus of $[C, p] \in \mathcal{M}_{3,1}$, where $p$ lies on a bitangent to a quartic plane curve, $\overline{H}$ is the closure of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_3$, and $\varphi : \overline{\mathcal{M}}_{3,1} \longrightarrow \overline{\mathcal{M}}_3$ simply forgets the marked point. We know that

$$\begin{aligned}
\overline{\Theta}_{3,1} &= 14\psi + 7\lambda - \delta_0 - 9\delta_1 - 5\delta_2,\\
\varphi^*\overline{H} &= 9\lambda - \delta_0 - 3\delta_1 - 3\delta_2,
\end{aligned}$$

where the class of $\overline{\Theta}_{3,1}$ was calculated by Farkas [F2], and the class of $\overline{H}$ is well known.

### 3.5. *Comparison with Brill–Noether Divisors*

Eisenbud and Harris [EH2] showed that the class of the closure of a pointed Brill–Noether divisor can be expressed as $\mu\mathcal{BN} + \nu W$, where

$$\mathcal{BN} = (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{g-1} i(g-i)\delta_i$$

is the pullback from $\overline{\mathcal{M}}_g$ of the Brill–Noether divisor, $W$ is the Weierstrass divisor, and $\mu$ and $\nu$ are nonnegative rational numbers. We observe that such a divisor satisfies

$$\mu = -\frac{6c_0}{g+1}$$

and

$$\nu = \frac{2c_\psi}{g(g+1)} = -\frac{6(g+3)}{g+1}c_0 - c_\lambda.$$

Hence, as divisors coming from the interior always have $c_\psi \geq 0$, we have the simple coefficient check

$$2c_\psi + 6(g+3)gc_0 + g(g+1)c_\lambda = 0.$$

Any class that violates this cannot be the class of a pointed Brill–Noether divisor. No class calculated in this section satisfies this relation (other than the Weierstrass divisor) and hence does not correspond to the class of a pointed Brill–Noether divisor.

## 4. Coupled Partition Divisors in $\overline{\mathcal{M}}_{g,n}$

In this section, we consider the divisors $D^n_{\underline{d},2^{g-1}}$ in $\overline{\mathcal{M}}_{g,n}$ for $g \geq 2$ with $\underline{d} = (d_1, \dots, d_n)$ for $\sum_i d_i = 0$. We refer to such a partition $(d_1, \dots, d_n, 2^{g-1})$ of $2g-2$ as a *coupled partition.* When $d_i$ are all even, there are two components based on even and odd spin structures. The coupled partition divisors present the simplest case to provide an exposition of our method as it pertains to strata of Abelian differentials with multiple components. We define $\underline{d}^-$ to be the vector containing only negative entries of $\underline{d}$. For any $S \subseteq \{1, \dots, n\}$, we define $d_S = \sum_{j \in S} d_j$. We record the results of this section in the following theorem.

THEOREM 4.1 (Coupled partition divisors).

$$D^2_{1,1,2^{g-2}} = 2^{g-3}\bigg(2^{g+1}\lambda + 2^{g-1}(\psi_1+\psi_2) - 2^{g-2}\delta_0 - \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1} 2^{g-1}\delta_{i:\{1\}}\bigg).$$

*For* $\underline{d}=(d_1,\dots,d_n)$ *such that* $\sum_{j=1}^n d_j=0$ *with* $\underline{d}^-\neq\{-2\}$,

$$D^n_{\underline{d},2^{g-1}}=2^{g-2}\Bigg(2^{g+1}\lambda+2^{g-1}\sum_{j=1}^n d_j^2\psi_j-2^{g-2}\delta_0 - \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g}2^{g-i+1}(2^i-1)\delta_{i:S}-2^{g-1}\sum_{\substack{|d_S|\geq 1\\ 1\notin S}}\sum_{i=0}^{g-1}d_S^2\delta_{i:S}\Bigg).$$

*If all* $d_j$ *are even, then*

$$D^{n,\mathrm{odd}}_{\underline{d},2^{g-1}}=2^{g-2}\Bigg((2^g-1)\lambda+\frac{2^g-1}{4}\sum_{j=1}^n d_j^2\psi_j-2^{g-3}\delta_0 - \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g}(2^i-1)(2^{g-i}+1)\delta_{i:S}-\frac{2^g-1}{4}\sum_{\substack{|d_S|\geq 2\\ 1\notin S}}\sum_{i=0}^{g-1}d_S^2\delta_{i:S}\Bigg)$$

*and*

$$D^{n,\mathrm{even}}_{\underline{d},2^{g-1}}=2^{g-2}\Bigg((2^g+1)\lambda+\frac{2^g+1}{4}\sum_{j=1}^n d_j^2\psi_j-2^{g-3}\delta_0 - \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g}(2^i-1)(2^{g-i}-1)\delta_{i:S}-\frac{2^g+1}{4}\sum_{\substack{|d_S|\geq 2\\ 1\notin S}}\sum_{i=0}^{g-1}d_S^2\delta_{i:S}\Bigg).$$

*For* $\underline{d}=(-2,1,1)$,

$$D^3_{\underline{d},2^{g-1}}=2^{g-3}\Bigg(2^{g+1}\lambda+2^{g+2}\psi_1+2^{g-1}(\psi_2+\psi_3)-2^{g-2}\delta_0 - \sum_{i=0}^{g-1}2^{i+1}(2^{g-i}-1)\delta_{i\{1,2,3\}}-\sum_{i=0}^{g-1}2^{i+1}(2^{g-i}+1)\delta_{i:\{2,3\}} - \sum_{i=0}^{g-1}2^{g-1}(\delta_{i:\{1,2\}}+\delta_{i:\{1,3\}})\Bigg).$$

*For* $\underline{d}=(-2,2)$,

$$D^2_{\underline{d},2^{g-1}}=2^{g-3}\Bigg(2^{g+1}\lambda+2^{g+2}\psi_1+2(2^g+1)\psi_2-2^{g-2}\delta_0 - \sum_{i=0}^{g-1}2^{i+1}(2^{g-i}-1)\delta_{i\{1,2\}}-2^{i+1}(2^{g-i}+1)\sum_{i=1}^{g-1}\delta_{i:\{2\}}\Bigg),$$

*with*

$$\begin{aligned} D^{2,\mathrm{odd}}_{\underline{d},2^{g-1}} &= \varphi_2^* D^{1,\mathrm{odd}}_{2^{g-1}} \\ &= 2^{g-3}\Bigg( (2^g-1)(\lambda + 2\psi_1 + 0\psi_2) - 2^{g-3}\delta_0 \\ &\quad - \sum_{i=0}^{g-1}(2^i+1)(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}(2^i-1)(2^{g-i}+1)\delta_{i:\{2\}}\Bigg), \end{aligned}$$

*where $\varphi_2 : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,1}$ forgets the second marked point, and*

$$\begin{aligned} D^{2,\mathrm{even}}_{\underline{d},2^{g-1}} &= 2^{g-3}\Bigg( (2^g+1)(\lambda + 2(\psi_1 + \psi_2)) - 2^{g-3}\delta_0 \\ &\quad - \sum_{i=0}^{g-1}(2^i-1)(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}(2^i+1)(2^{g-i}+1)\delta_{i:\{2\}}\Bigg). \end{aligned}$$

The following propositions provide the proof of this theorem.

PROPOSITION 4.2. *The class of the divisor $D^2_{1,1,2^{g-2}}$ in $\overline{\mathcal{M}}_{g,2}$ is*

$$\begin{aligned} D^2_{1,1,2^{g-2}} &= 2^{g-3}\Bigg( 2^{g+1}\lambda + 2^{g-1}(\psi_1+\psi_2) - 2^{g-2}\delta_0 \\ &\quad - \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1} 2^{g-1}\delta_{i:\{1\}}\Bigg). \end{aligned}$$

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C,y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general rational tail marked at three points $[X, x, q_1, q_2]$ as introduced in Section 2.7. Under this map, we have

$$\pi^* D^2_{1,1,2^{g-2}} = D^1_{2^{g-1}} = D^{1,\mathrm{odd}}_{2^{g-1}} + D^{1,\mathrm{even}}_{2^{g-1}}.$$

The two divisors on the RHS are known. Farkas [F2] calculated the divisor class of the closure of the locus of points in the support of odd theta characteristics in $\overline{\mathcal{M}}_{g,1}$. In our notation,

$$D^{1,\mathrm{odd}}_{2^{g-1}} = 2^{g-3}\Bigg( (2^g-1)(\lambda+2\psi) - 2^{g-3}\delta_0 - \sum_{i=1}^{g-1}(2^i+1)(2^{g-i}-1)\delta_i \Bigg). \quad (1)$$

This is the class of the odd spin structure component of $D^1_{2^{g-1}}$ which has two components by Konsevich and Zorich [KZ]. The class of the even spin structure component is

$$D^{1,\mathrm{even}}_{2^{g-1}} = 2^{g-3}\Bigg( (2^g+1)\lambda + 0\psi - 2^{g-3}\delta_0 - \sum_{i=1}^{g-1}(2^i-1)(2^{g-i}-1)\delta_i \Bigg). \quad (2)$$

An even theta characteristic of this type on a curve $C$ gives a degree $g-1$ cover from $C$ to $\mathbb{P}^1$. The ramification points of this cover place the curve $C$ in the divisor

$D^{\text{even}}_{4,2^{g-3}}$ in $\overline{\mathcal{M}}_g$. Hence if $\varphi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_g$ forgets the marked point, then we have $D^{1,\text{even}}_{2^{g-1}} = \varphi^* D^{\text{even}}_{4,2^{g-3}}$, where the divisor $D^{\text{even}}_{4,2^{g-3}}$ was originally calculated by Teixidor and Bigas [T] as the closure of the divisor of curves with a vanishing theta-null.

Hence we obtain the coefficients of $\lambda, \delta_0, \delta_{0:\{1,2\}}, \delta_{i:\{1,2\}}$ for $i > 0$. For the coefficients of $\psi_i$, consider the test curve $B$ defined by taking a general curve $C$ and marking a general point as the second point. Allow the first point to vary in the curve. We have

$$\begin{aligned} B \cdot D^2_{1,1,2^{g-2}} &= (2g-1)c_{\psi_1} + c_{\psi_2} + c_{0:\{1,2\}} \\ &= \mathrm{dJ}[g; 1, 2^{g-2}] = 2^{g-2}(2^{g-1}(g-2)+1). \end{aligned}$$

Hence by symmetry

$$c_{\psi_i} = \frac{1}{2g}(2^{g-2}(2^{g-1}(g-2)+1) + 2^{g-2}(2^g-1)) = 2^{2g-4}.$$

Finally, we need to calculate the coefficients of $\delta_{i:\{1\}}$. Consider the map $\pi : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g+h,2}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g,2}$ identifies the point $y_2$ with the point $x$ of a marked general genus $h$ curve $(X, x, q)$ as introduced in Section 2.7. Under this map, we have

$$\pi^* D^2_{1,1,2^{g-2}} = 4^h D^2_{1,1,2^{g-h-2}}.$$

The known $\psi_i$ coefficients then complete our calculation. □

REMARK 4.3. This formula is known in $g = 2$ and $g = 3$ by equations (4) and (3) (presented in Section 5), respectively.

We now specialize to the two cases where the signature has exactly one pole with multiplicity two.

PROPOSITION 4.4. *For* $\underline{d} = (-2, 1, 1)$,

$$\begin{aligned} D^3_{\underline{d},2^{g-1}} = 2^{g-3}(&2^{g+1}\lambda + 2^{g+2}\psi_1 + 2^{g-1}(\psi_2+\psi_3) - 2^{g-2}\delta_0 \\ &- \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}-1)\delta_{i\{1,2,3\}} - \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}+1)\delta_{i:\{2,3\}} \\ &- \sum_{i=0}^{g-1} 2^{g-1}(\delta_{i:\{1,2\}} + \delta_{i:\{1,3\}}). \end{aligned}$$

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,3} \longrightarrow \overline{\mathcal{M}}_{g+1,2}$ that for $[C, y_1, y_2, y_3] \in \overline{\mathcal{M}}_{g,3}$ identifies the point $y_1$ with the point $x$ of a general marked elliptic tail $[X, x]$ as introduced in Section 2.7. We have

$$\pi^* D^2_{1,1,2^{g-1}} = D^3_{-2,1,1,2^{g-1}} + 3\varphi_1^* D^2_{1,1,2^{g-2}},$$

where $\varphi_1 : \overline{\mathcal{M}}_{g,3} \longrightarrow \overline{\mathcal{M}}_{g,2}$ simply forgets the first marked point. The multiplicity 3 of the second component represents placing one of the unmarked double zeros at a distinct two-torsion point on the elliptic curve. By Proposition 4.2 we have that $\pi^* D^2_{1,1,2^{g-1}}$ equals

$$2^{g-2}\Bigg(2^{g+2}\lambda + 2^g(\psi_2+\psi_3) + 2^{g+1}\psi_1 - 2^{g-1}\delta_0 - \sum_{i=0}^{g-1} 2^{i+2}(2^{g-i}-1)\delta_{i:\{1,2,3\}} - \sum_{i=0}^{g-1} 2^{i+1}(2^{g+1-i}-1)\delta_{i:\{2,3\}} - \sum_{i=1}^{g-1} 2^g(\delta_{i:\{1,3\}}+\delta_{i:\{1,2\}})\Bigg)$$

and $3\varphi_1^* D^2_{1,1,2^{g-2}}$ equals

$$3\cdot 2^{g-3}\Bigg(2^{g+1}\lambda + 2^{g-1}(\psi_2+\psi_3) - 2^{g-2}\delta_0 - 2^{g-1}(\delta_{0:\{1,2\}}+\delta_{0:\{1,3\}}) - \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}-1)(\delta_{i:\{2,3\}}+\delta_{i:\{1,2,3\}}) - \sum_{i=1}^{g-1} 2^{g-1}(\delta_{i:\{1,3\}}+\delta_{i:\{1,2\}})\Bigg).$$

The proposition follows. □

PROPOSITION 4.5. *For $\underline{d}=(-2,2)$,*

$$D^2_{\underline{d},2^{g-1}} = 2^{g-3}\Bigg(2^{g+1}\lambda + 2^{g+2}\psi_1 + 2(2^g+1)\psi_2 - 2^{g-2}\delta_0 - \sum_{i=0}^{g-1} 2^{i+1}(2^{g-i}-1)\delta_{i\{1,2\}} - 2^{i+1}(2^{g-i}+1)\sum_{i=1}^{g-1}\delta_{i:\{2\}}\Bigg)$$

*with*

$$D^{2,\mathrm{odd}}_{\underline{d},2^{g-1}} = 2^{g-3}\Bigg((2^g-1)(\lambda+2\psi_1+0\psi_2) - 2^{g-3}\delta_0 - \sum_{i=0}^{g-1}(2^i+1)(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}(2^i-1)(2^{g-i}+1)\delta_{i:\{2\}}\Bigg)$$

*and*

$$D^{2,\mathrm{even}}_{\underline{d},2^{g-1}} = 2^{g-3}\Bigg((2^g+1)(\lambda+2(\psi_1+\psi_2)) - 2^{g-3}\delta_0 - \sum_{i=0}^{g-1}(2^i-1)(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}(2^i+1)(2^{g-i}+1)\delta_{i:\{2\}}\Bigg).$$

*Proof.* For $\underline{d}=(-2,2)$, consider the map $\pi:\overline{\mathcal{M}}_{g,2}\longrightarrow\overline{\mathcal{M}}_{g,3}$ that for $[C,y_1,y_2]\in\overline{\mathcal{M}}_{g,2}$ identifies the point $y_2$ with the point $x$ of a general rational tail marked at three points $[X, x, q_1, q_2]$ as introduced in Section 2.7. We have

$$\pi^* D^3_{-2,1,1,2^{g-1}} = D^2_{-2,2,2^{g-1}},$$

where the divisor on the LHS is known by Proposition 4.4.

To distinguish the components, consider the map $\pi : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g+1,1}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g,2}$ identifies the point $y_1$ with the point $x$ of a general marked elliptic tail $[X, x]$ as introduced in Section 2.7. We have

$$\pi^* D^1_{2^g} = D^2_{-2,2,2^{g-1}} + 3\varphi_1^* D^1_{2^{g-1}},$$

where $\varphi_1 : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,1}$ simply forgets the first marked point. The multiplicity 3 of the second component represents placing one of the unmarked double zeros at a two-torsion point to the node on the elliptic curve. On the components, this becomes

$$\pi^* D^{1,\mathrm{odd}}_{2^g} = D^{2,\mathrm{even}}_{-2,2,2^{g-1}} + 3\varphi_1^* D^{1,\mathrm{odd}}_{2^{g-1}}$$

and

$$\pi^* D^{1,\mathrm{even}}_{2^g} = D^{2,\mathrm{odd}}_{-2,2,2^{g-1}} + 3\varphi_1^* D^{1,\mathrm{even}}_{2^{g-1}},$$

where the divisors $D^{1,\mathrm{odd}}_{2^g}$ and $D^{1,\mathrm{even}}_{2^g}$ are known by equations (1) and (2), respectively. □

REMARK 4.6. Observe what may at first appear to be the curious consequence that $D^{2,\mathrm{odd}}_{-2,2,2^{g-1}} = \varphi_2^* D^{1,\mathrm{odd}}_{2^{g-1}}$. Recall our definition

$$D^{2,\mathrm{odd}}_{-2,2,2^{g-1}} := \overline{\{(C, p_1, p_2) \in \mathcal{M}_{g,2} \mid -p_1 + p_2 + s_1 + \cdots + s_{g-1} \sim \eta_C \text{ for } \eta_C \text{ odd and } p_i, s_j \text{ distinct}\}}.$$

If $h^0(\eta_C) = 1$ with $s'_1 + \cdots + s'_{g-1} \sim \eta_C$, then $h^0(\eta_C + x) \geq 1$ for any $x$ as $s'_1 + \cdots + s'_{g-1} + x$ is a section. However, this section does not satisfy our requirements, and hence we require $h^0(\eta_C + x) = 2$. The Riemann–Roch theorem then gives

$$h^0(\eta_C - x) = 1 - g + (g - 2) + h^0(\eta_C + x) = 1,$$

which explains this result.

REMARK 4.7. As a check, consider the map $\pi : \overline{\mathcal{M}}_{g-i,2} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g-i,2}$ identifies the point $y_1$ with the point $x$ of a general marked genus $i$ curve $[X, x, q]$ as introduced in Section 2.7. We obtain

$$\pi^* D^{2,\mathrm{odd}}_{-2,2,2^{g-1}} = 2^{i-1}(2^i + 1) D^{2,\mathrm{odd}}_{-2,2,2^{g-i-1}} + 2^{i-1}(2^i - 1) D^{2,\mathrm{even}}_{-2,2,2^{g-i-1}}$$

and

$$\pi^* D^{2,\mathrm{even}}_{-2,2,2^{g-1}} = 2^{i-1}(2^i - 1) D^{2,\mathrm{odd}}_{-2,2,2^{g-i-1}} + 2^{i-1}(2^i + 1) D^{2,\mathrm{even}}_{-2,2,2^{g-i-1}}.$$

REMARK 4.8. When $g = 2$, the pinch partition and coupled partition divisors coincide, and these results agree with the results in Section 5.

At this point, we provide a simple example of controlling the residues in a meromorphic differential on a rational curve that will prove important in our following divisor class calculations.

EXAMPLE 4.9. Consider a meromorphic differential on a rational curve with poles of multiplicities $j$ and $k$ at 0 and $\infty$, respectively, for $2 \le j \le k$ and zeros at 1 and $t$ of multiplicity $j+k-m-2$ and $m$, respectively, for $j-1 \le m \le j+k-3$. The differential is given locally at 0 by

$$c\frac{(z-1)^{j+k-m-2}(z-t)^m}{z^j}\,dz$$

for some constant $c \neq 0$. The residue at 0 is given by

$$c(-1)^{k-1}t^{m-j+1}\sum_{i=0}^{j-1}\binom{j+k-m-2}{i}\binom{m}{j-i-1}t^i.$$

Hence by investigating the polynomial

$$\sum_{i=0}^{j-1}\binom{j+k-m-2}{i}\binom{m}{j-i-1}t^i$$

we obtain the number of meromorphic differentials on a rational curve of signature $\kappa = (-j, -k, m, j+k-m-2)$ with zero residue at the poles.

For example, consider the meromorphic differentials on a rational curve of signature $\mu = (-4, -h, h, 2)$. From our discussion we see that the polynomial becomes

$$\binom{h}{1}t + 2\binom{h}{2}t^2 + \binom{h}{3}t^3,$$

which has two nonzero solutions when $h \ge 3$ and only one solution when $h = 2$. When $h = 1$, there are no solutions; indeed, the residue at a simple pole is necessarily nonzero.

PROPOSITION 4.10. *The class of the divisor $D^2_{-h,h,2^{g-1}}$ for $h \ge 3$ in $\overline{\mathcal{M}}_{g,2}$ is*

$$D^2_{-h,h,2^{g-1}} = 2^{g-2}\Bigg(2^{g+1}\lambda + 2^{g-1}h^2\psi_1 + 2^{g-1}h^2\psi_2 - 2^{g-2}\delta_0 - \sum_{i=0}^{g-1}2^{i+1}(2^{g-i}-1)\delta_{i:\{1,2\}} - \sum_{i=1}^{g-1}2^{g-1}h^2\delta_{i:\{2\}}\Bigg).$$

*When $h$ is even, this divisor has two components with classes*

$$D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} = 2^{g-2}\Bigg((2^g-1)\left(\lambda + \frac{h^2}{4}(\psi_1+\psi_2) - 2\delta_{0:\{1,2\}}\right) - 2^{g-3}\delta_0 - \sum_{i=1}^{g-1}(2^i+1)(2^{g-i}-1)\delta_{i,\{1,2\}} - \frac{2^g-1}{4}\sum_{i=1}^{g-1}h^2\delta_{i:\{2\}}\Bigg)$$

*and*

$$D^{2,\text{even}}_{-h,h,2^{g-1}} = 2^{g-2}\Bigg((2^g+1)\Big(\lambda + \frac{h^2}{4}(\psi_1+\psi_2) - 0\delta_{0:\{1,2\}}\Big) - 2^{g-3}\delta_0 - \sum_{i=1}^{g-1}(2^i-1)(2^{g-i}-1)\delta_{i,\{1,2\}} - \frac{2^g+1}{4}\sum_{i=1}^{g-1}h^2\delta_{i:\{2\}}\Bigg).$$

REMARK 4.11. Consider

$$D_\infty = \lim_{h\to\infty}\frac{1}{h^2}D^2_{-h,h,2^{g-1}} = 2^{2g-3}\Bigg(\psi_1+\psi_2-\sum_{i=1}^{g-1}\delta_{i:\{2\}}\Bigg)$$

with

$$D^{\text{odd}}_\infty = \frac{2^g-1}{2^{g+1}}D_\infty \quad\text{and}\quad D^{\text{even}}_\infty = \frac{2^g+1}{2^{g+1}}D_\infty.$$

Then we obtain

$$D^2_{-h,h,2^{g-1}} = D^2_{-2,2,2^{g-1}} + \varphi_1^*D^1_{2^{g-1}} + h^2D_\infty,$$

where $\varphi_1 : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,1}$ forgets the first marked point. This relation also holds in the odd and even spin structure components for even $h$.

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C,y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general marked rational tail $[X,x,q_1,q_2]$ as introduced in Section 2.7. We have

$$\pi^*D^2_{-h,h,2^{g-1}} = 2D^1_{2^{g-1}} = 2D^{1,\text{odd}}_{2^{g-1}} + 2D^{1,\text{even}}_{2^{g-1}}$$

for $h \geq 3$, as Example 4.9 shows that to obtain a zero residue at the node as required by the global residue condition, there are exactly two points to place the unmarked zero of multiplicity 2 on the rational tail if $h \geq 3$ and exactly one point if $h = 2$. The divisor classes on the right-hand side are given by equations (1) and (2), respectively. Hence we again obtain the coefficients of $\lambda, \delta_0, \delta_{0:\{1,2\}}, \delta_{i:\{1,2\}}$ for $i > 0$. For the coefficients of $\psi_i$, consider the test curve $B_i$ defined by taking a general curve $C$ and allowing the $i$th marked point to vary in the curve while fixing the other marked point at a general point. We have

$$\begin{aligned} B_1\cdot D^2_{-h,h,2^{g-1}} &= (2g-1)c_{\psi_1} + c_{\psi_2} + c_{0:\{1,2\}} = 2^{2g-2}gh^2 - 2^{g-1}(2^g-1) \\ &= 2^{g-2}(2^g gh^2 - 2^{g+1} + 2) \end{aligned}$$

by the Picard variety method introduced in Section 2.5, where the correction term is for the $2^{g-1}(2^g-1)$ solutions where the points are equal. These solutions violate the global residue condition. Each solution has multiplicity one. Similarly,

$$B_2\cdot D^2_{-h,h,2^{g-1}} = c_{\psi_1} + (2g-1)c_{\psi_2} + c_{0:\{1,2\}} = 2^{g-2}(2^g gh^2 - 2^{g+1} + 2).$$

Hence

$$2gc_{\psi_i} = 2^{g-2}(2^g gh^2 - 2^{g+1} + 2) + 2^{g-1}(2^g-1),$$

giving

$$c_{\psi_i} = 2^{2g-3}h^2.$$

Finally, we compute the coefficients of $\delta_{i:\{2\}}$ for $i > 0$. Consider the map $\pi : \overline{\mathcal{M}}_{g-i,2} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g-i,2}$ identifies the point $y_2$ with the point $x$ of a general marked genus $i$ curve $[X, x, q]$ as introduced in Section 2.7. We have

$$\pi^* D^2_{-h,h,2^{g-1}} = 4^i D^2_{-h,h,2^{g-i-1}}$$

and, similarly,

$$\pi^* D^2_{h,-h,2^{g-1}} = 4^i D^2_{h,-h,2^{g-i-1}}.$$

This agrees with all of our calculated coefficients and gives the final unknown coefficients

$$c_{i:\{2\}} = -4^i 2^{2(g-i)-3} h^2 = -2^{2g-3} h^2.$$

Next, we need to identify the components when $h$ is even. As discussed in Section 2.1, the divisor has two irreducible components in this case corresponding to odd and even spin structures. We use the same procedure to calculate the class of the components. Consider the map $\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C, y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general marked rational curve $[X, x, q_1, q_2]$ as introduced in Section 2.7. By our discussion of meromorphic differentials on a rational curve in Example 4.9 we see that there are two points on the rational tail to place the double zero to make the residue at the node zero and hence satisfy the global residue condition. Further, these will give limits of theta characteristics by Section 2.3, and under this map,

$$\pi^* D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} = 2D^{1,\mathrm{odd}}_{2^{g-1}} \quad \text{and} \quad \pi^* D^{2,\mathrm{even}}_{-h,h,2^{g-1}} = 2D^{1,\mathrm{even}}_{2^{g-1}},$$

where the divisor classes on the right-hand side are given by equations (1) and (2), respectively. Hence we again obtain the coefficients of $\lambda, \delta_0, \delta_{0:\{1,2\}}, \delta_{i:\{1,2\}}$ for $i > 0$. To obtain the coefficients of $\psi_i$, we need to distinguish which intersections with our test curves $B_1$ and $B_2$ belong to which component. We observe that by the Picard variety method, for any fixed theta characteristic $\eta_C$ on a general curve $C$, there are $gh^2/4$ solutions $p_i$ that satisfy

$$\frac{h}{2} p_1 + \sum_{j=2}^{g} p_j \sim \eta_C + \frac{h}{2} x$$

and $gh^2/4$ solutions $p_i$ that satisfy

$$-\frac{h}{2} p_1 + \sum_{j=2}^{g} p_j \sim \eta_C - \frac{h}{2} x$$

for any fixed general point $x \in C$. We observe that the solutions we discounted by were all odd theta characteristics; hence

$$\begin{aligned} B_1 \cdot D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} &= (2g-1)c_{\psi_1} + c_{\psi_2} + c_{0:\{1,2\}} \\ &= 2^{g-1}(2^g-1)g\left(\frac{h}{2}\right)^2 - 2^{g-1}(2^g-1) \end{aligned}$$

and, similarly,

$$B_2 \cdot D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} = c_{\psi_1} + (2g-1)c_{\psi_2} + c_{0:\{1,2\}}$$
$$= 2^{g-1}(2^g-1)g\left(\frac{h}{2}\right)^2 - 2^{g-1}(2^g-1).$$

This gives

$$c^{\mathrm{odd}}_{\psi_1} = c^{\mathrm{odd}}_{\psi_2} = 2^{g-4}(2^g-1)h^2.$$

Similarly,

$$B_1 \cdot D^{2,\mathrm{even}}_{-h,h,2^{g-1}} = (2g-1)c_{\psi_1} + c_{\psi_2} + c_{0:\{1,2\}} = 2^{g-3}(2^g+1)h^2 g$$

and

$$B_2 \cdot D^{2,\mathrm{even}}_{-h,h,2^{g-1}} = c_{\psi_1} + (2g-1)c_{\psi_2} + c_{0:\{1,2\}} = 2^{g-3}(2^g+1)h^2 g,$$

giving

$$c^{\mathrm{even}}_{\psi_1} = c^{\mathrm{even}}_{\psi_2} = 2^{g-4}(2^g+1)h^2.$$

Finally, consider the map $\pi : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g+j,2}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g,2}$ identifies the point $y_2$ with the point $x$ of a general marked genus $j$ curve $[X, x, q]$ as introduced in Section 2.7. We have

$$\pi^* D^{2,\mathrm{odd}}_{-h,h,2^{g+j-1}} = 2^{j-1}(2^j+1)D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} + 2^{j-1}(2^j-1)D^{2,\mathrm{even}}_{-h,h,2^{g-1}}$$

and

$$\pi^* D^{2,\mathrm{even}}_{-h,h,2^{g+j-1}} = 2^{j-1}(2^j-1)D^{2,\mathrm{odd}}_{-h,h,2^{g-1}} + 2^{j-1}(2^j+1)D^{2,\mathrm{even}}_{-h,h,2^{g-1}}.$$

Similarly,

$$\pi^* D^{2,\mathrm{odd}}_{h,-h,2^{g+j-1}} = 2^{j-1}(2^j+1)D^{2,\mathrm{odd}}_{h,-h,2^{g-1}} + 2^{j-1}(2^j-1)D^{2,\mathrm{even}}_{h,-h,2^{g-1}}$$

and

$$\pi^* D^{2,\mathrm{even}}_{h,-h,2^{g+j-1}} = 2^{j-1}(2^j-1)D^{2,\mathrm{odd}}_{h,-h,2^{g-1}} + 2^{j-1}(2^j+1)D^{2,\mathrm{even}}_{h,-h,2^{g-1}}.$$

This agrees with our calculated coefficients and gives the final unknown coefficients

$$c^{\mathrm{odd}}_{i:\{2\}} = -2^{g-4}(2^g-1)h^2 \quad \text{and} \quad c^{\mathrm{even}}_{i:\{2\}} = -2^{g-4}(2^g+1)h^2. \qquad \square$$

Next, we generalize to $\overline{\mathcal{M}}_{g,n}$.

PROPOSITION 4.12. *Consider* $\underline{d} = (d_1, \dots, d_n)$ *such that* $\sum_{j=1}^n d_j = 0$ *with* $\underline{d}^- \neq \{-2\}$. *Then*

$$D^n_{\underline{d},2^{g-1}} = 2^{g-2}\Bigg(2^{g+1}\lambda + 2^{g-1}\sum_{j=1}^n d_j^2\psi_j - 2^{g-2}\delta_0$$
$$- \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g} 2^{g-i+1}(2^i-1)\delta_{i:S} - 2^{g-1}\sum_{\substack{|d_S|\geq 1\\ 1\notin S}}\sum_{i=0}^{g-1} d_S^2\delta_{i:S}\Bigg).$$

*If all $d_j$ are even, then*

$$D^{n,\mathrm{odd}}_{\underline{d},2^{g-1}} = 2^{g-2}\bigg((2^g-1)\lambda + \frac{2^g-1}{4}\sum_{j=1}^{n} d_j^2\psi_j - 2^{g-3}\delta_0 - \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g}(2^i-1)(2^{g-i}+1)\delta_{i:S} - \frac{2^g-1}{4}\sum_{\substack{|d_S|\geq 2\\ 1\notin S}}\sum_{i=0}^{g-1} d_S^2\delta_{i:S}\bigg)$$

*and*

$$D^{n,\mathrm{even}}_{\underline{d},2^{g-1}} = 2^{g-2}\bigg((2^g+1)\lambda + \frac{2^g+1}{4}\sum_{j=1}^{n} d_j^2\psi_j - 2^{g-3}\delta_0 - \sum_{\substack{|d_S|=0\\ 1\notin S}}\sum_{i=0}^{g}(2^i-1)(2^{g-i}-1)\delta_{i:S} - \frac{2^g+1}{4}\sum_{\substack{|d_S|\geq 2\\ 1\notin S}}\sum_{i=0}^{g-1} d_S^2\delta_{i:S}\bigg).$$

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ that for $[C, y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general marked rational curve $[X, x, q_1, \ldots, q_n]$ as introduced in Section 2.7. We have

$$\pi^* D^n_{\underline{d},2^{g-1}} = 2D^1_{2^{g-1}},$$

where there are two points on the rational tail that make the residue at the node zero and hence satisfy the global residue condition. If all $d_i$ are even, then this relation also holds on the odd and even spin structure components. The classes of the divisors on the right-hand side of this equation in these cases are given by equations (1) and (2), respectively. This provides the coefficients for $\lambda$, $\delta_0$, and $\delta_{i:\{1,\ldots,n\}}$.

Now, for any $j \in \{1, \ldots, n\}$, consider the map $\pi : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,n}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g,2}$ identifies the point $y_2$ with the point $x$ of a general marked rational curve $[X, x, q_1, \ldots, q_{n-1}]$ and labels the points $q_i$ from $\{1, \ldots, n\} \setminus \{j\}$ as introduced in Section 2.7. For $|d_j| \geq 3$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^2_{d_j,-d_j,2^{g-1}};$$

for $d_j = 2$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^2_{2,-2,2^{g-1}} + \varphi_2^* D^1_{2^{g-1}};$$

and for $d_j = -2$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^2_{-2,2,2^{g-1}} + \varphi_1^* D^1_{2^{g-1}},$$

where $\varphi_i : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,1}$ forgets the $i$th point. For $d_j = 1$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = 2D^2_{1,1,2^{g-2}},$$

and for $d_j = -1$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = 2D^2_{1,1,2^{g-2}}.$$

When all $d_i$ are even, these relations also hold on the odd and even spin structure components. By the divisor classes presented in Propositions 4.2, 4.5, and 4.10 and by equations (1) and (2) these relations agree with the previously calculated coefficients and give us the coefficients for $\psi_j$ and $\delta_{i:\{j\}}$.

Now, for any $S \subseteq \{1, \dots, n\}$ with $2 \le |S| \le n-2$, consider the map $\pi : \overline{\mathcal{M}}_{g,n-|S|+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ that for $[C, y_1, \dots, y_{n-|S|+1}] \in \overline{\mathcal{M}}_{g,n-|S|+1}$ identifies the point $y_1$ with the point $x$ of a general marked rational curve $[X, x, q_1, \dots, q_{|S|}]$ and labels the $q_j$ from $S$ as introduced in Section 2.7. For $|d_S| \ge 3$, we have

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^{n-|S|+1}_{d_S,\underline{d}(S^C),2^{g-1}},$$

where, for any $T \subseteq \{1, \dots, n\}$, we define $\underline{d}(T)$ to be the truncated vector containing only the entries of $\underline{d}$ indexed by $T$. In fact, as $|S^C| \ge 2$ for $\{d_S, \underline{d}(S^C)\}^- \ne \{-2\}$, this relation holds where we use the convention

$$D^n_{0,\underline{d},2^{g-1}} = \varphi_1^* D^{n-1}_{\underline{d},2^{g-1}},$$

where $\varphi_i : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1}$ forgets the $i$th point. When all $d_i$ are even, this relation also holds for the odd and even spin structure components.

The final situation to consider is where $\{d_S, \underline{d}(S^C)\}^- = \{-2\}$. For $d_S = -2$, we have $\underline{d}(S^C) = \{1, 1\}$, and hence

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^3_{-2,1,1,2^{g-1}} + \varphi_1^* D^2_{1,1,2^{g-2}},$$

whereas if $d_S \ne -2$, then necessarily $\underline{d}(S^C) = \{1, -2\}$ and $d_S = 1$, and hence

$$\pi^* D^n_{\underline{d},2^{g-1}} = D^3_{1,1,-2,2^{g-1}} + \varphi_3^* D^2_{1,1,2^{g-2}}.$$

When all $d_j$ are even, these relations hold on the odd and even spin structure components. By these relations we obtain our remaining coefficients. □

REMARK 4.13. As a quick check of the formulas for simple poles, consider the divisor $D^4_{-1,-1,1,1,2^{g-1}}$ in $\overline{\mathcal{M}}_{g,4}$. Under the map $\pi : \overline{\mathcal{M}}_{g,3} \longrightarrow \overline{\mathcal{M}}_{g,4}$ that for $[C, y_1, y_2, y_3] \in \overline{\mathcal{M}}_{g,3}$ identifies the point $y_1$ with the point $x$ of the general marked rational curve $[X, x, q_1, q_2]$ as introduced in Section 2.7, we have

$$\pi^* D^4_{-1,-1,1,1,2^{g-1}} = D^3_{-2,1,1,2^{g-1}} + \varphi_1^* D^2_{1,1,2^{g-2}},$$

which agrees with our class calculation.

REMARK 4.14. Restricting to the $g = 2$ case, Boissy [B] showed that $\mathcal{P}(-1, -1, 2, 2)$ has two connected components based on hyperellipticity. In the case of the divisors, this gives

$$D^3_{-1,-1,2,2} = D^{3,\text{non-hyp}}_{-1,-1,2,2} + \varphi_3^* D^2_{1,1},$$

where $\varphi_3 : \overline{\mathcal{M}}_{2,3} \longrightarrow \overline{\mathcal{M}}_{2,2}$ forgets the final marked point.

## 5. Pinch Partition Divisors in $\overline{\mathcal{M}}_{g,n}$

Farkas and Verra [FV2] calculated the class of the divisor $D^{g-1}_{1^{2g-4},2}$, which they called the closure of the antiramification locus:

$$D^{g-1}_{1^{2g-4},2} = -4(g-7)\lambda + 4(g-2)\sum_{i=1}^{g-1}\psi_i - 2\delta_0 + \sum_{i=0}^{g}\sum_{s=0}^{i-1} c_{i:s}\delta_{i:S} \tag{3}$$

for $s = |S|$ with $s = 0, \ldots, i-1$, where

$$c_{i:s} = -(2g-3)s^2 + (4gi + 2g - 10i + 1)s - 2gi^2 + 7i^2 - 2gi - i - 2.$$

Note that $c_{0:s} = c_{g:g-s-1}$ for $s \geq 2$.

This can be generalized to holomorphic and meromorphic strata with the same signature of "pinched" unmarked points. We summarize the results of this section in the following theorem.

THEOREM 5.1 (Pinch partition divisors). *For $\underline{d} = (d_1, \ldots, d_n)$ with $d_i \geq 0$ and $\sum d_i = g-1$, we have, for $g \geq 3$,*

$$D^n_{\underline{d},1^{g-3},2} = -4(g-7)\lambda + \sum_{i=1}^{g-1}(2g(d_i+1) - 3d_i - 5)d_i\psi_i - 2\delta_0 + \sum_{i=0}^{g}\sum_{d_S=0}^{i-1} c_{i:S}\delta_{i:S},$$

*where*[5]

$$c_{i:S} = (3-2g)d_S^2 + (4gi + 2g - 10i + 1)d_S - 2gi^2 + 7i^2 - 2gi - i - 2$$

*for $d_S := \sum_{i \in S} d_i$.*

*For $\underline{d} = (d_1, \ldots, d_n)$ with $\sum d_i = g-2$, $d_j \leq -2$, and $d_i \geq 0$ for $i \neq j$, we have, for $d_j \leq -3$,*

$$D^n_{\underline{d},1^{g-2},2} = (26-4g)\lambda + \sum_{i=1}^{n} 2d_i((g-1)d_i + g - 2)\psi_i - 2\delta_0 + \sum_{i=0}^{g-1} c_{i:S}\delta_{i:S},$$

*where, for $j \notin S$ and $d_S \leq i-1$,*

$$c_{i:S} = (2-2g)d_S^2 + 2(2gi + g - 4i + 1)d_S - 2(gi^2 + gi - 3i^2 + i + 1),$$

*and, for $d_S \geq i$,*

$$c_{i:S} = (2-2g)d_S^2 + 2(2gi - g - 4i + 2)d_S - 2(gi^2 - 3i^2 - gi + 4i).$$

*For $d_j = -2$,*

$$\begin{aligned} D^n_{\underline{d},1^{g-2},2} &= (27-4g)\lambda + 4g\psi_j \\ &\quad + \sum_{i \neq j} \frac{(4g(d_i+1) - 5d_i - 9)d_i}{2}\psi_i - 2\delta_0 + \sum_{i=0}^{g-1} c_{i:S}\delta_{i:S}, \end{aligned}$$

[5] Note that $c_{0:S} = c_{g:g-d_S-1}$ for $|S| \geq 2$.

*where, for $j \notin S$ and $d_S \leq i-1$,*

$$c_{i:S} = \frac{1}{2}((5-4g)d_S^2 + (8gi+4g-18i+3)d_S - 4gi^2 - 4gi + 13i^2 - 3i - 4)$$

*and, for $d_S \geq i$,*

$$c_{i:S} = \frac{1}{2}((5-4g)d_S^2 + (8gi-4g-18i+9)d_S - 4gi^2 + 4gi + 13i^2 - 17i).$$

Let $\underline{d} = (d_1, \dots, d_n)$ be an $n$-tuple of integers satisfying $\sum d_j = g$ with $d_j \geq 0$. Logan [L] computed the class of the pointed Brill–Noether divisors, which from our perspective are the divisors

$$D^n_{\underline{d},1^{g-2}} = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \psi_j - 0 \cdot \delta_0 - \sum_{i,S} \binom{|d_S - i|+1}{2} \delta_{i:S} \tag{4}$$

in $\operatorname{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$, where $d_S := \sum_{j \in S} d_j$.

Let $\underline{d} = (d_1, \dots, d_n)$ be an $n$-tuple of integers satisfying $\sum d_j = g-1$ with some $d_j < 0$. Müller [Mü] computed the class of the closure of the pullback of the theta divisor under a map specified by $\underline{d}$ from $\mathcal{M}_{g,n}$ to the universal Picard variety using Porteous' formula and test curves. From our perspective these divisors are[6]

$$\begin{aligned} D^n_{\underline{d},1^{g-1}} = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \psi_j - 0 \cdot \delta_0 \\ - \sum_{\substack{i,S \\ S \subset S_+}} \binom{|d_S - i|+1}{2} \delta_{i:S} - \sum_{\substack{i,S \\ S \not\subset S_+}} \binom{d_S - i+1}{2} \delta_{i:S} \end{aligned} \tag{5}$$

in $\operatorname{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$, where $S_+ := \{j \mid d_j > 0\}$ and $d_S := \sum_{j \in S} d_j$. Grushevsky and Zakharov [GZ] reproduced this result using a different method of a systematic set of test curves.

The following propositions provide the proof of Theorem 5.1. We begin with the holomorphic case.

PROPOSITION 5.2. *Consider $\underline{d} = (d_1, \dots, d_n)$ with $d_i \geq 0$ and $\sum d_i = g-1$. Then we have, for $g \geq 3$,*

$$\begin{aligned} D^n_{(\underline{d},1^{g-3},2)} = -4(g-7)\lambda + \sum_{i=1}^{g-1} (2g(d_i+1) - 3d_i - 5)d_i \psi_i \\ - 2\delta_0 + \sum_{i=0}^{g} \sum_{d_S=0}^{i-1} c_{i:S}\delta_{i:S}, \end{aligned}$$

*where*

$$c_{i:S} = (3-2g)d_S^2 + (4gi+2g-10i+1)d_S - 2gi^2 + 7i^2 - 2gi - i - 2$$

[6] Note that, in this formula, $S \neq \{1, \dots, n\}$. In this case the coefficient is found by $S^C = \emptyset \subset S_+$. The condition on $S$ in the formula is separating the cases where all poles lie on the same component.

*for*

$$d_S := \sum_{i \in S} d_i$$

*and* $d_S = 0, \ldots, i-1$. *Note that* $c_{0:S} = c_{g:g-d_S-1}$ *for* $|S| \geq 2$.

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,g-1}$ that for $[C, y_1, \ldots, y_n] \in \overline{\mathcal{M}}_{g,n}$ identifies each $y_i$ with the point $x_i$ of a general marked rational curve $[X_i, x_i, q_1, \ldots, q_{d_i}]$ as introduced in Section 2.7. Clearly,

$$\pi^* D^{g-1}_{(1^{2g-4},2)} = D^n_{(\underline{d},1^{g-3},2)},$$

where the class of $D^{g-1}_{(1^{2g-4},2)}$ is given in equation (3). For $n = 1$ and $g \geq 3$, the divisor $D^1_{g-1,2,1^{g-3}}$ agrees with that in Theorem 3.1. For $g = 3$, the divisors $D^1_{2,2}$ and $D^2_{1,1,2}$ agree with the sums of the classes of equations (1) and (2), respectively, and with Theorem 4.1. □

To investigate the meromorphic case, we begin with $\overline{\mathcal{M}}_{g,2}$.

Proposition 5.3. *For $h \geq 3$ and $g \geq 2$, we have*

$$\begin{aligned} D^2_{(-h,g+h-2,1^{g-2},2)} = {} & (26-4g)\lambda + 2h(gh-g-h+2)\psi_1 \\ & + 2(g+h-2)(g^2+(h-2)g-h)\psi_2 \\ & - 2\delta_0 + \sum_{i=1}^{g-1} c_{i:\{1\}}\delta_{i:\{1\}} + \sum_{i=1}^{g} c_{i:\emptyset}\delta_{i:\emptyset}, \end{aligned}$$

*where* $\delta_{g:\emptyset} = \delta_{0:\{1,2\}}$ *and*

$$c_{i:\{1\}} = -2((g-3)i^2 + (2gh-4h-g+4)i + gh^2 - h^2 - gh + 2h)$$

*and*

$$c_{i:\emptyset} = -2((g-3)i^2 + (g+1)i + 1).$$

*For $h = 2$, we have*

$$\begin{aligned} D^2_{(-2,g,1^{g-2},2)} = {} & (27-4g)\lambda + 4g\psi_1 + \frac{g(4g^2-g-9)}{2}\psi_2 - 2\delta_0 \\ & + \sum_{i=1}^{g-1} c_{i:\{1\}}\delta_{i:\{1\}} + \sum_{i=1}^{g} c_{i:\emptyset}\delta_{i:\emptyset}, \end{aligned}$$

*where*

$$c_{i:\{1\}} = \frac{1}{2}((13-4g)i^2 + (17-12g)i - 8g)$$

*and*

$$c_{i:\emptyset} = \frac{1}{2}((13-4g)i^2 - (4g+3)i - 4).$$

*Proof.* Consider the map $\pi : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g+h,2}$ that for $[C, y_1, y_2] \in \overline{\mathcal{M}}_{g,2}$ identifies the point $y_1$ with the point $x$ of a general marked genus $h$ curve $[X, x, q]$ as introduced in Section 2.7. For $h = 1$, we have

$$\pi^* D^2_{1,g-1,1^{g-2},2} = \varphi_1^* D^1_{g-1,1^{g-3},2} + 4D^2_{1,g-1,1^{g-2}},$$

but all three classes here are known by Theorem 3 and agree with this relation.

For $h \geq 3$, we have

$$\pi^* D^2_{1,g+h-2,1^{g+h-3},2} = D^2_{-h,g+h-2,1^{g-2},2} + (4h-2)D^2_{-h+1,g+h-2,1^{g-1}},$$

where the multiplicity $4h - 2$ comes from a simple application of the Plücker formula. The first divisor class is known by Proposition 5.2, and the third class by equation (4). This proves the first equation in the proposition.

For $h = 2$, we have the relation

$$\pi^* D^2_{1,g,1^{g-1},2} = D^2_{-2,g,1^{g-2},2} + 7\varphi_1^* D^1_{g,1^{g-2}},$$

where $\varphi_1 : \overline{\mathcal{M}}_{g,2} \longrightarrow \overline{\mathcal{M}}_{g,1}$ forgets the first marked point. The multiplicity of the second term is due to the Picard variety method and represents the number of solutions to the equation

$$2p_1 + p_2 \sim K_C - x + 2y$$

for fixed general $x$ and $y$ on a general curve $C$ with genus $g(C) = 2$ where $p_1, p_2, x, y$ are distinct. The Picard variety method gives $2^2 \cdot 1^2 \cdot 2 - 1$, where the discounted solution is when $p_1 = y$ and $p_2 = x'$ is the conjugate point to $x$ under the hyperelliptic involution. This solution has multiplicity one as $x$ and $y$ were fixed general points. The first divisor class is known by Proposition 5.2, and the third class is the Weierstrass divisor given in Section 3.1. This proves the second and final equations in the proposition. □

REMARK 5.4. For $g \geq 3$ or $g = 2$ and $h$ odd, these divisors are irreducible. In the case that $g = 2$, these divisors correspond to coupled partition divisors. In this case the formula from the two perspectives agree, and further, for $g = 2$ and $h$ even, the class of the two irreducible components is given in Propositions 4.5 and 4.10.

REMARK 5.5. We can perform a quick check on the majority of the calculated coefficients of this proposition. Consider the map $\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_{g,2}$ that for $[C, y] \in \overline{\mathcal{M}}_{g,1}$ identifies the point $y$ with the point $x$ of a general marked rational curve $[X, x, q_1, q_2]$. For $h \geq 3$, we have

$$\pi^* D^2_{-h,g+h-2,1^{g-2},2} = D^1_{g-1,1^{g-3},2} + 2D^1_{g,1^{g-2}},$$

and, for $h = 2$, this becomes

$$\pi^* D^2_{-2,g,1^{g-2},2} = D^1_{g-1,1^{g-3},2} + D^1_{g,1^{g-2}}.$$

The change in the multiplicity of the Weierstrass divisor denoted $D^1_{g,1^{g-2}}$ here is due to the fact that in this case there is only one position on the rational tail where the double zero makes the residue at the node vanish. All classes on the right-hand side are known by Theorem 3 and agree with the classes of Proposition 5.3.

This result can be extended to the meromorphic case with exactly one pole.

PROPOSITION 5.6. *Consider* $\underline{d} = (d_1, \dots, d_n)$ *with* $\sum d_i = g-2$, $d_j \le -2$ *and* $d_i \ge 0$ *for* $i \ne j$. *Then, for* $d_j \le -3$,

$$D^n_{\underline{d},1^{g-2},2} = (26-4g)\lambda + \sum_{i=1}^{n} 2d_i((g-1)d_i + g - 2)\psi_i - 2\delta_0 + \sum_{i=0,S}^{g-1} c_{i:S}\delta_{i:S},$$

*where, for* $j \notin S$ *and* $d_S \le i-1$,

$$c_{i:S} = (2-2g)d_S^2 + 2(2gi + g - 4i + 1)d_S - 2(gi^2 + gi - 3i^2 + i + 1),$$

*and, for* $d_S \ge i$,

$$c_{i:S} = (2-2g)d_S^2 + 2(2gi - g - 4i + 2)d_S - 2(gi^2 - 3i^2 - gi + 4i).$$

*For* $d_j = -2$,

$$D^n_{\underline{d},1^{g-2},2} = (27-4g)\lambda + 4g\psi_j + \sum_{i \ne j} \frac{(4g(d_i+1) - 5d_i - 9)d_i}{2}\psi_i - 2\delta_0 + \sum_{i=0}^{g-1} c_{i:S}\delta_{i:S},$$

*where, for* $j \notin S$ *and* $d_S \le i-1$,

$$c_{i:S} = \frac{1}{2}((5-4g)d_S^2 + (8gi + 4g - 18i + 3)d_S - 4gi^2 - 4gi + 13i^2 - 3i - 4),$$

*and, for* $d_S \ge i$,

$$c_{i:S} = \frac{1}{2}((5-4g)d_S^2 + (8gi - 4g - 18i + 9)d_S - 4gi^2 + 4gi + 13i^2 - 17i).$$

*Proof.* Let $d_j = -h$. Consider the map $\pi : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g+h,n}$ that for $[C, y_1, \dots, y_n] \in \overline{\mathcal{M}}_{g,n}$ identifies the point $y_j$ with the point $x$ of a general marked genus $h$ curve $[X, x, q]$ as introduced in Section 2.7. Then, for $h \ge 3$, we have

$$\pi^* D^n_{\underline{d}',1^{g+h-3},2} = D^n_{\underline{d},1^{g-2},2} + (4h-2)D^n_{\underline{d}'',1^{g-1}},$$

where $\underline{d}'$ and $\underline{d}''$ are the vector $\underline{d}$ with the $j$th entry replaced by a 1 and $-h+1$, respectively.

For $h = 2$, we obtain

$$\pi^* D^n_{\underline{d}',1^{g-1},2} = D^n_{\underline{d},1^{g-2},2} + 7\varphi_j^* D^{n-1}_{\underline{d}'',1^{g-1}},$$

where $\underline{d}'$ and $\underline{d}''$ are the vector $\underline{d}$ with the $j$th entry replaced by 1 and the $j$th entry omitted, respectively. The multiplicity 7 is the result of an application of the Picard variety method as discussed in the proof of Proposition 5.3. The classes on the left-hand side are known by Proposition 5.3, whereas the other unknown classes are given in equation (5). □

REMARK 5.7. The first example of a pinch partition divisor is $D^{g-1}_{1^{2g-4},2}$ computed by Farkas and Verra [FV1], who showed this divisor to be extremal by a covering curve construction. The general idea can be generalized to meromorphic differentials. For any $\underline{d} = (d_1, \dots, d_g)$ with $\sum d_i = g-2$ and $d_i \in \mathbb{Z} \setminus \{0\}$, fix a general genus $g \geq 2$ curve $C$ and the map

$$\begin{array}{rccc} \varphi_{\underline{d}}: & C^g & \dashrightarrow & C_g = C^{g-1}/S_{g-1} \\ & (p_1, \dots, p_g) & \mapsto & [q_1, \dots, q_g], \end{array}$$

where this map is defined as $q_i \in C$ such that

$$\sum_{i=1}^{g} d_i p_i + \sum_{i=1}^{g} q_i \sim K_C.$$

Unfortunately, unlike in the holomorphic case, this finite map is not surjective. The locus of indeterminacy of this map is the codimension two locus where $h^0(K_C - \sum d_i p_i) \geq 2$ and, further, the image of the locus of indeterminacy under the resolution of $\varphi_{\underline{d}}$ is the points $q_i$ that are collinear in the canonical embedding, that is, $h^0(K_C - q_1 - \cdots - q_g) \geq 1$ giving a divisor in $C_g$. As a divisor is extracted by this resolution, we cannot conclude that the irreducible divisor equal to the pullback of the extremal and rigid divisor $\Delta = \{[q_1, \dots, q_g] \in C_g \mid q_1 = q_2\}$ in $C_g$ is also extremal.

## Appendix: The Residual Divisor by Alternate Methods

In this section, we reproduce the results of Section 3.2 by different and more labor intensive methods of Porteous' formula and test curves.

### *A.1. Appendix: Locating the Limits of Weierstrass and Residual Points on General Nodal Curves*

In this section, we use the results of Section 2.2 to locate the limits of Weierstrass and residual points on general nodal curves that will inform our later analysis.

*A.1.1. A Disconnecting Node with One Component of Genus $i$.* Consider the nodal curve obtained by attaching a general genus $g-i$ curve $Y$ at a non-Weierstrass point $y \in Y$ to a general genus $i$ curve $X$ at a non-Weierstrass point $x \in X$. We would like to locate the limits of Weierstrass and residual points on smooth curves degenerating to the nodal curve.

Twisted canonical divisors on this nodal curve of the type we are considering have either a point $p_1$ of multiplicity $g$ on the $X$ or $Y$ component or sitting on a rational bridge between $x$ and $y$. Let $p_j$ for $j = 2, \dots, g-1$ be the limits of residual points. If $p_1$ occurs on the $X$ component, then we have

$$(\omega_X) = gp_1 - (g-i+1)x + \sum_{j=2}^{i} p_j \sim K_X,$$

which has $g^2 i - i$ solutions by the Picard variety method discussed in Section 2.5, where we have discounted by the unique solution with $p_1 = x$, which has multiplicity $i$. In the $Y$ component, we have

$$(\omega_Y) = (g - i - 1)y + \sum_{j=i+1}^{g-1} p_j \sim K_Y,$$

which has a unique solution for a general point $y$. Hence we have $(g^2 - 1)i$ solutions of this type. If $p_1$ sits on $Y$, then by the same argument we have $(g^2 - 1)(g - i)$ solutions.

Further, as $x$ and $y$ are general points on general curves, it is not possible to have any $p_j$ on a rational bridge between $x$ and $y$. Any $p_j$ for $j = 2, \ldots, g - 1$ on such a rational bridge would contradict the curves $X$ and $Y$ or the points $x$ and $y$ being general. If $p_1$ lies on a rational bridge, then the only possibility is that the bridge contains one zero of multiplicity $g$ and poles at the nodes of multiplicity $-(i + 1)$ and $-(g - i + 1)$. By the cross ratio we can set the poles to 0 and $\infty$ and the zero to 1. The resulting differential is given locally at 0 by

$$c \frac{(1 - z)^g}{z^{i+1}} dz$$

for some constant $c \in \mathbb{C}^*$. The residues at the nodes then cannot be zero, and we have found all

$$(g^2 - 1)i + (g^2 - 1)(g - i) = (g + 1)g(g - 1)$$

sets of Weierstrass and residual points as expected.

As a cross-check, consider two test curves in $\overline{\mathcal{M}}_{g,1}$ constructed from the nodal curve we are considering. Attach a general genus $g - i$ curve $Y$ at a non-Weierstrass point $y \in Y$ to a general genus $i$ curve $X$ at a non-Weierstrass point $x \in X$. Let $B_X$ be the test curve formed by allowing the marked point to vary in the $X$ component, and let $B_Y$ be the test curve formed by allowing the marked point to vary in the $Y$ component. Consider the Weierstrass divisor calculated by Cukierman [Cu]

$$W = \frac{g(g+1)}{2}\psi - \lambda - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2}\delta_i.$$

This is the closure in $\overline{\mathcal{M}}_{g,1}$ of Weierstrass points, and hence intersecting this divisor with our test curves should verify the number of limits of Weierstrass points that we have found on each component of our nodal curve. Indeed,

$$B_X \cdot W = (2i - 1)c_\psi - c_i + c_{g-i} = (g + 1)(g - 1)(g - i)$$

and

$$B_Y \cdot W = (2(g - i) - 1)c_\psi + c_i - c_{g-i} = (g + 1)(g - 1)i,$$

which agree with our calculations.

*A.1.2. A Nondisconnecting Node.* Consider a general genus $g-1$ curve $X$ and identify two general non-Weierstrass points $x$ and $y$ to form a node. We would like to locate the limits of Weierstrass and residual points on smooth curves degenerating to the nodal curve. If all $p_j$ occur on $X$, then

$$(\omega_X) = -x - y + gp_1 + \sum_{j=2}^{g-1} p_j \sim K_X$$

with $p_j \neq x, y$. By the Picard variety method introduced in Section 2.5 we have $g^2(g-1)$ such solutions, but we must discount for any of these solutions where $p_j = x$ or $y$. As $K_X + x$ has a base point at $x$, we see that if any $p_j = y$, then this would cause some other $p_i = x$. If $j \neq 1$ and $i \neq 1$, then this causes the curve $X$ to have an exceptional Weierstrass point providing a contradiction with our assumption that $X$ is general. If $i$ or $j = 1$, then we have that $x$ or $y$ is a Weierstrass point, contradicting our assumptions.

The last possibility is that the limit of Weierstrass points specializes to the node. In this case, we can blow up and consider this case as $p_1$ sitting on a rational bridge between $x$ and $y$. In this case, on the $X$ component of our two component curve, we would have

$$(\omega_X) = (g-i-1)x + (i-1)y + \sum_{j=2}^{g-1} p_j \sim K_X$$

for $i = 1, \dots, g-1$. Hence in the canonical embedding of $X$, fixing such multiplicities at $x$ and $y$ specifies a plane and hence a unique solution. On the rational bridge, we have a $g^{g-1}_{2g-2}$ that adheres to the vanishing of sections in $|K_X + x + y|$ at $x$ and $y$. The limit $p_1$ in this situation is thus the ramification points of the $g^{g-1}_g$ created by imposing the vanishing multiplicities at the nodes in the rational curve. The Plücker formula shows that there are $g$ such points, and we obtain $g(g-1)$ solutions of this type. Hence we have found all

$$g^2(g-1) + g(g-1) = (g+1)g(g-1)$$

sets of Weierstrass and residual points as expected.

## *A.2. Porteous' Formula*

We calculate the $\lambda$ and $\psi$ coefficients of the residual divisor by realizing the locus of interest in $\mathcal{M}_{g,1}$ as the points at which a suitably chosen map between vector bundles drops dimension. The calculation of the class then becomes a well-treaded computation in the Chow ring. This method is known as Porteous' formula, and we follow the treatment and notation of Faber [F]. Let $\mathcal{C}^n_g$ denote the $n$-fold fiber product of $\mathcal{M}_{g,1}$ over $\mathcal{M}_g$. Consider $\pi_2 : \mathcal{C}^2_g \to \mathcal{C}^1_g$ that forgets the last point and $\pi_1 : \mathcal{C}^2_g \to \mathcal{C}^1_g$ that forgets the first point.

Let $\omega_i$ be the line bundle on $\mathcal{C}^n_g$ obtained by pulling back $\omega$ on $\mathcal{C}_g$ on the projection of the $i$th coordinate and denote its class as $K_i$ in Chow.

Let $E = \mathbb{E}$ be the Hodge bundle, and let $F$ be the bundle whose fibers are $H^0(K/(K - gp_1 - p_2))$. The bundle $F$ has rank $g + 1$. Then we have

$$c(F) = (1 + K_2 - g\Delta_2)(1 + K_1)(1 + 2K_1) \cdots (1 + gK_1).$$

We have the natural evaluation map

$$\varphi : \mathbb{E} \longrightarrow F,$$

where $\mathbb{E}$ is the Hodge bundle. The locus where this map drops dimension is exactly the points $(C, p_1, p_2)$ where $h^0(C, K_C(-gp_1 - p_2)) > 0$. We calculate the class of this locus via Porteous' formula. We know that

$$c(-\mathbb{E}) = c(\mathbb{E}^\vee) = 1 - \lambda_1 + \lambda_2 + \cdots + (-1)^g \lambda_g.$$

Hence by Porteous' formula we have that the class $Y$ of $(C, p_1, p_2)$ where $gp_1 + p_2$ is special is the locus where the map has rank $\leq g - 1$:

$$\Delta_{g+1-(g-1),g-(g-1)}(c(F)/c(\mathbb{E})) = \Delta_{2,1}(c(F) \cdot c(\mathbb{E}^\vee)) = c(F) \cdot c(\mathbb{E}^\vee)|_2.$$

First, we observe that

$$\begin{aligned}
c_1(F) &= K_2 - g\Delta_2 + \frac{g(g+1)}{2} K_1, \\
c_2(F) &= \sum_{i=1}^{g} i K_1 K_2 - g\left(\sum_{i=1}^{g} i\right)\Delta_2 K_1 + \sum_{i=1}^{g-1}\left(\sum_{j=i+1}^{g} ij\right) K_1^2 \\
&= \frac{g(g+1)}{2} K_1 K_2 - \frac{g^2(g+1)}{2}\Delta_2 K_1 + \frac{(g-1)g(g+1)(3g+2)}{24} K_1^2,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{i=1}^{g-1}\left(\sum_{j=i+1}^{g} ij\right) &= \sum_{i=1}^{g-1} i\left(\frac{g(g+1)}{2} - \frac{i(i+1)}{2}\right) \\
&= \frac{(g-1)g(g+1)(3g+2)}{24}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
[Y] = \lambda_2 - \lambda_1\left(K_2 - g\Delta_2 + \frac{g(g+1)}{2} K_1\right) + \frac{g(g+1)}{2} K_1 K_2 \\
- \frac{g^2(g+1)}{2}\Delta_2 K_1 + \frac{(g-1)g(g+1)(3g+2)}{24} K_1^2.
\end{aligned}$$

By Faber [F], for $\pi_d : \mathcal{C}_g^d \to \mathcal{C}_g^{d-1}$ forgetting the last point, we have

$$\begin{aligned}
\pi_{d*}(MD_{i,d}) &= M, \\
\pi_{d*}(MK_d^k) &= M \cdot \pi_*(\kappa_{k-1}),
\end{aligned}$$

where $M$ is a monomial of classes that are pulled back from $\mathcal{C}_g^{d-1}$. To put a class in a form like this, there are a few other useful relations:

$$\begin{aligned}
D_{i,j} D_{j,d} &= D_{i,j} D_{i,d} \quad \text{for } i < j < d, \\
D_{i,d}^2 &= -K_i D_{i,d} \quad \text{for } i < d,
\end{aligned}$$

$$K_d D_{i,d} = K_i D_{i,d}.$$

Now via Faber's algorithm for pushing down, forgetting the second point, we have

$$\begin{aligned} W &= \frac{1}{g-2}\pi_{2*}[Y] = \frac{1}{g-2}\left((g-\kappa_0)\lambda_1 + \frac{g(g+1)}{2}(\kappa_0 - g)K_1\right) \\ &= \frac{g(g+1)}{2}\psi - \lambda, \end{aligned}$$

as expected. Here we have used the fact that on $W$ we have that $\pi_{2*}$ is of degree $g-2$ (there are $g-2$ residual points for each Weierstrass point $p$). Forgetting the first point, we have

$$\begin{aligned} R &= \pi_{1*}[Y] \\ &= \left(g - \frac{g(g+1)}{2}\kappa_0\right)\lambda_1 + \frac{g(g+1)}{2}(\kappa_0 - g)K_2 \\ &\quad + \frac{(g-1)g(g+1)(3g+2)}{24}\kappa_1 \\ &= \frac{g(g+1)(g-2)}{2}\psi + \frac{g(3g^3-3g+2)}{2}\lambda, \end{aligned}$$

which agrees with our previous calculation of these coefficients in Section 3.2.

### *A.3. Test Curves*

By creating a number of curves in $\overline{\mathcal{M}}_{g,1}$ we can calculate the intersections with the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1}) \otimes \mathbb{Q}$ and the residual divisor directly to obtain a number of relationships between the coefficients of the residual divisor. With enough relationships, we can determine all coefficients. To this end, let the class of the residual divisor in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,1}) \otimes \mathbb{Q}$ be denoted

$$R = c_\psi \psi + c_\lambda \lambda + \sum_{i=0}^{g-1} c_i \delta_i.$$

*Test Curve A.* Consider a general genus $g$ curve $C$. Allow the marked point $q$ to vary in the curve. We observe that the intersection of this test curve with all boundary divisors and $\lambda$ is zero, and we have

$$A \cdot R = -(2-2g)c_\psi.$$

To find this intersection directly, we observe that, as the curve is general, there are $(g+1)g(g-1)$ normal Weierstrass points, each with $g-2$ residual points. Hence

$$A \cdot R = (g+1)g(g-1)(g-2),$$

and hence

$$c_\psi = \frac{(g+1)g(g-2)}{2}.$$

Hence we have verified this result from Porteous' formula via test curves.

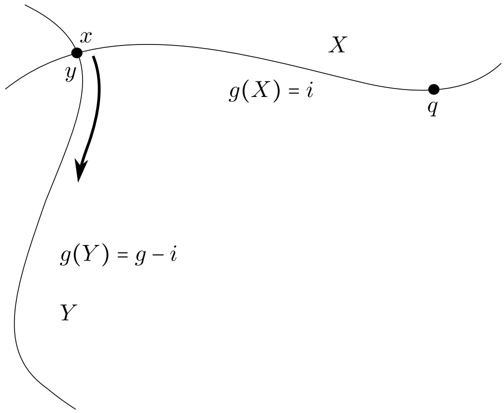


**Figure 4** Test curve $B_i$

*Test Curves $B_i$.* Let $Y$ be a genus $g-i$ curve, and let $X$ be a genus $i$ curve. Attach $X$ to $Y$ at a general point in $x \in X$ and allow the attaching point $y \in Y$ to vary. Mark a general point $q \in X$ as shown in Figure 4. We require $0 < i < g-1$.

We observe that $B_i \cdot R = (2-2(g-i))c_i$. To locate the limits of residual points in this test curve, there are two possible cases based on the order of vanishing at $y$. In the first case, we have that $y$ is a Weierstrass point, and in the $Y$ component, we have

$$(\omega_Y) = (g-i)y + \sum_{j=1}^{g-i-2} q_j \sim K_Y,$$

which has solutions where $y$ is a Weierstrass point and the $q_j$ residual in the $Y$ component. There are $(g-i+1)(g-i)(g-i-1)$ such solutions allowing for the ordering of the $q_j$. In the $X$ component, we have

$$(\omega_X) = -(g-i+2)x + q + gp + \sum_{j=g-i-1}^{g-3} q_j \sim K_X,$$

which gives

$$gp + \sum_{j=g-i-1}^{g-3} q_j \sim K_X + (g-i+2)x - q.$$

By the Picard variety method discussed in Section 2.5 we have $g^2 i - (i-1)$ solutions, where we have discounted for the unique solution where $p = x$ has multiplicity $(i-1)$. Hence solutions of this type contribute

$$(g-i+1)(g-i)(g-i-1)(g^2 i - (i-1))$$

to the intersection with the residual divisor.

If $y$ is not a Weierstrass point, then for any solution, in the $Y$ component, we must have

$$(\omega_Y) = gp - iy + \sum_{j=1}^{g-i-2} q_j \sim K_Y.$$

By the Picard variety method discussed in Section 2.5 we have

$$g^2 i^2 (g-i)(g-i-1) - (g-i+1)(g-i+1)(g-i)(g-i-1)$$

solutions, where we have discounted for the multiplicity $g-i+1$ solutions where $p = y$ is a Weierstrass point. We observe that this is consistent with the case $i = 1$, where there are no solutions. In the $X$ component, this corresponds to the unique solution

$$\sum_{j=g-i-1}^{g-3} q_j \sim K_X - q - (i-2)x.$$

Hence solutions of this type contribute

$$(g^2 i^2 - (g-i+1)^2)(g-i)(g-i-1)$$

to the intersection with the residual divisor, and we are left with

$$B_i \cdot R = g(g^2 i + gi - g + i - 1)(g-i)(g-i-1),$$

and hence

$$c_i = -\frac{g(g^2 i + gi - g + i - 1)(g-i)}{2}$$

for $1 \leq i \leq g-2$.

*Test Curves $C_i$.* Let $Y$ be a genus $g-i$ curve, and let $X$ be a genus $i$ curve. Attach $X$ to $Y$ at a general point in $x \in X$ and a general point $y \in Y$ to vary. Let the marked point $q$ vary in $X$ as shown in Figure 5. We observe that

$$C_i \cdot R = (2i-1)c_\psi - c_i + c_{g-i}.$$

But in Section A.1.1 we located the limits of residual points on a general nodal curve of the type we are considering here giving

$$C_i \cdot R = g(g^2-1)(i-1),$$

which agrees with our formula for $c_i$ in the last section and shows that it also applies to $i = g-1$.

*Test Curve $D$.* Take a smooth general genus $g-1$ curve $C$. Create a node by identifying one nonspecial fixed point $y$ on the curve with another point $x$ that varies in the curve. Mark a general point $q$ on the curve as shown in Figure 6.

We have

$$D \cdot R = c_\psi + (2-2g)c_0 + c_{g-1}.$$

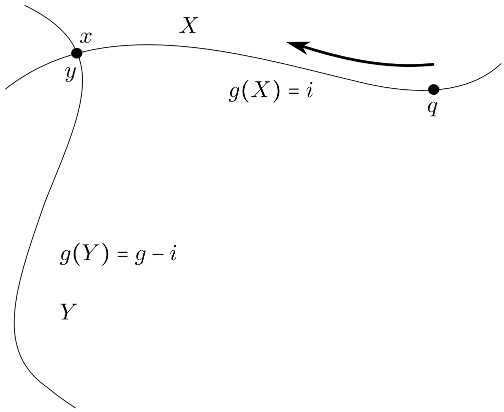


**Figure 5** Test curve $C_i$

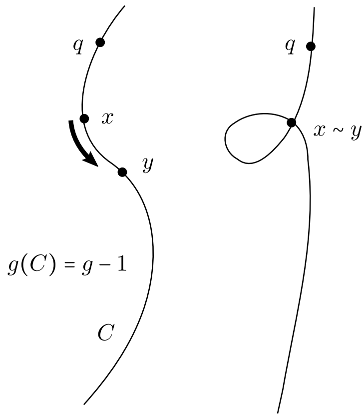


**Figure 6** Test curve $D$

To find the limits of residual points in this test curve, there are solutions of two types. The solutions of the first type are points $x, p, q_j$ that satisfy

$$(\omega_C) = -x - y + gp + q + \sum_{j=1}^{g-3} q_j \sim K_C,$$

which gives

$$gp + q + \sum_{j=1}^{g-3} q_j \sim K_C + x + y,$$

where $y$ and $q$ are fixed, and $x$ is varying. By the Picard variety method discussed in Section 2.5 there are $g^2(g-1)(g-2)$ solutions. There are no solutions to discount for with $x = p, q, y, q_j$ or $y = p, q_j$. If $x = p$, then we have

$$(g-1)x + q + \sum_{j=1}^{g-3} q_j \sim K_C + y.$$

As $y$ is a base point of $K_C + y$, we have either $y = q_j$ for some $j$ making $q$ a residual point or $y = x$ giving

$$(g-2)y + q + \sum_{j=1}^{g-3} \sim K_C.$$

Both cases contradict the assumption that $q$ and $y$ are general. Similarly, if $x = q, y, q_j$ for some $j$ or $y = p, q_j$, then we have a contradiction of $y$ and $q$ being general points or $C$ a general curve with only normal Weierstrass points.

The second way that limits of residual points can occur in our test curve is when the point approaches the node and $p$ actually sits on a rational bridge between $x$ and $y$. If this occurs, then we have

$$(\omega_C) = (i-1)x + (g-i-1)y + q + \sum_{j=1}^{g-3} q_j \sim K_C,$$

which becomes

$$(i-1)x + \sum_{j=1}^{g-3} q_j \sim K_C - q - (g-i-1)y$$

for $i = 2, \dots, g-1$. Such $x$ are the ramification points of $|K_C - q - (g-i-1)y|$, which is a $g^{i-2}_{g+i-4}$, and hence by the Plücker formula there are

$$(r+1)d + (r+1)r(g(C)-1) = (i-1)(gi-g-i)$$

such ramification points. Hence in total we have

$$\sum_{i=2}^{g-1} (i-1)(gi-g-i) = \frac{1}{6} g(2g^3 - 11g^2 + 19g - 10)$$

solutions. Each solution has multiplicity $g$ due to $g$ ramification points of a general $g^{g-1}_g$ on a rational bridge between $x$ and $y$.

This gives the relation

$$c_\psi + (2-2g)c_0 + c_{g-1} = g^2(g-1)(g-2) + \frac{1}{6} g^2(2g^3 - 11g^2 + 19g - 10),$$

and from the known values of $c_\psi$ and $c_{g-1}$ this gives

$$c_0 = \frac{g^2 - g^4}{6}.$$

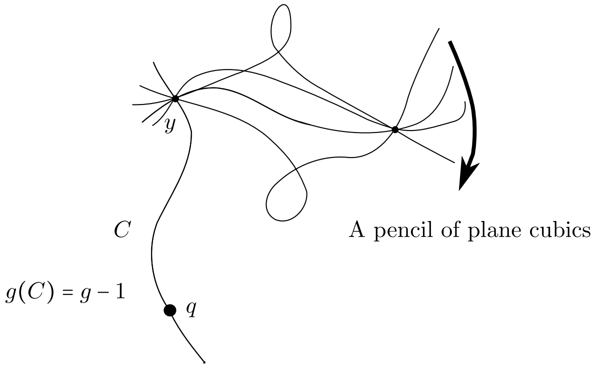


**Figure 7** Test curve $E$

*Test Curve E*. Take a pencil of plane cubics. Attach one base point to a general genus $g-1$ curve $C$ at a general point $y \in C$. Mark another general point $q$ on $C$ as shown in Figure 7.

This is a standard test curve, and it is well known [HMo] that $E \cdot \lambda = 1$, $E \cdot \delta_0 = 12$, $E \cdot \delta_{g-1} = -1$, giving

$$E \cdot R = c_\lambda + 12c_0 - c_{g-1}.$$

To find the intersection directly, we observe that, for any such solution, either the limit $p$ is in the $C$ component, or it is not. If the limit point $p$ lies on $C$, then we have

$$(\omega_C) = gp + q + (g-k-5)y + \sum_{j=1}^{k} q_j \sim K_C$$

for some $k = 1, \ldots, g-3$. Any such solution would contradict our assumption that $q$ and $y$ are general. If $p$ does not lie on $C$, then we have

$$(\omega_C) = q + (2g-k-5)y + \sum_{j=1}^{k} q_j \sim K_C$$

for some $k = 1, \ldots, g-3$. Again, we have a contradiction for any $k$ with the assumption that $q$ and $y$ are general points. Hence $E \cdot R = 0$, and by our previous test curve results we see

$$c_\lambda = \frac{g(3g^3 - 3g + 2)}{2},$$

which agrees with our Porteous' formula result.

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## References

[AC] E. Arbarello and M. Cornalba, *The Picard groups of the moduli space of curves,* Topology 26 (1987), 153–171.

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I,* Grundlehren Math. Wiss., 267, Springer-Verlag, New York, 1985.

[BCGGM] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Compactification of strata of Abelian differentials,* Duke Math. J. 167 (2018), 2347–2416.

[B] C. Boissy, *Connected components of the moduli space of meromorphic differentials,* Comment. Math. Helv. 90 (2015), no. 2, 255–286.

[C] D. Chen, *Degenerations of Abelian differentials,* J. Differential Geom. 107 (2017), 395–453.

[CT] D. Chen and N. Tarasca, *Loci of curves with subcanonical points in low genus,* Math. Z. 284 (2016), 683–714.

[Cool] J. L. Coolidge, *A treatise on algebraic plane curves,* Oxford University Press, 1931.

[Co] M. Cornalba, *Moduli of curves and theta-characteristics,* Lectures on Riemann surfaces, Trieste, 1987, pp. 560–589, World Sci. Publ., Teaneck, NJ, 1989.

[Cu] F. Cukierman, *Families of Weierstrass points,* Duke Math. J. 58 (1989), 317–346.

[D] S. Diaz, *Exceptional Weierstrass points and the divisor on moduli space that they define,* Mem. Amer. Math. Soc. 56 (1985), no. 327.

[EH1] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* $\geq 23$*,* Invent. Math. 90 (1987), no. 2, 359–387.

[EH2] ———, *Irreducibility of some families of linear series with Brill–Noether number* $-1$*,* Ann. Sci. Éc. Norm. Supér. (4) 22 (1989), no. 2, 33–53.

[EM] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the* $SL(2,\mathbb{R})$ *action on Moduli space,* arXiv:1302.3320.

[EMM] A. Eskin, M. Mirzakhani, and A. Mohammadi, *Isolation, equidistribution, and orbit closures for the* $SL(2,\mathbb{R})$ *action on moduli space,* Ann. of Math. 182 (2015), no. 2, 673–721.

[F] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves,* Moduli of curves and Abelian varieties (the Dutch intercity seminar on moduli), Aspects Math., E33, pp. 109–129, Vieweg, Wiesbaden, 1999.

[F1] G. Farkas, *The geometry of the moduli space of curves of genus* 23*,* Math. Ann. 318 (2000), no. 1, 43–65.

[F2] ———, *Brill–Noether geometry on moduli spaces of spin curves,* Classification of algebraic varieties, EMS Ser. Congr. Rep., pp. 259–276, Eur. Math. Soc., Zürich, 2011.

[FP] G. Farkas and R. Pandharipande, *The moduli space of twisted canonical divisors,* J. Inst. Math. Jussieu, 17 (2018), 615–672.

[FV1] G. Farkas and A. Verra, *The classification of universal Jacobians over the moduli space of curves,* Comment. Math. Helv. 180 (2013), no. 3, 587–611.

[FV2] ———, *The universal theta divisor over the moduli space of curves,* J. Math. Pures Appl. 100 (2013), no. 4, 591–605.

[G] Q. Gendron, *The Deligne–Mumford and the Incidence Variety Compactifications of the Strata of* $\Omega\mathcal{M}_g$*,* Ann. Inst. Fourier (Grenoble) 68 (2018), 1169–1240.

[GZ] S. Grushevsky and D. Zakharov, *The double ramification cycle and the theta divisor,* Proc. Amer. Math. Soc. 142 (2014), no. 12, 4053–4064.

[HMo] J. Harris and I. Morrison, *Moduli of curves,* Grad. Texts in Math., 187, Springer-Verlag, New York, 1998.

[HMu] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves,* Invent. Math. 67 (1982), no. 1, 23–88.

[K] M. Kontsevich, *Lyapunov exponents and Hodge theory,* The mathematical beauty of physics, Adv. Ser. Math. Phys., 24, pp. 318–332, World Sci. Publishing, River Edge, NJ, 1997.

[KZ] M. Kontsevich and A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities,* Invent. Math. 153 (2003), no. 3, 631–678.

[L] A. Logan, *The Kodaira dimension of moduli spaces of curves with marked points,* Amer. J. Math. 125 (2003), no. 1, 105–138.

[Mc] C. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces,* J. Amer. Math. Soc. 16 (2003), 857–885.

[M] S. Mullane, *Divisorial strata of Abelian differentials,* Int. Math. Res. Not. 6 (2017), 1717–1748.

[Mü] F. Müller, *The pullback of a theta divisor to $\overline{\mathcal{M}}_{g,n}$,* Math. Nachr. 286 (2013), no. 11–12, 1255–1266.

[Mu] D. Mumford, *Theta characteristics of an algebraic curve,* Ann. Sci. Éc. Norm. Supér. (4) 4 (1971), 181–192.

[T] M. Teixidor i Bigas, *The divisor of curves with a vanishing theta-null,* Compos. Math. 66 (1988), 15–22.

Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-8
60325 Frankfurt am Main
Germany

mullane@math.uni-frankfurt.de