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On a Conjecture of Sokal Concerning Roots of the Independence Polynomial

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Abstract. A conjecture of Sokal [24], regarding the domain of nonvanishing for independence polynomials of graphs, states that given any natural number $\Delta \geq 3$, there exists a neighborhood in $\mathbb{C}$ of the interval $[0, (\Delta-1)^{\Delta-1}/(\Delta-2)^{\Delta})$ on which the independence polynomial of any graph with maximum degree at most Δ does not vanish. We show here that Sokal's conjecture holds, as well as a multivariate version, and prove the optimality for the domain of nonvanishing. An important step is to translate the setting to the language of complex dynamical systems.

1. Introduction

For a graph $G = (V, E)$ and $\lambda = (\lambda_v)_{v \in V} \in \mathbb{C}^V$, the *multivariate independence polynomial* is defined as

$$Z_G(\lambda) := \sum_{\substack{I \subseteq V \\ \text{independent}}} \prod_{v \in I} \lambda_v.$$

We recall that a set $I \subseteq V$ is called *independent* if it does not span any edges of G. The *univariate independence polynomial,* which we also denote by $Z_G(\lambda)$, is obtained from the multivariate independence polynomial by plugging in $\lambda_v = \lambda$ for all $v \in V$.

In statistical physics the univariate independence polynomial is known as the partition function of the hardcore model. When $\lambda = 1$, $Z_G(\lambda)$ equals the number of independent sets in the graph G.

Motivated by applications in statistical physics, Sokal [24, Question 2.4] asked about domains of the complex plane where the independence polynomial does not vanish. Just below Question 2.4 in [24], Sokal conjectures: "*there is a complex domain D_Δ containing at least the interval $0 \leq \lambda < 1/(\Delta-1)$ of the real axis—and possibly even the interval $0 \leq \lambda < \lambda_\Delta := (\Delta-1)^{\Delta-1}/(\Delta-2)^{\Delta}$—on which $Z_G(\lambda)$ does not vanish for all graphs of maximum degree at most Δ*".

In this paper, we confirm the strong form of his conjecture for the univariate independence polynomial. In Section 4, we prove the following result.

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THEOREM 1.1. *Let $\Delta \in \mathbb{N}$ with $\Delta \geq 3$. Then there exists a complex domain D_Δ containing the interval $0 \leq \lambda < \lambda_\Delta$ such that, for any graph $G = (V, E)$ of maximum degree at most Δ and any $\lambda \in D_\Delta$, we have that $Z_G(\lambda) \neq 0$.*

If we allow ourselves an epsilon bit of room, then the same result also holds for multivariate independence polynomials. This is the contents of Theorem 4.2 in Section 4. We show in the Appendix that the literal statement of Theorem 1.1 does not hold in the multivariate setting.

It follows from nontrivial results in complex dynamical systems that the bound in Theorem 1.1 is in fact optimal in light of the following:

PROPOSITION 1.2. *Let $\Delta \in \mathbb{N}$ with $\Delta \geq 3$. Then there exist $\lambda \in \mathbb{C}$ arbitrarily close to λ_Δ for which there exists a graph G of maximum degree Δ with $Z_G(\lambda) = 0$.*

This result is a direct consequence of Proposition 2.1 in Section 2.1. We discuss the underlying results from the theory of complex dynamical systems in the Appendix.

Other results for the nonvanishing of the independence polynomial include a result of Shearer [22], which says that, for any graph $G = (V, E)$ of maximum degree at most Δ and any λ such that $|\lambda_v| \leq (\Delta - 1)^{\Delta-1}/\Delta^\Delta$ for each $v \in V$, we have $Z_G(\lambda) \neq 0$. See [21] for a slight improvement and extensions. Moreover, Chudnovsky and Seymour [11] proved that the univariate independence polynomial of a claw-free graph (a graph G is called *claw-free* if it does not contain four vertices that induce a tree with three leaves) has all its roots on the negative real axis.

Motivation. Another motivation for Theorem 1.1 comes from the design of efficient approximation algorithms for (combinatorial) partition functions. Weitz [25] showed that there is a (deterministic) fully polynomial time approximation algorithm (FPTAS) for computing $Z_G(\lambda)$ for any $0 \leq \lambda < \lambda_c(\Delta)$ for any graph of maximum degree at most Δ. His method is often called the *correlation decay method* and has subsequently been used and modified to design many other FPTASs for several other types of partition functions; see, for example, [7; 14; 17; 16]. More recently, Barvinok initiated a line of research that led to quasi-polynomial-time approximation algorithms for several types of partition functions and graph polynomials; see, for example, [1; 2; 6; 5; 20; 3] and Barvinok's recent book [4]. This approach is based on Taylor approximations of the log of the partition function/graph polynomial and allows us to give good approximations in regions of the complex plane where the partition function/polynomial does not vanish. In his recent book [4], Barvinok refers to this approach as the *interpolation method.* Patel and the second author [19] recently showed that the interpolation method in fact yields polynomial-time approximation algorithms for these partition functions/graph polynomials when restricted to bounded degree graphs.

In combination with the results in Section 4.2 from [19], Theorem 1.1 immediately implies that the interpolation methods yields a polynomial-time approximation algorithm for computing the independence polynomial at any fixed

$0 \leq \lambda < \lambda_\Delta$ on graphs of maximum degree at most Δ, thereby matching Weitz's result. In particular, Theorem 1.1 gives evidence for the usefulness of the interpolation method.

Preliminaries

We collect some preliminaries and notational conventions here. Graphs may be assumed to be simple, as vertices with loops attached to them can be removed from the graph and parallel edges can be replaced by single edges without affecting the independence polynomial. Let $G = (V, E)$ be a graph. For a subset $U \subseteq V$, we denote the graph induced by U by $G[U]$. For $U \subset V$, we denote by $G \setminus U$ the graph induced by $V \setminus U$; in case $U = \{u\}$ we just write $G - u$. For a vertex $v \in V$, we denote by $N[v] := \{u \in V \mid \{u, v\} \in E\} \cup \{v\}$ the *closed neighborhood* of v. The *maximum degree* of G is the maximum number of neighbors of a vertex over all vertices of G. This is denoted by $\Delta(G)$.

For $\Delta \in \mathbb{N}$ and $k \in \mathbb{N}$, we denote by $T_{\Delta,k}$ the rooted tree recursively defined as follows: for $k = 0$, $T_{\Delta,0}$ consists of a single vertex; for $k > 0$, $T_{\Delta,k}$ consists of the root vertex that is connected to the $\Delta - 1$ root vertices of $\Delta - 1$ disjoint copies of $T_{\Delta,k-1}$. We will sometimes, abusing terminology, refer to $T_{\Delta,k}$ as *regular trees.* Note that the maximum degree of $T_{\Delta,k}$ equals Δ when $k \geq 2$ and equals $\Delta - 1$ when $k = 1$.

Organization

The remainder of this paper is organized as follows. In the next section, we translate the setting to the language of complex dynamical systems and prove another nonvanishing result for the multivariate independence polynomial; see Theorem 2.3. Section 3 contains technical, yet elementary, derivations needed for the proof of our main result, which is given in Section 4. We conclude with some questions in Section 5. In the Appendix, we discuss results from complex dynamical systems theory needed to prove Proposition 1.2.

2. Setup

Let us fix a graph $G = (V, E)$, $\lambda = (\lambda_v)_{v \in V} \in \mathbb{C}^V$, and a vertex $v_0 \in V$. The fundamental recurrence relation for the independence polynomial is

$$Z_G(\lambda) = \lambda_{v_0} Z_{G \setminus N[v_0]}(\lambda) + Z_{G - v_0}(\lambda). \tag{1}$$

Let us define, assuming that $Z_{G-v_0}(\lambda) \neq 0$,

$$R_{G,v_0} = \frac{\lambda_{v_0} Z_{G \setminus N[v_0]}(\lambda)}{Z_{G-v_0}(\lambda)}. \tag{2}$$

In the case that $\lambda_v > 0$ for all $v \in V$, ratio (2) is always defined. This definition is inspired by Weitz [25]. Note that by (1)

$$R_{G,v_0} \neq -1 \quad \text{if and only if} \quad Z_G(\lambda) \neq 0. \tag{3}$$

So, for our purposes, it suffices to look at the ratio R_{G,v_0}.

2.1. Regular Trees

We now consider the univariate independence polynomial for the trees $T_{\Delta,k}$. Let v_k denote the root vertex of $T_{\Delta,k}$. Then for $k>0$, $T_{\Delta,k}-v_k$ is equal to the disjoint union of $\Delta-1$ copies of $T_{\Delta,k-1}$. Additionally, for $k>1$, $T_{\Delta,k}\setminus N[v_k]$ is equal to the disjoint union of $\Delta-1$ copies of $T_{\Delta,k-1}-v_{k-1}$. Using this, we note that, for $k>2$, (2) takes the following form:

$$
\begin{aligned}
R_{T_{\Delta,k},v_k} &= \lambda\left(\frac{Z_{T_{\Delta,k-1}-v_{k-1}}}{Z_{T_{\Delta,k-1}}}\right)^{\Delta-1} = \lambda\left(\frac{Z_{T_{\Delta,k-1}-v_{k-1}}}{\lambda Z_{T_{\Delta,k-1}\setminus N[v_{k-1}]}+Z_{T_{\Delta,k-1}-v_{k-1}}}\right)^{\Delta-1} \\
&= \frac{\lambda}{(1+R_{T_{\Delta,k-1},v_{k-1}})^{\Delta-1}}.
\end{aligned} \tag{4}
$$

We denote the extended complex plane $\mathbb{C}\cup\{\infty\}$ by $\widehat{\mathbb{C}}$. For $\lambda\in\mathbb{C}$ and $d\in\mathbb{N}$, we define $f_{d,\lambda}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ by

$$
f_{d,\lambda}(x)=\frac{\lambda}{(1+x)^d}.
$$

So (4) gives that $R_{T_{\Delta,k},v_k}=f_{\Delta-1,\lambda}(R_{T_{\Delta,k-1},v_{k-1}})$. Noting that $R_{T_{\Delta,v_0}}=\lambda$, we observe that $R_{T_{\Delta,k},v_0}=f^{\circ k}_{\Delta-1,\lambda}(\lambda)$. So to understand under which conditions $R_{T_{\Delta,k},v_k}$ equals -1 or not, it suffices to look at the orbits of $f_{\Delta-1,\lambda}$ with starting point λ or, equivalently, with starting point -1.

A somewhat similar relation between graphs and the iteration of rational maps was explored by Bleher, Roeder, and Lyubich in [10] and [9]. Whereas here one iteration of $f_{\Delta-1,\lambda}$ corresponds to adding an additional level to a tree, there one iteration corresponded to adding an additional refinement to a hierarchical lattice.

Let us denote by $U_d\subset\mathbb{C}$ the open set of parameters λ for which $f_{d,\lambda}$ has an attracting fixed point. Then

$$
U_d=\left\{\frac{-\alpha d^d}{(d+\alpha)^{d+1}}\;\middle|\;|\alpha|<1\right\}. \tag{5}
$$

Indeed, writing $f=f_{d,\lambda}$, we note that if x is a fixed point of f, then we have

$$
f'(x)=\frac{-d}{1+x}\frac{\lambda}{(1+x)^d}=\frac{-dx}{1+x}.
$$

Let $\alpha\in\mathbb{C}$. Then $f'(x)=\alpha$ if and only if $x=\frac{-\alpha}{d+\alpha}$, and consequently

$$
\lambda=x(1+x)^d=\frac{-\alpha d^d}{(d+\alpha)^{d+1}}.
$$

A fixed point $x=f(x)$ is attracting if and only if $|f'(x)|<1$, which implies description (5). For parameters λ in the boundary $\partial U_{\Delta-1}$, the function f has a neutral fixed point, and for a dense set of parameters $\lambda\in\partial U_{\Delta-1}$, the fixed point is *parabolic,* that is, the derivative at the fixed point is a root of unity. Classical results from complex dynamical systems allow us to deduce the following regarding the vanishing/nonvanishing of the independence polynomial.

PROPOSITION 2.1. *Let $\Delta \in \mathbb{N}$ be such that $\Delta \geq 3$. Then*

(i) *for all $k \in \mathbb{N}$ and $\lambda \in U_{\Delta-1}$, $Z_{T_{\Delta,k}}(\lambda) \neq 0$;*

(ii) *if $\lambda \in \partial U_{\Delta-1}$, then for any open neighborhood U of λ, there exist $\lambda' \in U$ and $k \in \mathbb{N}$ such that $Z_{T_{\Delta,k}}(\lambda') = 0$.*

We note that, for $\lambda = -(\Delta-1)^{\Delta-1}/\Delta^{\Delta}$, part (ii) was proved by Shearer [22]; see also [21]. Part (i) follows quickly from elementary results in complex dynamics, but the statements that imply part (ii) are less trivial. The necessary background from the complex dynamical systems, including the proof of Proposition 2.1 and a counterexample to the multivariate statement of Theorem 1.1, will be discussed in the Appendix. Note that Proposition 1.2 from the introduction is a particular case of Proposition 2.1.

So we can conclude that Sokal's conjecture is already proved for regular trees. We now move to general (bounded-degree) graphs.

2.2. *A Recursive Procedure for Ratios for All Graphs*

It is convenient to have an expression similar to (4) for all graphs. Let G be a graph with fixed vertex v_0. Let $v_1, \ldots, v_d$ be the neighbors of v_0 in G (in any order). Set $G_0 = G - v_0$ and for $i = 1, \ldots, d$, define $G_i := G_{i-1} - v_i$. Then $G_d = G \setminus N[v_0]$. The following lemma gives recursive relation for the ratios and has been used before over the real numbers in, for example, [16].

LEMMA 2.2. *Suppose $Z_{G_i}(\lambda) \neq 0$ for all $i = 0, \ldots, d$. Then*

$$R_{G,v_0} = \frac{\lambda_{v_0}}{\prod_{i=1}^{d}(1 + R_{G_{i-1},v_i})}. \tag{6}$$

Proof. Let us write

$$\frac{Z_{G-v_0}(\lambda)}{Z_{G\setminus N[v_0]}(\lambda)} = \frac{Z_{G_0}(\lambda)}{Z_{G_1}(\lambda)}\frac{Z_{G_1}(\lambda)}{Z_{G_2}(\lambda)}\cdots\frac{Z_{G_{d-1}}(\lambda)}{Z_{G_d}(\lambda)} = \prod_{i=1}^{d}\frac{Z_{G_i}(\lambda) + \lambda_{v_i} Z_{G_{i-1}\setminus N[v_i]}}{Z_{G_i}(\lambda)}$$

$$= \prod_{i=1}^{d}(1 + R_{G_{i-1},v_i}),$$

where in the second equality we use (1). As

$$R_{G,v_0} = \frac{\lambda_{v_0} Z_{G\setminus N[v_0]}(\lambda)}{Z_{G-v_0}(\lambda)} = \frac{\lambda_{v_0}}{Z_{G-v_0}(\lambda)/Z_{G\setminus N[v_0]}(\lambda)},$$

the lemma follows. □

As an illustration of Lemma 2.2, we will now prove a result that shows that $Z_G(\lambda)$ is nonzero as long as the norms and arguments of λ_v are small enough. This result is implied by our main theorem for angles that are much smaller still, but the following statement is not implied by our main theorem and is another contribution to Sokal's question [24, Question 2.4]. The proof, moreover, serves as warm up for the proof of our main result.

THEOREM 2.3. *Let $G = (V, E)$ be any graph of maximum degree at most $\Delta \geq 2$. Let $\varepsilon > 0$, and let $\lambda \in \mathbb{C}^V$ be such that $|\lambda_v| \leq \tan(\frac{\pi}{(2+\varepsilon)(\Delta-1)})$ and $|\arg(\lambda_v)| < \frac{\varepsilon/2}{2+\varepsilon}\pi$ for all $v \in V$. Then $Z_G(\lambda) \neq 0$.*

Proof. Since the independence polynomial is multiplicative over the disjoint union of graphs, we may assume that G is connected. Fix a vertex v_0 of G. We will show by induction that, for each subset $U \subseteq V \setminus \{v_0\}$, we have:

(i) $Z_{G[U]}(\lambda) \neq 0$,
(ii) if $u \in U$ has a neighbor in $V \setminus U$, then $|R_{G[U],u}| < \tan(\frac{\pi}{(2+\varepsilon)(\Delta-1)})$,
(iii) if $u \in U$ has a neighbor in $V \setminus U$, then $\Re(R_{G[U],u}) > 0$.

Clearly, if $U = \emptyset$, then all (i), (ii), and (iii) are true. Now suppose $U \subseteq V \setminus \{v_0\}$ and let $H = G[U]$. Let $u_0 \in U$ be such that u_0 has a neighbor in $V \setminus U$ (u_0 exists as G is connected). Let $u_1, \ldots, u_d$ be the neighbors of u_0 in H. Note that $d \leq \Delta - 1$. Define $H_0 = H - u_0$ and set $H_i = H_{i-1} - u_i$ for $i = 1, \ldots, d$. Then by induction we know that $Z_{H_i}(\lambda) \neq 0$ for $i = 0, \ldots, d$ and $\Re(R_{H_{i-1},u_i}) > 0$ for $i \geq 1$, implying that $|1 + R_{H_{i-1},u_i}| \geq 1$. So by Lemma 2.2 we know that

$$|R_{H,u_0}| = \frac{|\lambda_{u_0}|}{\prod_{i=1}^{d} |1 + R_{H_{i-1},u_i}|} < |\lambda_{u_0}| \leq \tan\left(\frac{\pi}{(2+\varepsilon)(\Delta-1)}\right),$$

showing that (ii) holds for U.

To see that (iii) holds, we look at the angle α that R_{H,u_0} makes with the positive real axis. It suffices to show that $|\alpha| < \pi/2$. Since by induction $\Re(R_{H_{i-1},u_i}) > 0$ and $|R_{H_{i-1},u_i}| \leq \tan(\frac{\pi}{(2+\varepsilon)(\Delta-1)})$, we see that the angle α_i that $1 + R_{H_{i-1},u_i}$ makes with the positive real axis satisfies $|\alpha_i| \leq \frac{\pi}{(2+\varepsilon)(\Delta-1)}$. This implies by Lemma 2.2 that

$$|\alpha| < (\Delta - 1)\frac{\pi}{(2+\varepsilon)(\Delta-1)} + \frac{(\varepsilon/2)\pi}{2+\varepsilon} = \pi/2,$$

showing (iii).

As by (iii), R_{H,u_0} has strictly positive real part and hence does not equal -1, we conclude by (3) that $Z_H(\lambda) \neq 0$. So we conclude that (i), (ii), and (iii) hold for all $U \subseteq V \setminus \{v_0\}$.

To conclude the proof, it remains to show that $Z_G(\lambda) \neq 0$. Let $v_1, \ldots, v_d$ be the neighbors of v_0. Let G_i, $i = 0, \ldots, d$, be defined as the graphs H_i before. Then by (i) and (ii) we know that $Z_{G_i}(\lambda) \neq 0$ for $i = 0, \ldots, d$ and $\Re(R_{G_{i-1},v_i}) > 0$ for $i \geq 1$. So, as before, we have that the angle α_i that $1 + R_{G_{i-1},v_i}$ makes with the positive real line satisfies $|\alpha_i| \leq \frac{\pi}{(2+\varepsilon)\Delta-1}$. So by Lemma 2.2 the absolute value of the argument of R_{G,v_0} is bounded by

$$\frac{(\varepsilon/2)\pi}{2+\varepsilon} + \Delta\frac{\pi}{(2+\varepsilon)(\Delta-1)} \leq \frac{(2+\varepsilon/2)\pi}{2+\varepsilon} < \pi,$$

using that $\frac{\Delta}{\Delta-1} \leq 2$. This implies by (3) that $Z_G(\lambda) \neq 0$ and finishes the proof. □

For $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}$, define the map $F_{d,\lambda} : \widehat{\mathbb{C}}^d \to \widehat{\mathbb{C}}$ by

$$(x_1, \dots, x_d) \mapsto \frac{\lambda}{\prod_{i=1}^d (1 + x_i)}.$$

Given $\epsilon > 0$, the proof of Theorem 2.3 consisted mainly of finding a domain $D \subset \widehat{\mathbb{C}}$ not containing -1 such that if $x_1, \dots, x_d \in D$, then $F_{d,\lambda}(x_1, \dots, x_d) \in D$ for all $0 \le d \le \Delta - 1$.

To prove Theorem 1.1, we will similarly construct, for each Δ, a domain D that contains the interval $[0, \lambda_\Delta]$ but not the point -1 and is mapped inside itself by $f_{d,\lambda}$ for all $0 \le d \le \Delta - 1$ and all λ in a sufficiently small complex neighborhood of the interval $[0, (1 - \epsilon)\lambda_\Delta)$. Had these functions $f_{d,\lambda}$ all been strict contractions on the interval $[0, \lambda_\Delta]$, the existence of such a domain D would have been immediate. Unfortunately, the functions $f_{d,\lambda}$ are typically not contractions, even for real-valued λ. However, since the positive real line is contained in the basin of an attracting fixed point, it follows from basic theory of complex dynamical systems [18] that each $f_{d,\lambda}$ is strictly contracting on $[0, \lambda_\Delta)$ with respect to the Poincaré metric of the corresponding attracting basin. Although these Poincaré metrics vary with λ and d, this observation does give hope for finding coordinates with respect to which all the maps $f_{d,\lambda}$ are contractions.

In the next section, we introduce explicit coordinates with respect to which $f_{\Delta-1,\lambda_\Delta}$ becomes a contraction and then show that, for $d \le \Delta - 1$ and $\lambda \in [0, \lambda_\Delta)$, the maps $f_{d,\lambda}$ are all strict contractions with respect to the same coordinates. We then utilize these coordinates to give a proof of Theorem 1.1 in Section 4.

3. Change of Coordinates

Our aim in this section is to find a coordinate change for each $\Delta \ge 3$ so that the maps $f_{d,\lambda}$ are contractions in these coordinates for any $0 \le d \le \Delta - 1$ and $0 \le \lambda \le \lambda_\Delta$.

3.1. The Case $d = \Delta - 1$ and $\lambda = \lambda_\Delta$

We consider the coordinate changes

$$z = \varphi_y(x) = \log(1 + y \log(1 + x))$$

with $y > 0$. We note that a similar coordinate change using a double logarithm was used in [16]. The best argument for using this specific form is that it seems to fit our purposes.

Our initial goal is to pick y, depending on Δ, such that the *parabolic* map $f(x) := f_{\Delta-1,\lambda_\Delta}(x)$ becomes a contraction with respect to the new coordinates. Note that we call f parabolic if $\lambda = \lambda_\Delta$. In this case the fixed point of f is given by

$$x_\Delta = \frac{1}{\Delta - 2} = \frac{1}{d - 1},$$

and since $f'(x_\Delta) = -1$, it is thus parabolic. In the z-coordinates, we consider the map

$$g(z) = g_{\Delta-1,\lambda_\Delta}(z) = \varphi_y \circ f \circ \varphi_y^{-1}.$$

Note that the function $\varphi_y : \mathbb{R}_+ \to \mathbb{R}_+$ is bijective and $\mathbb{R}_+$ is forward invariant under f. It follows that the composition g is well defined on $\mathbb{R}_+$. We write $z_\Delta := \varphi_y(x_\Delta)$. Then z_Δ is fixed under g, and we immediately obtain $g'(z_\Delta) = -1$. Thus, in order that $|g'| \le 1$, we in particular need that $g''(z_\Delta) = 0$.

Let us start by computing g' and g''. Writing $x_1 = f(x_0)$ and $z_0 = \varphi_y(x_0)$, we note that

$$\begin{aligned} g'(z_0) &= \varphi_y'(x_1) \cdot f'(x_0) \cdot (\varphi_y^{-1})'(z_0) \\ &= \frac{\varphi_y'(x_1)}{\varphi_y'(x_0)} \cdot f'(x_0) \\ &= \frac{1 + y\log(1+x_0)}{1 + y\log(1+x_1)} \cdot \frac{1+x_0}{1+x_1} \cdot \frac{-dx_1}{1+x_0} \\ &= \frac{1 + y\log(1+x_0)}{1 + y\log(1+x_1)} \cdot \frac{-dx_1}{1+x_1}. \end{aligned} \tag{7}$$

Now note that

$$g'' = \frac{\partial g'}{\partial x_0} \cdot \frac{\partial x_0}{\partial z_0},$$

and since $\partial x_0/\partial z_0 > 0$, we look for points z_0 where $\partial g'/\partial x_0(z_0) = 0$. We obtain

$$\begin{aligned} \frac{\partial g'}{\partial x_0}(z_0) = {} & \frac{y/(1+x_0)}{1 + y\log(1+x_1)} \cdot \frac{-dx_1}{1+x_1} \\ & + (1 + y\log(1+x_0)) \cdot \frac{\partial}{\partial x_1}\left(\frac{1}{1 + y\log(1+x_1)} \cdot \frac{-dx_1}{1+x_1}\right) \cdot \frac{\partial x_1}{\partial x_0}. \end{aligned}$$

By considering x_1 as a variable depending on x_0, and thus also on z_0, the presentation of the calculations here and later in this section becomes significantly more succinct. Since

$$\frac{\partial x_1}{\partial x_0} = \frac{-dx_1}{1+x_0}$$

and since

$$\frac{\partial}{\partial x_1}\left(\frac{1}{1 + y\log(1+x_1)} \cdot \frac{-dx_1}{1+x_1}\right) = d \cdot \frac{x_1 y - (1 + y\log(1+x_1))}{(1+x_1)^2(1 + y\log(1+x_1))^2}, \tag{8}$$

we obtain

$$\begin{aligned} \frac{\partial g'}{\partial x_0}(z_0) = {} & \frac{y/(1+x_0)}{1 + y\log(1+x_1)} \cdot \frac{-dx_1}{1+x_1} \\ & + (1 + y\log(1+x_0))\frac{-d^2x_1}{1+x_0} \cdot \frac{x_1 y - (1 + y\log(1+x_1))}{(1+x_1)^2(1 + y\log(1+x_1))^2}. \end{aligned} \tag{9}$$

PROPOSITION 3.1. *The only value of $y > 0$ for which $g''(z_\Delta) = 0$ is given by*

$$y = y_\Delta := \frac{1}{2x_\Delta - \log(1+x_\Delta)}.$$

Proof. Noting that $x_1 = x_0$ and $dx_1/(1+x_1) = 1$ when $x_0 = x_\Delta$, we obtain

$$\frac{\partial g'}{\partial x_0}(z_\Delta) = d \cdot \frac{1 + y\log(1+x_\Delta) - 2x_\Delta y}{(1+y\log(1+x_\Delta))(1+x_\Delta)^2}.$$

Thus $g''(z_\Delta) = 0$ if and only if

$$y = y_\Delta := \frac{1}{2x_\Delta - \log(1+x_\Delta)}.$$

□

From now on we assume that $y = y_\Delta$.

COROLLARY 3.2. *We have that* $|g'(z)| \leq 1$ *for all* $z \geq 0$.

Proof. Since

$$\lim_{z\to+\infty} |g'(z)| = 0,$$

it suffices to show that $|g'(0)| < 1$, which follows if we show that $g''(0) < 0$, for which it is sufficient to show that $\partial g'/\partial x_0(0) < 0$.

Plugging in $x_0 = 0$ into (9), we get

$$\frac{\partial g'}{\partial x_0}(0) = \frac{dx_1(d - y(1+x_1))(1 + y\log(1+x_1) - dx_1 y)}{(1+y\log(1+x_1))^2(1+x_1)^2}$$

with $x_1 = f(0) = \lambda$. Hence we can complete the proof by showing that

$$d - y(1+\lambda) = d - y - y\lambda < 0. \tag{10}$$

Using that $1/y = 2/(d-1) - \log(d/(d-1))$, we observe that

$$1/y < \frac{1}{d-1} + \frac{1}{2(d-1)^2}$$

and hence $y > (d-1)^2/(d-1/2)$. From this we obtain

$$\begin{aligned} d - y - y\lambda &< \frac{d(d-1/2) - (d-1)^2 - (d-1)(d/(d-1))^d)}{d-1/2} \\ &< \frac{d(d-1/2) - (d-1)^2 - (d-1)(1 + d/(d-1))}{d-1/2} = \frac{-d/2}{d-1/2} < 0, \end{aligned}$$

which completes the proof. □

In particular, it follows that, for all $x \geq 0$, we have that $f^{\circ n}(x) \to x_\Delta$.

3.2. *Smaller Values of λ and d*

We now consider the case where $\lambda < \lambda_\Delta$ and the map f has degree $d \leq \Delta - 1$. We again consider the map

$$g_{d,\lambda}(z) = \varphi_y \circ f_{d,\lambda} \circ \varphi_y^{-1}.$$

Again, we will often just write g instead of $g_{d,\lambda}$. Our goal is to show that $|g'(z_0)| < 1$ for all $z_0 \geq 0$.

To do so, we consider g' as a function of λ, d, and z_0. We first look at the case where λ is fixed and d is varying.

LEMMA 3.3. *Let $\Delta \in \mathbb{N}$ with $\Delta \geq 3$. Let $0 \leq \lambda \leq \lambda_\Delta$, and let $d \in \{0, 1, \ldots, \Delta - 1\}$. Let $z_0 \geq 0$ be such that $g''_{d,\lambda}(z_0) = 0$. Then we have $0 \geq g'_{d,\lambda}(z_0) \geq g'_{\Delta-1,\lambda}(z_0)$.*

Proof. We consider the derivative of g' with respect to d in the points z_0 where $g''(z_0) = 0$. By (9), $g''(z_0)$ is a multiple of

$$\frac{y}{1 + y\log(1 + x_1)} \cdot \frac{1}{1 + x_1} \quad + \\ - d(1 + y\log(1 + x_0)) \cdot \frac{1 + y\log(1 + x_1) - x_1 y}{(1 + x_1)^2(1 + y\log(1 + x_1))^2}.$$

Since $g''(z_0) = 0$, we obtain

$$y(1 + x_1)(1 + y\log(1 + x_1)) = d(1 + y\log(1 + x_0)) \cdot (1 + y\log(1 + x_1) - x_1 y).$$

In particular, we get that

$$1 + y\log(1 + x_1) - x_1 y > 0 \tag{11}$$

and

$$d\log(1 + x_0) = \frac{(1 + x_1)(1 + y\log(1 + x_1))}{1 + y\log(1 + x_1) - x_1 y} - \frac{d}{y}. \tag{12}$$

Now notice that by (7) we have that $\frac{\partial}{\partial d} g'$ is a positive multiple of

$$\frac{-x_1}{(1 + x_1)(1 + y\log(1 + x_1))} + \frac{\partial x_1}{\partial d} \cdot \frac{\partial}{\partial x_1}\left(\frac{1}{1 + y\log(1 + x_1)} \cdot \frac{-d x_1}{1 + x_1}\right),$$

which by (8) is a positive multiple of

$$-(1 + x_1)(1 + y\log(1 + x_1)) + d\log(1 + x_0) \cdot (1 + y\log(1 + x_1) - x_1 y).$$

When we plug in equation (12) to eliminate x_0 from this expression, we note that the term $(1 + x_1)(1 + y\log(1 + x_1))$ cancels, and we obtain that $\frac{\partial}{\partial d} g'$ is a positive multiple of

$$-\frac{d}{y}(1 + y\log(1 + x_1) - x_1 y),$$

which is negative as observed in (11).

So, we see that as we decrease d, the value of $g'(z_0)$ increases, and hence it follows that $0 \geq g'_{d,\lambda}(z_0) \geq g'_{\Delta-1,\lambda}(z_0)$, as desired. □

We next compute the derivative of g' with respect to λ. Note that x_1 depends on λ, but x_0 does not, and hence

$$\frac{\partial g'}{\partial \lambda}(z_0) = (1 + y\log(1 + x_0)) \cdot \frac{\partial}{\partial \lambda}\left(\frac{-d x_1}{(1 + x_1)(1 + y\log(1 + x_1))}\right) \\ = (1 + y\log(1 + x_0)) \cdot \frac{\partial x_1}{\partial \lambda} \cdot \frac{\partial}{\partial x_1}\left(\frac{-d x_1}{(1 + x_1)(1 + y\log(1 + x_1))}\right).$$

Thus $\partial g'/\partial\lambda(z_0) = 0$ if and only if

$$\frac{\partial}{\partial x_1}\left(\frac{-d x_1}{(1 + x_1)(1 + y\log(1 + x_1))}\right) = 0,$$

which by (8) is the case if and only if

$$x_1 y - (1 + y\log(1 + x_1)) = 0. \tag{13}$$

LEMMA 3.4. *Let $\Delta \geq 5$. For any $\lambda \leq \lambda_\Delta$ and $0 \leq d \leq \Delta - 1$, we have*

$$x_1 y - (1 + y\log(1 + x_1)) < 0.$$

In particular, $g'(z_0)$ is decreasing in λ for any $z_0 \geq 0$.

Proof. We note that $x_1 y - (1 + y\log(1 + x_1))$ is increasing in x_1 for $x_1 > 0$. So it suffices to plug in $\lambda = \lambda_\Delta$ and $x_0 = 0$, that is, plug in $x_1 = \lambda_\Delta$. Note that this makes it independent of d.

Plugging in $x_1 = \lambda_\Delta$, we get

$$\lambda_\Delta y - (1 + y\log(1 + \lambda_\Delta) = y(\lambda_\Delta - (1/y + \log(1 + \lambda_\Delta)),$$

So, as $y > 0$, is suffices to show that

$$c(\Delta) := \lambda_\Delta - (1/y + \log(1 + \lambda_\Delta) < 0. \tag{14}$$

By a direct computer calculation we obtain that (14) holds for $\Delta \in \{5, 6, 7\}$. (See Table 1).

Using that $x - x^2/2 \leq \log(1 + x) \leq x$ for all $x \geq 0$, we obtain

$$\begin{aligned}\lambda_\Delta - (1/y + \log(1 + \lambda_\Delta)) &\leq \lambda_\Delta - (x_\Delta + \lambda_\Delta - \lambda_\Delta^2/2)\\ &= \lambda_\Delta^2/2 - \frac{1}{\Delta - 2}.\end{aligned}$$

Using that

$$\lambda_\Delta = \frac{\Delta - 1}{(\Delta - 2)^2}\left(\frac{\Delta - 1}{\Delta - 2}\right)^{\Delta - 2} \leq \frac{e(\Delta - 1)}{(\Delta - 2)^2},$$

we obtain that

$$\lambda_\Delta - (1/y + \log(1 + \lambda_\Delta)) \leq \frac{e^2(1 + \frac{1}{\Delta - 2})^2 - 2(\Delta - 2)}{2(\Delta - 2)^2}. \tag{15}$$

Since the right-hand side of (15) is negative for $\Delta = 8$ and since the numerator is clearly decreasing in Δ, we conclude that (14) is true for all $\Delta \geq 8$. This concludes the proof. □

Table 1

	Δ		
	5	6	7
$c(\Delta)$	−0.0450	−0.0809	−0.0887

LEMMA 3.5. *Let* $\Delta \in \{3,4\}$. *Let* $z_0 > 0$ *and* $\lambda_0 > 0$ *be such that*

$$\frac{\partial}{\partial \lambda} g'_{\Delta-1,\lambda}(z_0) = 0$$

for $\lambda = \lambda_0$. *Then* $g'_{\Delta-1,\lambda_0}(z_0) \geq -0.92$.

Proof. By assumption we have $\partial g'(z_0)/\partial\lambda = 0$. Thus (13) implies that

$$x_1 y = 1 + y\log(1+x_1). \tag{16}$$

This implies that for x_1 to be a solution to (16), we need that $x_1 \geq x_\Delta$. Indeed, suppose that $x_1 < x_\Delta$. Then we have from (16) that

$$x_1 y = 1 + y\log(1+x_1) > 1 + yx_1 - yx_1^2/2,$$

from which we obtain $yx_1^2 > 2$. However, as $y < 1/x_\Delta$, we have $yx_1^2 < yx_\Delta^2 < x_\Delta < 2$, a contradiction.

Now (16) combined with (7) gives

$$\begin{aligned} g'(z_0) &= \frac{1+y\log(1+x_0)}{yx_1} \cdot \frac{-(\Delta-1)x_1}{1+x_1} \\ &= \frac{-(\Delta-1)(1+y\log(1+x_0))}{y(1+x_1)}. \end{aligned} \tag{17}$$

Now recall that $y = y_\Delta$ satisfies

$$2x_\Delta y = 1 + y\log(1+x_\Delta). \tag{18}$$

Now using that $x_1 \geq x_\Delta$ and combining (16) and (18), we obtain

$$x_1 = \frac{1+y\log(1+x_1)}{y} = 2x_\Delta\left(\frac{1+y\log(1+x_1)}{1+y\log(1+x_\Delta)}\right) \geq 2x_\Delta.$$

Using this, we obtain

$$\begin{aligned} x_1 &= 2x_\Delta\left(\frac{1+y\log(1+x_1)}{1+y\log(1+x_\Delta)}\right) \geq 2x_\Delta\left(\frac{1+y\log(1+2x_\Delta)}{1+y\log(1+x_\Delta)}\right) \\ &= 2x_\Delta\left(\frac{1+\log(1+2x_\Delta)/(2x_\Delta-\log(1+x_\Delta))}{1+\log(1+x_\Delta)/(2x_\Delta-\log(1+x_\Delta))}\right) \\ &= 2x_\Delta + \log\left(\frac{1+2x_\Delta}{1+x_\Delta}\right) = \alpha_\Delta x_\Delta, \end{aligned}$$

where $\alpha_3 = 2+\log(3/2) \approx 2.405$ and $\alpha_4 = 2 + 2\log(4/3)) \approx 2.575$. This then implies that

$$1 + x_0 \leq (\lambda_\Delta/x_1)^{1/(\Delta-1)} \leq \alpha_\Delta^{-1/(\Delta-1)}(1+x_\Delta).$$

Since (17) is decreasing in x_0 and increasing in x_1, we can plug in $x_0 = \alpha_\Delta^{-1/(\Delta-1)}(1+x_\Delta)$ and $x_1 = \alpha_\Delta x_\Delta$ to obtain

$$g'_{\Delta-1,\lambda_0}(z_0) > \frac{-(\Delta-1)(1+y\log(\alpha_\Delta^{-1/(\Delta-1)}(1+x_\Delta))}{y(1+\alpha_\Delta x_\Delta)}$$
$$= \frac{-2(\Delta-1)x_\Delta y + y\log(\alpha_\Delta)}{y(1+\alpha_\Delta x_\Delta)}$$
$$= \frac{-2(\Delta-1)/(\Delta-2)+\log(\alpha_\Delta)}{(\Delta-2+\alpha_\Delta)/(\Delta-2)} \approx \begin{cases} -0.9168 & \text{if } \Delta = 3, \\ -0.8979 & \text{if } \Delta = 4. \end{cases}$$

This finishes the proof. □

We can now finally show that the coordinate change works for all values of the parameters we are interested in.

PROPOSITION 3.6. *Let $\Delta \geq 3$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $0 \leq \lambda < (1-\varepsilon)\lambda_\Delta$, then $|g'_{d,\lambda}(z_0)| < 1-\delta$ for all $z_0 \geq 0$ and $d \in \{0, 1, \ldots, \Delta-1\}$.*

Proof. Let $J = [0, (1-\varepsilon)\lambda_\Delta]$, and let

$$M := \min_{z_0 \geq 0, \lambda \in J, d = 0, \ldots, \Delta-1} g'_{d,\lambda}(z_0).$$

Since, for any $\lambda \in J$, we have that $\lim_{z_0 \to \infty} g'_{d,\lambda}(z_0) = 0$ and since $g''(0) < 0$ by the proof of Corollary 3.2 (which remains valid as (10) is decreasing in d), it follows that we may assume that M is attained at some triple (z_0, λ_0, d) with $z_0 > 0$, $\lambda_0 \in J$, and $d \in \{0, \ldots, \Delta-1\}$. This then implies that $g''_{d,\lambda_0}(z_0) = 0$, and hence by Lemma 3.3 we know that $g'_{d,\lambda_0}(z_0) \geq g'_{\Delta-1,\lambda_0}(z_0)$, that is, we have that $d = \Delta - 1$.

If $g'_{d,\lambda_0}(z_0)$ attains its minimum (as a function of λ) at some $\lambda < \lambda_\Delta$, then $\frac{\partial}{\partial\lambda} g'(z_0) = 0$. So by Lemma 3.4 we know that $\Delta \in \{3, 4\}$. Then Lemma 3.5 implies that $M \geq -0.92$. So we may assume that g' is strictly decreasing as a function of λ on $[0, \lambda_\Delta]$. This then implies that $\lambda_0 = (1-\varepsilon)\lambda_\Delta$, and so there exists $\delta > 0$ (we may assume $\delta < 0.08$) such that

$$M = g'_{d,\lambda_0}(z_0) > (1-\delta) g'_{d,\lambda_\Delta}(z_0) \geq -1 + \delta,$$

where the last inequality is by Corollary 3.2. This finishes the proof. □

4. Proof of Theorem 1.1

Our proof essentially follows the same pattern as the proof of Theorem 2.3, but instead of working with the function $F_{d,\lambda}$, we now need to work with a conjugation of $F_{d,\lambda}$. Let $\Delta \geq 3$. Recall from the previous section the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $z = \varphi(x) = \log(1 + y\log(1+x))$ with $y = y_\Delta$. We now extend the function φ to a neighborhood $V \subset \mathbb{C}$ of $\mathbb{R}_+$ by taking the branch for both logarithms that is real for $x > 0$. By making V sufficiently small we can guarantee that φ is invertible. Now define, for $d = 0, \dots, \Delta - 1$, the map $G_{d,\lambda} : \varphi(V)^d \to \widehat{\mathbb{C}}$ by

$$(z_1, \dots, z_d) \mapsto \varphi\left(\frac{\lambda}{\prod_{i=1}^d 1 + \varphi^{-1}(z_i)}\right).$$

For a set $A \subset \mathbb{C}$ and $\varepsilon > 0$, we write $\mathcal{N}(A, \varepsilon) := \{z \in \mathbb{C} \mid |z - a| < \varepsilon \text{ for some } a \in A\}$. Now define for $\varepsilon > 0$ the set $D(\varepsilon) \subset \mathbb{C}$ by

$$D(\varepsilon) := \mathcal{N}([0, \varphi(\lambda_\Delta)], \varepsilon).$$

We collect a very useful property:

LEMMA 4.1. *Let $\Delta \geq 3$ and $\varepsilon > 0$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that, for any $\lambda \in \Lambda(\varepsilon_2) := \mathcal{N}([0, (1-\varepsilon)\lambda_\Delta], \varepsilon_2)$, $d = 0, \dots, \Delta - 1$, and $z_1, \dots, z_d \in D(\varepsilon_1)$, we have $G_{d,\lambda}(z_1, \dots, z_d) \in D(\varepsilon_1)$.*

Proof. We first prove this for the particular case $z_1 = z_2 = \cdots = z_d = z$. In this case, we have $G_{d,\lambda}(z_1, \dots, z_d) = g_{d,\lambda}(z)$. By Proposition 3.6 we know that there exists $\delta > 0$ such that, for any $d = 0, \dots, \Delta - 1$, we have

$$|g'_{d,\lambda}(z)| < 1 - \delta \quad \text{for all } \lambda \in [0, (1-\varepsilon)\lambda_\Delta] \text{ and } z \in [0, \varphi(\lambda_\Delta)].$$

By the continuity of g' as a function of z and λ there exist $\varepsilon_1, \varepsilon_2 > 0$ such that, for all $d = 0, \dots, \Delta - 1$ and each $(z, \lambda) \in D(\varepsilon_1) \times \Lambda(\varepsilon_2)$, we have

$$|g'_{d,\lambda}(z)| \leq 1 - \delta/2.$$

We may assume that ε_2 is small enough so that, for any d,

$$\sup_{\lambda \in \Lambda(\varepsilon_2), z \in [-\varepsilon_1, \varphi(\lambda_\Delta)]} \left| \frac{\partial}{\partial \lambda} g_{d,\lambda}(z) \right| \leq \frac{\delta \varepsilon_1}{2\varepsilon_2}.$$

Fix now $\lambda \in \Lambda(\varepsilon_2)$ and d and let $z \in D(\varepsilon_1)$. Let $z' \in [0, \varphi(\lambda_\Delta)]$ be such that $|z - z'| < \varepsilon_1$, and let $\lambda' \in [0, (1-\varepsilon)\lambda_\Delta))$ be such that $|\lambda - \lambda'| < \varepsilon_2$ Then

$$\begin{aligned} |g_{d,\lambda}(z) - g_{d,\lambda'}(z')| &\leq |g_{d,\lambda}(z) - g_{d,\lambda}(z')| + |g_{d,\lambda}(z') - g_{d,\lambda'}(z')| \\ &< (1 - \delta/2)\varepsilon_1 + \varepsilon_1 \delta/2 < \varepsilon_1, \end{aligned}$$

implying that the distance of $g_{d,\lambda}(z)$ to $[0, \varphi(\lambda_\Delta)]$ is at most ε_1, as $g_{d,\lambda'}(z') \in [0, \varphi(\lambda_\Delta)]$. Hence $g_{d,\lambda}(z) \in D(\varepsilon_1)$, which proves the lemma for $z_1 = z_2 = \cdots = z_d = z$.

For the general case, fix d, let $\lambda \in \Lambda(\varepsilon_2)$, and consider $x = \prod_{i=1}^d (1 + \varphi^{-1}(z_i))$ for certain $z_i \in D = D(\varepsilon_1)$. We want to show that $x = \prod_{i=1}^d (1 + \varphi^{-1}(z)) = (1 + \varphi^{-1}(z))^d$ for some $z \in D$. First of all, note that

$$1 + \varphi^{-1}(z_i) = \exp\left(\frac{\exp(z_i) - 1}{y}\right).$$

Then

$$x = \exp\left(\sum_{i=1}^d \left(\frac{\exp(z_i) - 1}{y}\right)\right),$$

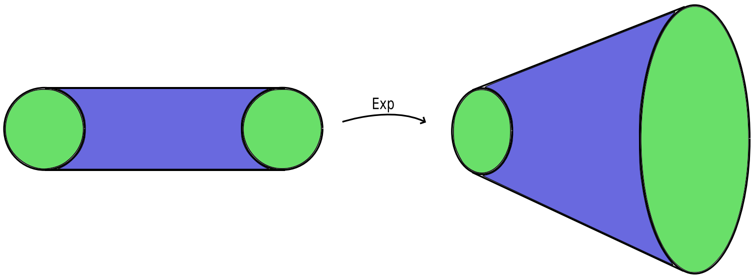

Figure 1 The set D and its image under the exponential map

which is equal to $(1+\varphi^{-1}(z))^d$ for some $z \in D$, provided that

$$\frac{1}{d}\sum_{i=1}^{d} \exp(z_i) = \exp(z) \tag{19}$$

for some $z \in D$. Consider the image of D under the exponential map. The domain D is smoothly bounded; its boundary consists of two arbitrarily small half-circles and two parallel horizontal intervals. Recall that the exponential imagine of a disk of radius less than 1 is strictly convex, a fact that can easily be checked by computing that the curvature of its boundary has constant sign. Therefore $\exp(D)$ is a smoothly bounded domain whose boundary consists of two radial intervals and two strictly convex curves, and hence $\exp(D)$ must also be convex. See Figure 1 for a sketch of the domain D and its image under the exponential map. It follows that the convex combination $\frac{1}{d}\sum_{i=1}^{d}\exp(z_i)$ is contained in the image of D. In other words, there exists $z \in D$ such that (19) is satisfied. This now implies that $G_{d,\lambda}(z_1, \ldots, z_d) = g_{d,\lambda}(z) \in D$, as desired. □

4.1. Proof of Theorem 1.1

We first state and prove a more precise version of Theorem 1.1 for the multivariate independence polynomial.

THEOREM 4.2. *Let $\Delta \in \mathbb{N}$ with $\Delta \geq 3$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any graph $G = (V, E)$ of maximum degree at most Δ and any $\lambda \in \mathbb{C}^V$ satisfying $\lambda_v \in \mathcal{N}([0, (1-\varepsilon)\lambda_\Delta), \delta)$ for each $v \in V$, we have that $Z_G(\lambda) \neq 0$.*

Proof. Let ε_1 and ε_2 be the two constants from Lemma 4.1, where ε_1 is chosen sufficiently small. Let $D = D(\varepsilon_1)$ and $\delta = \varepsilon_2$. Let G be a graph of maximum degree at most Δ. Since the independence polynomial is multiplicative over the disjoint union of graphs, we may assume that G is connected. Fix a vertex v_0 of G. We will show by induction that, for each subset $U \subseteq V \setminus \{v_0\}$, we have:

(i) $Z_{G[U]}(\lambda) \neq 0$,
(ii) if $u \in U$ has a neighbor in $V \setminus U$, then $\varphi(R_{G[U],u}) \in D$,

Clearly, if $U = \emptyset$, then both (i) and (ii) are true.

Now suppose $U \subseteq V \setminus \{v_0\}$ is nonempty and let $H = G[U]$. Let $u_0 \in U$ be such that u_0 has a neighbor in $V \setminus U$ (u_0 exists as G is connected). Let $u_1, \ldots, u_d$ be the neighbors of u_0 in H. Note that $d \leq \Delta - 1$. Define $H_0 = H - u_0$ and set $H_i = H_{i-1} - u_i$ for $i = 1, \ldots, d$. Then by induction we know that $Z_{H_i}(\lambda) \neq 0$ for $i = 0, \ldots, d$, so the ratios R_{H_{i-1},u_i} are well defined for $i \geq 1$, and by induction they satisfy $\varphi(R_{H_{i-1},u_i}) \in D$. By Lemma 2.2

$$R_{H,u_0} = \frac{\lambda_{u_0}}{\prod_{i=1}^{d}(1 + R_{H_{i-1},u_i})}.$$

Since $\varphi(R_{H_{i-1},u_i}) \in D$ for $i = 1, \ldots, d$, we have by Lemma 4.1 that $\varphi(R_{H,u_0}) \in D$. From this we conclude that $R_{H,u_0} \neq -1$, as $-1 \notin \varphi^{-1}(D)$. So by (3) $Z_H(\lambda) \neq 0$. This shows that (i) and (ii) hold for all subsets $U \subseteq V \setminus \{v_0\}$.

To conclude the proof, we need to show that $Z_G(\lambda) \neq 0$. Let $v_1, \ldots, v_d$ be the neighbors of v_0 (in any order). Define $G_0 = G - v_0$ and set $G_i = G_{i-1} - v_i$ for $i = 1, \ldots, d$. Then by (i) we know that $Z_{G_i}(\lambda) \neq 0$ for $i = 0, \ldots, d$, so the ratios R_{G_{i-1},v_i} are well defined for $i \geq 1$, and by (ii) they satisfy $\varphi(R_{G_{i-1},v_i}) \in D$. Write for convenience $z_i = R_{G_{i-1},v_i}$ for $i = 1, \ldots, d$. Then, by the same reasoning as before, we have

$$R_{G,v_0}(1 + z_d) = \frac{\lambda_{v_0}}{\prod_{i=1}^{d-1}(1 + z_i)} \in \varphi^{-1}(D).$$

This implies that R_{G,v_0} is not equal to -1, for if this were the case, we would have $-1 \in z_d + \varphi^{-1}(D)$. However, $z_d \in \varphi^{-1}(D)$, and for ε_1 small enough, $\varphi^{-1}(D)$ will have real part bounded away from $-1/2$, a contradiction. We conclude that $Z_G(\lambda) \neq 0$. □

Theorem 1.1 is now an easy consequence.

Proof of Theorem 1.1. Let for $\varepsilon > 0$, $\delta(\varepsilon)$ be the associated $\delta > 0$ from Theorem 4.2. Consider a sequence $\epsilon_i \to 0$ and define

$$D_\Delta := \bigcup \mathcal{N}([0, (1 - \varepsilon_i)\lambda_\Delta), \delta(\varepsilon_i)).$$

The set D_Δ is clearly open and contains $[0, \lambda_\Delta)$. Moreover, for any graph G of maximum degree at most Δ and $\lambda \in D_\Delta$, we have $Z_G(\lambda) \neq 0$, as $\lambda \in \mathcal{N}([0, (1 - \varepsilon)\lambda_\Delta), \delta(\varepsilon))$ for some $\varepsilon > 0$. □

Let us recall that the literal statement of Theorem 1.1 is false in the multivariate setting, as we will prove in the Appendix. However, by the same reasoning as before we immediately obtain the following:

COROLLARY 4.3. *Let $\Delta \in \mathbb{N}$ with $\Delta \geq 3$, and let $n \in \mathbb{N}$. Then there exists a complex domain D_Δ containing $[0, \lambda_\Delta)^n$ such that, for any graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ of maximum degree at most Δ and any $\lambda \in D_\Delta$, we have that $Z_G(\lambda) \neq 0$.*

We remark that the difference between Corollary 4.3 and Theorem 1.1 is subtle. The set D_Δ is chosen of the form

$$D_\Delta := \bigcup \mathcal{N}([0, (1-\varepsilon_i)\lambda_\Delta), \delta(\varepsilon_i))^n$$

as before. In particular, the set D_Δ is not of the form D^n for some open set D containing $[0, \lambda_\Delta)$, and hence in this sense it is not a literal generalization of Theorem 1.1.

5. Concluding Remarks and Questions

In this paper, we have shown that Sokal's conjecture is true. By the results from [19] this gives as a direct application the existence of an efficient algorithm (different from Weitz's algorithm [25]) for approximating the independence polynomial at any fixed $0 < \lambda < \lambda_\Delta$. By a result of Sly and Sun [23] it is known that, unless NP = RP, there does not exist an efficient approximation algorithm for computing the independence polynomial at $\lambda > \lambda_\Delta$ for graphs of maximum degree at most Δ. Very recently, it was shown by Galanis, Goldberg, and Štefankovič [13], building on locations of zeros of the independence polynomial for certain trees, that it is NP-hard to approximate the independence polynomial at $\lambda < -(\Delta-1)^{\Delta-1}/\Delta^\Delta$ for graphs of maximum degree at most Δ. Recall from Proposition 2.1 that, at any λ contained in

$$U_{\Delta-1} = \left\{ \lambda_\Delta(\alpha) = \frac{-\alpha(\Delta-1)^{\Delta-1}}{(\Delta-1+\alpha)^\Delta} \;\middle|\; |\alpha| < 1 \right\},$$

the independence polynomial for regular trees does not vanish and that, for any $\lambda \in \partial(U_{\Delta-1})$, there exists λ' arbitrarily close to λ for which there exists a regular tree T such that $Z_T(\lambda') = 0$. This naturally leads two the following two questions.

QUESTION 1. Let $\alpha \in \mathbb{C}$ be such that $|\alpha| > 1$. Let $\varepsilon > 0$ and $\Delta \in \mathbb{N}$. Is it true that it is NP-hard to compute an ε-approximation[1] of the independence polynomial at $\lambda_\Delta(\alpha)$ for graphs G of maximum degree at most Δ?

This question has recently been answered positively, in a strong form, by Bezáková, Galanis, Goldberg, and Štefankovič [8]. They in fact showed that it is even #P hard to approximate the independence polynomial at nonpositive λ contained in the complement of the closure of $U_{\Delta-1}$.

QUESTION 2. Is it true that, for any graph G of maximum degree at most $\Delta \geq 3$ and any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$, we have $Z_G(\lambda_\Delta(\alpha)) \neq 0$? The same question is also interesting for the multivariate independence polynomial.

We note that if this question also had a positive answer, it would lead to a complete understanding of the complexity of approximating the independence polynomial of graphs at any complex number λ in terms of the maximum degree.

[1] By an ε-approximation of $Z_G(\lambda)$ we mean a nonzero number $\zeta \in \mathbb{C}$ such that $e^{-\varepsilon} \leq |Z_G(\lambda)/\zeta| \leq e^{\varepsilon}$ and such that the angle between $Z_G(\lambda)$ and ζ is at most ε.

Appendix: Parabolic Bifurcations in Complex Dynamical Systems and Proposition 2.1

The proof of Proposition 2.1 follows from results well known to the complex dynamical systems community but not easily found in textbooks. In this appendix, we give a short overview of the results needed and outline how Proposition 2.1 can be deduced from these results. The presentation is aimed at researchers who are not experts on parabolic bifurcations. Details of proofs will be given only in the simplest setting. The readers interested in working out the general setting are encouraged to look at the provided references.

We consider iteration of the rational function

$$f_\lambda(z) = \frac{\lambda}{(1+z)^d},$$

where $\lambda \in \mathbb{C}$ and $d \geq 2$. We note that f_λ has two critical points, -1 and ∞, and that $f_\lambda(-1) = \infty$.

LEMMA A.1. *If f_λ has an attracting or parabolic periodic orbit $\{x_1, \ldots, x_k\}$, then the orbits of -1 and ∞ both converge to this orbit.*

This statement is an immediate consequence of the following classical result, which can, for example, be found in [18].

THEOREM A.2. *Let f be a rational function of degree $d \geq 2$ with an attracting or parabolic cycle. Then the corresponding immediate basin must contain at least one critical point.*

Let us say a few words about how to prove this result in the parabolic case. Recall that a period orbit is called *parabolic* if its multiplier, the derivative in case of a fixed point, is a root of unity. We consider the model case, where 0 is a parabolic fixed point with derivative 1, and f has the form

$$z_1 = z_0 - z_0^2 + h.o.t.$$

Considering the change of coordinates $u = \frac{1}{z}$, we obtain

$$u_1 = u_0 + 1 + O\left(\frac{1}{u_0}\right),$$

and we observe that if $r > 0$ is chosen sufficiently small, then the orbits of all initial values $z \in D(r, r) = \{|z - r| < r\}$ converge to the origin tangent to the positive real axis. In fact, after a slightly different change of coordinates, we can obtain the simpler map

$$u_1 = u_0 + 1.$$

These coordinates on $D(r, r)$ are usually denoted by $u = \phi^i(z)$ and are referred to as the *incoming Fatou coordinates*. The Fatou coordinates are invertible on a sufficiently small disk $D(r, r)$ and can be holomorphically extended to the whole parabolic basin by using the functional equation $\phi^i(f(z)) = \phi^i(z) + 1$.

Considering the inverse map $z_1 = z_0 + z_0^2 + h.o.t.$, we similarly obtain the outgoing Fatou coordinates ϕ^o defined on a small disk $D(-r, r)$. It is often convenient to use the inverse map of ϕ^o, which we will denote by ψ^o. This inverse map can again be extended to all of $\mathbb{C}$ by using the functional equation $\psi^o(\zeta - 1) = f(\psi^o(\zeta))$.

Now let f be a rational function of degree at least 2 and suppose that the parabolic basin does not contain a critical point. Then ϕ^i extends to a biholomorphic map from $\mathbb{C}$ to the parabolic basin. This gives a contradiction, as a parabolic basin must be a hyperbolic Riemann surface, thats is, its covering space is the unit disk and therefore cannot be equivalent to $\mathbb{C}$. A similar argument can be given to deduce that any attracting basin must contain a critical point.

Let us return to the original maps f_λ. Recall that, for fixed $d \geq 2$, we denote the region in parameter space $\mathbb{C}_\lambda$ for which f_λ has an attracting fixed point by U_d. The set U_d is an open and connected neighborhood of the origin. An immediate corollary of the discussion is the following:

COROLLARY A.3. *For each $\lambda \in U_d$, the orbit of the initial value*

$$z_0 = f_\lambda^{\circ 2}(\infty) = \lambda$$

avoids the point -1.

In fact, it turns out that we can prove the following stronger statement.

LEMMA A.4. *The region U_d is a maximal open set of parameters λ for which the orbit of z_0 avoids the critical point -1.*

Observe that Lemma A.4 directly implies Proposition 2.1.

Note that the parameters λ for which there is a parabolic fixed point form a dense subset of ∂U_d. Hence, to obtain Lemma A.4, it suffices to prove that, for any parabolic parameter $\lambda_0 \in \partial U_d$ and any neighborhood $\mathcal{N}(\lambda_0)$, there exist a parameter $\lambda \in \mathcal{N}(\lambda_0)$ and $N \in \mathbb{N}$ for which $f_\lambda^{\circ N}(z_0) = -1$. The fact that such λ and N exist follows from the following result regarding parabolic bifurcations.

THEOREM A.5. *Let f_ϵ be a one-parameter family of rational functions that vary holomorphically with ϵ. Assume that $f = f_0$ has a parabolic periodic cycle and that this periodic cycle bifurcates for ϵ near 0. Denote one of the corresponding parabolic basins by $\mathcal{B}_f$, let $z_0 \in \mathcal{B}_f$, and let $w \in \hat{\mathbb{C}} \setminus \mathcal{E}_f$. Then there exists a sequence of $\epsilon_j \to 0$ and $N_j \to \infty$ for which $f_{\epsilon_j}^{\circ N_j}(z_0) = w$.*

Here $\mathcal{E}_f$ denotes the exceptional set, the largest finite completely invariant set, which by Montel's theorem contains at most two points; see [18]. Since the set $\{-1, \infty\}$ containing the two critical points of the rational functions $f_\lambda : z \mapsto \lambda/(1+z)^d$ does not contain periodic orbits, it quickly follows that the exceptional set of these functions is empty. Lemma A.4 follows from Theorem A.5 by taking $w = -1$ and considering a sequence (λ_j) that converges to a parabolic parameter $\lambda_0 \in \partial U_d$.

Perturbations of parabolic periodic points play a central role in complex dynamical systems and have been studied extensively; see, for example, the classical works of Douady [12] and Lavaurs [15]. We only give an indication of how to prove Theorem A.5, by discussing again the simplest model, $f(z) = z - z^2 + h.o.t.$ and $f_\epsilon(z) = f(z) + \epsilon^2$. For $\epsilon \neq 0$, the unique parabolic fixed point $0 = f(0)$ splits up into two fixed points. For $\epsilon > 0$ small, these two fixed points are both close to the imaginary axis, forming a small "gate" for orbits to pass through.

For $\epsilon > 0$ small enough, the orbit of an initial value $z_0 \in \mathcal{B}_f$, converging to 0 under the original map f, will pass through the gate between these two fixed points, from the right to the left half-plane. The time it takes to pass through the gate is roughly π/ϵ. The following more precise statement was proved in [15].

THEOREM A.6 (Lavaurs, 89'). *Let $\alpha \in \mathbb{C}$, Consider sequences (ϵ_j) of complex numbers satisfying $\epsilon_j \to 0$ and positive integers (n_j) for which*

$$\frac{\pi}{\epsilon_j} - n_j \to \alpha.$$

Then the maps $f_{\epsilon_j}^{\circ n_j}$ converge, uniformly on compact subsets of $\mathcal{B}_f$, to the map $\mathcal{L}_\alpha = \psi^o \circ T_\alpha \circ \phi^i$, where T_α denotes the translation $x \mapsto x + \alpha$.

Let $w \in \hat{\mathbb{C}} \setminus \mathcal{E}$, and let $\zeta_0 \in \mathbb{C}$ for which $\psi^o(\zeta_0) = w$. Let $\alpha \in \mathbb{C}$ be given by

$$\alpha = \zeta_0 - \phi^i(z_0)$$

such that $\mathcal{L}_\alpha(z_0) = w$. Fix $\rho > 0$ small and, for $\theta \in [0, 2\pi]$, write

$$\alpha(\theta) = \alpha + \rho e^{i\theta}$$

and

$$\epsilon_n(\theta) = \frac{\pi}{\alpha(\theta) + n}.$$

It follows that

$$f_{\epsilon_n(\theta)}^{\circ n}(z_0) \longrightarrow \mathcal{L}_{\alpha(\theta)} := \psi^o \circ T_{\alpha(\theta)} \circ \phi^i(z_0)$$

uniformly over all $\theta \in [0, 2\pi]$ as $n \to \infty$. Since the curve given by $\theta \mapsto \mathcal{L}_{\alpha(\theta)}(z_0)$ winds around -1, it follows that, for n sufficiently large, there exists $\alpha'_n \in \mathcal{N}(\alpha, \rho)$ for which

$$f_{\epsilon'_n}^{\circ n}(z_0) = w$$

is satisfied for

$$\epsilon'_n = \frac{\pi}{\alpha'_n + n}.$$

The general proof of Theorem A.5 follows the same outline.

We end by proving that the literal statement of Theorem 1.1 is false in the multivariate setting.

THEOREM A.7. *Let $\Delta \geq 3$, and let D_Δ be any neighborhood of the interval $[0, \lambda_\Delta)$. Then there exists a graph $G = (V, E)$ of maximum degree at most Δ and $\lambda \in D_\Delta^V$ such that $Z_G(\lambda) = 0$.*

We will in fact use regular trees G for which all vertices on a a given level will have the same values λ_{v_i}. In this setting, we deal with a nonautonomous dynamical system given by the sequence

$$x_k = \frac{\lambda_k}{(1+x_{k-1})^{\Delta-1}}$$

with $x_0 = 0$, where each $\lambda_k \in D_\Delta$. Hence Theorem A.7 is implied by the following proposition.

PROPOSITION A.8. *Given Δ and D_Δ as in Theorem A.7, there exist an integer $N \in \mathbb{N}$ and $\lambda_0, \dots, \lambda_N \in D_\Delta$ that give $x_N = -1$.*

The proof follows from the following lemma, which can be found in [18] and is a direct consequence of Montel's theorem.

LEMMA A.9. *Let f be a rational function of degree at least 2, let x lie in the Julia set of f, and let V be a neighborhood of x. Then*

$$\bigcup_{n\in\mathbb{N}} f^n(V) = \hat{\mathbb{C}} \setminus \mathcal{E}_f,$$

where $\mathcal{E}_f$ is the exceptional set of f.

Let $\Delta \geq 3$ and $\lambda \neq 0$. As noted before in this appendix, the exceptional set of the function $f_{\Delta-1,\lambda}$ is empty. Thus, by the compactness of the Riemann sphere it follows that, for any neighborhood V of a point in the Julia set, there exists $N \in \mathbb{N}$ such that $f^N_{\Delta-1,\lambda}(V) = \hat{\mathbb{C}}$.

To prove Proposition A.8, let us denote the set of all possible values of points x_N by A. Then A contains D_Δ and, in particular, a neighborhood V of the parabolic fixed point x_Δ of the function $f_{\Delta-1,\lambda_\Delta}$.

The parabolic fixed point x_Δ is contained in the Julia set of $f_{\Delta-1,\lambda_\Delta}$. Thus it follows that there exists $N \in \mathbb{N}$ for which $f^N_{\Delta-1,\lambda_\Delta}(V) = \hat{\mathbb{C}}$. Then $f^N_{\Delta-1,\lambda}(V) = \hat{\mathbb{C}}$ holds for $\lambda \in D$ sufficiently close to λ_Δ, and thus $A = \hat{\mathbb{C}}$, so that $-1 \in A$, which completes the proof of Proposition A.8.

Note that in this construction the λ_i take exactly two distinct values. On the lowest level of the tree, they are very close to x_Δ, and on all other levels, they are very close to λ_Δ. The thinner the set D_Δ, the more levels the tree needs to have.

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References

[1] A. Barvinok, *Computing the permanent of (some) complex matrices,* Found. Comput. Math. (2014), 1–14.

[2] ______, *Computing the partition function for cliques in a graph,* Theory Comput. 11 (2015), 339–355, Article 13.

[3] ______, *Approximating permanents and hafnians of positive matrices,* Discrete Anal. (2017), no. 2, 34 pp.

[4] ______, *Combinatorics and complexity of partition functions,* Algorithms Combin., 30, Springer, Cham, Switzerland, 2017.

[5] A. Barvinok and P. Soberón, *Computing the partition function for graph homomorphisms with multiplicities,* J. Combin. Theory Ser. A 137 (2016), 1–26.

[6] ______, *Computing the partition function for graph homomorphisms,* Combinatorica 37 (2017), 633–650.

[7] M. Bayati, D. Gamarnik, D. Katz, C. Nair, and P. Tetali, *Simple deterministic approximation algorithms for counting matchings,* Proceedings of the thirty-ninth annual ACM symposium on theory of computing, pp. 122–127, ACM, 2007.

[8] I. Bezakóva, A. Galanis, L. A. Goldberg, and D. Štefankovič, *Inapproximability of the independent set polynomial in the complex plane,* Proceedings of the 50th annual ACM SIGACT symposium on theory of computing 1234–1240, ACM, 2018 and arXiv preprint, 2018 Feb 18, arXiv:1711.00282.

[9] P. Bleher, M. Lyubich, and R. Roeder, *Lee–Yang–Fisher zeros for DHL and 2D rational dynamics, II. Global pluripotential interpretation,* preprint, 2011, available online at arXiv:1107.5764, 36 pages.

[10] ______, *Lee–Yang zeros for DHL and 2D rational dynamics, I. Foliation of the physical cylinder,* J. Math. Pures Appl. 107 (2017), 491–590.

[11] M. Chudnovsky and P. Seymour, *The roots of the independence polynomial of a clawfree graph,* J. Combin. Theory Ser. B 97 (2007), 350–357.

[12] A. Douady, *Does a Julia set depend continuously on the polynomial?* Complex dynamical systems, Cincinnati, OH, 1994, Proc. Sympos. Appl. Math., 49, pp. 91–138, Amer. Math. Soc., Providence, RI, 1994.

[13] A. Galanis, L. A. Goldberg, and D. Štefankovič, *Inapproximability of the independent set polynomial below the Shearer threshold,* LIPIcs–Leibniz international proceedings in informatics, 80, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2017, and arXiv preprint, 2016 Dec 17, arXiv:1612.05832.

[14] D. Gamarnik and D. Katz, *Correlation decay and deterministic FPTAS for counting list-colorings of a graph,* J. Discrete Algorithms 12 (2012), 29–47.

[15] P. Lavaurs, *Systèmes dynamiques holomorphiques: explosion des points périodiques,* PhD Thesis, Univesité Paris-Sud, 1989.

[16] J. Liu and P. Lu, *FPTAS for # BIS with degree bounds on one side,* Proceedings of the forty-seventh annual ACM on symposium on theory of computing, 2015 Jun 14, pp. 549–556, ACM.

[17] P. Lu and Y. Yin, *Improved FPTAS for multi-spin systems,* Approximation, randomization, and combinatorial optimization. Algorithms and techniques, pp. 639–654, Springer, Berlin Heidelberg, 2013.

[18] J. Milnor, *Dynamics in one complex variable,* Third edition, Ann. of Math. Stud., 160, Princeton University Press, Princeton, NJ, 2006.

[19] V. Patel and G. Regts, *Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials,* SIAM J. Comput. 46 (2017), 1893–1919.

[20] G. Regts, *Zero-free regions of partition functions with applications to algorithms and graph limits,* Combinatorica 38 (2018), 987–1015.

[21] A. D. Scott and A. D. Sokal, *The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma,* J. Stat. Phys. 118 (2005), 1151–1261.

[22] J. B. Shearer, *On a problem of Spencer,* Combinatorica 5 (1985), 241–245.
[23] A. Sly and N. Sun, *Counting in two-spin models on d-regular graphs,* Ann. Probab. 42 (2014), 2383–2416.
[24] A. Sokal, *A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models,* Markov Process. Related Fields 7 (2001), 21–38.
[25] D. Weitz, *Counting independent sets up to the tree threshold,* Proceedings of the thirty-eighth annual ACM symposium on theory of computing, STOC 06, pp. 140–149, ACM, New York, NY, USA, 2006.

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