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Relative $\mathbb{Q}$-Gradings from Bordered Floer Theory

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Abstract. In this paper, we show how to recover the relative $\mathbb{Q}$-grading in Heegaard Floer homology from the noncommutative grading on bordered Floer homology.

1. Introduction

Heegaard Floer homology, introduced by the second author and Z. Szabó, is an invariant for a closed connected oriented three-manifold equipped with a spin^c structure [12]. Heegaard Floer homology is defined as Lagrangian intersection Floer homology groups of certain Lagrangians in a symmetric product of a Riemann surface and as such is most naturally only relatively cyclicly graded. Indeed, the Heegaard Floer homology of a three-manifold equipped with the spin^c structure $\mathfrak{s}$ is graded by the group $\mathbb{Z}/\operatorname{div}(c_1(\mathfrak{s}))$, where $\operatorname{div}(c_1(\mathfrak{s}))$ denotes the divisibility of the first Chern class of the spin^c structure $\mathfrak{s}$. In particular, if the first Chern class of $\mathfrak{s}$ is torsion, then the corresponding Heegaard Floer homology is relatively $\mathbb{Z}$-graded.

With the help of the functoriality properties of Heegaard Floer homology, the relative $\mathbb{Z}$-grading on Heegaard Floer homology can be lifted to an absolute $\mathbb{Q}$-grading when the underlying spin^c structure is torsion [14]. (Compare Frøyshov [2].) This absolute $\mathbb{Q}$-grading contains subtle topological information; for a beautiful recent application, see Greene's paper [1]. See Gripp and Huang [4] for an interpretation of the absolute $\mathbb{Q}$-grading in terms of homotopy classes of 2-plane fields; cf. [5]. Although, by works of Sarkar and Wang [17] and Sarkar [16], the absolute $\mathbb{Q}$-grading is algorithmically computable, it remains somewhat mysterious and hard to compute.

By contrast, the relative $\mathbb{Q}$-grading induced by the absolute $\mathbb{Q}$-grading is much simpler. In this paper, we show how to use bordered Floer homology to compute this relative $\mathbb{Q}$-grading between different torsion spin^c structures, by decomposing a 3-manifold along a connected surface; it turns out that the noncommutative grading on bordered Floer homology contains the necessary information. (Another way of computing the relative $\mathbb{Q}$-grading, using covering spaces, was given by D. Lee and the first author [6].)

Finally, note that Heegaard Floer homology has several variants, $\widehat{HF}$, HF^-, HF^∞, and HF^+. Although we focus on the relative $\mathbb{Q}$-grading on $\widehat{HF}$ (as that is

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the version with a corresponding bordered theory), this determines the relative $\mathbb{Q}$-grading on HF^+, HF^-, and HF^∞ via the exact triangles

$$\cdots \longrightarrow \widehat{HF}(Y) \longrightarrow HF^+(Y) \xrightarrow{\cdot U} HF^+(Y) \xrightarrow{[1]} \cdots$$

and

$$\cdots \longrightarrow HF^-(Y) \longrightarrow HF^\infty(Y) \longrightarrow HF^+(Y) \xrightarrow{[1]} \cdots.$$

Remark 1.1. Rustamov computed the Euler characteristic of the reduced Heegaard Floer homology $HF_{\mathrm{red}}(Y)$ of a rational homology sphere Y in terms of the absolute grading and the Casson–Walker invariant [15, Theorem 3.3]:

$$\sum_{\mathfrak{s}\in \mathrm{spin}^c(Y)} \left[\chi(HF_{\mathrm{red}}(Y,\mathfrak{s})) - \frac{d(Y,\mathfrak{s})}{2}\right] = |H_1(Y)|\lambda(Y).$$

(Here, $d(Y,\mathfrak{s})$ is the correction term from [11], which is determined by $HF^+(Y,\mathfrak{s})$ as an absolutely $\mathbb{Q}$-graded $\mathbb{F}_2[U]$-module.) Turning this around, given $HF^+(Y)$ as a relatively $\mathbb{Q}$-graded $\mathbb{F}_2[U]$-module, and the Casson–Walker invariant of Y, we can compute the absolute $\mathbb{Q}$-grading on Y. (This observation was pointed out by the referee.)

2. Background

2.1. The Relative $\mathbb{Q}$-Grading

The absolute $\mathbb{Q}$-grading on $\widehat{HF}(Y,\mathfrak{s})$ is defined as follows. Choose a spin^c cobordism $(W^4,\mathfrak{t})$ from $(S^3,\mathfrak{s}_0)$ to $(Y,\mathfrak{s})$. Associated with the cobordism W is a map $\hat{F}_{W,\mathfrak{s}}\colon \widehat{HF}(S^3,\mathfrak{s}_0) \to \widehat{HF}(Y,\mathfrak{s})$. The absolute grading on $\widehat{HF}(Y,\mathfrak{s})$ is characterized by the property that the generator of $\widehat{HF}(S^3) \cong \mathbb{Z}$ lies in degree 0 and the map F has degree

$$\frac{c_1(\mathfrak{t})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

(Actually, since $\hat{F}_{W,\mathfrak{s}}$ might be trivial on homology, it is more accurate to say the grading is characterized by the property that Maslov index 0 triangles in the definition of $\hat{F}_{W,\mathfrak{s}}$ have this degree.) See [14, Section 7].

The paper [6] shows that the relative $\mathbb{Q}$-grading can be reformulated as follows. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are generators for $\widehat{CF}(Y)$ (computed via some pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$) so that $c_1(\mathfrak{s}(\mathbf{x}))$ and $c_1(\mathfrak{s}(\mathbf{y}))$ are torsions. Then there is a finite-order covering space $p\colon \widetilde{Y} \to Y$ such that $p^*\mathfrak{s}(\mathbf{x}) = p^*\mathfrak{s}(\mathbf{y})$ [6, Corollary 2.10]. The Heegaard diagram $\mathcal{H}$ for Y lifts to a (multipointed) Heegaard diagram $\widetilde{\mathcal{H}}$ for $\widetilde{Y}$. The generators $\mathbf{x}$ and $\mathbf{y}$ have preimages $p^{-1}(\mathbf{x})$ and $p^{-1}(\mathbf{y})$ in $\widetilde{\mathcal{H}}$, so that $\mathfrak{s}(p^{-1}(\mathbf{x})) = p^*\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(p^{-1}(\mathbf{y})) = p^*\mathfrak{s}(\mathbf{y})$. Thus, $p^{-1}(\mathbf{x})$ and $p^{-1}(\mathbf{y})$ have a well-defined $\mathbb{Z}$-grading difference. Then

$$\mathrm{gr}_{\mathbb{Q}}(\mathbf{y}) - \mathrm{gr}_{\mathbb{Q}}(\mathbf{x}) = (1/n)\,\mathrm{gr}_{\mathbb{Z}}(p^{-1}(\mathbf{x}), p^{-1}(\mathbf{y})),$$

where n is the order of the cover $\widetilde{Y} \to Y$.

More concretely, let $\pi_2(\mathbf{x},\mathbf{y})$ denote the set of domains connecting $\mathbf{x}$ and $\mathbf{y}$ with multiplicity 0 at the basepoint z. Even though $\mathbf{x}$ and $\mathbf{y}$ may not be connected by a domain (i.e., $\pi_2(\mathbf{x},\mathbf{y})$ may be empty), if we allow rational multiples of the regions in Σ, then they can be connected. That is, let $\pi_2^{\mathbb{Q}}(\mathbf{x},\mathbf{y})$ denote the set of rational linear combinations B of components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ not containing the basepoint z and satisfying the condition

$$\partial(\partial B \cap \boldsymbol{\alpha}) = -\partial(\partial B \cap \boldsymbol{\beta}) = \mathbf{y} - \mathbf{x}.$$

Elements of $\pi_2^{\mathbb{Q}}(\mathbf{x},\mathbf{y})$ are *rational domains* connecting $\mathbf{x}$ to $\mathbf{y}$ and satisfying $n_z = 0$. If $c_1(\mathfrak{s}(\mathbf{x})) - c_1(\mathfrak{s}(\mathbf{y}))$ is a torsion, then $\pi_2^{\mathbb{Q}}(\mathbf{x},\mathbf{y})$ is nonempty, and

$$\mathrm{gr}_{\mathbb{Q}}(\mathbf{y}) - \mathrm{gr}_{\mathbb{Q}}(\mathbf{x}) = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B)$$

for any $B \in \pi_2^{\mathbb{Q}}(\mathbf{x},\mathbf{y})$. Here, $e(B)$ denotes the Euler measure of B, and $n_{\mathbf{x}}(B)$ and $n_{\mathbf{y}}(B)$ denote the point measure of B at the points $\mathbf{x}$ and $\mathbf{y}$; compare [7, Section 4.2]. This agrees with the formulas in [6, Section 2.3].

2.2. *The Structure of Bordered Floer Theory*

Bordered Floer theory assigns to a connected oriented surface $F = F(\mathcal{Z})$ represented by a *pointed matched circle* $\mathcal{Z}$ (Figure 1) a *dg* algebra $\mathcal{A} = \mathcal{A}(\mathcal{Z})$ [10, Chapter 3]. With a connected oriented 3-manifold Y with boundary parameterized by $F(\mathcal{Z})$ it associates invariants $\widehat{CFA}(Y)_{\mathcal{A}(\mathcal{Z})}$, a right $\mathcal{A}_\infty$-module over $\mathcal{A}(\mathcal{Z})$, and ${}^{\mathcal{A}(-\mathcal{Z})}\widehat{CFD}(Y)$, a left projective *dg* module over $\mathcal{A}(-\mathcal{Z})$ [10, Chapters 6, 7]. Both $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ are well-defined up to homotopy equivalence. These modules are related to the invariants of a closed 3-manifold by a *pairing theorem:*

THEOREM 2.1 ([10, Theorem 1.3]). *If Y_1 and Y_2 are 3-manifolds with boundaries parameterized by $F(\mathcal{Z})$ and $-F(\mathcal{Z})$, respectively, then*

$$\widehat{CF}(Y_1 \cup_F Y_2) \simeq \widehat{CFA}(Y_1)\, \widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})}\, \widehat{CFD}(Y_2).$$

Here $\widetilde{\otimes}_{\mathcal{A}(\mathcal{Z})}$ denotes the $\mathcal{A}_\infty$-tensor product over $\mathcal{A}(\mathcal{Z})$. There is a particularly convenient model $\boxtimes$ for the $\mathcal{A}_\infty$-tensor product, so that $\widehat{CF}(Y_1 \cup_F Y_2)$ is in fact isomorphic as an $\mathbb{F}_2$-vector space to $\widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$ (for corresponding choices of auxiliary data, as discussed further).

The isomorphism in Theorem 2.1 is, in an appropriate sense, an isomorphism of relatively graded groups. This will be discussed in Section 2.3.

For the purposes of this paper, we will use the following basic facts about $\mathcal{A}(\mathcal{Z})$, $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$:

- The invariants $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ are defined in terms of a bordered Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y. Here Σ is a compact orientable surface of some genus g with one boundary component; $\boldsymbol{\alpha}$ consists of pairwise-disjoint embedded arcs $\boldsymbol{\alpha}^a$ and circles $\boldsymbol{\alpha}^c$ in Σ with $\partial\boldsymbol{\alpha}^a \subset \partial\Sigma$, whereas $\boldsymbol{\beta}$ consists of embedded circles only; and z is a basepoint in $\partial\Sigma$, not lying on any α-arc. The boundary $\partial\mathcal{H}$ of a bordered Heegaard diagram is a pointed matched circle. See Figure 2 for an example and [10, Chapter 4] for more detail.

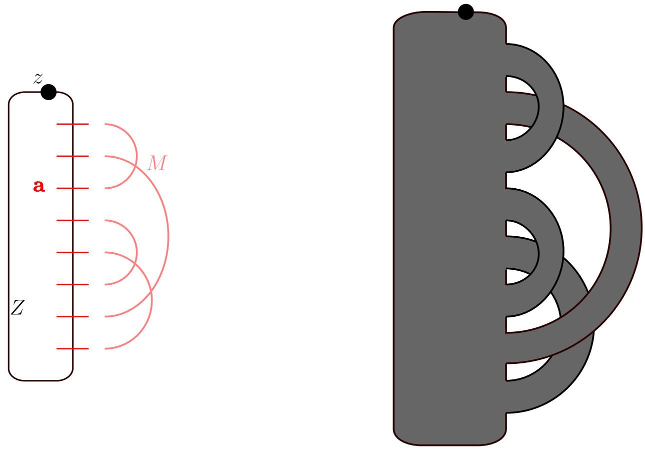

Figure 1 The surface represented by a pointed matched circle. Left: a pointed matched circle $\mathcal{Z}$, consisting of a circle Z, $4k$ points $\mathbf{a}$ (the case $k = 2$ is shown) matched in pairs via a matching M, and with a basepoint z. Right: the surface with boundary $F^\circ(\mathcal{Z})$ represented by $\mathcal{Z}$. The surface $F(\mathcal{Z})$ represented by $\mathcal{Z}$ is obtained by gluing a disk to $F^\circ(\mathcal{Z})$. As such, it contains a distinguished disk and a basepoint on the boundary of that disk

- If the bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ represent Y_1 and Y_2, respectively, and $\partial Y_1 = F(\mathcal{Z}) = -\partial Y_2$, then $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$ represents $Y = Y_1 \cup_\partial Y_2$.
- Given a genus g bordered Heegaard diagram $\mathcal{H}$ representing Y, the modules $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$ are generated by all sets $\mathbf{x} = \{x_1, \dots, x_g\}$ of g points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$ so that exactly one x_i lies on each α- or β-circle and at most one x_i lies on each α-arc. (Again, see Figure 2.) Let $\mathfrak{S}(\mathcal{H})$ denote the set of generators $\mathbf{x}$ in $\mathcal{H}$.
- Given generators $\mathbf{x}$ and $\mathbf{y}$ in $\mathfrak{S}(\mathcal{H})$, a *domain connecting* $\mathbf{x}$ *to* $\mathbf{y}$ is a linear combination B of components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ such that $\partial((\partial B) \cap \boldsymbol{\beta}) = \mathbf{x} - \mathbf{y}$, $\partial((\partial B) \cap (\boldsymbol{\alpha} \cup \partial\Sigma)) = \mathbf{y} - \mathbf{x}$, and multiplicity 0 at z. (See Figure 2.) Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$. For $B \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\partial^\alpha B = (\partial B) \cap \boldsymbol{\alpha}$, $\partial^\beta B = (\partial B) \cap \boldsymbol{\beta}$, and $\partial^\partial B = (\partial B) \cap (\partial\Sigma)$.
- Given a bordered Heegaard diagram $\mathcal{H}$ for Y, associated with each generator $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ is a spin^c structure $\mathfrak{s}(\mathbf{x})$ on Y. The modules $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$ decompose according to these spin^c structures, $\widehat{CFD}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{spin}^c(Y)} \widehat{CFD}(Y, \mathfrak{s})$ and $\widehat{CFA}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{spin}^c(Y)} \widehat{CFA}(Y, \mathfrak{s})$. Let $\mathfrak{S}(\mathcal{H}, \mathfrak{s}) = \{\mathbf{x} \in \mathfrak{S}(\mathcal{H}) \mid \mathfrak{s}(\mathbf{x}) = \mathfrak{s}\}$ denote the set of generators for $\widehat{CFA}(Y, \mathfrak{s})$ and $\widehat{CFD}(Y, \mathfrak{s})$.
- Given bordered Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$, let $\mathcal{H} = \mathcal{H}_1 \cup_\partial \mathcal{H}_2$. There is an obvious embedding $\mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$ of the set of generators $\mathfrak{S}(\mathcal{H})$ of $\widehat{CF}(\mathcal{H})$. The image of this embedding is the set of pairs

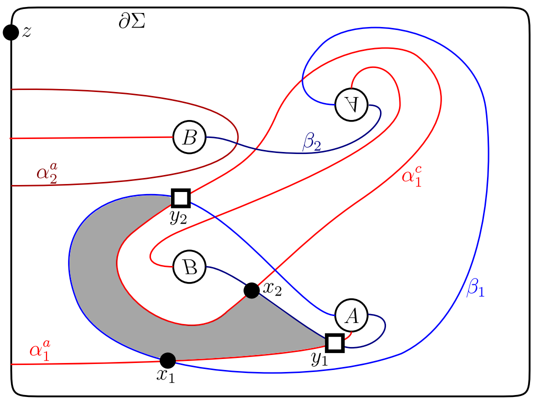

Figure 2 A bordered Heegaard diagram. The circles labeled A and B are connected by tubes. This bordered Heegaard diagram represents the trefoil complement (with a particular parameterization of its boundary). Two generators in $\mathfrak{S}(\mathcal{H})$ are marked, one by solid disks and the other by empty squares. A domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ connecting $\mathbf{x}$ and $\mathbf{y}$ is also shown, shaded

$(\mathbf{x}_1, \mathbf{x}_2) \in \mathfrak{S}(\mathcal{H}_1) \times \mathfrak{S}(\mathcal{H}_2)$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ occupy complementary α-arcs. It turns out that these are exactly the generators of $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2)$.

There is an extension of bordered Floer theory to manifolds with two boundary components, which are assigned various types of bimodules [8]. The generalizations of the results of this paper to the bimodule case are straightforward, and we shall not discuss them.

2.3. The (Noncommutative) Grading in Bordered Floer

As noted in the Introduction, the grading on bordered Floer homology is noncommutative.

Definition 2.2. Let G be a group, and let λ a central element in G. If $\mathcal{A}$ is a differential algebra, then a *grading of $\mathcal{A}$ by G* consists of a decomposition (as Abelian groups) $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ of $\mathcal{A}$ into homogeneous parts so that, for any homogeneous algebra elements a and b,

$$\mathrm{gr}(ab) = \mathrm{gr}(a)\,\mathrm{gr}(b) \quad \text{if } ab \neq 0, \tag{2.1}$$

$$\mathrm{gr}(\partial a) = \lambda^{-1}\,\mathrm{gr}(a) \quad \text{if } \partial a \neq 0. \tag{2.2}$$

Let S be a left G-set. If ${}_{\mathcal{A}}M$ is a left differential $\mathcal{A}$-module, then a *grading of ${}_{\mathcal{A}}M$ by S* consists of a decomposition (as Abelian groups) $M = \bigoplus_{s\in S} M_s$ of M into homogeneous parts such that, for homogeneous elements $x \in M$ and $a \in \mathcal{A}$,

$$\mathrm{gr}(ax) = \mathrm{gr}(a)\,\mathrm{gr}(x) \quad \text{if } ax \neq 0, \tag{2.3}$$

$$\mathrm{gr}(\partial x) = \lambda^{-1}\,\mathrm{gr}(x) \quad \text{if } \partial x \neq 0. \tag{2.4}$$

More generally, if S is an $\mathcal{A}_\infty$-module over $\mathcal{A}$, equations (2.3) and (2.4) become

$$\operatorname{gr}(m_{k+1}(a_1,\dots,a_k,x)) = \lambda^{k-1}\operatorname{gr}(a_1)\cdots\operatorname{gr}(a_k)\operatorname{gr}(x),$$

if $m_{k+1}(a_1,\dots,a_k,x)\neq 0$.

Gradings on right modules by right G-sets are defined similarly.

In the case of bordered Floer homology, for each pointed matched circle $\mathcal{Z} = (Z,\mathbf{a},M,z)$, there is a group $G'(\mathcal{Z})$ such that $\mathcal{A}(\mathcal{Z})$ is graded by $G'(\mathcal{Z})$. Let $Z' = Z\setminus\{z\}$, so Z' is an oriented interval, and the points in $\mathbf{a}$ inherit an ordering from the orientation on Z'. The group $G'(\mathcal{Z})$ is a $\mathbb{Z}$-central extension of $H_1(Z',\mathbf{a})$. To specify it, given a point $p\in\mathbf{a}$ and homology class $c\in H_1(Z',\mathbf{a})$, define $\mu(c,p)$ to be the average local multiplicity of c near p. Extend μ to a bilinear map $H_1(Z',\mathbf{a})\otimes H_0(\mathbf{a})\to\frac{1}{2}\mathbb{Z}$. Then G' is defined by the commutation relation

$$g\cdot h = \lambda^{2\mu([h],\partial[g])}h\cdot g,$$

where $[\cdot]\colon G'(\mathcal{Z})\to H_1(Z',\mathbf{a})$ is the canonical projection, and λ is a generator of the central $\mathbb{Z}$.

Explicitly, we can write elements of $G'(\mathcal{Z})$ as certain pairs (m,α) where $m\in\mathbb{Q}$ and $\alpha\in H_1(Z',\mathbf{a})$, with multiplication given by

$$(m_1,\alpha_1)\cdot(m_2,\alpha_2) = (m_1+m_2+\mu(\alpha_2,\partial\alpha_1),\alpha_1+\alpha_2). \tag{2.5}$$

Let $[i,i+1]$ denote the interval in Z' between the ith and $(i+1)$th points in $\mathbf{a}$. Define $G'(\mathcal{Z})$ to be the group generated by the element $\lambda = (1,0)$ and the $|\mathbf{a}|-1$ elements $(-\frac{1}{2},[i,i+1])$, with multiplication given by formula (2.5).

The grading on $\widehat{CFA}(\mathcal{H},\mathfrak{s})$ is given as follows. Given a Heegaard diagram $\mathcal{H}$ with $\partial\mathcal{H} = \mathcal{Z}$ and a spin^c structure $\mathfrak{s}$ on $Y = Y(\mathcal{H})$, fix a base generator $\mathbf{x}_0\in\mathfrak{S}(\mathcal{H},\mathfrak{s})$. For a domain $B\in\pi_2(\mathbf{x},\mathbf{y})$, define

$$g'(B) = (-e(B)-n_{\mathbf{x}}(B)-n_{\mathbf{y}}(B),\partial^\partial B)\in G'(\mathcal{Z}). \tag{2.6}$$

If $B_1\in\pi_2(\mathbf{x},\mathbf{y})$ and $B_2\in\pi_2(\mathbf{y},\mathbf{z})$, let $B_1*B_2\in\pi_2(\mathbf{x},\mathbf{z})$ denote the concatenation of B_1 and B_2. Then $g'(B_1*B_2) = g'(B_1)\cdot g'(B_2)$ [10, Lemma 10.4]. In particular, $P'(\mathbf{x}_0) = \{g'(B)\mid B\in\pi_2(\mathbf{x}_0,\mathbf{x}_0)\}$ is a subgroup of $G'(\mathcal{Z})$. The module $\widehat{CFA}(\mathcal{H},\mathfrak{s})$ is graded by the right $G'(\mathcal{Z})$-set $G'_A(\mathcal{H},\mathfrak{s}) := P'(\mathbf{x}_0)\backslash G'(\mathcal{Z})$. (This construction depends on $\mathbf{x}_0$, but different choices of $\mathbf{x}_0$ give canonically isomorphic grading sets; see [10, Lemma 10.14].) The grading of an element $\mathbf{x}\in\mathfrak{S}(\mathcal{H},\mathfrak{s})$ is given by $g'(B)$ for any $B\in\pi_2(\mathbf{x}_0,\mathbf{x})$, thought of as an element of the coset space $G'_A(\mathcal{H},\mathfrak{s}) := P'(\mathbf{x}_0)\backslash G'(\mathcal{Z})$.

The invariant $\widehat{CFD}(\mathcal{H})$ is a module over $\mathcal{A}(-\partial\mathcal{H})$ rather than $\partial\mathcal{H}$. So, in grading $\widehat{CFD}(\mathcal{H})$, we will use the antihomomorphism $R\colon G'(-\mathcal{Z})\to G'(\mathcal{Z})$ given by $R(j,\alpha) = (j,r_*(\alpha))$, where $r\colon -Z\to Z$ is the (orientation-reversing) identity map. The grading on $\widehat{CFD}(\mathcal{H},\mathfrak{s})$ is then defined similarly to the grading on $\widehat{CFA}(\mathcal{H},\mathfrak{s})$, except that the left module $\widehat{CFD}(\mathcal{H},\mathfrak{s})$ is graded by the left G'-set $G'_D(\mathcal{H},\mathfrak{s}) := G'(\mathcal{Z})/R(P'(\mathbf{x}_0))$, where $\mathcal{Z} = -\partial\mathcal{H}$, and the grading of an element $x\in\mathfrak{S}(\mathcal{H},\mathfrak{s})$ is given by (the equivalence class of) $R(g'(B))$ for any $B\in\pi_2(\mathbf{x}_0,\mathbf{x})$.

The tensor product $\widehat{CFA}(\mathcal{H}_1,\mathfrak{s}_1)\boxtimes\widehat{CFD}(\mathcal{H}_2,\mathfrak{s}_2)$ is graded by the amalgamated product of the grading sets $G'_A(\mathcal{H}_1)\times_{G'} G'_D(\mathcal{H}_2)$; the grading of $\mathbf{x}_1\otimes\mathbf{x}_2$ is $\mathrm{gr}'(\mathbf{x}_1\otimes\mathbf{x}_2)=(\mathrm{gr}'(\mathbf{x}_1),\mathrm{gr}'(\mathbf{x}_2))$. (In fact, certain results are cleaner if we work instead with a certain subset of this amalgamated product that contains the gradings of all tensor products of generators; compare Theorem 2.3.) Note that since λ is central in G', the set $G'_A(\mathcal{H}_1)\times_{G'} G'_D(\mathcal{H}_2)$ retains an action by λ, which we will think of as a $\mathbb{Z}$-action.

A graded version of the pairing theorem states the following:

THEOREM 2.3 ([10, Theorem 10.42]). *If Y_1 and Y_2 are 3-manifolds with boundaries parameterized by F and $-F$, respectively, then there is a map*

$$\Phi\colon \widehat{CFA}(Y_1)\boxtimes\widehat{CFD}(Y_2)\to\widehat{CF}(Y_1\cup_F Y_2)$$

such that:

(1) *Φ is a homotopy equivalence.*
(2) *Given generators $\mathbf{x}_1\otimes\mathbf{x}_2$ and $\mathbf{y}_1\otimes\mathbf{y}_2$ for $\widehat{CFA}(Y_1)\boxtimes\widehat{CFD}(Y_2)$, $\mathfrak{s}(\Phi(\mathbf{x}_1\otimes\mathbf{x}_2))=\mathfrak{s}(\Phi(\mathbf{y}_1\otimes\mathbf{y}_2))$ if and only if*:
   - $\mathfrak{s}(\mathbf{x}_1)=\mathfrak{s}(\mathbf{y}_1)=:\mathfrak{s}_1$,
   - $\mathfrak{s}(\mathbf{x}_2)=\mathfrak{s}(\mathbf{y}_2)=:\mathfrak{s}_2$, *and*
   - *the generators $g'(\mathbf{x}_1)\times_{G'} g'(\mathbf{x}_2)$ and $g'(\mathbf{y}_1)\times_{G'} g'(\mathbf{y}_2)$ lie in the same $\mathbb{Z}$-orbit of $G'_A(\mathcal{H}_1,\mathfrak{s}_1)\times_{G'} G'_D(\mathcal{H}_2,\mathfrak{s}_2)$.*
(3) *If $\mathfrak{s}(\Phi(\mathbf{x}_1\otimes\mathbf{x}_2))=\mathfrak{s}(\Phi(\mathbf{y}_1\otimes\mathbf{y}_2))$, then*

$$g'(\mathbf{y}_1)\times_{G'} g'(\mathbf{y}_2)=\lambda^{\mathrm{gr}(\Phi(\mathbf{x}_1\otimes\mathbf{x}_2),\Phi(\mathbf{y}_1\otimes\mathbf{y}_2))}g'(\mathbf{x}_1)\times_{G'} g'(\mathbf{x}_2).$$

For this paper, we will use a slightly larger grading group and the corresponding grading sets. Given a pointed matched circle $\mathcal{Z}=(Z,\mathbf{a},M,z)$, let $G'_{\mathbb{Q}}(\mathcal{Z})$ denote the $\mathbb{Q}$-central extension of $H_1(Z',\mathbf{a};\mathbb{Q})$ with multiplication given by

$$(m_1,\alpha_1)\cdot(m_2,\alpha_2)=(m_1+m_2+\mu(\alpha_2,\partial\alpha_1),\alpha_1+\alpha_2),$$

that is, the same formula as equation (2.5).

There is an obvious inclusion $G'\to G'_{\mathbb{Q}}$, so the G'-grading on $\mathcal{A}(\mathcal{Z})$ induces a $G'_{\mathbb{Q}}$-grading on $\mathcal{A}(\mathcal{Z})$. Also, note that, for $g\in G'_{\mathbb{Q}}$ and $q\in\mathbb{Q}$, there is a well-defined element $q\cdot g\in G'_{\mathbb{Q}}$ obtained by multiplying all the coefficients in g by q.

If $\mathcal{H}$ is a Heegaard diagram with $\partial\mathcal{H}=\mathcal{Z}$ (respectively, $\partial\mathcal{H}=-\mathcal{Z}$), then we can define a $G'_{\mathbb{Q}}$-grading on $\widehat{CFA}(\mathcal{H})$ (respectively, $\widehat{CFD}(\mathcal{H})$) using formula (2.6). Given $\mathbf{x}\in\mathfrak{S}(\mathcal{H})$, let $P'_{\mathbb{Q}}(\mathbf{x})$ denote the subgroup of $G'_{\mathbb{Q}}$ generated by $\{q\cdot g'(B)\mid B\in\pi_2(\mathbf{x},\mathbf{x}),\ q\in\mathbb{Q}\}$. Fix a base generator $\mathbf{x}_0\in\mathfrak{S}(\mathcal{H},\mathfrak{s})$. For any $\mathbf{x}\in\mathfrak{S}(\mathcal{H},\mathfrak{s})$, choose $B\in\pi_2(\mathbf{x}_0,\mathbf{x})$ and define $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{x})=g'(B)$ (respectively, $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{x})=R(g'(B))$), viewed as an element of $G'_{A,\mathbb{Q}}(\mathcal{H},\mathfrak{s}):=P'_{\mathbb{Q}}(\mathbf{x}_0)\backslash G'_{\mathbb{Q}}(\mathcal{Z})$ (respectively, $G'_{D,\mathbb{Q}}(\mathcal{H},\mathfrak{s}):=G'_{\mathbb{Q}}(\mathcal{Z})/R(P'_{\mathbb{Q}}(\mathbf{x}_0))$).

There is also a refined grading on the algebra by a group G that is a $\mathbb{Z}$-central extension of $H_1(F(\mathcal{Z}))$ and the corresponding gradings on the modules; see [10, Section 3.3] or [8, Section 3.2.1]. Generally, we will work with the larger grading group in this paper, but see also Remark 3.4.

REMARK 2.4. Recall that the grading in monopole Floer homology of Y is by homotopy classes of 2-plane fields on Y [5]. The refined grading group for $\mathcal{Z}$ can be understood in terms of homotopy classes of 2-plane fields on $[0,1] \times F(\mathcal{Z})$. In fact, it was proved recently that the G-set gradings on the modules can also be understood in terms of 2-plane fields [3].

3. From Bordered Floer to the Relative $\mathbb{Q}$-Grading

THEOREM 3.1. *Suppose that Y is a closed 3-manifold decomposed along a connected surface as $Y = Y_1 \cup_F Y_2$. Let $\mathcal{H} = \mathcal{H}_1 \cup_{\mathcal{Z}} \mathcal{H}_2$ be the corresponding decomposition of a Heegaard diagram for Y. Suppose that $\mathbf{x}, \mathbf{y} \in \mathfrak{S}(\mathcal{H})$ are such that $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(\mathbf{y})$ are torsion and $\mathfrak{s}(\mathbf{x})|_{Y_i} = \mathfrak{s}(\mathbf{y})|_{Y_i} =: \mathfrak{s}_i$ for $i = 1, 2$. Write $\mathbf{x} = \Phi(\mathbf{x}_1 \otimes \mathbf{x}_2)$ and $\mathbf{y} = \Phi(\mathbf{y}_1 \otimes \mathbf{y}_2)$, where $\mathbf{x}_i$ and $\mathbf{y}_i$ are in $\mathfrak{S}(\mathcal{H}_i)$. Then*

(1) *the generators $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_2)$ and $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_2)$ lie in the same $\mathbb{Q}$-orbit of $G'_{A,\mathbb{Q}}(\mathcal{H}_1, \mathfrak{s}_1) \times_{G'_{\mathbb{Q}}} G'_{D,\mathbb{Q}}(\mathcal{H}_2, \mathfrak{s}_2)$, and*

(2) $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_2) = \lambda^{\mathrm{gr}_{\mathbb{Q}}(\mathbf{x},\mathbf{y})}\, \mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_2)$.

Proof. Since the statements are independent of the base generators used to define the grading sets for $\widehat{CFA}(\mathcal{H}_1, \mathfrak{s}_1)$ and $\widehat{CFD}(\mathcal{H}_2, \mathfrak{s}_2)$, we may choose $\mathbf{x}_i$ to be the base generator for $\mathcal{H}_i$.

Since $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(\mathbf{y})$ are torsions, it follows from [6] (cf. Section 2.1) that there is a rational domain $B \in \pi_2^{\mathbb{Q}}(\mathbf{x}, \mathbf{y})$ connecting $\mathbf{x}$ and $\mathbf{y}$. Intersecting B with $\mathcal{H}_1$ and $\mathcal{H}_2$, we obtain rational domains $B_i \in \pi_2^{\mathbb{Q}}(\mathbf{x}_i, \mathbf{y}_i)$.

We argue that the rational domain B_i can be used to compute the grading of $\mathbf{y}_i$ (which was originally defined using integral domains). Since $\mathfrak{s}(\mathbf{x}_i) = \mathfrak{s}(\mathbf{y}_i)$, $\pi_2^{\mathbb{Q}}(\mathbf{x}_i, \mathbf{y}_i) = \pi_2(\mathbf{x}_i, \mathbf{y}_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. That is, any rational domain B_i connecting $\mathbf{x}_i$ and $\mathbf{y}_i$ can be written as

$$B_i = q_{i,1} C_{i,1} + \cdots + q_{i,\ell} C_{i,\ell},$$

where $q_{i,j} \in \mathbb{Q}$ and $C_{i,j} \in \pi_2(\mathbf{x}_i, \mathbf{y}_i)$. (To see this, note that $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$ is an affine copy of $H_2(Y_i, \partial Y_i; \mathbb{Z})$, whereas $\pi_2^{\mathbb{Q}}(\mathbf{x}_i, \mathbf{y}_i)$ is an affine copy of $H_2(Y_i, \partial Y_i; \mathbb{Q})$.) Consequently, B_i differs from any integral domain in $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$ by a rational periodic domain and hence has the same image in $G'_{A,\mathbb{Q}}(\mathcal{H}_1, \mathfrak{s}_1)$ or $G'_{D,\mathbb{Q}}(\mathcal{H}_2, \mathfrak{s}_2)$. So, as an element of $G'_{A,\mathbb{Q}}(\mathcal{H}_1, \mathfrak{s}_1)$,

$$g'(B_1) = (-e(B_1) - n_{\mathbf{x}_1}(B_1) - n_{\mathbf{y}_1}(B_1), \partial^{\partial} B_1) = \mathrm{gr}'(\mathbf{y}_1),$$

and, as an element of $G'_{D,\mathbb{Q}}(\mathcal{H}_2, \mathfrak{s}_2)$,

$$R(g'(B_2)) = (-e(B_2) - n_{\mathbf{x}_2}(B_2) - n_{\mathbf{y}_2}(B_2), r_*(\partial^{\partial} B_2)) = \mathrm{gr}'(\mathbf{y}_2).$$

Note also that $\partial^{\partial} B_2 = -\partial^{\partial} B_1$.

Thus, with our choice of base generator, $\mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{x}_2) = 0$, whereas

$$\begin{aligned} &\mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_1) \times_{G'_{\mathbb{Q}}} \mathrm{gr}'_{\mathbb{Q}}(\mathbf{y}_2) \\ &\quad = (-e(B_1) - n_{\mathbf{x}_1}(B_1) - n_{\mathbf{y}_1}(B_1) - e(B_2) - n_{\mathbf{x}_2}(B_2) - n_{\mathbf{y}_2}(B_2), 0) \end{aligned}$$

$$= (-e(B) - n_{\mathbf{x}}(B) - n_{\mathbf{y}}(B), 0)$$
$$= \lambda^{\mathrm{gr}_{\mathbb{Q}}(\mathbf{x},\mathbf{y})},$$

as desired. □

To complete the computation of the relative $\mathbb{Q}$-grading on $\widehat{CF}$, we observe that it is always possible to find a splitting satisfying the conditions of Theorem 3.1. One way to do so is to take a Heegaard splitting:

LEMMA 3.2. *Let $Y = Y_1 \cup_F Y_2$ be a Heegaard splitting of a 3-manifold Y. If $\mathfrak{s}$ and $\mathfrak{s}'$ are torsion* spin^c*-structures on Y, then $\mathfrak{s}|_{Y_1} = \mathfrak{s}'|_{Y_1}$ and $\mathfrak{s}|_{Y_2} = \mathfrak{s}'|_{Y_2}$.*

Proof. Since Y_i is a handlebody, $H^2(Y_i) = 0$. Thus, there is a unique spin^c-structure on Y_i. The result follows. □

COROLLARY 3.3. *The G'-set grading* gr' *defined in* [10] *determines the relative $\mathbb{Q}$-grading on $\widehat{HF}$.*

Proof. By definition, the grading gr' determines $\mathrm{gr}'_{\mathbb{Q}}$, which in turn, by Lemma 3.2 and Theorem 3.1, determines the relative $\mathbb{Q}$ grading. □

REMARK 3.4. It is sometimes convenient to work with the smaller grading group G from [10], rather than with G'. To obtain a G-set grading on $\widehat{CFD}$ and $\widehat{CFA}$, we conjugate by grading refinement data; see [8, Section 3.2.1]. In the proof of Theorem 3.1, since we work with the same grading refinement data on the two sides, it cancels out in the computation. Thus, Theorem 3.1 holds with respect to the small grading group, as well.

REMARK 3.5. In [9], we give an algorithm for computing $\widehat{HF}(Y)$ by taking a Heegaard decomposition of Y and factoring the gluing map into arc-slides. For such a decomposition, the hypotheses of Theorem 3.1 are automatically satisfied. Thus, keeping track of the $G'_{\mathbb{Q}}$-gradings along the way, [9] automatically computes the relative $\mathbb{Q}$-grading on $\widehat{HF}(Y)$.

REMARK 3.6. Instead of defining a $G'_{\mathbb{Q}}$-grading on $\widehat{CFD}$ by (roughly) tensoring the G'-grading with $\mathbb{Q}$ as before, we could instead use rational domains to induce a G'-grading. The resulting relative grading agrees with the one above when the one above is defined, but it is defined more often. Theorem 3.1 then no longer needs the hypothesis that $\mathfrak{s}(\mathbf{x})|_{Y_i} = \mathfrak{s}(\mathbf{y})|_{Y_i}$. The drawback is that, for this definition, $\mathrm{gr}'_{\mathbb{Q}}$ is no longer induced from gr', so we would not obtain Corollary 3.3.

4. Examples

We give an application of Theorem 3.1 to computing the $\mathbb{Q}$-graded Heegaard Floer homology groups of surgeries on some knots in S^3. Our knots are rather simple (the unknot and the trefoil), and hence the graded Heegaard Floer homology groups on their surgeries have been known for some time; however, these computations do give a nice illustration of the theorem.

To start, let Y denote the (-2)-framed complement of the left-handed trefoil T. By [10, Theorem 11.27], $\widehat{CFD}(Y)$ is given by

$$\begin{tikzcd} x_3 \arrow[d, "\rho_1"'] & & \\ y_2 & & \\ x_2 \arrow[u, "\rho_{123}"] & y_1 \arrow[l, "\rho_2"] & x_1. \arrow[l, "\rho_3"] \arrow[uull, "\rho_{12}"'] \end{tikzcd} \tag{4.1}$$

If we take x_3 as the base generator, then the gradings lie in $G'/\langle(-3/2; -1, 1, 2)\rangle$ and are given by:

$$\begin{aligned} \mathrm{gr}'(x_1) &= (1; 0, 2, 2)/\langle(-3/2; -1, 1, 2)\rangle, \\ \mathrm{gr}'(x_2) &= (1/2; 0, 1, 1)/\langle(-3/2; -1, 1, 2)\rangle, \\ \mathrm{gr}'(x_3) &= (0; 0, 0, 0)/\langle(-3/2; -1, 1, 2)\rangle, \\ \mathrm{gr}'(y_1) &= (3/2; 0, 2, 1)/\langle(-3/2; -1, 1, 2)\rangle, \\ \mathrm{gr}'(y_2) &= (-1/2; -1, 0, 0)/\langle(-3/2; -1, 1, 2)\rangle \end{aligned}$$

(compare [10, Section 11.9]). To see this, use the fact that the type D structure is graded, so if $a \otimes y$ occurs in $\delta^1(x)$, then

$$\mathrm{gr}'(a) \cdot \mathrm{gr}'(y) = (-1; 0, 0, 0) \cdot \mathrm{gr}'(x) \in G'_D. \tag{4.2}$$

So, start at the base generator x_3 and walk around the diagram (4.1) using formula (4.2) to determine the gradings of the other generators. On returning to x_3, we have the equation $\mathrm{gr}'(x_3) = \mathrm{gr}'(x_3) \cdot (-3/2; -1, 1, 2)$, so $(-3/2; -1, 1, 2)$ is the grading of the periodic domain.

Let $\mathcal{H}_0$ denote the ∞-framed solid torus. Then $\widehat{CFA}(\mathcal{H}_0)$ has one generator n with $m_3(n, \rho_2, \rho_1) = n$. In particular,

$$\begin{aligned} \mathrm{gr}'(n) &= \mathrm{gr}'(n)\,\mathrm{gr}'(\rho_2)\,\mathrm{gr}'(\rho_1)\lambda \\ &= \mathrm{gr}'(n)(-1/2; 0, 1, 0)(-1/2; 1, 0, 0)\lambda \\ &= \mathrm{gr}'(n)(-1/2; 1, 1, 0). \end{aligned}$$

So, $\mathrm{gr}'(n)$ lies in $\langle(-1/2; 1, 1, 0)\rangle\backslash G'$.

Tensoring the two together, we find that $\widehat{CFA}(\mathcal{H}_0) \boxtimes \widehat{CFD}(Y)$ is generated by $n \otimes y_1$ and $n \otimes y_2$ with no differential. It follows at once that $\widehat{HF}(S^3_{-2}(T)) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$, that is, $S^3_{-2}(T)$ has the same (ungraded) Heegaard Floer homology as a lens space; this was, of course, known before [13].

So far, we have found that the ungraded Heegaard Floer homology of -2 surgery on the trefoil and the unknot are the same. They are, however, distinguished by their relative $\mathbb{Q}$-gradings, which we can recover from the bordered invariants as follows.

The previous computation gives

$$\mathrm{gr}'(n \otimes y_1) = \langle(-1/2; 1, 1, 0)\rangle\backslash(3/2; 0, 2, 1)/\langle(-3/2; -1, 1, 2)\rangle,$$
$$\mathrm{gr}'(n \otimes y_2) = \langle(-1/2; 1, 1, 0)\rangle\backslash(-1/2; -1, 0, 0)/\langle(-3/2; -1, 1, 2)\rangle.$$

Working in $G'_{\mathbb{Q}}$, we can rewrite the first of these equations as

$$\begin{aligned}\mathrm{gr}'_{\mathbb{Q}}(n \otimes y_1) &= \langle(-1/2; 1, 1, 0)\rangle\backslash(3/4; -3/2, -3/2, 0)\cdot(3/2; 0, 2, 1)\\ &\qquad\cdot(3/4; 1/2, -1/2, -1)/\langle(-3/2; -1, 1, 2)\rangle\\ &= \langle(-1/2; 1, 1, 0)\rangle\backslash(1; -1, 0, 0)/\langle(-3/2; -1, 1, 2)\rangle.\end{aligned}$$

Consequently, the grading difference between $n \otimes y_1$ and $n \otimes y_2$ is $3/2$.

By contrast, the invariant of the -2-framed unknot has three generators:

$$\begin{tikzcd} b_1 & \\ a \arrow[u, "\rho_1"] \arrow[r, "\rho_3"'] & b_2. \arrow[ul, "\rho_{23}"'] \end{tikzcd}$$

If we take a as the base generator, the gradings lie in $G'/\langle(1/2; -1, 1, 2)\rangle$, and are given by

$$\mathrm{gr}'(a) = (0; 0, 0, 0)/\langle(1/2; -1, 1, 2)\rangle,$$
$$\mathrm{gr}'(b_1) = (-1/2; -1, 0, 0)/\langle(1/2; -1, 1, 2)\rangle,$$
$$\mathrm{gr}'(b_2) = (-1/2; 0, 0, -1)/\langle(1/2; -1, 1, 2)\rangle.$$

This gives

$$\mathrm{gr}'(n \otimes b_1) = \langle(-1/2; 1, 1, 0)\rangle\backslash(-1/2; -1, 0, 0)/\langle(1/2; -1, 1, 2)\rangle,$$
$$\mathrm{gr}'(n \otimes b_2) = \langle(-1/2; 1, 1, 0)\rangle\backslash(-1/2; 0, 0, -1)/\langle(1/2; -1, 1, 2)\rangle.$$

Working in $G'_{\mathbb{Q}}$, we can rewrite the first of these equations as

$$\mathrm{gr}'_{\mathbb{Q}}(n \otimes b_1) = \langle(-1/2; 1, 1, 0)\rangle\backslash(-1; 0, 0, -1)/\langle(1/2; -1, 1, 2)\rangle.$$

Consequently, the grading difference between $n \otimes b_1$ and $n \otimes b_2$ is $1/2$. Thus we see that the relative $\mathbb{Q}$-grading distinguishes the Heegaard Floer homology of -2 surgery on the trefoil from -2 surgery on the unknot.

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