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Simultaneous Flips on Triangulated Surfaces

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Abstract. We investigate a type of distance between triangulations on finite-type surfaces where one moves between triangulations by performing simultaneous flips. We consider triangulations up to homeomorphism, and our main results are upper bounds on the distance between triangulations that only depend on the topology of the surface.

1. Introduction

The general theme of defining and measuring distances between triangulations on surfaces plays a role in the study of geometric topology, the geometric group theory perspective of mapping class groups, and in combinatorial geometry.

A usual measure of distance is the flip distance measured by the number of flip moves necessary to go from one triangulation to another. With this measure we associate *flip graphs,* where vertices are triangulations, and there is an edge between vertices if the corresponding triangulations differ by a flip. These graphs appear in a number of contexts, most famously perhaps when the underlying surface is a polygon, and in this case the flip graph is the 1-skeleton of a polytope (the associahedron) [9; 10]; these graphs are finite, and their diameters are now completely known [7; 8]. In general, provided that the surface has enough topology, flip graphs are not finite and are combinatorial models for homeomorphism groups acting on surfaces. A natural finite graph associated with a surface is its *modular flip graph,* where we consider triangulations up to a homeomorphism. This graph (when defined properly and up to a few exceptions) is exactly the quotient of the flip graph by its graph automorphisms [6; 4].

In this paper, we consider a natural variant by measuring the distance between triangulations by considering the minimal number of *simultaneous* flip moves necessary between them. So in this case, flips that are made on disjoint quadrilaterals can be performed simultaneously. The simultaneous flip distance has been studied in the case of plane triangulations [1] (note there is slight difference in the definition of a triangulation) but also finds its roots in related problems in Teichmüller theory. A related problem in surface theory is to measure the distance between surfaces, and when these surfaces are hyperbolic and have the same topology, these distances and the related metric spaces give rise to the geometric

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study of Teichmüller and moduli spaces. In these spaces, several of the important metrics (namely the Teichmüller and Thurston metrics) are ℓ^∞ metrics. The simultaneous flip metric can be thought of as a combinatorial analogue to these metrics.

Our main goal is to study the diameters of modular flip graphs of finite-type orientable surfaces endowed with this distance. In particular, we are interested in how these diameters grow as functions of the number of marked points (or punctures) and the genus of the underlying surface. There are two possible types of marked points depending on whether they are labeled or not. This is equivalent to asking whether we consider the quotient by surface homeomorphisms, which fix marked points either individually or globally. Our methods allow us to show the following:

THEOREM 1.1. *There exists an explicit constant $U > 0$ such that the following holds. Let $\Sigma_{g,n}$ be a surface of genus g with n labeled marked points. Then any two triangulations of $\Sigma_{g,n}$ up to homeomorphism are related by at most*

$$U(\log(g+n))^2$$

simultaneous flip moves.

In other terms, this quantity is an upper bound on the diameter $\operatorname{diam}(\mathcal{MF}^s(\Sigma_{g,n}))$ of the modular flip graph. We prove this result in different contexts and with different explicit constants in front of the leading term; although the constants are explicit, we insist on the fact that it is really the order of growth that we have focused on.

We point out that we do not know whether the growth rate is optimal; the best lower bounds we know are of order $\log(\kappa)$, where κ is the arc complexity of the surface. It does not seem a priori obvious how to fill the gap nor even what to conjecture as the right rate of growth (see Section 6). However, in the case of unlabeled marked points, we show that the growth is at most $\log(n)$ in terms of the number of marked points. As in the case of triangulations of planar configurations of points, we cannot hope for better (see Section 6).

Organization

In the next section, we introduce the objects we will work with and prove two lemmas. In Section 3, we prove the main theorem for punctured spheres, and in Section 4, we prove it for genus g surfaces with a single puncture. These results allow us to deduce the general upper bound in Section 5. In the final section, we discuss lower bounds and further questions.

2. Preliminaries

In our setup, Σ is a topological orientable connected finite-type surface with a finite set of marked points on it. Its boundary can consist of marked points and possibly boundary curves, with the additional condition that each boundary curve

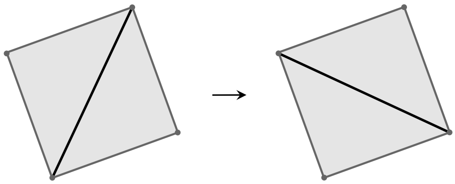


Figure 1 A flip

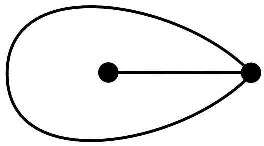


Figure 2 The central arc is not flippable

has at least one marked point on it. Marked points can be *labeled* or *unlabeled.* Sometimes we will call *punctures* the marked points that do not lie on a boundary curve. We will be interested in the combinatorics of arcs and triangulations of Σ. The arcs we consider are isotopy classes of simple arcs based at the marked points of Σ. A *multiarc* is a union of distinct isotopy classes of arcs disjoint except, possibly, at their endpoints. A *triangulation* of Σ is a maximal multiarc on Σ (note that this definition is not standard everywhere). The triangulations we consider here are allowed to contain loops and multiple edges; in particular, triangles may share more than a single vertex or a boundary arc.

The arc complexity $\kappa(\Sigma)$ is the number of arcs in (any) triangulation of Σ. The Euler characteristic tells us that $\kappa(\Sigma) = 6g + 3b + 3s + p - 6$, where g is the genus of Σ, s is the number of punctures, b is the number of boundary curves, and p is the number of marked points on the boundary curves.

The *modular flip graph* $\mathcal{MF}(\Sigma)$ is a graph whose vertices are triangulations of Σ with vertices in the set of marked points of Σ up to homeomorphism. The homeomorphisms we consider here preserve the set of marked points; in particular, they fix the set of the labeled marked points pointwise, and they are allowed to permute the unlabeled marked points. Two vertices of $\mathcal{MF}(\Sigma)$ are joined by an edge if the two underlying triangulations differ by exactly one arc; equivalently, two triangulations are joined by an edge if they differ by a *flip,* that is, the operation of replacing one diagonal with the other one in a quadrilateral (see Figure 1).

An arc that can be flipped is called *flippable,* and all arcs are flippable except those contained in a punctured monogon (see Figure 2).

The modular flip graph $\mathcal{MF}(\Sigma)$ can also be described as the quotient of the flip graph of Σ modulo the action of the mapping class group (see [5; 6]).

In this paper, we are interested in the *modular simultaneous flip graph* $\mathcal{MF}^s(\Sigma)$. This is also a graph whose vertices are the triangulations of Σ up to homeomorphisms. Here two vertices are joined by an edge if the two underlying triangulations differ by a finite number of flips supported on disjoint quadrilaterals on Σ, that is, a finite number of flips that can be performed simultaneously on Σ.

The following result by Bose, Czyzowicz, Gao, Morin, and Wood is Theorem 4.4 in [1]. It is both a prototype for what will be explored and a tool that we shall exploit.

THEOREM 2.1. *There exists a constant $K > 0$ such that the following is true. Let P_n be a polygon with n vertices, and let T and T' be two triangulations of P_n. Then it is possible to relate T to T' in at most $K\log(n)$ simultaneous flips.*

The constant K is computable, and in [1] it is shown that K can be taken to be less than 44. We further will not be particularly concerned in optimizing constants since the main point is the order of growth. However, the constants will be computable and we will indicate exact upper bounds that follow from our methods.

An obvious consequence of the theorem stated is the following. Given a triangulation T of P_n, let T_v be the unique triangulation of P_n with maximal degree in v. Then the simultaneous flip distance between T and T_v is at most $K\log(n)$. A result of this type is true in any context, as stated in the following lemma.

LEMMA 2.2. *Let v be a puncture on a surface Σ, and let T be a triangulation of Σ. Then there exists a sequence of at most $H\log(\kappa(\Sigma))$ simultaneous flips applied to T such that on the resulting triangulation, v is of maximal degree. The constant H can be taken equal to* 100.

Proof. When Σ is a polygon, this is a consequence of the previous theorem (with a better constant). We can thus suppose that Σ has some topology.

We begin by cutting Σ along a multiarc made of $2g+n-1$ arcs of T such that the resulting surface is a connected polygon with $4g+2n-2$ sides.

We now choose a copy of v and apply Theorem 2.1 to increase the degree until it is maximal within the polygon. This step requires at most $K\log(4g+2n-2)$ flips.

We now return to the full surface. Note that every triangle now has v as a vertex. With one simultaneous flip move, we can ensure that every triangle has v as two of its vertices. To do this, consider a triangle with only one copy of v as a vertex: exactly one of its three arcs does not have v_0 as an endpoint. This arc is flippable, since otherwise it surrounds a monogon as in Figure 2, and thus there is a triangle without v as any of its vertices. So the triangles with the property of having an arc without v as an endpoint come in pairs and form quadrilaterals together. These arcs can all be flipped simultaneously.

Now it is not difficult to see that with a final simultaneous flip we can ensure that all triangles have *only* v as vertices or are what we call *petals* based in v. A petal is a triangle like in Figure 2, and its base is the exterior vertex. We thus have reached a desired triangulation as the degree is maximal in v.

We can now quantify the procedure: the number of simultaneous flip moves is bounded above by

$$K\log(4g+2n-2)+2.$$

Finally, note that when $\kappa(\Sigma)\geq 2$, we have

$$K\log(4g+2n-2)+2\leq 100\log(\kappa(\Sigma)),$$

and this completes the proof. □

We recall that the *intersection number* $i(a,b)$ between two arcs a and b is defined to be the minimum number of intersection points between two arcs in the homotopy classes of a and b (the homotopies are relative to endpoints). The intersection number of two multiarcs A and B is defined as

$$i(A,B)=\sum_{b\in B}\sum_{a\in A} i(a,b).$$

LEMMA 2.3. *Let a be an arc, and let T be a triangulation of Σ such that $i(a,b)\leq 1$ for all $b\in T$. Then T can be moved in at most $L\log(i(a,T)+1)$ simultaneous flips to a triangulation containing a. The constant L can be taken equal to* 100.

Proof. Assume that $i(a,T)\geq 1$. Consider the set of all triangles of T through which a passes. They can be assembled into a polygon P, and because a only intersects every arc of T at most once, it intersects each of the triangles of P once. Thus a is a diagonal of this polygon. The polygon has complexity $\kappa=i(a,T)$ by construction and so has $i(a,T)+3$ vertices. Consider any triangulation T_a of P containing a: we now apply Theorem 2.1 to pass from T to T_a in at most $K\log(i(a,T)+3)<100\log(i(a,T)+1)$ moves. □

3. Punctured Spheres

In this section, we focus our attention on finding upper bounds on the simultaneous distance between triangulations of punctured spheres and disks with a single marked point on the boundary.

We begin by proving the following theorem for Ω_n', a punctured disk with n marked points inside and a single marked point on the boundary.

THEOREM 3.1. *There exists $A>0$ such that* $\operatorname{diam}(\mathcal{MF}^s(\Omega_n'))<A(\log(n+1))^2$. *The constant A can be taken equal to* 1,000.

Proof. Consider $T,T'\in\mathcal{MF}^s(\Omega_n')$ and denote by v_0 the boundary vertex of Ω_n'.

We begin by flipping both T and T' until the degree of v_0 is maximal. By Lemma 2.2 this step requires at most $H\log(\kappa(\Omega_n'))=H\log(3n-2)$ moves for each triangulations.

The result is a triangulation in which every puncture has an arc joining it to v_0, which in turn is surrounded by an arc. As previously, we call the unique triangle containing a given puncture a *petal* (see Figure 3), and the complement of the union of the petals is an $(n+1)$-gon with $n+1$ copies of v_0 as its vertices.

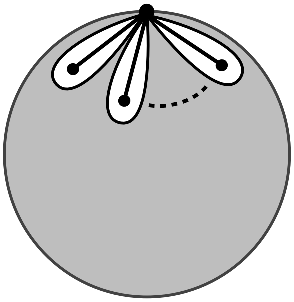


Figure 3 The shaded area is triangulated (so arcs have both end-points on v_0)

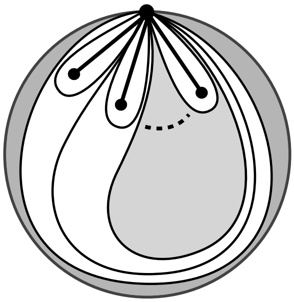


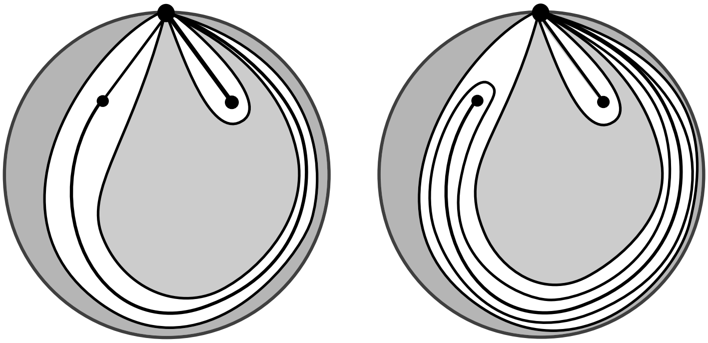
Figure 4 The shaded areas are triangulated in the same fashion around each petal

For each of our two triangulations, we will now perform the same procedure. We begin by looking at the polygon; one of the edges corresponds to the boundary arc of Ω_n', say a. We give the vertices of the polygon a cyclic order with p_0 being on the left of a, and p_n on the right.

By Theorem 2.1 any two triangulations of the polygon are at the distance roughly $\log(n)$ apart, and we will use that to obtain a special type of triangulation. More precisely, we move until the degree of p_n is maximal. By Theorem 2.1 this step takes at most $K\log(n+1)$ flips.

We now return to the petals. Figure 4 represents the result of the previous step around a petal. The goal is to split the vertices into two groups, both surrounded by an arc: one group with all vertices v_1 to $v_{\lfloor n/2\rfloor}$ and the other group with the remaining vertices. This can be done in two steps.

The first step takes two moves: flip (simultaneously) all arcs surrounding the petals containing vertices v_1 to $v_{\lfloor n/2\rfloor}$ and then flip all arcs between v_0 and v_k for $k=1,\dots,\lfloor\frac{n}{2}\rfloor$. Note that there is one possible restriction: if one of these vertices is the rightmost (where right and left are as in Figure 4), then it should not be moved at all. After this process, it will be the leftmost vertex among all vertices v_0 to $v_{\lfloor n/2\rfloor}$ but to the right of all other vertices.



(a) Breaking the petal... (b)... and building it elsewhere

Figure 5 The first step

The result around an individual petal is illustrated in Figure 5(a).

We then flip symmetrically as in Figure 5(b).

The result is again a triangulation with petals, but this time the petals with vertices v_1 to $v_{\lfloor n/2 \rfloor}$ are grouped together with respect to the left-right order.

The second step is to move in the polygon on complement of the petals to create a triangulation that contains two special arcs b, c: one that surrounds the petals containing v_1 to $v_{\lfloor n/2 \rfloor}$ and the other that surrounds the remaining petals. Note that a, b, c are the arcs of a triangle. How the rest of the triangulation looks like is irrelevant. By Theorem 2.1 this step takes at most $K \log(n+1)$ flips.

Now we move (simultaneously) inside each arc b and c, which surround respectively $\lfloor \frac{n}{2} \rfloor$ and $n - \lfloor \frac{n}{2} \rfloor$ vertices. Denote by Ω_b and Ω_c the two subsurfaces bounded by b and c. By induction on n the number of flips inside each of the two subsurfaces is at most

$$A \log^2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right).$$

The distance between T and T' is at most

$$\begin{aligned} d(T, T') &\leq A \log^2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + 2(2K \log(n+1) + H \log(3n-2)) + 4 \\ &\leq A \log^2(n+1). \end{aligned}$$

A direct computation proves that when A is large enough (for example $A =$ 1,000), the last inequality holds for every $n \geq 1$. $\square$

From Theorem 3.1 the same type result can be deduced for a punctured sphere.

THEOREM 3.2. *Let Ω_n be a sphere with n labeled punctures. Then there exists $B > 0$ such that* $\operatorname{diam}(\mathcal{MF}^s(\Omega_n)) < B(\log(n))^2$. *The constant B can be taken equal to* 1,100.

Proof. For $n \leq 3$, the result is immediate since Ω_n has at most six triangulations. We will now assume that $n \geq 4$.

We can prove the theorem analogously to the previous theorem, but for simplicity, we will use the previous result directly.

Let us denote by $v_0, \dots, v_{n-1}$ the punctures of Ω_n. Given two triangulations T, T', we begin by flipping them to increase the valency of v_0 until it is maximal. By Lemma 2.2 this step takes at most $2H\log(\kappa(\Omega_n)) = 2H\log(3n-2)$ moves. As a result, we obtain two triangulations, say $\tilde{T}, \tilde{T}'$, which have all other vertices joined to v_0 by an unflippable arc. Consider the petal surrounding v_{n-1}; the complementary region to it is a triangulation of a disk with a single marked vertex (namely v_0) on its boundary and with $n-2$ interior vertices. Theorem 3.1 tells us that $\tilde{T}$ and $\tilde{T}'$ are at most $A\log^2(n-1)$ apart. We thus have

$$\begin{aligned} d(T,T') &\le d(T,\tilde{T}) + d(T',\tilde{T}') + d(\tilde{T},\tilde{T}') \\ &\le 2H\log(3n-2) + A\log^2(n-1) \\ &\le 200\log(3n-2) + 1{,}000\log^2(n-1) \\ &\le B\log^2(n). \end{aligned}$$

A direct computation proves that when B is large enough (for example, any $B \ge 1{,}100$ works), the last inequality holds for every $n \ge 4$. $\square$

REMARK 3.3. The case where the punctures of Ω'_n are unlabeled is easier. Consider T, S in $\mathcal{MF}^s(\Omega'_n)$ and denote v_0 the boundary vertex of Ω'_n. We begin by flipping to increase the valence of v_0 until it is maximal. By Lemma 2.2 this step requires at most $H\log(\kappa(\Omega'_n)) = H\log(3n-2)$. Now, up to homeomorphism, the two triangulations differ only in an $(n+1)$-gon (the shaded area of Figure 3). By Theorem 2.1 the triangulations T and S differ by at most

$$2H\log(3n-2) + K\log(n+1) < 400\log(n)$$

simultaneous flips. We have thus proved the following:

THEOREM 3.4. *Let Ω'_n be a disk with n unlabeled punctures. There exists $B > 0$ such that* $\operatorname{diam}(\mathcal{MF}^s(\Omega'_n)) < A\log(n)$. *The constant A can be taken equal to* 400.

REMARK 3.5. The previous proof applies word-by-word for unlabeled punctured spheres Ω_n. We thus have the following:

THEOREM 3.6. *Let Ω_n be a sphere with n unlabeled punctures. There exists $B > 0$ such that* $\operatorname{diam}(\mathcal{MF}^s(\Omega_n)) < B\log(n)$. *The constant B can be taken equal to* 400.

4. Surfaces with Genus

In this section, we prove our upper bounds in terms of genus.

For technical reasons, we begin by proving a theorem for surfaces of genus g with a single boundary component with a marked point on it.

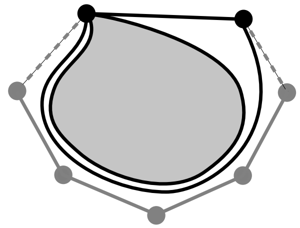


Figure 6 The shaded region is of genus $g - \lfloor \frac{g}{2} \rfloor$

THEOREM 4.1. *Let Γ'_g be a surface of genus g with a single boundary component with a marked point on it. Then*

$$\operatorname{diam}(\mathcal{MF}^s(\Gamma'_g)) < C(\log(g+1))^2,$$

where C can be taken equal to 3,000.

We use the technique introduced in Disarlo–Parlier [5], and before proceeding to the proof, we state two lemmas we will need. Proofs can be found in Disarlo–Parlier [5] (Lemmas 4.4 and 4.5).

LEMMA 4.2. *Let T be a triangulation of Λ, a genus $g \geq 1$ surface with a single boundary curve and k marked points all on the boundary. Then there exists $a \in T$ such that $\Lambda \setminus a$ is connected and of genus $g - 1$.*

LEMMA 4.3. *Let T be a triangulation of Λ, a genus $g \geq 0$ surface with two boundary curves, both with marked points, and all marked points on the boundary. Then there exists $a \in T$ such that $\Lambda \setminus a$ has only one boundary component.*

We can now proceed to the proof of the theorem.

Proof of Theorem 4.1. We begin by checking the result for $g = 1$, namely when the diameter is at most $2{,}000 \log(2) > 5$. Indeed, a one-holed torus has at most five possible triangulations, so the result is true.

Now suppose that $g \geq 2$.

Denote by v the boundary vertex of Γ'_g. Given triangulations S, T of Γ'_g, flip both until the valence of v is maximal and denote by S_v, T_v the triangulations obtained. By Lemma 2.2 each step takes at most $H \log(\kappa(\Gamma'_g))$ flips. Now we proceed as in the proof of Theorem 4.3 in Disarlo–Parlier [5]. We successively apply the previous lemmas to find a collection of $2\lfloor \frac{g}{2} \rfloor$ arcs along which we can cut so that the resulting surface has genus $g - \lfloor \frac{g}{2} \rfloor$ and a single boundary component with $1 + 4\lfloor \frac{g}{2} \rfloor$ arcs. (Note that so far we have not applied a single flip to either S_v or T_v.)

Our aim is now to introduce two special arcs that are essentially parallel to the single boundary of the surface we have obtained by cutting along the arcs (see Figure 6).

We describe the process we will apply to both triangulations S_v, T_v. If we consider the arc a that is a boundary arc of Γ'_g, then note that the arcs b, c we want to introduce form a triangle with a, and both cut Γ'_g into surfaces. The arc b cuts off a surface $\Gamma^{(1)}$ of genus $\lfloor \frac{g}{2} \rfloor$, and c cuts off a surface $\Gamma^{(2)}$ of genus $g - \lfloor \frac{g}{2} \rfloor$. They also have the nice property of intersecting any arc of the triangulation (either S_v or T_v) at most once. In addition, this means that they intersect the full triangulation at most $\kappa(\Gamma'_g)$ times.

We can now use Lemma 2.3 from the preliminaries, which tells us that the arcs can be introduced in at most $L \log(\kappa(\Gamma'_g)+1) = L\log(6g-1)$ moves. We denote by S', T' the new triangulations obtained, both containing the arcs b, c. Denote by $S'_k$and T'_k the restrictions of S' and T' to $\Gamma^{(k)}$ for $k = 1, 2$. Now flip S'_k and T'_k inside $\Gamma^{(k)}$ for $k = 1, 2$. Once the triangulations coincide on both $\Gamma^{(1)}$ and $\Gamma^{(2)}$, they will coincide on Γ'_g.

By induction on g the following holds:

$$\begin{aligned} d(S'_1, T'_1) &\le C \log^2\left(\left\lfloor \frac{g}{2} \right\rfloor + 1\right) \le C \log^2\left(\frac{g}{2} + 1\right), \\ d(S'_2, T'_2) &\le C \log^2\left(g - \left\lfloor \frac{g}{2} \right\rfloor + 1\right) \le C \log^2\left(\frac{g+1}{2} + 1\right). \end{aligned}$$

Putting this all together, we have

$$\begin{aligned} d(S, T) &\le d(S, S') + d(T, T') + \max\{d(S'_1, T'_1), d(S'_2, T'_2)\} \\ &\le 2(L \log(6g-1) + H \log(6g-2)) + C \log^2\left(\frac{g+1}{2} + 1\right) \\ &\le C \log^2(g+1). \end{aligned}$$

A direct computation proves that the last inequality holds for every $g \ge 2$ when C is large enough (for instance, $C = 3{,}000$). □

We can use the previous theorem to show an analogous result for a genus g surface with a single puncture.

THEOREM 4.4. *Let Γ_g be a surface of genus g with a single marked point. Then*

$$\operatorname{diam}(\mathcal{MF}^s(\Gamma_g)) < C(\log(g+1))^2.$$

The constant C can be taken equal to 3,000.

Proof. We argue as in the previous theorem by considering for any triangulation a collection of $2\lfloor \frac{g}{2} \rfloor$ arcs that, when cut along, give a surface of genus $g - \lfloor \frac{g}{2} \rfloor$ with a single boundary component with $4\lfloor \frac{g}{2} \rfloor$ arcs. As in the previous proof, we can introduce an arc that separates the surface into two subsurfaces of genus $\lfloor \frac{g}{2} \rfloor$ and $g - \lfloor \frac{g}{2} \rfloor$. Then we apply the previous theorem to both to obtain the result. □

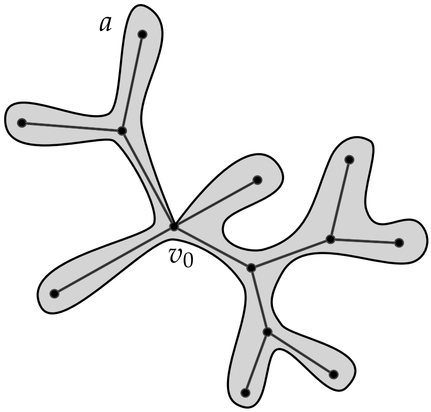


Figure 7 A spanning tree of the vertices and the arc a

5. Hybrid Surfaces

In this section, we prove our most general upper bound, which works for surfaces with marked points and genus.

THEOREM 5.1. *Let $\Sigma_{g,n}$ be a surface of genus g with n labeled marked points. Then*

$$\operatorname{diam}(\mathcal{MF}^s(\Sigma_{g,n})) < D(\log(g+n))^2.$$

The constant D can be taken equal to 4,500.

Proof. Consider a triangulation of $\Sigma := \Sigma_{g,n}$ and a spanning tree of its 1-skeleton. Note that a spanning tree contains exactly $n-1$ arcs. Consider a marked vertex v_0 and the loop a based in v_0 obtained by leaving from v_0 and following the spanning tree along an arc (leaving the spanning tree, say, to the left) and going around the entire tree before returning to v_0 (see Figure 7).

The arc a is separating and leaves the genus to one side and the punctures to other (except for the point v_0, which lies on the arc itself). We claim that it can be introduced in the triangulation in at most $(H+L)\log(\kappa(\Sigma))$ moves.

To do so, we can proceed as follows. Cutting along the arcs of the spanning tree, we find a surface Σ^β of genus g and a single polygonal boundary component β with all marked points now on the boundary. A marked point of degree d in the spanning tree appears on the boundary component β exactly d times, and β is a polygon of $2n-2$ arcs (twice the number of arcs of the spanning tree).

Note that the arc a also lives on Σ^β and is a loop parallel to β with its basepoint a copy of v_0. We now flip the restriction of the triangulation to increase the valence of the basepoint of a until it is maximal. By Lemma 2.2 this step requires at most $H\log(\kappa(\Sigma)-(n-1)) < H\log(\kappa(\Sigma))$ simultaneous flips. The arc a now intersects any arc in the triangulation at most once and thus by Lemma 2.3 can be introduced in at most $L\log(\kappa(\Sigma))$ moves.

This can be done to any triangulation, so now considering two triangulations T and S, we perform such a process on both. The new triangulations obtained, say S' and T', possibly differ in "the genus part" Γ'_g or the "puncture part" Ω'_{n-1}, but by applying Theorems 3.1 and 4.1 we can conclude that they lie at the distance at most

$$\begin{aligned} d(S,T) &\le d(S,S') + d(T,T') + d(S',T') \\ &\le 2(H+L)\log(\kappa(\Sigma)) + \max\{\operatorname{diam}(\mathcal{MF}^s(\Gamma'_g)), \operatorname{diam}(\mathcal{MF}^s(\Omega'_{n-1}))\} \\ &\le 2(H+L)\log(6g+3n-6) + C\log^2(g+n) \\ &\le D\log^2(g+n). \end{aligned}$$

A direct computation proves that the last inequality holds for every g, n such that $g+n \ge 2$, provided that D is large enough (for example, $D = 4{,}500$). □

REMARK 5.2. We can apply the same proof as before to the case of a surface with unlabeled marked points. We have to be careful because in the estimates we are trying to capture the cases where both the genus and number of points are increasing, possibly at different rates. Our previous upper bounds for spheres with unlabeled marked points grow like $\log(n)$; in combination with the given proof, this implies that for fixed genus, we can again obtain an upper bound on order $\log(n)$ with an additive constant that depends on the genus. Again, all constants can be made explicit, but for simplicity, we do not discuss this in detail.

6. Lower Bounds and Further Questions

For surfaces with genus and labeled marked points, our upper bounds grow roughly like $(\log(\kappa))^2$ in the arc complexity κ of the surface. It is not clear whether this order of growth is optimal.

An immediate lower bound can be deduced from known bounds on the diameters of the usual flip graphs. In those cases, lower (and upper) bounds are known to grow like $g\log(g) + n\log(n)$ (see Theorem 1.4 and Corollary 4.19 of [5]). As at most a linear number of flips in terms of the complexity can be performed simultaneously, this implies a lower bound of order $\log(\kappa)$. The counting argument used to provide this bound is pretty simple, especially compared to our upper bounds, and it does not seem particularly adapted to simultaneous flips. In terms of unlabeled marked points, the lower bound on the order of growth is also $\log(\kappa)$. It seems surprising that there is no difference in order of growth between labeled and unlabelled marked points. All of these things seem to indicate that a better lower bound may be achievable.

On the other hand, there are some indications that an upper bound of order $\log(\kappa)$ may be possible. A seemingly related problem to estimating distances in the flip graph is the problem of estimating the distances between 3-regular graphs using Whitehead moves. These graphs are dual to a triangulation, and a flip on a triangulation corresponds to a Whitehead move. Triangulations are really different though; first of all, they really correspond to ribbon graphs and not to 3-regular graphs. Second, only certain Whitehead moves on a 3-regular graph can be emulated by flips. In particular, it is not possible to deduce results about flip distances from estimates on Whitehead moves or vice versa. Although the relationship is not direct, there have been a number of recent results that seem to indicate similar behaviors. The $\kappa \log(\kappa)$ behavior discussed previously for modular flip graphs is also present for Whitehead moves on graphs (see, for instance, [2; 3]). Simultaneous flip moves are thus related to simultaneous Whitehead moves, and Rafi and Tao have shown that the growth for graphs behaves like $\log(\kappa)$. This seems to indicate that perhaps our upper bounds may be improvable. A further indication that this order of growth may be correct are the results in [1], which were among the tools needed for our upper bounds.

In short, we now know that the rough behavior in terms of either the genus or number of labeled marked points is bounded below and above by a function of type $\log(\kappa)^\alpha$ for $\alpha \in [1, 2]$, and determining the exact behavior seems to be an interesting problem.

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