Sparse Domination Theorem for Multilinear Singular Integral Operators with L^r-Hörmander Condition

KANGWEI LI

ABSTRACT. In this note, we show that if T is a multilinear singular integral operator associated with a kernel satisfies the so-called multilinear L^r-Hörmander condition, then T can be dominated by multilinear sparse operators.

1. Introduction and Main Results

This note is devoted to obtain a sparse domination formula for a class of multilinear singular integral operators. Dominating Calderón–Zygmund operators by sparse operators starts from Lerner [19], who obtained the following result:

$$\|Tf(x)\|_X \leq C_T \sup_{\mathcal{D},\mathcal{S}} \left\| \sum_{Q \in \mathcal{S}} \left(\fint_Q |f| \right) \chi_Q(x) \right\|_X,$$

where the supremum is taken over all the sparse families $\mathcal{S} \subset \mathcal{D}$ (see further for the definition) and all the dyadic grids $\mathcal{D}$, and X is an arbitrary Banach function space. Then the A_2 theorem (Hytönen [12]) follows as an easy consequence.

   Later, this result was refined by a pointwise control independently and simultaneously by Conde-Alonso and Rey [6] and Lerner and Nazarov [21]. All the results mentioned require the kernel satisfying the log-Dini condition. Finally, Lacey [17] relaxed the log-Dini condition to the Dini condition, and then Hytönen, Roncal, and Tapiola [14] refined the proof by tracking the precise dependence on the constants. Very recently, Lerner [20] also provided a new proof for this result, which also works for some more general operators. To be precise, Lerner showed the following result.

THEOREM A ([20, Thm. 4.2]). *Let T be a sublinear operator of weak type (q, q) such that the corresponding grand maximal truncated operator $\mathcal{M}_T$ is of weak type (r, r), where $1 \leq q \leq r < \infty$. Then, for every compactly supported*

$f \in L^r(\mathbb{R}^n)$, *there exists a sparse family* $\mathcal{S}$ *such that for a.e.* $x \in \mathbb{R}^n$,

$$|T(f)(x)| \le c_{n,q,r}(\|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \to L^{r,\infty}}) \sum_{Q \in \mathcal{S}} \left(\fint_Q |f|^r \right)^{\frac{1}{r}} \chi_Q(x).$$

Let us recall the notations in this result. We say that $\mathcal{S}$ is a sparse family if for all cubes $Q \in \mathcal{S}$, there exist $E_Q \subset Q$ that are pairwise disjoint and $|E_Q| \ge \gamma |Q|$, where $0 < \gamma < 1$. For a given operator T, the so-called "grand maximal truncated" operator $\mathcal{M}_T$ is defined by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \operatorname{ess\ sup}_{\xi \in Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|.$$

It is shown in [20] that Calderón-Zygmund operators with Dini continuous kernel satisfy the assumption in Theorem A with $q = r = 1$. In this short note, we give a multilinear analogue of Theorem A. Then as an application, we give a sparse domination formula for multilinear singular integral operators with kernels $K(x, y_1, \ldots, y_m)$ satisfying the so-called m-linear L^r-Hörmander condition

$$K_r := \sup_Q \sup_{x,z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^{\frac{m}{r}} \left(\int_{(2^k Q)^m \setminus (2^{k-1} Q)^m} |K(x, y_1, \ldots, y_m) \right.$$
$$\left. - K(z, y_1, \ldots, y_m)|^{r'} \, d\vec{y} \right)^{\frac{1}{r'}} < \infty,$$

where $Q^m = \underbrace{Q \times \cdots \times Q}_{m \text{ times}}$ and $1 \le r < \infty$. When $r = 1$, this formula is understood as

$$K_1 := \sup_Q \sup_{x,z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^m \operatorname{ess\ sup}_{\vec{y} \in (2^k Q)^m \setminus (2^{k-1} Q)^m} |K(x, y_1, \ldots, y_m)$$
$$- K(z, y_1, \ldots, y_m)| < \infty.$$

Bernicot, Frey, and Petermichl [1] showed that a large class of singular non-integral operators can be dominated by sparse operators (in fact, in the bilinear form sense). Even in the linear case, the L^r-Hörmander condition is beyond the "off-diagonal estimate" assumption in [1]. We also remark that our assumption is weaker than assumption (H2) used in [2] (see Proposition 3.2). It is also easy to see that our assumption is weaker than the Dini condition used in [9] (see Proposition 3.1).

Now to state our main result, we need a multilinear analogue of grand maximal truncated operator. Given an operator T, define

$$\mathcal{M}_T(f_1, \ldots, f_m)(x)$$
$$= \sup_{Q \ni x} \operatorname{ess\ sup}_{\xi \in Q} |T(f_1, \ldots, f_m)(\xi) - T(f_1\chi_{3Q}, \ldots, f_m\chi_{3Q})(\xi)|.$$

Note that we do not require T to be multisublinear. Given a cube Q_0, for $x \in Q_0$, we also define a local version of $\mathcal{M}_T$ by

$$
\begin{aligned}
\mathcal{M}_{T,Q_0}&(f_1, \ldots, f_m)(x) \\
&= \sup_{Q \ni x, Q \subset Q_0} \operatorname*{ess\,sup}_{\xi \in Q} |T(f_1 \chi_{3Q_0}, \ldots, f_m \chi_{3Q_0})(\xi) \\
&\qquad - T(f_1 \chi_{3Q}, \ldots, f_m \chi_{3Q})(\xi)|.
\end{aligned}
$$

Our first result reads as follows.

THEOREM 1.1. *Assume that T is bounded from $L^q \times \cdots \times L^q$ to $L^{q/m,\infty}$ and $\mathcal{M}_T$ is bounded from $L^r \times \cdots \times L^r$ to $L^{r/m,\infty}$, where $1 \le q \le r < \infty$. Then, for compactly supported functions $f_i \in L^r(\mathbb{R}^n)$, $i = 1, \ldots, m$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$
\begin{aligned}
|T(f_1, &\ldots, f_m)(x)| \\
&\le c_{n,q,r}(\|T\|_{L^q \times \cdots \times L^q \to L^{q/m,\infty}} + \|\mathcal{M}_T\|_{L^r \times \cdots \times L^r \to L^{r/m,\infty}}) \\
&\quad \times \sum_{Q \in \mathcal{S}} \prod_{i=1}^{m} \left(\fint_Q |f_i|^r \right)^{\frac{1}{r}} \chi_Q(x).
\end{aligned}
$$

As a consequence, we show the following:

THEOREM 1.2. *Let T be a multilinear singular integral operator that is bounded from $L^r \times \cdots \times L^r$ to $L^{r/m,\infty}$ and suppose that its kernel satisfies the m-linear L^r-Hörmander condition. Then, for compactly supported functions $f_i \in L^r(\mathbb{R}^n)$, $i = 1, \ldots, m$, there exists a sparse family $\mathcal{S}$ such that for a.e. $x \in \mathbb{R}^n$,*

$$
|T(f_1, \ldots, f_m)(x)| \le c_{n,r}(\|T\|_{L^r \times \cdots \times L^r \to L^{r/m,\infty}} + K_r)
$$
$$
\times \sum_{Q \in \mathcal{S}} \prod_{i=1}^{m} \left(\fint_Q |f_i|^r \right)^{\frac{1}{r}} \chi_Q(x).
$$

We would like to remark that recently a different approach to sparse domination of singular integrals not relying on weak endpoint bounds for grand maximal functions has been developed by many authors; see [5; 8; 15; 18].

In the next section, we give a proof for Theorems 1.1 and 1.2. In Section 3, we give some remarks about the m-linear L^r-Hörmander condition.

2. Proof of Theorems 1.1 and 1.2

The proof of Theorem 1.1 will follow the same idea used in [20], but with proper changes for the multilinear case. For simplicity, we only prove both results in the case of $m = 2$, and the general case can be proved similarly. First, we prove the following lemma.

LEMMA 2.1. *Let T be bounded from $L^q \times L^q \to L^{q/2,\infty}$, $1 \le q < \infty$. Then for a.e. $x \in Q_0$,*

$$|T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)|$$
$$\le c_n \|T\|_{L^q \times L^q \to L^{\frac{q}{2},\infty}} |f_1(x) f_2(x)| + \mathcal{M}_{T,Q_0}(f_1, f_2)(x).$$

Proof. Suppose that $x \in \text{int } Q_0$ and let x be a point of approximate continuity of $T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})$ (see e.g. [10, p. 46]). Then for every $\epsilon > 0$, the sets

$$E_s(x) := \{y \in B(x, s) : |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(y) - T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)| < \epsilon\}$$

satisfy $\lim_{s \to 0} \frac{|E_s(x)|}{|B(x,s)|} = 1$. Denote by $Q(x, s)$ the smallest cube centered at x and containing $B(x, s)$. Let $s > 0$ be so small that $Q(x, s) \subset Q_0$. Then for a.e. $y \in E_s(x)$,

$$|T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)|$$
$$< |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(y)| + \epsilon$$
$$\le |T(f_1 \chi_{3Q(x,s)}, f_2 \chi_{3Q(x,s)})(y)|$$
$$+ \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \epsilon.$$

It follows that

$$|T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)|$$
$$\le \operatorname*{ess\,inf}_{y \in E_s(x)} |T(f_1 \chi_{3Q(x,s)}, f_2 \chi_{3Q(x,s)})(y)|$$
$$+ \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \epsilon$$
$$\le |E_s(x)|^{-\frac{2}{q}} \|T(f_1 \chi_{3Q(x,s)}, f_2 \chi_{3Q(x,s)})\|_{L^{q/2,\infty}}$$
$$+ \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \epsilon$$
$$\le \|T\|_{L^q \times L^q \to L^{\frac{q}{2},\infty}} \frac{1}{|E_s(x)|^{2/q}} \prod_{i=1}^{2} \left(\int_{3Q(x,s)} |f_i(y)|^q \, dy \right)^{\frac{1}{q}}$$
$$+ \mathcal{M}_{T,Q_0}(f_1, f_2)(x) + \epsilon.$$

Assuming additionally that x is a Lebesgue point of $|f_1|^q$ and $|f_2|^q$ and letting subsequently $s \to 0$ and $\epsilon \to 0$ will conclude the proof. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix a cube $Q_0 \subset \mathbb{R}^n$. We shall prove the following recursive inequality:

$$|T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)| \chi_{Q_0}$$
$$\le c_{n,T} \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}} + \sum_j |T(f_1 \chi_{3P_j}, f_2 \chi_{3P_j})(x)| \chi_{P_j}, \qquad (2.1)$$

where P_j are disjoint dyadic subcubes of Q_0, that is, $P_j \in \mathcal{D}(Q_0)$, and moreover $\sum_j |P_j| \le \frac{1}{2} |Q_0|$. Observe that for arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

we have

$$|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0}$$

$$= |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\bigcup_j P_j} + \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{P_j}$$

$$\leq |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\bigcup_j P_j} + \sum_j |T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}$$

$$+ \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}.$$

Hence, to prove the recursive claim, it suffices to show that we can select pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and such that for a.e. $x \in Q_0$,

$$\sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}$$

$$+ |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0\setminus\bigcup_j P_j} \leq c_{n,T}\langle|f_1|^r\rangle_{3Q_0}^{\frac{1}{r}}\langle|f_2|^r\rangle_{3Q_0}^{\frac{1}{r}}.$$

By our assumption, $\mathcal{M}_T$ is bounded from $L^r \times L^r$ to $L^{r/2,\infty}$. Therefore there is a sufficiently large c_n such that the set

$$E := \{x \in Q_0 : |f_1(x)f_2(x)| > c_n\langle|f_1|^r\rangle_{3Q_0}^{\frac{1}{r}}\langle|f_2|^r\rangle_{3Q_0}^{\frac{1}{r}}\}$$

$$\cup \{x \in Q_0 :$$

$$\mathcal{M}_{T,Q_0}(f_1, f_2)(x) > c_n\|\mathcal{M}_T\|_{L^r \times L^r \to L^{r/2,\infty}}\langle|f_1|^r\rangle_{3Q_0}^{\frac{1}{r}}\langle|f_2|^r\rangle_{3Q_0}^{\frac{1}{r}}\}$$

will satisfy $|E| \leq \frac{1}{2^{n+2}}|Q_0|$. The Calderón–Zygmund decomposition applied to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| < \frac{1}{2}|P_j|$$

and $|E \setminus \bigcup_j P_j| = 0$. It follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap E^c \neq \emptyset$. Therefore

$$\operatorname*{ess\,sup}_{\xi \in P_j} |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(\xi) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(\xi)|$$

$$\leq c_n\|\mathcal{M}_T\|_{L^r \times L^r \to L^{r/2,\infty}}\langle|f_1|^r\rangle_{3Q_0}^{\frac{1}{r}}\langle|f_2|^r\rangle_{3Q_0}^{\frac{1}{r}}.$$

On the other hand, by Lemma 2.1, for a.e. $x \in Q_0 \setminus \bigcup_j P_j$, we have

$$|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|$$

$$\leq c_n(\|T\|_{L^q \times L^q \to L^{q/2,\infty}} + \|\mathcal{M}_T\|_{L^r \times L^r \to L^{r/2,\infty}})\langle|f_1|^r\rangle_{3Q_0}^{\frac{1}{r}}\langle|f_2|^r\rangle_{3Q_0}^{\frac{1}{r}}.$$

Therefore, combining the estimates, we arrive at (2.1) with

$$c_{n,T} \approx \|T\|_{L^q \times L^q \to L^{q/2,\infty}} + \|\mathcal{M}_T\|_{L^r \times L^r \to L^{r/2,\infty}}.$$

Now with (2.1), the rest of the argument is the same as that in [20], and we complete the proof. $\qquad\square$

Next, we turn to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove that $\mathcal{M}_T$ is bounded from $L^r \times L^r$ to $L^{r/2,\infty}$. Indeed, let $x, x', \xi \in Q \subset \frac{1}{2} \cdot 3Q$. We have

$$|T(f_1, f_2)(\xi) - T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(\xi)|$$

$$= \left| \iint_{(\mathbb{R}^n)^2 \backslash (3Q)^2} K(\xi, y_1, y_2) f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right|$$

$$\leq \left| \iint_{(\mathbb{R}^n)^2 \backslash (3Q)^2} (K(\xi, y_1, y_2) - K(x', y_1, y_2)) f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right|$$

$$+ |T(f_1, f_2)(x')| + |T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(x')|. \tag{2.2}$$

By the bilinear L^r-Hörmander condition we have

$$\left| \iint_{(\mathbb{R}^n)^2 \backslash (3Q)^2} (K(\xi, y_1, y_2) - K(x', y_1, y_2)) f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \right|$$

$$\leq \sum_{k=1}^{\infty} \iint_{(2^k 3Q)^2 \backslash (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) - K(x', y_1, y_2)|$$

$$\cdot |f_1(y_1)| \cdot |f_2(y_2)| \, dy_1 \, dy_2$$

$$\leq \sum_{k=1}^{\infty} \left(\iint_{(2^k 3Q)^2 \backslash (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) - K(x', y_1, y_2)|^{r'} \, dy_1 \, dy_2 \right)^{\frac{1}{r'}}$$

$$\times \left(\iint_{(2^k 3Q)^2} |f_1(y_1) f_2(y_2)|^r \, dy_1 \, dy_2 \right)^{\frac{1}{r}}$$

$$= \sum_{k=1}^{\infty} |2^k 3Q|^{\frac{2}{r}} \left(\iint_{(2^k 3Q)^2 \backslash (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) \right.$$

$$- K(x', y_1, y_2)|^{r'} \, dy_1 \, dy_2 \Big)^{\frac{1}{r'}}$$

$$\times \left(\frac{1}{|2^k 3Q|^2} \iint_{(2^k 3Q)^2} |f_1(y_1) f_2(y_2)|^r \, dy_1 \, dy_2 \right)^{\frac{1}{r}}$$

$$\leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}.$$

Then, taking $L^{r/4}$ average over $x' \in Q$ on both sides of (2.2), we obtain

$$|T(f_1, f_2)(\xi) - T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(\xi)|$$

$$\leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + \left(\frac{1}{|Q|} \int_Q |T(f_1, f_2)(x')|^{\frac{r}{4}} \, dx' \right)^{\frac{4}{r}}$$

$$+ \left(\frac{1}{|Q|} \int_Q |T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(x')|^{\frac{r}{4}} \, dx' \right)^{\frac{4}{r}}$$

$$\leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + M_{r/4}(T(f_1, f_2))(x)$$

$$+ c_r \|T(f_1 \chi_{3Q}, f_2 \chi_{3Q})\|_{L^{\frac{r}{2},\infty}(Q, \frac{dx'}{|Q|})}$$

$$\leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + M_{r/4}(T(f_1, f_2))(x)$$

$$+ c_r \|T\|_{L^r \times L^r \to L^{r/2,\infty}} \left(\frac{1}{|Q|} \int_{3Q} |f_1|^r \right)^{\frac{1}{r}} \cdot \left(\frac{1}{|Q|} \int_{3Q} |f_2|^r \right)^{\frac{1}{r}}$$

$$\leq (K_r + c_{n,r} \|T\|_{L^r \times L^r \to L^{r/2,\infty}})(\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}$$

$$+ M_{r/4}(T(f_1, f_2))(x),$$

where the bilinear maximal function $\mathcal{M}$ is defined as

$$\mathcal{M}(f, g)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f_1| \right) \left(\frac{1}{|Q|} \int_Q |f_2| \right).$$

So we conclude that

$$\mathcal{M}_T(f_1, f_2)(x) \leq (K_r + c_{n,r} \|T\|_{L^r \times L^r \to L^{r/2,\infty}})(\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}$$

$$+ M_{r/4}(T(f_1, f_2))(x).$$

It is obvious that $(\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}$ is bounded from $L^r \times L^r$ to $L^{r/2,\infty}$. On the other hand, since it is well known that $M_{r/4}$ is bounded from $L^{r/2,\infty}$ to $L^{r/2,\infty}$, by the assumption that T is bounded from $L^r \times L^r$ to $L^{r/2,\infty}$ we obtain

$$\|M_{r/4}(T(f_1, f_2))\|_{L^{r/2,\infty}} \leq c_{n,r} \|T\|_{L^r \times L^r \to L^{r/2,\infty}} \|f_1\|_{L^r} \|f_2\|_{L^r}.$$

Therefore $\mathcal{M}_T$ is bounded from $L^r \times L^r \to L^{r/2,\infty}$, and the desired result follows from Theorem 1.1. $\qquad\square$

3. Some Remarks

In this section, we give some remarks about the L^r-Hörmander condition and some applications of our main result. It is well known that the Hörmander condition

$$\sup_{x,z \in \mathbb{R}^n} \int_{|y-x|>2|x-z|} |K(x, y) - K(z, y)| \, dy < \infty$$

is not sufficient for

$$\int_{\mathbb{R}^n} T(f)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} M_r(f)^p w(x) \, dx, \quad w \in A_\infty, \qquad (3.1)$$

for any $r \geq 1$ and $0 < p < \infty$ (see [25, Thm. 3.1]). So we cannot expect the sparse domination theorem for singular integral operators with this kernel. This is because once we have

$$|T(f)(x)| \leq C_{T,r} \sum_{Q \in \mathcal{S}} \left(\fint_Q |f|^r \right)^{\frac{1}{r}} \chi_Q(x)$$

for some $r \geq 1$, then (3.1) holds for $p = 1$ (see [13, Lemma 4.1]) and therefore for all $0 < p < \infty$ (see [7, Thm. 1.1]). Then it is reasonable to consider somewhat stronger condition such as our L^r-Hörmander condition. For more background, see [16; 25; 26].

Now we shall briefly show that our conditions are weaker than the Dini condition, which is used in [9] by Damián, Hormozi, and the author. Recall that the Dini condition is defined by

$$
\begin{aligned}
|K(x + h, y_1, \dots, y_m) &- K(x, y_1, \dots, y_m)| \\
&+ |K(x, y_1 + h, \dots, y_m) - K(x, y_1, \dots, y_m)| + \cdots \\
&+ |K(x, y_1, \dots, y_m + h) - K(x, y_1, \dots, y_m)| \\
&\leq \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^{2n}} \omega \left(\frac{|h|}{\sum_{i=1}^{m} |x - y_i|} \right)
\end{aligned}
$$

whenever $|h| \leq \frac{1}{2} \max\{|x - y_i| : i = 1, \dots, m\}$, where ω is increasing, $\omega(0) = 0$, and $\|\omega\|_{\mathrm{Dini}} = \int_0^1 w(t) \, dt / t < \infty$.

PROPOSITION 3.1. *The m-linear Dini condition implies the m-linear L^r-Hörmander condition.*

Proof. Again, we just prove the case $m = 2$. It is obvious that we just require regularity in the x variable. Fix $x, z \in \frac{1}{2} Q$. Since $|x - z| < \frac{1}{2} \sqrt{n} \ell(Q)$, for $k > \log_2(1 + 4\sqrt{n})$ and $(y_1, y_2) \in (2^k Q)^2 \setminus (2^{k-1} Q)^2$, we have $|x - z| \leq \frac{1}{2} \max\{|x - y_1|, |x - y_2|\}$. Therefore

$$
\begin{aligned}
\sum_{k > \log_2(1 + 4\sqrt{n})} &|2^k Q|^{\frac{2}{r}} \left(\int_{(2^k Q)^2 \setminus (2^{k-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, d\bar{y} \right)^{\frac{1}{r'}} \\
&\leq \sum_{k > \log_2(1 + 4\sqrt{n})} |2^k Q|^{\frac{2}{r}} \cdot w\left(\frac{2\sqrt{n}}{2^k - 1} \right) \cdot \frac{1}{(2^{k-2} - \frac{1}{4})^{2n} \ell(Q)^{2n}} \cdot |2^k Q|^{\frac{2}{r'}} \\
&\lesssim_n \sum_{k > \log_2(1 + 4\sqrt{n})} w\left(\frac{4\sqrt{n}}{2^k} \right) \lesssim_n \|\omega\|_{\mathrm{Dini}}.
\end{aligned}
$$

It remains to consider $1 \leq k \leq \log_2(1 + 4\sqrt{n})$. We should be careful because we do not assume any size condition. In this case, since

$$
\max\{|y - y_1|, |y - y_2|\} \geq \frac{1}{4} \ell(Q), \quad \forall y \in \frac{1}{2} Q \text{ and } (y_1, y_2) \in (2^k Q)^2 \setminus (2^{k-1} Q)^2,
$$

we select $4\lceil \sqrt{n} \rceil$ points $x_1, \dots, x_{4\lceil \sqrt{n} \rceil}$ in the segment between x and z such that

$$
|x - x_1|, |x_i - x_{i+1}|, |x_{4\lceil \sqrt{n} \rceil} - z| \leq \frac{1}{8} \ell(Q), \quad i = 1, \dots, 4\lceil \sqrt{n} \rceil - 1.
$$

For convenience, denote $x_0 = x$ and $x_{4\lceil \sqrt{n} \rceil + 1} = z$. Then we have

$$
|K(x, y_1, y_2) - K(z, y_1, y_2)| \leq \sum_{i=0}^{4\lceil \sqrt{n} \rceil} |K(x_i, y_1, y_2) - K(x_{i+1}, y_1, y_2)|
$$

$$
\leq c_n \omega\left(\frac{1}{2} \right) \ell(Q)^{-2n}.
$$

Consequently,

$$\sum_{1 \le k \le \log_2(1+4\sqrt{n})} |2^k Q|^{\frac{2}{r}} \left(\int_{(2^k Q)^2 \setminus (2^{k-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$\lesssim_n \omega\left(\frac{1}{2}\right) \lesssim \|\omega\|_{\text{Dini}}.$$

This completes the proof. □

Next, we will show that the L^r-Hörmander condition is also weaker than the regularity assumption used in [2] (which was originally introduced in [3]). Recall that the regularity assumption in [2] reads as follows.

(**H2**): There exists $\delta > n/r$ such that

$$\left(\int_{S_{j_m}(Q)} \cdots \int_{S_{j_1}(Q)} |K(x, y_1, \ldots, y_m) - K(z, y_1, \ldots, y_m)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$\le C \frac{|x - z|^{m(\delta - n/r)}}{|Q|^{m\delta/n}} 2^{-m\delta j_0}$$

for all cubes Q, all $x, z \in \frac{1}{2}Q$, and all $(j_1, \ldots, j_m) \ne (0, \ldots, 0)$, where $j_0 = \max\{j_i\}_{1 \le i \le m}$, and $S_j(Q) = 2^j Q \setminus 2^{j-1} Q$ if $j \ge 1$ and, otherwise, $S_j(Q) = Q$.

PROPOSITION 3.2. *Assumption* (H2) *implies the m-linear L^r-Hörmander condition.*

Proof. Again, we just prove the bilinear case. Observe that

$$(2^j Q)^2 \setminus (2^{j-1} Q)^2$$
$$= (2^j Q \setminus 2^{j-1} Q)^2 \cup (2^j Q \times (2^j Q \setminus 2^{j-1} Q)) \cup ((2^j Q \setminus 2^{j-1} Q) \times 2^j Q)$$
$$= (S_j(Q))^2 \cup \left(\bigcup_{l \le j} S_l(Q) \times S_j(Q) \right) \cup \left(\bigcup_{l \le j} S_j(Q) \times S_l(Q) \right).$$

By the triangle inequality we have

$$\sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \left(\int_{(2^j Q)^2 \setminus (2^{j-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$\le \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \left(\int_{(S_j(Q))^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$+ \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \sum_{l=0}^{j} \left(\int_{S_j(Q) \times S_l(Q)} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$+ \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \sum_{l=0}^{j} \left(\int_{S_l(Q) \times S_j(Q)} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} \, \mathrm{d}\vec{y} \right)^{\frac{1}{r'}}$$

$$\lesssim \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \frac{1}{|Q|^{2/r}} 2^{-2\delta j}(1+j)$$
$$< \infty,$$

where the last inequality holds since $\delta > n/r$. This completes the proof. $\square$

REMARK 3.3. For $r > 1$, let T be a linear Fourier multiplier with Hörmander condition with parameter $n/2 < s < n$ (see (3.2) for the definition). It is shown in [16, Thm. 3] that T is not bounded on $L^p(w)$ for some $w \in A_p$ when $p < n/s$ or $p > (\frac{n}{s})'$. This means that (H2) (which is a consequence of the assumption described as before; see [2]) and therefore the L^r-Hörmander condition are not sufficient for the Dini condition.

For $r = 1$, recall that we only need the regularity on the x-variable; in this sense, the L^1-Hörmander condition is strictly weaker than the Dini condition. However, the full regularity in the Dini condition is to ensure the weak endpoint boundedness. In fact, we can show that there is only a tiny difference between the L^1-Hörmander and Dini conditions in the x-variable. To see this, define

$$\omega^{x,z}(t) := \sup_{\vec{y}: \frac{t}{2} \leq \frac{|x-z|}{\sum_{i=1}^{2} |x-y_i|} \leq t} |K(x, y_1, y_2) - K(z, y_1, y_2)| \left(\sum_{i=1}^{2} |x - y_i| \right)^{2n}.$$

Then

$$\sup_{x,z} \sum_{k=1}^{\infty} \omega^{x,z}(2^{-k})$$

$$\lesssim \sup_{Q} \sup_{x,z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^2 \operatorname*{ess\,sup}_{\vec{y} \in (2^k Q)^2 \setminus (2^{k-1}Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|$$

$$\leq K_1,$$

where $\frac{1}{2}Q$ is a cube containing both x and z with $\ell(\frac{1}{2}Q) = \|x - z\|_\infty$. However, the Dini condition in the x-variable can be written as

$$\sum_{k=1}^{\infty} \sup_{x,z} \omega^{x,z}(2^{-k}) < \infty.$$

It is hard to find an example to differentiate these two conditions.

REMARK 3.4. We claim that actually the L^r-Hörmander condition is strictly weaker than (H2). Indeed, (H2) is essentially of Hölder type, whereas the L^r-Hörmander condition is essentially of Dini type. To prove our claim, we borrow the example from [25]. To make things easier we only consider the linear case in one dimension. Define

$$K(x) = |x - 4|^{-\frac{1}{r'}} \left(\log \frac{e}{|x-4|} \right)^{-\frac{1+\beta}{r'}} \chi_{\{3 < x < 5\}}(x).$$

It is easy to check that $K \in L^{r'} \cap L^1$. Then $T : f \to K * f$ is bounded on L^p for all $1 \leq p \leq \infty$. It is already proved in [25] that K satisfies the L^r-Hörmander condition. Define

$$K_\ell(x) = \begin{cases} K(x), & x \in \bigcup_{k=0}^{2^{\ell+1}-1}(3+\frac{k}{2^\ell}, 3+\frac{3k+1}{3\cdot2^\ell}], \\ 0 & \text{otherwise.} \end{cases}$$

Similar argument as that in [25] shows that K_ℓ satisfies the L^r-Hörmander condition uniformly, that is, $\sup_\ell (K_\ell)_r < \infty$. Let $x = 0$, $z = 2^{-\ell-1}$, and $I_\ell = [0, 2^{-\ell})$. We need to analyze

$$\left(\int_{S_j(I_\ell)} |K_\ell(y-x) - K_\ell(y-z)|^{r'} \, dy\right)^{\frac{1}{r'}}.$$

Observe that only $j = \ell+2$ and $j = \ell+3$ are nonzero terms. We have

$$\left(\int_{2^{\ell+2}I_\ell \backslash 2^{\ell+1}I_\ell} |K_\ell(y) - K_\ell(y - 2^{-\ell-1})|^{r'} \, dy\right)^{\frac{1}{r'}}$$

$$\geq \left(\sum_{k=0}^{2^\ell-1} \int_{3+\frac{k}{2^\ell}}^{3+\frac{3k+1}{3\cdot2^\ell}} |K_\ell(y)|^{r'} \, dy\right)^{\frac{1}{r'}}$$

$$\gtrsim \|K\|_{L^{r'}} \gtrsim 2^{(\delta-\frac{1}{r})\ell} \|K\|_{L^{r'}} \frac{|x-z|^{(\delta-\frac{1}{r})}}{|I_\ell|^\delta} 2^{-(\ell+2)\delta}.$$

This shows that there exists a sequence of operators $T_\ell : f \to K_\ell * f$ whose kernels satisfy the L^r-Hörmander condition uniformly. However, since $\delta > \frac{1}{r}$, the constant C in (H2) tends to infinity as $\ell \to \infty$, which means that these two conditions cannot be equivalent. In other words, the L^r-Hörmander condition is strictly weaker than (H2).

Finally, we give an application of our result for multilinear Fourier multipliers

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} a(\vec{y}) e^{2\pi i x \cdot (\sum_{i=1}^m y_i)} \prod_{i=1}^m \widehat{f}(y_i) \, d\vec{y}.$$

It is shown in [3] that the multilinear Mihlin condition implies assumption (H2). However, for the multilinear Hörmander condition ([11]), that is,

$$\sup_{R>0} \|a(R\xi)\chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{mn})} < \infty, \quad \frac{mn}{2} < s \leq mn, \tag{3.2}$$

which is weaker than the multilinear Mihlin condition, it is unknown. Very recently, Chaffee, Torres, and Wu [4] showed that (3.2) implies the multilinear L^r-Hörmander condition with $r = mn/s$. Therefore, Theorem 1.2 also applies to multilinear Fourier multipliers whose symbols satisfy (3.2).

As a consequence of our sparse domination theorem, we can give quantitative weighted bounds for them. We have the following:

THEOREM 3.5. *Let T satisfy the assumption in Theorem* 1.2. *Then*

$$\|T\|_{L^{p_1}(w_1)\times\cdots\times L^{p_m}(w_m)\to L^p(v_{\vec{w}})} \le c_{m,n,T,\vec{p}}[\vec{w}]_{A_{\vec{P}/r}}^{\max\{1,\max_i \frac{(p_i/r)'}{p}\}},$$

where

$$[\vec{w}]_{A_{\vec{P}/r}} := \sup_Q \left(\fint_Q \prod_{i=1}^m w_i^{\frac{p}{p_i}} \right) \prod_{i=1}^m \left(\fint_Q w_i^{-\frac{r}{p_i-r}} \right)^{\frac{p(p_i-r)}{p_i r}}.$$

For the proof, we refer the readers to [2]. We can also follow the maximal function trick used in [9] and then utilize the result in [22]. We can also obtain the $A_p - A_\infty$ type bounds; see [23; 9] for details. As we have discussed, all these estimates apply to the multilinear Fourier multipliers with symbols satisfying (3.2). Notice that the qualitative result was obtained by the author and Sun in [24].

References

[1] F. Bernicot, D. Frey, and S. Petermichl, *Sharp weighted norm estimates beyond Calderón–Zygmund theory,* Anal. PDE 9 (2016), 1079–1113.

[2] T. A. Bui, J. Conde-Alonso, X. T. Duong, and M. Hormozi, *A note on weighted bounds for singular operators with nonsmooth kernels,* Studia Math. 236 (2017), 245–269.

[3] T. A. Bui and X. T. Duong, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers,* Bull. Sci. Math. 137 (2013), no. 1, 63–75.

[4] L. Chaffee, R. Torres, and X. Wu, *Multilinear weighted norm inequalities under integral type regularity conditions,* Harmonic Analysis, Partial Differential Equations and Applications, Part of the series Applied and Numerical Harmonic Analysis, pp. 193–216.

[5] J. Conde-Alonso, A. Culiuc, F. Di Plinio, and Y. Ou, *A sparse domination principle for rough singular integrals,* Anal. PDE 10 (2017), 1255–1284.

[6] J. Conde-Alonso and G. Rey, *A pointwise estimate for positive dyadic shifts and some applications,* Math. Ann. 365 (2016), 1111–1135.

[7] D. Cruz-Uribe, J. M. Martell, and C. Pérez, *Extrapolation from A_∞ weights and applications,* J. Funct. Anal. 213 (2004), 412–439.

[8] A. Culiuc, F. Di Plinio, and Y. Ou, *Domination of multilinear singular integrals by positive sparse forms,* preprint, available at arXiv:1603.05317.

[9] W. Damián, M. Hormozi, and K. Li, *New bounds for bilinear Calderón–Zygmund operators and applications,* Rev. Mat. Iberoam. (to appear), available at arXiv:1512.02400.

[10] L. Evans and R. Gariepy, *Measure theory and fine properties of functions,* Stud. Adv. Math., viii+268 pp., CRC Press, Boca Raton, FL, 1992.

[11] L. Grafakos and Z. Si, *The Hörmander type multiplier theorem for multilinear operators,* J. Reine Angew. Math. 668 (2012), 133–147.

[12] T. Hytönen, *The sharp weighted bound for general Calderón–Zygmund operators,* Ann. of Math. (2) 175 (2012), no. 3, 1473–1506.

[13] T. Hytönen and C. Pérez, *The $L(\log L)^\epsilon$ endpoint estimate for maximal singular integral operators,* J. Math. Anal. Appl. 428 (2015), 605–626.

[14] T. Hytönen, L. Roncal, and O. Tapiola, *Quantitative weighted estimates for rough homogeneous singular integrals,* Israel J. Math. 218 (2017), 133–164.

[15] B. Krause and M. Lacey, *Sparse bounds for random discrete Carleson theorems,* preprint, available at arXiv:1609.08701.

[16] D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers,* Trans. Amer. Math. Soc. 255 (1979), 343–362.

[17] M. Lacey, *An elementary proof of the A_2 bound,* Israel J. Math. 217 (2017), 181–195.

[18] M. Lacey and S. Spencer, *Sparse bounds for oscillatory and random singular integrals,* New York J. Math. 23 (2017), 119–131.

[19] A. Lerner, *A simple proof of the A_2 conjecture,* Int. Math. Res. Not. IMRN 14 (2013), 3159–3170.

[20] _____, *On pointwise estimates involving sparse operators,* New York J. Math. 22 (2016), 341–349.

[21] A. Lerner and F. Nazarov, *Intuitive dyadic calculus: the basics,* preprint, 2015, available at arXiv:1508.05639.

[22] K. Li, K. Moen, and W. Sun, *The sharp weighted bound for multilinear maximal functions and Calderón–Zygmund operators,* J. Fourier Anal. Appl. 20 (2014), 751–765.

[23] K. Li and W. Sun, *Weak and strong type weighted estimates for multilinear Calderón–Zygmund operators,* Adv. Math. 254 (2014), 736–771.

[24] _____, *Weighted estimates for multilinear Fourier multipliers,* Forum Math. 27 (2015), 1101–1116.

[25] J. Martell, C. Pérez, and R. Trujillo-González, *Lack of natural weighted estimates for some singular integral operators,* Trans. Amer. Math. Soc. 357 (2005), 385–396.

[26] D. Watson, *Weighted estimates for singular integrals via Fourier transform estimates,* Duke Math. J. 60 (1990), 389–399.

Department of Mathematics and
  Statistics
P.O.B. 68
(Gustaf Hällströmin katu 2b)
FI-00014 University of Helsinki
Finland

kangwei.nku@gmail.com

Current address
BCAM, Basque Center for Applied
  Mathematics
Alameda de Mazarredo 14
48009 Bilbao, Bizkaia
Basque Country
Spain

kli@bcamath.org