

# Bernstein–Sato Polynomials on Normal Toric Varieties

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ABSTRACT. We generalize the Bernstein–Sato polynomials of Budur, Mustață, and Saito to ideals in normal semigroup rings. In the case of monomial ideals, we also relate the roots of the Bernstein–Sato polynomial to the jumping coefficients of the corresponding multiplier ideals. To prove the latter result, we obtain a new combinatorial description for the multiplier ideals of a monomial ideal in a normal semigroup ring.

## 1. Introduction

Let  $f \in \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$  be a nonconstant polynomial, and let  $D = \mathbb{C}[x, \partial_x]$  be the Weyl algebra. The Bernstein–Sato polynomial (or  $b$ -function) of  $f$ , introduced independently in [Ber72] and [SS72], is the monic polynomial  $b_f(s) \in \mathbb{C}[s]$  of smallest degree such that there exists  $P(s) \in D[s] = D \otimes_{\mathbb{C}} \mathbb{C}[s]$  satisfying the functional equation  $P(s) \cdot f^{s+1} = b_f(s) f^s$ . It is well known that  $b_f(s) = s + 1$  if and only if the hypersurface defined by  $f$  is nonsingular. Moreover, the roots of  $b_f(s)$  are conjecturally related to the eigenvalues of the Milnor monodromy of  $f$  [Mal83] and the poles of the local zeta function associated with  $f$  [Igu00].

Another important singularity invariant are the multiplier ideals  $\mathcal{J}(\mathbb{C}^n, \alpha \langle f \rangle)$  associated with the hypersurface defined by  $f$  in  $\mathbb{C}^n$  [Laz04]. They can be defined via an embedded log resolution of the pair  $(\mathbb{C}^n, \langle f \rangle)$ . When the coefficient  $\alpha$  varies, the multiplier ideal  $\mathcal{J}(\mathbb{C}^n, \alpha \langle f \rangle)$  jumps. In fact, the smallest jumping coefficient of  $(\mathbb{C}^n, \langle f \rangle)$  (i.e. the log-canonical threshold of the hypersurface defined by  $f$ ) is the smallest root of  $b_f(-s)$  [Yan83; Lic89; Kol97]. Generalizing this result, one of the main theorems in [ELSV04] states that if  $\xi$  is a jumping coefficient in  $(0, 1]$  of the pair  $(\mathbb{C}^n, \langle f \rangle)$ , then it is a root of  $b_f(-s)$ . Along the same lines, Budur, Mustață, and Saito extended the notion of  $b$ -functions to the case of arbitrary ideals on smooth affine varieties [BMS06a]. Utilizing the theory of  $V$ -filtrations from [Kas83] and [Mal83], they showed that the  $b$ -function of an ideal is independent of the choice of generators. Furthermore, they generalized the connection established in [ELSV04] between the roots of their  $b$ -functions and the jumping coefficients of multiplier ideals to this more general setting.

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**THEOREM 1.1** ([BMS06a]). *Let  $I$  be an ideal on a smooth affine variety  $X$  over  $\mathbb{C}$ . Then the log-canonical threshold of the pair  $(X, I)$  coincides with the smallest root  $\alpha_I$  of the  $b$ -function  $b_I(-s)$ , and any jumping coefficient of  $(X, I)$  in  $[\alpha_I, \alpha_I + 1)$  is a root of  $b_I(-s)$ .*

As the theory of multiplier ideals is generalized to the case of arbitrary ideals on normal varieties [dFH09], it is interesting to explore possible generalizations of Theorem 1.1. To this end, the first task is to look for a suitable candidate that plays the role of the Weyl algebra. In the current work, we restrict to the case where the ambient variety  $X$  is an affine normal toric variety. There exists an explicit combinatorial description of Grothendieck’s ring of differential operators  $D_X$  on the toric variety  $X$  [Mus87; Jon94; ST01], serving as an analog of Weyl algebra in the case of  $X = \mathbb{C}^n$ . We apply this description of  $D_X$  and the results in [BMS06a; BMS06b] to construct the  $b$ -function  $b_I(s)$  for an ideal  $I$  on the toric variety  $X$ . To state our results more explicitly, let us set up some notations.

**NOTATION 1.2.** 1. Throughout this article, we work over an algebraically closed field  $\mathbb{k}$  of characteristic 0 instead of the complex number field  $\mathbb{C}$ . Our results are valid in this setting. The coordinate ring of the affine normal toric variety  $X$  is represented by the semigroup ring  $\mathbb{k}[\mathbb{N}A]$  generated by the columns  $a_1, \dots, a_m$  of a rank  $d$  matrix  $A \in \mathbb{Z}^{d \times m}$ . We assume that  $\mathbb{Z}A = \mathbb{Z}^d$ , the cone  $C = \mathbb{R}_{\geq 0}A$  in  $\mathbb{R}^d$  over  $A$  is strongly convex, and the semigroup  $\mathbb{N}A$  is normal, meaning that  $C \cap \mathbb{Z}^d = \mathbb{N}A$ . The faces of  $C$  are denoted by greek letters  $\sigma, \tau$ , and so on. This may refer to an index set, to a collection of columns of  $A$ , or to the actual face of the cone. If  $\sigma$  is a *facet* of  $C$ , then define its *primitive integral support function*  $F_\sigma : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

- (a)  $F_\sigma(\mathbb{Z}A) = \mathbb{Z}$ ,
- (b)  $F_\sigma(a_i) \geq 0$  for  $i = 1, \dots, n$ , and
- (c)  $F_\sigma(a_i) = 0$  for  $a_i \in \sigma$ .

Define the linear map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{F}}$  by  $F(p) = (F_\sigma(p))_{\sigma \in \mathcal{F}}$ , where  $\mathcal{F}$  is the collection of facets of  $C$ .

- 2. We consider  $\mathbb{k}[\mathbb{N}A]$  as a subring of the Laurent polynomial ring  $\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$ . Then the ring of differential operators  $D_A$  on the toric variety  $X$  can be represented as a subring of the linear partial differential operators on  $\partial_{y_1}, \dots, \partial_{y_d}$  with Laurent polynomial coefficients,

$$D_A = \bigoplus_{u \in \mathbb{Z}^d} y^u \{ f(\theta) \in \mathbb{k}[\theta_1, \dots, \theta_d] \mid f \text{ vanishes on } \mathbb{N}A \setminus (-u + \mathbb{N}A) \},$$

where  $\theta_1, \dots, \theta_d$  are the commuting operators  $y_1 \partial_{y_1}, \dots, y_d \partial_{y_d}$ .

- 3. The linear map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{F}}$  induces a ring homomorphism

$$\mathbb{k}[\mathbb{N}A] \rightarrow \mathbb{k}[\mathbb{N}^{\mathcal{F}}] \quad \text{via} \quad \sum_{j=1}^k \lambda_j y^{\beta_j} \mapsto \sum_{j=1}^k \lambda_j x^{F(\beta_j)},$$

where  $\beta_1, \dots, \beta_j \in \mathbb{N}A$ , and  $(x_\sigma)_{\sigma \in \mathcal{F}}$  denote algebraically independent variables, so that  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}] = \mathbb{k}[x_\sigma \mid \sigma \in \mathcal{F}]$ . For an ideal  $I$  in  $\mathbb{k}[\mathbb{N}A] \subset \mathbb{C}$

$\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$ , we abuse notation to denote

$$F(I) = \left\{ \sum \lambda_j x^{F(\beta_j)} \in \mathbb{k}[\mathbb{N}^{\mathcal{F}}] \mid \sum_j \lambda_j y^{\beta_j} \in I \right\}.$$

Note that  $F(I)$  is an ideal in the semigroup ring  $\mathbb{k}[F(\mathbb{N}A)]$ . The ideal in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$  generated by  $F(I)$  is denoted by  $J = \langle F(I) \rangle$ .

4. For a monomial ideal  $I$  in  $\mathbb{k}[\mathbb{N}A] \subset \mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$ , denote by  $P_I$  the Newton polyhedron of  $I$ , which by definition is the convex hull of  $\bigcup_{v: y^v \in I} (v + \mathbb{N}A)$  in  $\mathbb{R}^d$ . The relative interior of  $P_I$  is denoted by  $\text{int } P_I$ .

The first goal of this paper is to extend the notion of  $b$ -functions to the toric setting. We use the same definition as in [BMS06a] except that the Weyl algebra  $D$  is replaced by the ring of differential operators  $D_A$  on  $X$ . Analyzing the combinatorics of the toric algebras, we obtain the following result.

**THEOREM (Theorem 3.4).** *The  $b$ -function  $b_I(s)$  of an ideal  $I$  on the toric variety  $X$  coincides with the  $b$ -function  $b_J(s)$  of the ideal  $J = \langle F(I) \rangle$  in the polynomial ring  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ .*

In particular, avoiding the study of the  $V$ -filtration in the toric case, we can still conclude that the toric  $b$ -function  $b_I(s)$  is independent of the choice of generators of  $I$ .

To generalize Theorem 1.1 to the toric setting, we further restrict to the case where  $I$  is a monomial ideal (i.e. an ideal defining a torus-invariant subvariety of  $X$ ). In this case, the multiplier ideals of the pair  $(X, I)$  were described in [Bli04], generalizing the work in [How01] on the multiplier ideals of monomial ideals in polynomial algebras. Using these formulas, we obtain a new expression of the multiplier ideals of monomial ideals on toric varieties using the map  $F$  (Proposition 4.2),

$$\mathcal{J}(X, \alpha I) = \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid F(v) + e \in \text{int}(\alpha P_J) \rangle,$$

where  $e \in \mathbb{N}^{\mathcal{F}}$  is such that  $e_\sigma = 1$  for all  $\sigma \in \mathcal{F}$ , and  $\text{int}(P_J)$  is the relative interior of the Newton polyhedron  $P_J$  of the monomial ideal  $J$  in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ . In particular, we obtain the identity

$$\mathcal{J}(X, \alpha I) = \mathbb{k}[\mathbb{N}A] \cap \mathcal{J}(\mathbb{k}^{\mathcal{F}}, \alpha J),$$

which relates the jumping coefficients of the pair  $(X, I)$  to those of  $(\mathbb{k}^{\mathcal{F}}, J)$ . Since the jumping coefficients of  $(\mathbb{k}^{\mathcal{F}}, J)$  are related to the roots of  $b_J(-s)$  by Theorem 1.1, our arguments give the following theorem.

**THEOREM (Theorem 4.4).** *Let  $I$  be a monomial ideal on an affine normal toric variety  $X$  over an algebraically closed field  $\mathbb{k}$  of characteristic 0. Then the log-canonical threshold of the pair  $(X, I)$  coincides with the smallest root  $\alpha_I$  of the  $b$ -function  $b_I(-s)$ , and all jumping coefficients of  $(X, I)$  in  $[\alpha_I, \alpha_I + 1)$  are roots of  $b_I(-s)$ .*

To end this introduction, we mention the work of Álvarez-Montaner, Huneke, and Núñez-Betancourt [ÅHN16], where  $b$ -functions are studied in the context of direct summand rings.

## 2. Bernstein–Sato Polynomials of Budur, Mustață, and Saito

We recall relevant background on Bernstein–Sato polynomials for ideals from the work of Budur, Mustață, and Saito [BMS06a]. Here we concentrate on the case of subvarieties of affine spaces, but the definition works for subvarieties of a smooth variety.

Let  $I$  be an ideal in  $\mathbb{k}[x_1, \dots, x_n]$  generated by  $f_1, \dots, f_r$ . Denote by  $D$  the Weyl algebra on  $x_1, \dots, x_n$ . If  $s_1, \dots, s_r$  are indeterminates, then consider

$$\mathbb{k}[x_1, \dots, x_n] \left[ \prod_{i=1}^r f_i^{-1}, s_1, \dots, s_r \right] \prod_{i=1}^r f_i^{s_i}.$$

This is a  $D[s_{ij}]$ -module, where  $s_{ij} = s_i t_i^{-1} t_j$ , and the action of the operator  $t_i$  is given by  $t_i(s_j) = s_j + \delta_{ij}$  (the Kronecker delta).

**DEFINITION 2.1.** The *Bernstein–Sato polynomial* or  *$b$ -function* associated with  $f = (f_1, \dots, f_r)$  is defined to be the monic polynomial of lowest degree in  $s = \sum_{i=1}^r s_i$  satisfying a relation of the form

$$b_f(s) \prod_{i=1}^r f_i^{s_i} = \sum_{k=1}^r P_k t_k \prod_{i=1}^r f_i^{s_i}, \quad (2.1)$$

where  $P_1, \dots, P_r \in D[s_{ij} \mid i, j \in \{1, \dots, r\}]$ .

In [BMS06a, Section 2], Budur, Mustață and Saito show that:

1.  $b_f(s)$  is a nonzero polynomial,
2.  $b_f(s)$  is independent of the generating set of  $I$ .

### 2.1. An Alternative Way to Define the Bernstein–Sato Polynomial

In [BMS06a, Section 2.10], Budur, Mustață and Saito give the following way to compute the  $b$ -function.

For  $c = (c_1, \dots, c_r) \in \mathbb{Z}^r$ , define  $\text{nsupp}(c) = \{i \mid c_i < 0\}$ . The Bernstein–Sato polynomial  $b_f(s)$  is the monic polynomial of the smallest degree such that  $b_f(s) \prod_{i=1}^r f_i^{s_i}$  belongs to the  $D[s_1, \dots, s_r]$ -submodule generated by

$$\prod_{i \in \text{nsupp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i + c_i}, \quad (2.2)$$

where  $c = (c_1, \dots, c_r)$  runs over the elements of  $\mathbb{Z}^r$  such that  $\sum_{i=1}^r c_i = 1$ . Here  $s = \sum_{i=1}^r s_i$  and  $\binom{s_i}{m} = s_i(s_i - 1) \cdots (s_i - m + 1)/m!$ .

The advantage of this approach is that we now work in the ring  $D[s_1, \dots, s_r]$  and need no longer consider the operators  $s_{ij}$  or  $t_k$ .

## 2.2. A Monomial-Ideal-Specific Way to Define the Bernstein–Sato Polynomial

The definition given in the previous subsection can be further specialized to the case of monomial ideals.

Let  $\mathfrak{a}$  be a monomial ideal in  $\mathbb{k}[x_1, \dots, x_n]$  whose minimal monomial generators are  $x^{\alpha_1}, \dots, x^{\alpha_r}$ . Set  $\ell_\alpha(s_1, \dots, s_r) = s_1\alpha_1 + \dots + s_r\alpha_r$ , and let  $\ell_i(s)$  be the  $i$ th coordinate of this vector. For  $v \in \mathbb{Z}^r$ , let  $\text{psupp}(v) = \{i \mid v_i > 0\}$ .

For  $c = (c_1, \dots, c_r) \in \mathbb{Z}^r$  such that  $\sum_{i=1}^r c_i = 1$ , set

$$g_c(s_1, \dots, s_r) = \prod_{j \in \text{supp}(c)} \binom{s_j}{-c_j} \cdot \prod_{i \in \text{psupp}(\ell_\alpha(c))} \binom{\ell_i(s_1, \dots, s_r) + \ell_i(c)}{\ell_i(c)},$$

and define  $I_\alpha$  to be the ideal of  $\mathbb{Q}[s_1, \dots, s_r]$  generated by the polynomials  $g_c$ .

Then [BMS06a, Prop. 4.2] asserts that  $b_f(s)$  is the monic polynomial of smallest degree such that  $b_f(\sum_{i=1}^r s_i)$  belongs to the ideal  $I_\alpha$ .

## 3. Bernstein–Sato Polynomials on Toric Varieties

The goal of this section is to extend Definition 2.1 to the toric setting. We first recall some background on the ring of differential operators of a toric variety.

### 3.1. Rings of Differential Operators on Toric Varieties

For a commutative algebra  $R$  over a commutative ring  $k$ , Grothendieck’s ring of  $k$ -linear differential operators  $D_k(R)$  of  $R$  [Gro67] is the  $R$ -subalgebra of  $\text{Hom}_k(R, R)$  defined by

$$D_k(R) = \bigcup_{i \in \mathbb{N} \cup \{0\}} D_i,$$

where  $D_0 = R$  and  $D_i = \{f \in \text{Hom}_k(R, R) \mid fr - rf \in D_{i-1} \text{ for all } r \in R\}$  for  $i \geq 1$ . When  $R$  is the polynomial ring in  $n$  variables over an algebraically closed field  $k$  of characteristic 0, the ring  $D_k(R)$  coincides with the  $n$ th Weyl algebra. Moreover, it is well known that if  $R$  is regular over  $k$ , then  $D_k(R)$  is the  $R$ -algebra generated by the  $k$ -derivations of  $R$ . The most influential applications of  $D$ -module theory and, in particular, the  $b$ -functions of [BMS06a] are built in this setting.

When  $R$  is not regular, the ring  $D_k(R)$  is not well-behaved [BGG72]. Nonetheless, when  $R = \mathbb{k}[\mathbb{N}\mathbb{A}]$  is a toric algebra over an algebraically closed field  $\mathbb{k}$  of characteristic 0, there exists an explicit combinatorial description of  $D_{\mathbb{k}}(\mathbb{k}[\mathbb{N}\mathbb{A}])$ . This expression of  $D_{\mathbb{k}}(\mathbb{k}[\mathbb{N}\mathbb{A}])$  was obtained independently in [Mus87] and [Jon94] when  $\mathbb{N}\mathbb{A}$  is normal and was further extended to the not necessarily normal case in [ST01]. We denote the ring of differential operators of the toric algebra  $\mathbb{k}[\mathbb{N}\mathbb{A}]$  by

$$D_A = D_{\mathbb{k}}(\mathbb{k}[\mathbb{N}\mathbb{A}]).$$

One way to understand  $D_A$  is to identify  $\mathbb{k}[\mathbb{N}\mathbb{A}]$  as a subring of the Laurent polynomial ring  $\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$  and study the  $\mathbb{k}$ -linear differential operators of

$\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]$ . It is known that if  $S$  is a multiplicatively closed subset of a  $k$ -algebra  $R$ , then  $D_k(S^{-1}R) \cong S^{-1}R \otimes_R D_k(R)$ . It follows that

$$D_{\mathbb{k}}(\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]) = \mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}] \langle \partial_{y_1}, \dots, \partial_{y_d} \rangle$$

is a localization of the  $n$ th Weyl algebra. Moreover, one can realize

$$D_A = \{ \delta \in D_{\mathbb{k}}(\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}]) \mid \delta(\mathbb{k}[\mathbb{N}A]) \subseteq \mathbb{k}[\mathbb{N}A] \}$$

as a subring of  $D_{\mathbb{k}}(\mathbb{k}[y_1^{\pm 1}, \dots, y_d^{\pm 1}])$ . In particular, we have the  $\mathbb{Z}^d$ -graded expression

$$D_A = \bigoplus_{u \in \mathbb{Z}^d} y^u \{ f(\theta) \in \mathbb{k}[\theta_1, \dots, \theta_d] \mid f \text{ vanishes on } \mathbb{N}A \setminus (-u + \mathbb{N}A) \}, \quad (3.1)$$

where  $\theta_1, \dots, \theta_d$  are the commuting operators  $y_i \partial_{y_i}$  for  $i = 1, \dots, d$ . Since we are only concerned with the case where  $\mathbb{N}A$  is normal, for a given  $u \in \mathbb{Z}^d$ , the ideal

$$\{ f(\theta) \in \mathbb{k}[\theta_1, \dots, \theta_d] \mid f \text{ vanishes on } \mathbb{N}A \setminus (-u + \mathbb{N}A) \}$$

is the principal ideal in  $\mathbb{C}[\theta_1, \dots, \theta_d]$  generated by

$$\prod_{F_\sigma(u) > 0} \prod_{j=0}^{F_\sigma(u)-1} (F_\sigma(\theta_1, \dots, \theta_d) - j), \quad (3.2)$$

where the product runs through all facets  $\sigma$  (with  $F_\sigma(u) > 0$ ) of the cone  $C$  associated with  $\mathbb{N}A$ , and  $F_\sigma$  is the primitive integral support function of  $\sigma$  as defined in Notation 1.2.1.

We are now ready to define the  $b$ -function for an ideal  $I$  in  $\mathbb{k}[\mathbb{N}A]$ . To make the exposition more transparent, we first treat the case where  $I$  is a monomial ideal.

### 3.2. Monomial Ideals in Semigroup Rings

Let  $\mathfrak{a}$  be a monomial ideal in  $\mathbb{k}[\mathbb{N}A]$  whose minimal monomial generators are Laurent monomials  $y^{\beta_1}, \dots, y^{\beta_r}$ . For  $f = (y^{\beta_1}, \dots, y^{\beta_r})$ , we define the Bernstein–Sato polynomial  $b_f(s)$  exactly as in Definition 2.1, except that the Weyl algebra  $D$  is replaced by the ring of differential operators  $D_A$  on  $\mathbb{k}[\mathbb{N}A]$ . If no polynomial satisfying the definition exists, then we make the convention that  $b_f(s)$  is zero; however, we will prove later that this is never the case.

In Section 2.1, we recalled how in [BMS06a] the computation of  $b$ -functions is reduced to a computation over  $D[s_1, \dots, s_r]$  (rather than over  $D[s_{ij}]$ ). The same argument can be applied to the toric setting. This implies that Bernstein–Sato polynomials over normal toric varieties can be computed using only operators from  $D_A[s_1, \dots, s_r]$ , without needing to work over the larger ring  $D_A[s_{ij}]$ .

Our task becomes to translate the situation from Section 2.2 to the new setting with monomial ideals in semigroup rings, rather than with monomial ideals in polynomial rings.

Let  $c \in \mathbb{Z}^r$  whose coordinates sum to 1, and for  $\beta = (\beta_1, \dots, \beta_r) \in (\mathbb{Z}^d)^r$ , set  $\ell_\beta(s_1, \dots, s_r) = s_1 \beta_1 + \dots + s_r \beta_r$ .

To find the Bernstein–Sato polynomial, we need to apply an operator in  $D_A[s_1, \dots, s_r]$  to  $\prod_{i \in \text{supp}(c)} \binom{s_1}{-c_i} \cdot \prod_{i=1}^r (y^{\beta_i})^{s_i+c_i}$  in such a way that the outcome is a multiple of  $y^{\sum_{i=1}^r s_i \beta_i} = y^{\ell_\beta(s_1, \dots, s_r)}$ .

If we apply the operator from (3.2) with  $u = \ell_\beta(c)$  to  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r (y^{\beta_i})^{s_i+c_i}$ , then we obtain

$$\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \prod_{F_\sigma(\ell_\beta(c)) > 0} \prod_{j=0}^{F_\sigma(\ell_\beta(c))-1} (F_\sigma(\ell_\beta(s_1, \dots, s_r) + \ell_\beta(c)) - j) \cdot y^{\ell_\beta(s_1, \dots, s_r)}.$$

**PROPOSITION 3.1.** *We use the notation introduced before. The Bernstein–Sato polynomial  $b_f(s)$  for  $f = (y^{\beta_1}, \dots, y^{\beta_r})$  considered as a monomial ideal in  $\mathbb{k}[\mathbb{N}A]$  is the monic polynomial of smallest degree such that  $b_f(s_1 + \dots + s_r)$  belongs to the ideal generated by*

$$\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \prod_{F_\sigma(\ell_\beta(c)) > 0} \binom{F_\sigma(\ell_\beta(s_1, \dots, s_r) + \ell_\beta(c))}{F_\sigma(\ell_\beta(c))}, \tag{3.3}$$

where  $c \in \mathbb{Z}^r$  has the coordinate sum 1.

*Proof.* It remains to be shown that, for any other operator (in  $D_A[s_1, \dots, s_r]$ ) that applied to  $\prod_{i \in \text{supp}(c)} \binom{s_1}{-c_i} \cdot \prod_{i=1}^r (y^{\beta_i})^{s_i+c_i}$  yields a multiple of  $y^{\ell(s_1, \dots, s_r)}$ , this multiple belongs to the ideal generated by (3.3).

This follows from the description of  $D_A$  given by (3.1) and (3.2). □

Recall that  $\mathcal{F}$  is the collection of facets of the cone  $C$  of  $\mathbb{N}A$  and that the map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{F}}$  defined by  $F(p) = (F_\sigma(p))_{\sigma \in \mathcal{F}}$  from Notation 1.2.1 induces an inclusion  $\mathbb{k}[\mathbb{N}A] \rightarrow \mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ .

**PROPOSITION 3.2.** *Denote by  $F(f)$  the sequence of monomials  $(x^{F(\beta_1)}, \dots, x^{F(\beta_r)})$  in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ . The Bernstein–Sato polynomial  $b_{F(f)}(s)$  of  $F(f)$  in the polynomial ring  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$  coincides with the Bernstein–Sato polynomial  $b_f(s)$  of  $f = (x^{\beta_1}, \dots, x^{\beta_r})$  in  $\mathbb{k}[\mathbb{N}A]$ . In particular, the latter polynomial  $b_f(s)$  is nonzero, and its roots can be computed using the combinatorial description [BMS06b, Thm. 1.1] applied to  $\langle x^{F(\beta_1)}, \dots, x^{F(\beta_r)} \rangle \subseteq \mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ .*

*Proof.* First, we observe that the minimal monomial generators of  $\langle x^{F(\beta_1)}, \dots, x^{F(\beta_r)} \rangle \subseteq \mathbb{k}[\mathbb{N}^{\mathcal{F}}]$  are the monomials  $x^{F(\beta_1)}, \dots, x^{F(\beta_r)}$ . To see this, note that  $x^{F(\beta_i)}$  divides  $x^{F(\beta_j)}$  is equivalent to  $F_\sigma(\beta_i) \leq F_\sigma(\beta_j)$  for all facets  $\sigma$  of the cone  $C$ . This implies that  $\beta_j - \beta_i$  lies in the cone  $C$ . Since  $\beta_j - \beta_i \in \mathbb{Z}^r$  and  $\mathbb{N}A$  is normal, we see that  $\beta_j - \beta_i \in \mathbb{N}A$ , and therefore  $y^{\beta_i}$  divides  $y^{\beta_j}$  in  $\mathbb{k}[\mathbb{N}A]$ .

Let  $\alpha_i = F(\beta_i)$  for  $i = 1, \dots, r$ , let  $\ell_\alpha(s_1, \dots, s_r) = s_1 \alpha_1 + \dots + s_r \alpha_r$ , and denote by  $\ell_i(s_1, \dots, s_r)$  the  $i$ th coordinate of the vector  $\ell_\alpha(s_1, \dots, s_r)$ . By [BMS06a, Prop. 4.2] the Bernstein–Sato polynomial of  $\langle x^{F(\beta_1)}, \dots, x^{F(\beta_r)} \rangle$  is the monic polynomial  $p$  of smallest degree such that  $p(s_1 + \dots + s_r)$  lies in the

ideal generated by

$$\prod_{j \in \text{supp}(c)} \binom{s_j}{-c_j} \prod_{i \in \text{supp}(\ell_\alpha(c))} \binom{\ell_i(s_1, \dots, s_r) + \ell_i(c)}{\ell_i(c)} \tag{3.4}$$

for  $c \in \mathbb{Z}^r$  with coordinate sum 1. But note that by construction  $\ell_\alpha = F \circ \ell_\beta$ , so that the generators (3.4) coincide exactly with the generators (3.3). As a reality check, note that both sets of generators belong to the ring  $\mathbb{k}[s_1, \dots, s_r]$ , which does not depend on the ambient rings of the monomial ideals involved.  $\square$

### 3.3. The General Case

Again, recall that we have a linear map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{F}}$  with  $F(\mathbb{N}A) \subseteq \mathbb{N}^{\mathcal{F}}$ . We construct a ring homomorphism

$$\mathbb{k}[\mathbb{N}A] \rightarrow \mathbb{k}[\mathbb{N}^{\mathcal{F}}] \quad \text{via} \quad \sum_{j=1}^k \lambda_j y^{\beta_j} \mapsto \sum_{j=1}^k \lambda_j x^{F(\beta_j)}, \tag{3.5}$$

where  $\beta_1, \dots, \beta_k \in \mathbb{N}A$ . That this is a homomorphism follows from the linearity of  $F$ , as  $x^{F(\beta+\beta')} = x^{F(\beta)+F(\beta')} = x^{F(\beta)} x^{F(\beta')}$ . We abuse notation and denote the homomorphism (3.5) by  $F$ .

If  $I \subset \mathbb{k}[\mathbb{N}A]$  is an ideal, then its image  $F(I)$  is not an ideal in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ ; we denote by  $\langle F(I) \rangle$  the ideal in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$  generated by  $F(I)$ .

LEMMA 3.3. *Let  $I \subset \mathbb{k}[\mathbb{N}A]$  be the ideal generated by  $g_1, \dots, g_r \in \mathbb{k}[\mathbb{N}A]$ . Then  $F(g_1), \dots, F(g_r)$  generate  $\langle F(I) \rangle$ .*

*Proof.* Since  $g_1, \dots, g_r$  generate  $I$ , and  $F$  is a ring homomorphism, then any element of  $F(I)$  is obtained as a combination of  $F(g_1), \dots, F(g_r)$  with coefficients in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ . This implies that the polynomials  $F(g_1), \dots, F(g_r)$  generate  $\langle F(I) \rangle$ .  $\square$

We apply Definition 2.1 to a sequence of polynomials in  $\mathbb{k}[\mathbb{N}A]$  using the ring of differential operators  $D_A$  instead of  $D$ . The following is the main result of this section.

THEOREM 3.4. *Let  $I \subset \mathbb{k}[\mathbb{N}A]$  be an ideal, and let  $g_1, \dots, g_r$  be generators for  $I$ . The Bernstein–Sato polynomial of  $g = (g_1, \dots, g_r)$  in  $\mathbb{k}[\mathbb{N}A]$  equals the Bernstein–Sato polynomial of  $f = (f_1, \dots, f_r) = F(g) = (F(g_1), \dots, F(g_r))$  in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ . Consequently, this Bernstein–Sato polynomial is nonzero and depends only on  $I$ , not on the particular set of generators chosen.*

*Proof.* We know that the Bernstein–Sato polynomial  $b_g(s)$  is the monic polynomial of smallest degree such that, for  $s = s_1 + \dots + s_r$ ,  $b_g(s) \prod_{i=1}^r g_i^{s_i}$  belongs to the  $D_A[s_1, \dots, s_r]$ -submodule generated by

$$\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r g_i^{s_i+c_i} \quad \text{for } (c_1, \dots, c_r) \in \mathbb{Z}^r \text{ with } \sum_{i=1}^r c_i = 1,$$



whereas  $b_f(s)$  is the monic polynomial of smallest degree such that  $b_f(s) \prod_{i=1}^r f_i^{s_i}$  belongs to the  $D[s_1, \dots, s_r]$ -submodule generated by

$$\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i} \quad \text{for } (c_1, \dots, c_r) \in \mathbb{Z}^r \text{ with } \sum_{i=1}^r c_i = 1.$$

Here  $D$  is the ring of differential operators on  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$ , and  $D_A$  is the ring of differential operators on  $\mathbb{k}[\mathbb{N}A]$ .

Fix  $c \in \mathbb{Z}^r$  such that  $\sum c_i = 1$ , and let  $P \in D[s_1, \dots, s_r]$  be such that  $P[\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i}]$  is a polynomial in  $s_1, \dots, s_r$  times  $\prod_{i=1}^r f_i^{s_i}$ . Applying the description given by (3.1) and (3.2) to  $D$  (instead of  $D_A$ ), we see that we can write  $P$  as a finite sum

$$P = \sum_{u \in \mathbb{Z}^{\mathcal{F}}} q_u(s_1, \dots, s_r) x^u \left[ \prod_{u_\sigma < 0} \prod_{j=0}^{-u_\sigma-1} ((\theta_x)_\sigma - j) \right] p_u(\theta_x),$$

where the  $p_u$  are polynomials in  $|\mathcal{F}|$  indeterminates with coefficients in  $\mathbb{k}$ .

Note that, by construction, the element  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i}$  is  $F(\mathbb{N}A)$ -graded. Since  $P[\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i}]$  is a multiple of  $\prod_{i=1}^r f_i^{s_i}$ , we may assume that the operator  $P$  is  $F(\mathbb{N}A)$ -graded as well. In other words, we may assume that  $u \in F(\mathbb{Z}^d)$  if  $q_u p_u \neq 0$ .

Thus we rewrite

$$P = \sum_{u=F(v) \in F(\mathbb{Z}^d)} q_u(s_1, \dots, s_r) x^u \left[ \prod_{u_\sigma < 0} \prod_{j=0}^{-u_\sigma-1} ((\theta_x)_\sigma - j) \right] p_u(\theta_x),$$

and if we denote

$$\hat{P} = \sum_{v \in \mathbb{Z}^d} q_{F(v)}(s_1, \dots, s_r) y^v \left[ \prod_{F_\sigma(v) < 0} \prod_{j=0}^{-F_\sigma(v)-1} (F_\sigma(\theta_y) - j) \right] p_{F(v)}((F_\sigma(\theta_y))_{\sigma \in \mathcal{F}}),$$

then  $\hat{P}$  is an element of  $D_A[s_1, \dots, s_r]$ , and  $\hat{P}$  applied to  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r g_i^{s_i+c_i}$  is a polynomial of  $s_1, \dots, s_r$  times  $\prod_{i=1}^r g_i^{s_i}$ , and this is the same polynomial that we obtain when we apply  $P$  to  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i}$  (and divide by  $\prod f_i^{s_i}$ ).

Conversely, if we apply an element of  $D_A[s_1, \dots, s_r]$  to  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r g_i^{s_i+c_i}$  and obtain a polynomial in  $s_1, \dots, s_r$  times  $\prod_{i=1}^r g_i^{s_i}$ , then we can obtain an element of  $D[s_1, \dots, s_r]$  that applied to  $\prod_{i \in \text{supp}(c)} \binom{s_i}{-c_i} \cdot \prod_{i=1}^r f_i^{s_i+c_i}$  gives exactly the same polynomial in  $s_1, \dots, s_r$  times  $\prod_{i=1}^r f_i^{s_i}$ . To do this, we write the element of  $D_A[s_1, \dots, s_r]$  as a sum of terms of the form

$$q_v(s_1, \dots, s_r) y^v \left[ \prod_{F_\sigma(v) < 0} \prod_{j=0}^{-F_\sigma(v)-1} (F_\sigma(\theta_y) - j) \right] p_v(\theta_y),$$

then express (nonuniquely) the polynomial  $p_v$  as sums of powers of the linear forms  $F_\sigma(\theta_y)$ , and then apply  $F$  in the obvious way. Nonuniqueness comes because we need to choose  $d$  linearly independent forms  $F_\sigma$  in order to get  $\mathbb{k}$ -algebra generators of  $\mathbb{k}[\theta_y]$ ; however, since the image of  $F$  as a linear map is  $d$ -dimensional, the choice does not affect the image.  $\square$

EXAMPLE 3.5. In [BMS06a], their  $b$ -function is defined for any scheme  $Z$ . Up to a shift by codimension, it is independent of the choice of the embedding of  $Z$  into a smooth variety. We provide a simple example showing that their  $b$ -functions are quite different from ours. This was suggested to us by Nero Budur.

Consider the affine line  $Z = \{u = v = 0\}$  embedded in the toric hypersurface  $X = \{uw - v^2 = 0\}$ . The  $b$ -function of the scheme  $Z$  in the sense of Budur, Mustață and Saito is  $b_Z(s) = s$ . However, using our definition and Theorem 3.4, we see that the toric  $b$ -function of  $Z$  embedded in  $X$  is the same as the Budur–Mustață–Saito  $b$ -function of the subscheme  $\{x^2 = xy = 0\}$  of the affine plane, which is equal to  $(s + 1)^2(s + 3/2)$ .

#### 4. Multiplier Ideals on Normal Toric Varieties

We recall basic definitions of multiplier ideals that can be found in Lazarsfeld’s text [Laz04].

Let  $X$  be a smooth variety over an algebraically closed field  $\mathbb{k}$  of characteristic 0. Let  $\mathcal{I} \subseteq \mathcal{O}_X$  be an ideal sheaf, and let  $\alpha > 0$  be a rational number. Fix a log resolution  $\mu : X' \rightarrow X$  of  $\mathcal{I}$  with  $\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$ . The multiplier ideal of the pair  $(X, \alpha\mathcal{I})$  is defined as

$$\mathcal{J}(X, \alpha\mathcal{I}) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \alpha \cdot E \rfloor),$$

where  $K_{X'/X} = K_{X'} - \mu^* K_X$  is the relative canonical divisor of  $\mu$ , which is an effective divisor supported on the exceptional locus of  $\mu$  whose local equation is given by the determinant of the derivative  $d\mu$ . The definition does not depend on the choice of log resolution. One of the important features of multiplier ideals is that  $\mathcal{J}(X, \alpha\mathcal{I})$  measures the singularity of the pair  $(X, \alpha\mathcal{I})$ : a smaller multiplier ideal corresponds to a worse singularity.

Notice that  $\mathcal{J}(X, \alpha\mathcal{I})$  becomes smaller as  $\alpha$  increases. The jumping coefficients of  $(X, \alpha\mathcal{I})$  are the positive real numbers  $0 < \alpha_1 < \alpha_2 < \dots$  such that  $\mathcal{J}(X, \alpha_j\mathcal{I}) = \mathcal{J}(X, \alpha\mathcal{I}) \neq \mathcal{J}(X, \alpha_{j+1}\mathcal{I})$  for  $\alpha_j \leq \alpha < \alpha_{j+1}$  ( $j \geq 0$ ) where  $\alpha_0 = 0$ . When  $X$  is affine and  $I$  is the ideal in the coordinate ring  $\mathbb{k}[X]$  corresponding to the sheaf  $\mathcal{I}$ , Budur, Mustață and Saito proved that the jumping coefficients of  $(X, \alpha I)$  in  $[\alpha_f, \alpha_f + 1)$  are roots of  $b_f(-s)$ , where  $f = (f_1, \dots, f_r)$  is a set of generators for  $I$ ,  $b_f(s)$  is the Bernstein–Sato polynomial of  $I$ , and  $\alpha_f$  is the smallest root of  $b_f(-s)$ . One of our goals is to generalize this correspondence between  $b$ -function roots and jumping coefficients to monomial ideals on affine normal toric varieties.

In the particular case that  $I$  is a monomial ideal in the polynomial ring  $\mathbb{k}[x_1, \dots, x_m]$ , Howald [How01] gave the following combinatorial formula for

the multiplier ideals of the pair  $(\mathbb{A}^m, \alpha I)$ :

$$\mathcal{J}(\mathbb{A}^m, \alpha I) = \langle x^v \mid v + e \in \text{int}(\alpha P_I) \rangle,$$

where  $P_I$  is the Newton polyhedron of  $I$ , and  $e$  is the vector  $(1, 1, \dots, 1)$  in  $\mathbb{N}^m$ . In particular, a rational number  $\alpha > 0$  is a jumping coefficient of  $(\mathbb{A}^m, \alpha I)$  if and only if the boundary of  $-e + \text{int}(\alpha P_I)$  contains a lattice point in  $\mathbb{N}^m$ .

The notion of multiplier ideal can be generalized to the case that  $\mathcal{I}$  is an ideal sheaf on a normal variety  $X$  over  $\mathbb{k}$  as follows. Let  $\Delta$  be an effective  $\mathbb{Q}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. Such  $\Delta$  is called a boundary divisor. Let  $\mu : X' \rightarrow X$  be a log resolution of the triple  $(X, \Delta, \mathcal{I})$ , and let  $\alpha > 0$  be a rational number. Suppose that  $\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$ . Then we can define the multiplier ideal  $\mathcal{J}(X, \Delta, \alpha \mathcal{I})$  associated with the triple  $(X, \Delta, \alpha \mathcal{I})$  as

$$\mathcal{J}(X, \Delta, \alpha \mathcal{I}) = \mu_* \mathcal{O}_{X'}(K_{X'} - \lfloor \mu^*(K_X + \Delta) + \alpha E \rfloor).$$

Again, this definition is not dependent on the choice of  $\mu$ . De Fernex and Hacon [dFH09] have given a definition of  $\mathcal{J}(X, \alpha Z)$  for non- $\mathbb{Q}$ -Gorenstein  $X$  without using the boundary divisor  $\Delta$ . They showed that there exists a boundary divisor  $\Delta$  such that the multiplier ideal  $\mathcal{J}(X, \Delta, \alpha Z)$  coincides with their multiplier ideal  $\mathcal{J}(X, \alpha Z)$  and that  $\mathcal{J}(X, \alpha Z)$  is the unique maximal element of the set

$$\{\mathcal{J}(X, \Delta, \alpha Z) \mid \Delta \text{ is a boundary divisor}\}.$$

#### 4.1. A New Expression for Multiplier Ideals on Toric Varieties

Let us explain how we use the map  $F$  to understand the multiplier ideals of de Fernex and Hacon in the case of monomial ideals on normal toric varieties.

Let  $\mathbb{k}[\mathbb{N}A] \subset \mathbb{k}[y_1^\pm, \dots, y_d^\pm]$  be a normal semigroup ring in our setting, and let  $X = \text{Spec}(\mathbb{k}[\mathbb{N}A])$ . Each facet  $\sigma \in \mathcal{F}$  corresponds to a torus invariant prime Weil divisor  $D_\sigma$  on  $X$ . The canonical class of  $X$  is represented by the torus invariant canonical divisor  $K_X = -\sum_{\sigma \in \mathcal{F}} D_\sigma$ . A boundary divisor  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, which means there exist  $l \in \mathbb{Z}$  and  $u \in \mathbb{Z}^d$  such that  $l(K_X + \Delta) = \text{div}(y^u)$ . Denote  $w_\Delta = \frac{u}{l}$ . Notice that

$$\Delta = \sum_{\sigma \in \mathcal{F}} (1 + F_\sigma(w_\Delta)) D_\sigma,$$

so the effectivity of  $\Delta$  is equivalent to the condition that  $F_\sigma(w_\Delta) \geq -1$  for all  $\sigma \in \mathcal{F}$ . Conversely, each  $w \in \mathbb{Q}^d$  such that  $F_\sigma(w) \geq -1$  for all  $\sigma \in \mathcal{F}$  gives rise to a boundary divisor  $\Delta_w$  on  $X$  by the same formula

$$\Delta_w = \sum_{\sigma \in \mathcal{F}} (1 + F_\sigma(w)) D_\sigma.$$

Let  $I$  be a monomial ideal in  $\mathbb{k}[\mathbb{N}A]$ , and let  $\alpha > 0$  be a rational number. Blickle [Bli04] showed that the multiplier ideal  $\mathcal{J}(X, \Delta, \alpha I)$  is equal to the ideal  $\langle y^v \in \mathbb{k}[\mathbb{N}A] \mid v - w_\Delta \in \text{int}(\alpha P_I) \rangle$ , generalizing Howald’s description of multiplier ideals [How01] in the case of monomial ideals in polynomial rings. Be

cautioned about the mistake of the sign of  $w_\Delta$  in Blickle’s original description. For  $w \in \mathbb{Q}^d$ , denote

$$\Omega_{w,\alpha I} = [w + \text{int}(\alpha P_I)] \quad \text{and} \quad \Omega_{\alpha I} = \bigcup_{w \in \mathbb{Q}^d: F_\sigma(w) \geq -1 \ \forall \sigma \in \mathcal{F}} \Omega_{w,\alpha I}. \quad (4.1)$$

Then the multiplier ideal of de Fernex and Hacon is

$$\mathcal{J}(X, \alpha I) = \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid v \in \Omega_{\alpha I} \rangle. \quad (4.2)$$

We claim that  $\mathcal{J}(X, \alpha I)$  can be computed using an analog of Howald’s formula. Recall that the linear map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^{\mathcal{F}}$  induces ring homomorphisms

$$\mathbb{k}[\mathbb{N}A] \xrightarrow{\sim} \mathbb{k}[F(\mathbb{N}A)] \rightarrow \mathbb{k}[\mathbb{N}^{\mathcal{F}}].$$

By abusing notation denote

$$F(I) = \mathbb{k}[F(\mathbb{N}A)] \cdot \langle x^{F(v)} \mid y^v \in I \rangle$$

the ideal in  $\mathbb{k}[F(\mathbb{N}A)]$  obtained from the semigroup isomorphism  $F : \mathbb{N}A \xrightarrow{\sim} F(\mathbb{N}A)$ . The monomial ideal in  $\mathbb{k}[\mathbb{N}^{\mathcal{F}}]$  generated by  $F(I)$  is denoted by

$$J = \mathbb{k}[\mathbb{N}^{\mathcal{F}}] \cdot \langle F(I) \rangle.$$

The Newton polyhedron of  $F(I)$  is denoted by  $P_{F(I)}$ ; this is the convex hull of  $\{F(v) \mid y^v \in I\}$  in  $F(\mathbb{R}^d)$ . We have

$$P_{F(I)} = \text{conv} \left[ \bigcup_{v: y^v \in I} (F(v) + F(\mathbb{N}A)) \right].$$

On the other hand, the Newton polyhedron of  $J$  in  $\mathbb{R}^{\mathcal{F}}$  is

$$P_J = \text{conv} \left[ \bigcup_{v: y^v \in I} (F(v) + \mathbb{N}^{\mathcal{F}}) \right].$$

Note that if  $y^{\beta_1}, \dots, y^{\beta_r}$  is the set of minimal monomial generators of  $I$ , then  $\{\beta_1, \dots, \beta_r\}$  (respectively,  $\{F(\beta_1), \dots, F(\beta_r)\}$ ) is exactly the set of vertices of the Newton polyhedron  $P_I$  (respectively,  $P_{F(I)}$  or  $P_J$ ). We have the following lemma that compares the relative interiors of  $\alpha P_{F(I)}$  and  $\alpha P_J$ .

LEMMA 4.1. *For general  $\mathbb{N}A$ , the relative interiors of  $P_{F(I)}$  and  $P_J$  satisfy*

$$\text{int}(\alpha P_{F(I)}) = F(C) \cap \text{int}(\alpha P_J).$$

*Proof.* A point  $u$  lies in  $\text{int}(P_{F(I)})$  if and only if  $u - v \in \text{int}(F(C))$  for some  $v$  on a bounded face of  $P_{F(I)}$ . Similarly, a point  $u$  lies in  $\text{int}(P_J)$  if and only if  $u - v \in \text{int} \mathbb{R}_{\geq 0}^{\mathcal{F}}$  for some  $v$  on a bounded face of  $P_J$ . Since  $F(I)$  and  $J$  have the same minimal monomial generators, the bounded faces of  $P_{F(I)}$  and  $P_J$  coincide. Therefore, it suffices to show that

$$\text{int}(F(C)) = F(C) \cap \text{int}(\mathbb{R}_{\geq 0}^{\mathcal{F}}).$$

But for  $p \in C$ ,  $F(p) \in \text{int} F(C)$  if and only if  $p$  is not contained in any facet  $\sigma \in \mathcal{F}$ , which means exactly  $F_\sigma(p) > 0$ . □

Let  $e$  be the element in  $\mathbb{R}^{\mathcal{F}}$  such that  $e_\sigma = 1$  for all  $\sigma \in \mathcal{F}$ .

PROPOSITION 4.2. *The multiplier ideal  $\mathcal{J}(X, \alpha I)$  is equal to the ideal*

$$\langle y^v \in \mathbb{k}[\mathbb{N}A] \mid F(v) + e \in \text{int}(\alpha P_J) \rangle.$$

*Proof.* By (4.1) and (4.2) it suffices to show that

$$F(\Omega_{\alpha I} \cap \mathbb{N}A) = (-e + \text{int}(\alpha P_J)) \cap F(\mathbb{N}A).$$

For the containment  $F(\Omega_{\alpha I} \cap \mathbb{N}A) \subseteq (-e + \text{int}(\alpha P_J)) \cap F(\mathbb{N}A)$ , it is enough to verify that

$$F(w + \text{int}(\alpha P_I)) \subseteq [-e + \text{int}(\alpha P_J)]$$

for any  $w \in \mathbb{Q}^d$  satisfying  $F_\sigma(w) \geq -1$  for all  $\sigma \in \mathcal{F}$ . Any such  $w$  satisfies  $F(w) + e \in \mathbb{R}_{\geq 0}^{\mathcal{F}}$ , so by Lemma 4.1

$$\begin{aligned} F(w + \text{int}(\alpha P_I)) &= [F(w) + \text{int}(\alpha F(P_I))] \\ &= [F(w) + \text{int}(\alpha P_{F(I)})] \\ &\subseteq [F(w) + \text{int}(\alpha P_J)] \\ &= [-e + (F(w) + e) + \text{int}(\alpha P_J)] \\ &\subseteq [-e + \mathbb{R}_{\geq 0}^{\mathcal{F}} + \text{int}(\alpha P_J)] \\ &\subseteq [-e + \text{int}(\alpha P_J)]. \end{aligned}$$

For the other containment  $F(\Omega_{\alpha I} \cap \mathbb{N}A) \supseteq (-e + \text{int}(\alpha P_J)) \cap F(\mathbb{N}A)$ , let  $v \in \mathbb{N}A$  be such that  $F(v) + e \in \text{int}(\alpha P_J)$ . Then there exists  $u$  lying on a bounded face of  $\alpha P_I$  such that

$$F(v) + e - F(u) \in \text{int}(\mathbb{R}_{\geq 0}^{\mathcal{F}}).$$

In particular,  $F_\sigma(v - u) > -1$  for all  $\sigma \in \mathcal{F}$ . Take any  $p \in \text{int} C$  and  $\epsilon > 0$  small enough so that

$$w := v - u - \epsilon p \in \mathbb{Q}^d \quad \text{and} \quad F_\sigma(w) > -1.$$

Then  $F(v) - F(u) - F(w) = F(\epsilon p) \in \text{int} C$  where  $F(u)$  lies on a bounded face of  $\alpha P_{F(I)}$ . Therefore, we have  $F(v) - F(w) \in \text{int}(\alpha P_{F(I)})$ , and hence  $F(v) \in F(\Omega_{w, \alpha I} \cap \mathbb{N}A) \subseteq F(\Omega_{\alpha I} \cap \mathbb{N}A)$ , as desired.  $\square$

REMARK 4.3. Notice that  $e$  may not be in  $F(\mathbb{N}A)$  and that  $e \in F(\mathbb{N}A)$  if and only if  $X$  is Gorenstein. Even if we extend to rational coefficients, then the element  $e$  may not be in  $F(\mathbb{Q} \otimes \mathbb{N}A) = F(\mathbb{Q}^d)$ . The condition  $e \in F(\mathbb{Q}^d)$  holds exactly when  $X$  is  $\mathbb{Q}$ -Gorenstein, namely  $K_X$  is a  $\mathbb{Q}$ -Cartier divisor. This is the case where  $X$  is  $\mathbb{Q}$ -factorial (or, equivalently, the semigroup  $\mathbb{N}A$  is simplicial).

The new expression of multiplier ideals in Proposition 4.2 is motivated by an observation in the special case where  $X$  is  $\mathbb{Q}$ -Gorenstein. In that case, we have  $e \in F(\mathbb{Q}^d)$  and  $\Omega_{\alpha I} = \Omega_{F^{-1}(-e), \alpha I}$ . So  $w = F^{-1}(-e)$  corresponds to the  $\Delta$  of de Fernex and Hacon. In fact, the divisor  $\Delta_{F^{-1}(-e)} = 0$  coincides with the

canonical choice of boundary divisor for the pair  $(X, \alpha I)$  in the  $\mathbb{Q}$ -Gorenstein case. Therefore, by (4.2) the multiplier ideal

$$\begin{aligned} \mathcal{J}(X, \alpha I) &= \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid v \in \Omega_{\alpha I} \rangle \\ &= \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid v \in \Omega_{F^{-1}(-e), \alpha I} \rangle \\ &= \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid F(v) + e \in \text{int}(\alpha F(P_I)) \rangle \\ &= \langle y^v \in \mathbb{k}[\mathbb{N}A] \mid F(v) + e \in \text{int}(\alpha P_{F(I)}) \rangle. \end{aligned}$$

#### 4.2. Roots–Jumping Coefficients Correspondence on Toric Varieties

Now, we combine the previous observations to establish the following theorem.

**THEOREM 4.4.** *Let  $\mathbb{N}A$  be a normal semigroup, let  $X = \text{Spec}(\mathbb{k}[\mathbb{N}A])$  be its associated affine toric variety, and let  $I$  be a monomial ideal on  $X$ . Suppose  $\alpha_I$  is the smallest root of  $b_I(-s)$  where  $b_I(s)$  is the Bernstein–Sato polynomial of  $I$  in  $\mathbb{k}[\mathbb{N}A]$ . Then all jumping coefficients of the pair  $(X, I)$  in  $[\alpha_I, \alpha_I + 1)$  are roots of  $b_I(-s)$ . Moreover, the number  $\alpha_I$  is the smallest jumping coefficient (i.e. the log-canonical threshold) of  $(X, I)$ .*

*Proof.* By Proposition 3.2 the Bernstein–Sato polynomial  $b_I(s)$  of  $I$  coincides with the Bernstein–Sato polynomial of the monomial ideal  $J = \mathbb{k}[\mathbb{N}^{\mathcal{F}}] \cdot \langle F(I) \rangle$ . Thus by [BMS06a, Thm. 2], the jumping coefficients of the pair  $(\mathbb{A}^{\mathcal{F}}, J)$  in  $[\alpha_I, \alpha_I + 1)$  are roots of  $b_I(-s)$ . By Howald’s formula a number  $\alpha$  is a jumping coefficient of  $(\mathbb{A}^{\mathcal{F}}, J)$  exactly when the boundary of  $(-e + \alpha P_J)$  contains a lattice point in  $\mathbb{N}^{\mathcal{F}}$ . Also, according to Proposition 4.2, a number  $\alpha$  is a jumping coefficient of  $(X, I)$  exactly when the boundary of  $(-e + \alpha P_J)$  contains a lattice point in  $F(\mathbb{N}A)$ . Therefore, the jumping coefficients of  $(X, I)$  are jumping coefficients of  $(\mathbb{A}^{\mathcal{F}}, J)$ , and the first statement of this theorem follows.

To prove  $\alpha_I$  is the log-canonical threshold of  $(X, I)$ , it suffices to show that the boundary of  $(-e + \alpha_I P_J)$  intersects  $F(\mathbb{N}A)$ . Since  $\alpha_I$  is the log-canonical threshold of  $(\mathbb{A}^{\mathcal{F}}, J)$ , we have

$$1 \in \mathbb{k}[\mathbb{N}^{\mathcal{F}}] = \mathcal{J}(\mathbb{A}^{\mathcal{F}}, \alpha_I J), \quad \text{and hence } F(0) = 0 \in (-e + \alpha_I P_J) \cap F(\mathbb{N}A). \quad \square$$

**EXAMPLE 4.5.** Let  $A = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{smallmatrix}\right)$  and  $I = \langle y_1 y_2, y_1 y_2^2 \rangle$  a monomial ideal in  $\mathbb{k}[\mathbb{N}A]$ . In this case, the semigroup ring  $\mathbb{k}[\mathbb{N}A]$  is simplicial and hence  $\mathbb{Q}$ -Gorenstein. The linear mapping  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is represented by the matrix  $\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$ . The subsemigroup  $\mathbb{k}[F(\mathbb{N}A)]$  of  $\mathbb{k}[x_1, x_2]$  is generated by  $x_1^3, x_1^2 x_2, x_1 x_2^2$ , and  $x_2^3$  and by the monomial ideal  $J = \mathbb{k}[x_1, x_2] \langle x_1^2 x_2, x_1 x_2^2 \rangle$ . Using the method of Budur, Mustařa and Saito as discussed in Section 2.2, we can compute the Bernstein–Sato polynomial  $b_J(s) = (s + 1)^2(3s + 2)(3s + 4)$ . (We point out that  $b$ -function algorithms have been developed and implemented; see [BL10].) Moreover, by Proposition 3.2 we have  $b_I(s) = (s + 1)^2(3s + 2)(3s + 4)$  as well. On the

other hand, using Howald’s formula, we find that  $\frac{2}{3}$ , 1, and  $\frac{4}{3}$  are jumping coefficients of  $(\mathbb{A}^2, J)$ , but only  $\frac{2}{3}$  and 1 are jumping coefficients of  $(X, I)$  according to Proposition 4.2.

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## References

- [ÀHN16] J. Àlvarez Montaner, C. Huneke, and L. Núñez-Betancourt, *D-modules, Bernstein–Sato polynomials and F-invariants of direct summands*, preprint, 2016, [arXiv:1611.04412](https://arxiv.org/abs/1611.04412).
- [BL10] C. Berkesch and A. Leykin, *Algorithms for Bernstein–Sato polynomials and multiplier ideals*, ISSAC 2010—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation, pp. 99–106, ACM, New York, 2010.
- [Ber72] I. N. Bernšteĭn, *Analytic continuation of generalized functions with respect to a parameter*, Funktsional. Anal. i Prilozhen. 6 (1972), no. 4, 26–40.
- [BGG72] I. N. Bernšteĭn, I. M. Gel’fand, and S. I. Gel’fand, *Differential operators on a cubic cone*, Uspekhi Mat. Nauk 27 (1972), no. 1(163), 185–190.
- [Bli04] M. Blickle, *Multiplier ideals and modules on toric varieties*, Math. Z. 248 (2004), no. 1, 113–121.
- [BMS06a] N. Budur, M. Mustața, and M. Saito, *Bernstein–Sato polynomials of arbitrary varieties*, Compos. Math. 142 (2006), no. 3, 779–797.
- [BMS06b] ———, *Combinatorial description of the roots of the Bernstein–Sato polynomials for monomial ideals*, Comm. Algebra 34 (2006), no. 11, 4103–4117.
- [dFH09] T. de Fernex and C. D. Hacon, *Singularities on normal varieties*, Compos. Math. 145 (2009), no. 2, 393–414.
- [ELSV04] L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Math. J. 123 (2004), no. 3, 469–506.
- [Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Publ. Math. Inst. Hautes Études Sci. 32 (1967), 361.
- [How01] J. A. Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. 353 (2001), no. 7, 2665–2671 (electronic).
- [Igu00] J. Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Stud. Adv. Math., 14, Amer. Math. Soc., Providence, RI; International Press, Cambridge, MA, 2000.
- [Jon94] A. G. Jones, *Rings of differential operators on toric varieties*, Proc. Edinb. Math. Soc. (2) 37 (1994), no. 1, 143–160.
- [Kas83] M. Kashiwara, *Vanishing cycle sheaves and holonomic systems of differential equations*, Algebraic geometry, Tokyo/Kyoto, 1982, Lecture Notes in Math., 1016, pp. 134–142, Springer, Berlin, 1983.
- [Kol97] J. Kollár, *Singularities of pairs*, Algebraic Geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, pp. 221–287, Amer. Math. Soc., Providence, RI, 1997.

- [Laz04] R. Lazarsfeld, *Positivity for vector bundles, and multiplier ideals, Positivity in algebraic geometry. II*, *Ergeb. Math. Grenzgeb. (3). A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 49, Springer, Berlin, 2004.
- [Lic89] B. Lichtin, *Poles of  $|f(z, w)|^{2s}$  and roots of the  $b$ -function*, *Ark. Mat.* 27 (1989), no. 2, 283–304.
- [Mal83] B. Malgrange, *Polynômes de Bernstein–Sato et cohomologie évanescence, analysis and topology on singular spaces, II, III*, *Astérisque*, 101, pp. 243–267, Soc. Math. France, Paris, 1983.
- [Mus87] I. M. Musson, *Rings of differential operators on invariant rings of tori*, *Trans. Amer. Math. Soc.* 303 (1987), no. 2, 805–827.
- [ST01] M. Saito and W. N. Traves, *Differential algebras on semigroup algebras*, *Symbolic Computation: Solving Equations In Algebra, Geometry, And Engineering* (South Hadley, MA, 2000), *Contemp. Math.*, 286, pp. 207–226, Amer. Math. Soc., Providence, RI, 2001.
- [SS72] M. Sato and T. Shintani, *On zeta functions associated with prehomogeneous vector spaces*, *Proc. Natl. Acad. Sci. USA* 69 (1972), 1081–1082.
- [Yan83] T. Yano,  *$b$ -functions and exponents of hypersurface isolated singularities, Singularities, Part 2* (Arcata, Calif., 1981), *Proc. Sympos. Pure Math.*, 40, pp. 641–652, Amer. Math. Soc., Providence, RI, 1983.

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