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A Skoda-Type Integrability Theorem for Singular Monge–Ampère Measures

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Abstract. Let φ be a plurisubharmonic function defined on an open subset of $\mathbb{C}^n$. We give a sufficient condition for the local integrability of $e^{-\varphi}$ with respect to a Monge–Ampère measure with Hölder-continuous potential $\mu = (dd^c u)^n$. This condition is expressed in terms of the Lelong numbers of φ and the Hölder exponent of u.

1. Introduction and Main Result

Let Ω be an open subset of $\mathbb{C}^n$. Recall that a function $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* (p.s.h. for short) if φ is upper semicontinuous, not identically $-\infty$ in some connected component of Ω, and if for every complex line $L \subset \mathbb{C}^n$, the function $\varphi|_{\Omega \cap L}$ is either subharmonic in $\Omega \cap L$ or identically $-\infty$.

The *Lelong number* of φ at $a \in \Omega$ can be defined as

$$\nu(\varphi; a) := \liminf_{z \to a, z \neq a} \frac{\varphi(z)}{\log|z-a|},$$

and it somewhat measures the singularity of φ at a. This number can be characterized as $\nu(\varphi; a) = \sup\{\gamma : \varphi(z) \leq \gamma \log|z-a| + O(1) \text{ as } z \to a\}$ and is one of the most basic quantities associated with the pole of φ at a. The function $z \mapsto \nu(\varphi; z)$ is upper semicontinuous with respect to the usual topology, and a deep theorem of Siu [Siu74] states that this function is also upper semicontinuous with respect to the Zariski topology, that is, for every $c > 0$, the set $\{a \in \Omega; \nu(\varphi, a) \geq c\}$ is a closed analytic subvariety of Ω. For further properties and equivalent definitions of the Lelong number, the reader may consult [Dem] and [Hör07].

Another way of measuring the singularity of a p.s.h. function at a point was introduced by Demailly and Kollár [DK01] (see also [Tia87]) as a tool to study several types of algebraic and analytic objects, such as holomorphic functions, divisors, coherent ideal sheaves, positive closed currents, and so on. They define the *integrability index* of φ at a by

$$c(\varphi; a) = \sup\{c \geq 0 : e^{-2c\varphi} \text{ is Lebesgue integrable in a neighborhood of } a\}.$$

A classical theorem of Skoda [Sko72] states that if $\nu(\varphi; a) < 2$, then $e^{-\varphi}$ is integrable in a neighborhood of a with respect to the Lebesgue measure. In terms of the quantities defined, this translates as $c(\varphi; a) \geq \nu(\varphi; a)^{-1}$. In the other direction, we also have $c(\varphi; a) \leq n\nu(\varphi; a)^{-1}$.

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The aim of this paper is to give a generalization of Skoda's integrability theorem with the Lebesgue measure replaced by a general Monge–Ampère mass with a Hölder-continuous local potential (see Section 2.1 for the definition and some examples). This class of measures is important in many applications of pluripotential theory to complex geometry and dynamics and has been the topic of recent research. The reader may consult [DN14] for a complete characterization of these measures and [DDG+14] for some of its properties.

THEOREM 1.1. *Let* $0 < \alpha \le 1$, *let* u *be an* α*-Hölder continuous p.s.h. function in the domain* $\Omega \subset \mathbb{C}^n$, *and let* $z \in \Omega$. *If* φ *is a p.s.h. function in* Ω *and if* $\nu(\varphi; z) < \frac{2\alpha}{\alpha+n(2-\alpha)}$, *then there is a neighborhood* $K \subset \Omega$ *of* z *such that the integral*

$$\int_K e^{-\varphi}(\mathrm{dd}^c u)^n$$

is finite. In other words, $e^{-\varphi}$ *is locally integrable in* $U := \{\xi \in \Omega; \nu(\varphi;\xi) < \frac{2\alpha}{\alpha+n(2-\alpha)}\}$ *with respect to the positive measure* $(\mathrm{dd}^c u)^n$.

As a corollary, we obtain an estimate of the mass of the measure $\mu = (\mathrm{dd}^c u)^n$ over the sublevel sets of a p.s.h. function. In the particular case where μ is the Lebesgue measure, this kind of estimate is useful in applications to complex geometry and dynamics; see Corollary 4.3.

The proof of Theorem 1.1 is inspired by the methods of [DNS10] and consists of approximating u by smooth potentials u_ε and applying successive integration by parts in order to replace $(\mathrm{dd}^c u)^n$ with $(\mathrm{dd}^c u_\varepsilon)^n$. At each step, we use Skoda's theorem in the form of an estimate of the volume of the sublevel sets of φ due to Kiselman and the Dinh–Nguyên–Sibony theorem stating that Monge–Ampère measures with Hölder-continuous potentials are locally moderate.

2. Preliminaries

2.1. Monge–Ampère Measures with Hölder-Continuous Potentials

Let X be a complex manifold of dimension n. Recall that a k-current on X is a continuous linear form on the space of compactly supported differential forms of degree $(2n-k)$. Such objects generalize k-forms with coefficients in L^1_{loc} and submanifolds of real codimension k. For the basic theory of currents in complex manifolds, see [Dem].

The existence of a complex structure implies, by duality, that every k-current decomposes as a sum of (p,q)-currents with $p+q=k$. It is a remarkable fact that real currents of type (p,p) carry a notion of positivity (see [Dem; Lel98]). Examples of positive currents include Kähler forms ($p=1$), currents of integration along complex submanifolds of X, and positive measures ($p=n$).

The operators ∂, $\bar\partial$, d, and $\mathrm{d}^c = \frac{i}{2\pi}(\bar\partial - \partial)$ extend to currents by duality, and the dd^c-Poincaré lemma states that every positive closed $(1,1)$-current T can be written locally as $T = \mathrm{dd}^c u$ where u is a p.s.h. function, called a local potential of T.

If T is a positive closed current and u is a locally bounded p.s.h. function, then the current uT is well defined, and the product (or intersection) current $dd^c u \wedge T$ can be defined by the formula

$$dd^c u \wedge T \stackrel{\text{def}}{=} dd^c(uT).$$

We may then define the product $S \wedge T$ when S is a positive closed $(1,1)$-current with bounded local potential: just write $S = dd^c u$ locally and use the last formula. We note however that the wedge product of general currents is not always well defined.

By induction the product $dd^c u_1 \wedge \cdots \wedge dd^c u_p$ is well defined for every locally bounded p.s.h. functions $u_1, \dots, u_p$. In particular, if u is a locally bounded p.s.h. function, then the current $(dd^c u)^n$ is well defined. It is a positive measure called the *Monge–Ampère measure* associated with u. See [Kli91] or the original paper [BT82] for some basic properties of the Monge–Ampère operator $u \mapsto (dd^c u)^n$.

Let now Ω be a domain in $\mathbb{C}^n$, and let u be an α-Hölder continuous p.s.h. function on Ω for some $0 < \alpha \leq 1$. From the previous discussion we can define the positive measure

$$\mu = (dd^c u)^n.$$

We call such a measure a *Monge–Ampère measure with Hölder-continuous potential.*

EXAMPLE 2.1. Measures that are absolutely continuous with respect to the Lebesgue measure with a density in L^p for some $p > 1$ are examples of Monge–Ampère measures with Hölder-continuous potential.

More precisely, let Ω be a bounded strongly pseudoconvex domain of $\mathbb{C}^n$. Denote by λ_{2n} the Lebesgue measure on $\mathbb{C}^n$. Let $\phi \in \mathcal{C}^{1,1}(\partial\Omega)$ and $f \in L^p(\Omega)$. By a result of Guedj, Zeriahi, and Kolodziej [GKZ08] the Dirichlet problem

$$\begin{cases} (dd^c u)^n = f\lambda_{2n} & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ which is α-Hölder continuous on $\overline{\Omega}$. The exponent α is explicit in terms of p and n. For a global counterpart of this result, see [DDG+14].

EXAMPLE 2.2. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$. Let T be its Green current, and $\mu = T^k$ be its Green measure (see [DS10] for the definitions). The current T has Hölder-continuous local potentials. Therefore μ locally is a Monge–Ampère measure with Hölder-continuous potential.

In the proof of Theorem 1.1, we will need the following simple regularization result, which can be proved using standard convolution with a cut-off function. We denote by B_r the ball of radius r centered at the origin of $\mathbb{C}^n$.

LEMMA 2.3. *Let* $r > 0$, *and let* u *be an* α*-Hölder-continuous p.s.h. function in* B_r *for some* $0 < \alpha \leq 1$. *Fix* $\rho > 0$ *such that* $\rho < r$. *Then, for every* $\varepsilon > 0$ *sufficiently small, there are a smooth p.s.h. function* u_ε *defined on* $B_{r-\rho}$ *and constants*

$c_1, c_2 > 0$ *independent of* ε *such that*

$$\|u - u_\varepsilon\|_{L^\infty(B_{r-\rho})} \leq c_1 \varepsilon^\alpha \quad \textit{and} \quad \|u_\varepsilon\|_{\mathcal{C}^2(B_{r-\rho})} \leq c_2 \varepsilon^{\alpha-2}.$$

2.2. *Locally Moderate Currents*

We now introduce the notion of a locally moderate current, which is the main tool in the proof of the Theorem 1.1. Recall that the set of all p.s.h. functions in X is closed in the space $L^1_{\mathrm{loc}}(X)$ and that every family of p.s.h. functions that is bounded in $L^1_{\mathrm{loc}}(X)$ is relatively compact in $L^1_{\mathrm{loc}}(X)$ (see Theorem 3.2.12 in [Hör07]). For simplicity, such a family is called a compact family.

The concept of locally moderate measures was introduced by Dinh and Sibony [DS06] and is very useful in the study of complex dynamical systems (see also [DS10]).

DEFINITION 2.4. A measure μ on a complex manifold X is called *locally moderate* if for any open set $U \subset X$, any compact set $K \subset U$, and any compact family $\mathcal{F}$ of p.s.h. functions on U, there are constants $\beta > 0$ and $C > 0$ such that

$$\int_K e^{-\beta\psi}\, d\mu \leq C \quad \text{for every } \psi \in \mathcal{F}.$$

It follows immediately from the definition that, for any $\mathcal{F}$ and μ as in the definition, $\mathcal{F}$ is bounded in $L^p_{\mathrm{loc}}(\mu)$ for $1 \leq p < \infty$ and μ does not charge pluripolar sets.

A positive closed current S of type (p, p) on X is said to be *locally moderate* if the trace measure $\sigma_S = S \wedge \omega^{n-p}$ is locally moderate. Here $n = \dim X$, and ω is the fundamental form of a fixed Hermitian metric on X.

THEOREM 2.5 (Dinh, Nguyên, and Sibony [DNS10]). *If* $1 \leq p \leq n$ *and* $u_1, \dots, u_p$ *are Hölder-continuous p.s.h. functions on* X, *then the Monge–Ampère current* $\mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_p$ *is locally moderate.*

This result says, in particular, that measures of the form $\mu = (\mathrm{dd}^c u)^n$ with a Hölder-continuous p.s.h. function u are locally moderate. It seems interesting to know whether the converse is true, that is, whether every locally moderate measure is locally of the form $\mu = (\mathrm{dd}^c u)^n$ for some Hölder-continuous p.s.h. function u. The answer is positive whenever the measure μ is smooth outside a finite set and, around these singular points, it has radial or toric symmetries; see [DDG+14].

3. Proof of the Integrability Theorem

This section is devoted to the proof of Theorem 1.1 and some related results.

We first state two lemmas used in the proof of Theorem 2.5 that will also be useful for us. We denote by B_r the ball of radius r centered at the origin of $\mathbb{C}^n$ and fix a Hermitian form ω as before.

LEMMA 3.1 ([DNS10]). *Let S be a locally moderate positive closed current of type $(n-1,n-1)$ on B_r. If $\mathcal{G}$ is a compact family of p.s.h. functions on B_r, then $\mathcal{G}$ is bounded in $L^1_{\mathrm{loc}}(\sigma_S)$. Moreover, the masses of the measures $\mathrm{dd^c}\varphi \wedge S$, $\varphi \in \mathcal{G}$, are locally bounded in B_r uniformly for φ.*

Proof. Let K be a compact subset of B_r. After subtracting a fixed constant, we may assume that every element of $\mathcal{G}$ is negative on K. Since σ_S is locally moderate, we can choose $\beta, C > 0$ such that $\int_K e^{-\beta\varphi}\, d\sigma_S \le C$ for every $\varphi \in \mathcal{G}$. We thus have $\int_K \beta|\varphi|\, d\sigma_S \le \int_K e^{-\beta\varphi}\, d\sigma_S \le C$ for every $\varphi \in \mathcal{G}$, which proves the first assertion.

For the second assertion, let K be a compact subset of B_r and consider a cut-off function χ that is equal to 1 in a neighborhood of K and is supported on a larger compact $L \subset B_r$. We have, for $\varphi \in \mathcal{G}$,

$$\int_K \mathrm{dd^c}\varphi \wedge S \le \int_L \chi\,\mathrm{dd^c}\varphi \wedge S = \int_L \mathrm{dd^c}\chi \wedge \varphi S \le \|\chi\|_{\mathcal{C}^2} \int_L |\varphi|\, d\sigma_S,$$

which is uniformly bounded by the first part of the lemma. □

LEMMA 3.2 ([DNS10]). *Let $r > 0$, S be a locally moderate positive closed current of type $(n-1,n-1)$ on B_{2r}, and u be an α-Hölder-continuous p.s.h. function on B_r that is smooth on $B_r \setminus B_{r-4\rho}$ for some $0 < \rho < r/4$. Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \le \chi \le 1$, and $\chi \equiv 1$ on $B_{r-2\rho}$.*

If φ is a p.s.h. function on B_{2r}, then

$$\int_{B_r} \chi\varphi\mathrm{dd^c}(uS) = -\int_{B_r\setminus B_{r-3\rho}} \mathrm{dd^c}\chi \wedge \varphi u S - \int_{B_r\setminus B_{r-3\rho}} \mathrm{d}\chi \wedge \varphi \mathrm{d^c}u \wedge S + \int_{B_r\setminus B_{r-3\rho}} \mathrm{d^c}\chi \wedge \varphi \mathrm{d}u \wedge S + \int_{B_{r-\rho}} \chi u \mathrm{dd^c}\varphi \wedge S.$$

Notice that the smoothness of u in $B_r \setminus B_{r-4\rho}$ makes the second and third integrals in the right-hand side meaningful.

Idea of the proof. The case where φ is smooth follows from a direct computation using integration by parts. The general case follows by approximating φ by a decreasing sequence of smooth p.s.h. functions. See [DNS10] for the complete proof. □

We also need a volume estimate of the sublevel sets of p.s.h. functions obtained by Kiselman [Kis00]. We include Kiselman's argument here for the the reader's convenience.

LEMMA 3.3 ([Kis00]). *Let φ be a p.s.h. function on an open set $\Omega \subset \mathbb{C}^n$, and $K \subset \Omega$ be a compact subset. Then, for every $\gamma < 2/\sup_{z\in K} \nu(\varphi; z)$, there is a constant $C_\gamma = C_\gamma(\varphi, \Omega, K)$ such that*

$$\lambda_{2n}(K \cap \{\varphi \le -M\}) \le C_\gamma e^{-\gamma M}, \quad M \in \mathbb{R},$$

where λ_{2n} denotes the Lebesgue measure in $\mathbb{C}^n$.

Proof. Since $e^{\gamma(-M-\varphi)} \geq 1$ on $K \cap \{\varphi \leq -M\}$, we have

$$\lambda_{2n}(K \cap \{\varphi \leq -M\}) \leq \int_K e^{\gamma(-M-\varphi(z))}\, d\lambda_{2n}(z) = e^{-\gamma M} \int_K e^{-\gamma\varphi}\, d\lambda_{2n}.$$

It suffices then to take $C_\gamma = \int_K e^{-\gamma\varphi}\, d\lambda_{2n}$, which is finite by Skoda's theorem since $\nu(\gamma\varphi; z) < 2$ for every $z \in K$. □

Before getting to the proof of Theorem 1.1, we need the following simple extension of the second part of Lemma 3.1.

LEMMA 3.4. *Let $1 \leq p \leq n$, and let S be a locally moderate positive closed current of type $(n-p-1, n-p-1)$ on B_r. If $\mathcal{G}$ is a compact family of p.s.h. functions on B_r and $\mathcal{H}$ is a locally uniformly bounded family of p.s.h. functions on B_r, then the masses of the measures $\mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S$, $\varphi \in \mathcal{G}$, $u \in \mathcal{H}$, are locally bounded in B_r uniformly in φ and u.*

Proof. Fix a compact subset K of B_r and let $L_0 = K, L_1, \ldots, L_p$ be compact subsets of B_r such that L_i is contained in the interior of L_{i+1}. Let χ_i, $i = 1, \ldots, p$, be smooth cut-off functions such that $0 \leq \chi_i \leq 1$, $\chi_i \equiv 1$ in L_{i-1}, and χ_i is supported in L_i. Then, for $\varphi \in \mathcal{G}$ and $u \in \mathcal{H}$, the mass of $\mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S$ over K is bounded by

$$\begin{aligned}
&\int_{L_1} \chi_1 \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^p \wedge S \\
&\quad = \int_{L_1} u(\mathrm{dd}^c\chi_1) \wedge \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \\
&\quad \leq \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_1} \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \wedge \omega \\
&\quad \leq \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_2} \chi_2 \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-1} \wedge S \wedge \omega \\
&\quad = \|\chi_1\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \int_{L_2} u \mathrm{dd}^c\chi_2 \wedge \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-2} \wedge S \wedge \omega \\
&\quad \leq \|\chi_1\|_{\mathcal{C}^2} \|\chi_2\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \|u\|_{L^\infty(L_2)} \int_{L_2} \mathrm{dd}^c\varphi \wedge (\mathrm{dd}^c u)^{p-2} \wedge S \wedge \omega^2 \\
&\quad \leq \cdots \\
&\quad \leq \|\chi_1\|_{\mathcal{C}^2} \cdots \|\chi_p\|_{\mathcal{C}^2} \|u\|_{L^\infty(L_1)} \cdots \|u\|_{L^\infty(L_p)} \int_{L_p} \mathrm{dd}^c\varphi \wedge S \wedge \omega^p,
\end{aligned}$$

where $\omega = \mathrm{dd}^c\|z\|^2$ is the standard fundamental form on $\mathbb{C}^n$. The result now follows from Lemma 3.1 and the fact that $\|u\|_{L^\infty(L_i)}$ is bounded independently of u. □

Proof of Theorem 1.1. There is no loss of generality in assuming that $z = 0$. Since φ is locally bounded from above, we may also assume that φ is negative. As before, $\omega = \mathrm{dd}^c\|z\|^2$.

The proof is inspired by the methods in [DNS10, Theorem 1.1]. It consists of successive applications of integration-by-parts formulas (Lemma 3.2) together with a regularization procedure.

For $N > 0$, define $\varphi_N = \max\{\varphi, -N\}$ and $\psi_N = \varphi_{N-1} - \varphi_N$. Notice that $0 \leq \psi_N \leq 1$, ψ_N is supported in $\{\varphi < -N+1\}$, and $\psi_N \equiv 1$ in $\{\varphi < -N\}$.

Observe that

$$\begin{aligned}\int e^{-\varphi}(\mathrm{dd}^c u)^n &= \sum_{N=0}^{\infty} \int_{\{-N \leq \varphi < -N+1\}} e^{-\varphi}(\mathrm{dd}^c u)^n \\ &\leq \sum_{N=0}^{\infty} e^N \int_{\{-N \leq \varphi < -N+1\}} (\mathrm{dd}^c u)^n \\ &\leq \sum_{N=0}^{\infty} e^N \int \psi_{N-1}(\mathrm{dd}^c u)^n. \end{aligned} \tag{3.1}$$

By the hypothesis that $\nu(\varphi; 0) < \frac{2\alpha}{\alpha+n(2-\alpha)}$ and the upper semicontinuity of the function $z \mapsto \nu(\varphi; z)$ there is $r > 0$ such that $\sup_{z \in B_{2r}} \nu(\varphi, z) \leq \frac{2\alpha}{\alpha+n(2-\alpha)} - \sigma$ for a small constant $\sigma > 0$. From Lemma 3.3 we get that

$$\lambda_{2n}(B_{2r} \cap \{\varphi \leq -N+1\}) \lesssim e^{-((\alpha+n(2-\alpha))/\alpha+\delta)N} = e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}, \tag{3.2}$$

where $\delta > 0$ is a small constant (depending on φ). Here and in what follows, the symbol $\lesssim$ means that the left-hand side is smaller than or equal to a constant times the right-hand side, the constant being independent from N.

Taking a smaller r if necessary, we may assume that u is defined on B_{2r}. Subtracting a constant, we may assume that $u \leq -1$. Consider the function $v(z) = \max(u(z), A \log \|z\|)$. Choosing $A > 0$ sufficiently small, we see that v coincides with u near the origin and that $v(z) = A \log \|z\|$ near the boundary of B_r. This allows us to assume that $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$ for some fixed $\rho < r/4$. Notice that, in particular, u is smooth on $B_r \setminus B_{r-4\rho}$.

Fix a smooth cut-off function χ with compact support in $B_{r-\rho}$, $0 \leq \chi \leq 1$, and $\chi \equiv 1$ on $B_{r-2\rho}$. Applying Lemma 3.2 to ψ_{N-1} and $(\mathrm{dd}^c u)^{n-1}$ and noticing that $(\mathrm{dd}^c u)^n = \mathrm{dd}^c(u(\mathrm{dd}^c u)^{n-1})$, we get

$$\begin{aligned}\int_{B_r} \chi \psi_{N-1}(\mathrm{dd}^c u)^n = &- \int_{B_r \setminus B_{r-3\rho}} \mathrm{dd}^c \chi \wedge \psi_{N-1} u (\mathrm{dd}^c u)^{n-1} \\ &- \int_{B_r \setminus B_{1-3\rho}} \mathrm{d}\chi \wedge \psi_{N-1} \mathrm{d}^c u \wedge (\mathrm{dd}^c u)^{n-1} \\ &+ \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}^c \chi \wedge \psi_{N-1} \mathrm{d}u \wedge (\mathrm{dd}^c u)^{n-1} \\ &+ \int_{B_{r-\rho}} \chi u \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1}. \end{aligned} \tag{3.3}$$

Observing that u is smooth in $B_r \setminus B_{r-3\rho}$ and that the support of ψ_{N-1} is contained in $\{\varphi \leq -N+1\}$ and using the volume estimate (3.2), we get that the absolute values of the first three integrals on the right-hand side are $\leq c_1 e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}$, where $c_1 > 0$ does not depend on N.

For $N \geq 1$, set $\varepsilon = \varepsilon(N) = e^{-(1/\alpha + c)N}$, where $0 < c < \frac{\delta}{n(2-\alpha)}$. Using Lemma 2.3, we can find, for N large, a regularization u_ε of u defined on $B_{r-\rho}$ such that

$$\|u - u_\varepsilon\|_\infty \lesssim \varepsilon^\alpha = e^{-(1+c\alpha)N} \quad \text{and} \quad \|u_\varepsilon\|_{\mathcal{C}^2} := \|u_\varepsilon\|_{\mathcal{C}^2(B_{r-\rho})} \lesssim \varepsilon^{\alpha-2}.$$

Writing $u = u_\varepsilon + (u - u_\varepsilon)$, the last integral in (3.3) is equal to

$$\int_{B_{r-\rho}} \chi u_\varepsilon \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1} + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1}.$$

Since $\{\varphi_N\}_{N\geq 0}$ is a compact family of p.s.h. functions and since the current $(\mathrm{dd}^c u)^{n-1}$ is locally moderate (Theorem 2.5), we see from Lemma 3.1 that the absolute value of the second integral is less than $c_2 \|u - u_\varepsilon\|_\infty \leq c_2' e^{-(1+c\alpha)N}$, where $c_2' > 0$ does not depend on N.

To deal with the remaining integral, we apply Lemma 3.2 for u_ε instead of u. Noticing that $\mathrm{dd}^c(u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}) = \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}$, we get

$$\begin{aligned}\int_{B_{r-\rho}} \chi u_\varepsilon \mathrm{dd}^c \psi_{N-1} \wedge (\mathrm{dd}^c u)^{n-1} = &\int_{B_r \setminus B_{r-3\rho}} \mathrm{dd}^c \chi \wedge \psi_{N-1} u_\varepsilon (\mathrm{dd}^c u)^{n-1} \\ &+ \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}\chi \wedge \psi_{N-1} \mathrm{d}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1} \\ &- \int_{B_r \setminus B_{r-3\rho}} \mathrm{d}^c \chi \wedge \psi_{N-1} \mathrm{d} u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1} \\ &+ \int_{B_r} \chi \psi_{N-1} \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1}.\end{aligned}$$

Since $u(z) = A \log \|z\|$ on $B_r \setminus B_{r-4\rho}$, the $\mathcal{C}^2$ norm of u_ε on $B_r \setminus B_{r-3\rho}$ does not depend on $\varepsilon = \varepsilon(N)$. Together with the volume estimate (3.2), this implies that the first three integrals in the right-hand side have absolute values less than $c_3 e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}$, where $c_3 > 0$ does not depend on N.

For the last integral, we write $\mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-1} = \mathrm{dd}^c(u(\mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2})$ and apply Lemma 3.2 for $S = \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2}$. This gives us four integrals. Three of them are integrals over $B_r \setminus B_{r-3\rho}$ involving u, u_ε, ψ_{N-1}, and its derivatives. As before, the absolute values of all them are $\lesssim e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}$. The remaining integral is

$$\int_{B_{r-\rho}} \chi u \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2},$$

which we write again as

$$\begin{aligned}&\int_{B_{r-\rho}} \chi u_\varepsilon \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2} \\ &\quad + \int_{B_{r-\rho}} \chi (u - u_\varepsilon) \mathrm{dd}^c \psi_{N-1} \wedge \mathrm{dd}^c u_\varepsilon \wedge (\mathrm{dd}^c u)^{n-2}.\end{aligned} \tag{3.4}$$

Since u_ε converges to u in L^∞, Lemma 3.4 implies that the masses of $dd^c\psi_{N-1} \wedge dd^c u_\varepsilon \wedge (dd^c u)^{n-2}$ are bounded independently of N and ε. Therefore, the modulus of the second integral is less than $c_4\|u - u_\varepsilon\|_\infty \le c_4' e^{-(1+c\alpha)N}$, where $c_4' > 0$ does not depend on N.

To deal with the the first integral in (3.4), we apply Lemma 3.2, obtaining three integrals over $B_r \setminus B_{r-3\rho}$ with absolute values $\lesssim e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}$ and the integral

$$\int_{B_{r-\rho}} \chi\psi_{N-1}(dd^c u_\varepsilon)^2 \wedge (dd^c u)^{n-2}.$$

We can repeat the procedure in order "move" the dd^c's from u to u_ε. At each step, we get integrals with absolute values $\lesssim e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N}$ or $\lesssim e^{-(1+c\alpha)N}$ (where the constants involved do not depend on N), and, at the final step, we get the integral

$$\int_{B_{r-\rho}} \chi\psi_{N-1}(dd^c u_\varepsilon)^n$$

of absolute value less than

$$\begin{aligned} c_5\|u_\varepsilon\|^n_{\mathcal{C}^2} \cdot \lambda_{2n}(B_r \cap \{\varphi \le -N+1\}) &\le c_5' \varepsilon^{n(\alpha-2)} e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N} \\ &= c_5' e^{(cn(2-\alpha)-(1+\delta))N} \end{aligned}$$

with c_5' independent from N.

Altogether these estimates yield

$$\int_{B_{r-\rho}} \chi\psi_{N-1}(dd^c u)^n \lesssim e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N} + e^{-(1+c\alpha)N} + e^{(cn(2-\alpha)-(1+\delta))N}.$$

Inserting these estimates into (3.1), we finally get

$$\begin{aligned} &\int_{B_r} e^{-\varphi}(dd^c u)^n \\ &\quad\lesssim \sum_{N=0}^{\infty} e^N [e^{-(1+\delta)N} e^{-(n(2-\alpha)/\alpha)N} + e^{-(1+c\alpha)N} + e^{(cn(2-\alpha)-(1+\delta))N}] \\ &\quad= \sum_{N=0}^{\infty} [e^{-\delta N-(n(2-\alpha)/\alpha)N} + e^{-c\alpha N} + e^{(cn(2-\alpha)-\delta)N}]. \end{aligned}$$

By the choice of c all the factors of N in the exponentials are negative, so the series converges, and hence the integral $\int e^{-\varphi}(dd^c u)^n$ is finite. □

REMARK 3.5. Using the same kind of computation as in the proof, we can obtain an analogous result for mixed Monge–Ampère masses of the type $dd^c u_1 \wedge \cdots \wedge dd^c u_n$. We can show that if all u_j are α_j-Hölder-continuous p.s.h. functions ($0 < \alpha_j \le 1$), then the integral

$$\int e^{-\varphi} dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

is locally finite in the set of points $z \in \Omega$ satisfying $\nu(\varphi; z) < \frac{2}{1-n+2(1/\alpha_1+\cdots+1/\alpha_n)}$

.

4. Related Results and Examples

We now give some results related to Theorem 1.1 and study some concrete examples.

A theorem of Zeriahi [Zer01] improves Kiselman's result (Lemma 3.3) by making the volume estimates of the sets $K \cap \{\varphi \le -M\}$ uniform in φ when these vary in a compact family of p.s.h. functions. Using this fact we can prove a uniform version of Theorem 1.1. The proof follows from Zeriahi's result and the fact that for a compact family of p.s.h. functions $\mathcal{F}$, the family $\{\varphi_N : \varphi \in \mathcal{F}, N \in \mathbb{N}\}$ is again compact, where $\varphi_N = \max\{\varphi, -N\}$. I would like to thank A. Zerahi for bringing [Zer01] to my attention and for raising the question of the uniform bound provided by the following extension of Theorem 1.1.

THEOREM 4.1. *Let* Ω, $0 < \alpha \le 1$, *and* u *be as in Theorem 1.1. Let* $\mathcal{F}$ *be a compact family of p.s.h. functions in* Ω, *and* $K \subset \Omega$ *a compact subset. Set* $\nu \doteq \sup\{\nu(\varphi; z) : z \in K, \varphi \in \mathcal{F}\}$. *If* $\nu < \frac{2\alpha}{\alpha + n(2-\alpha)}$, *then there are a neighborhood* E *of* K *and a constant* $C > 0$ *such that*

$$\int_E e^{-\varphi} (dd^c u)^n \le C$$

for every $\varphi \in \mathcal{F}$.

REMARK 4.2. Let Ω, u, $\mathcal{F}$, and K be as before and set $\mu = (dd^c u)^n$. Since μ is locally moderate (Theorem 2.5), there are constants $\beta > 0$ and $C > 0$ such that $\int_K e^{-\beta\psi}\, d\mu \le C$ for every $\psi \in \mathcal{F}$. The theorem allows us to estimate the value of the constant β in terms of α and the Lelong numbers of the functions of $\mathcal{F}$ over K.

From Theorem 1.1 and a computation analogous to that the proof of Lemma 3.3 there follows an estimate of the mass of the sublevel sets of p.s.h. functions with respect to Monge–Ampère measures with Hölder-continuous potential. Information about the size of sublevel sets of p.s.h. functions in terms of measures and capacities is useful in many applications (see [Koł98; Kis00; FG01; FJ03; Taf11]). As before, this estimate can be made uniform in φ as long as these vary in a compact family.

COROLLARY 4.3. *Let* φ *be a p.s.h. function on an open set* $\Omega \subset \mathbb{C}^n$, *and* $\mu = (dd^c u)^n$ *a Monge–Ampère mass on* Ω *with* u *an* α*-Hölder continuous p.s.h. function. If* $K \subset \Omega$ *is a compact subset, then for every* $\gamma < \frac{2\alpha}{\alpha + n(2-\alpha)} \frac{1}{\sup_{z \in K} \nu(\varphi; z)}$, *there is a constant* $C_\gamma = C_\gamma(\varphi, \Omega, K)$ *such that*

$$\mu(K \cap \{\varphi \le -M\}) \le C_\gamma e^{-\gamma M}, \quad M \in \mathbb{R}.$$

Another theorem of Skoda concerns the nonintegrability of a p.s.h. function with large Lelong number: if $\nu(\varphi; 0) > 2n$, then $e^{-\varphi}$ is not integrable in any neighborhood of the origin with respect to the Lebesgue measure (see [Hör07, Lemma 4.3.1]). We cannot hope for a similar result with respect to every Monge–Ampère measure with Hölder-continuous potential because the measure

$\mu = (dd^c u)^n$ can be arbitrarily small near 0 (and even zero), making the integral $\int e^{-\varphi}\, d\mu$ finite. We may note however the following fact.

PROPOSITION 4.4. *Fix $0 < \alpha \leq 1$. There exists a Monge–Ampère mass $\mu = (dd^c u)^n$ where u is an α-Hölderian p.s.h. function such that, for every p.s.h. function φ defined near 0 with $\nu(\varphi; 0) > n\alpha$, we have $\int_K e^{-\varphi}\, d\mu = +\infty$ for every neighborhood K of the origin.*

Proof. Let $\gamma = \nu(\varphi; 0) > n\alpha$. Since $\varphi(z) \leq \gamma \log \|z\| + O(1)$ near 0 (see Introduction), we have that $e^{-\varphi(z)} \geq C \frac{1}{\|z\|^{\gamma}} \geq C \frac{1}{\|z\|^{n\alpha}}$. If we take $u(z) = \|z\|^{\alpha}$, then a direct computation shows that $(dd^c u)^n = C^{st} \|z\|^{n(\alpha-2)} \cdot \lambda_{2n}$ in the sense of currents. We thus have

$$\int e^{-\varphi} (dd^c u)^n \geq C^{st} \int \frac{1}{\|z\|^{n\alpha}} \frac{1}{\|z\|^{n(2-\alpha)}}\, d\lambda_{2n} = C^{st} \int \frac{1}{\|z\|^{2n}}\, d\lambda_{2n},$$

and the last integral diverges in any neighborhood of the origin. □

REMARK 4.5. For $n = 1$, the condition on the Lelong number of φ in the hypothesis of Theorem 1.1 is $\nu(\varphi; z) < \alpha$. This bound is sharp as Proposition 4.4 shows.

REMARK 4.6. For $n \geq 2$, the condition $\nu(\varphi; z) < \frac{2\alpha}{\alpha + n(2-\alpha)}$ in Theorem 1.1 is probably no longer optimal, as the following example suggests.

Let $0 < \alpha \leq 1$ and $c > 0$. Consider the potential $u(z) = |z_1|^{\alpha} + \cdots + |z_n|^{\alpha}$ and the p.s.h. function $\varphi(z) = c \log |z_1|$ defined on $\mathbb{C}^n$. We then have that

$$(dd^c u)^n = n! \left(\frac{\alpha}{2}\right)^{2n} |z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2} \cdot \lambda_{2n}$$

as measures on $\mathbb{C}^n$. Notice that this expression makes sense since $|z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2} \in L^1_{\text{loc}}(\mathbb{C}^n)$ for $\alpha > 0$.

We thus have

$$\begin{aligned}
\int_{B_1} e^{-\varphi} (dd^c u)^n &= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^c} |z_1|^{\alpha-2} \cdots |z_n|^{\alpha-2}\, d\lambda_{2n} \\
&= n! \left(\frac{\alpha}{2}\right)^{2n} \int_{B_1} \frac{1}{|z_1|^{c-\alpha+2}} |z_2|^{\alpha-2} \cdots |z_n|^{\alpha-2}\, d\lambda_{2n},
\end{aligned}$$

which is finite if and only if $\nu(\varphi; 0) = c < \alpha$.

Let us finish with some other examples where the critical integrability exponents can be explicitly computed.

EXAMPLE 4.7 (Haar measure on the real torus). Consider the function $u(z) = \log^+ |z_1| + \cdots + \log^+ |z_n|$ in $\mathbb{C}^n$. It is a Lipschitz p.s.h. function, and $(dd^c u)^n = \mu_{\text{Haar}}$ is the Haar measure on the n-dimensional torus $\mathbb{T}^n = \{|z_1| = \cdots = |z_n| = 1\} = \{(e^{\theta_1}, \ldots, e^{\theta_n}) : \theta_j \in [-\pi, \pi]\}$. Theorem 1.1 states that the integral $\int e^{-\varphi}\, d\mu_{\text{Haar}}$ near a point a is finite as long as $\nu(\varphi; a) < \frac{2}{1+n}$.

Let $p=(1,\dots,1)$ and consider the p.s.h. function $\psi(z)=c\log|z_1-1|$. Its restriction to $\mathbb{T}^n$ is given by $\psi(\theta_1,\dots,\theta_n)=c\log|e^{i\theta_1}-1|$, so that

$$\int e^{-\psi}\,d\mu_{\mathrm{Haar}}=\int\frac{d\theta_1\cdots d\theta_n}{|e^{i\theta_1}-1|^c}=\int\frac{d\theta_1\cdots d\theta_n}{2^c\sin(\theta_1/2)^c},$$

which is finite if and only if $\nu(\varphi;p)=c<1$.

EXAMPLE 4.8. Consider the function $u(z,w)=|z|(1+|w|^2)$ in $\mathbb{C}^2$. It is a Lipschitz continuous p.s.h. function, and the associated Monge–Ampère measure $\mu=(\mathrm{dd}^c u)^2$ is a multiple of the Lebesgue measure in $\mathbb{C}^2$, so the critical exponents in this case are given by Skoda's theorem. In other words, the integral $\int e^{-\varphi}\,d\mu$ converges around a if $\nu(\varphi;a)<2$ and diverges if $\nu(\varphi;a)>4$.

EXAMPLE 4.9. There are even less regular p.s.h. functions whose associated Monge–Ampère measures are smooth. A particular case of the following example can be found in [BD11]. In $\mathbb{C}^n$, write $z=(z',z'')$ with $z'\in\mathbb{C}^{n-k}$ and $z''\in\mathbb{C}^k$ for some $1\le k\le n-1$. Consider the function

$$u_k(z)=\|z'\|^{2-2(k/n)}(1+\|z''\|^2).$$

Then u_k is a continuous p.s.h. function in $\mathbb{C}^n$ (because $\log u$ is p.s.h.), and

$$(\mathrm{dd}^c u_k)^n=c_{n,k}(1+\|z''\|^2)^{n-k-1}\lambda_{2n},$$

where $c_{n,k}$ is some positive constant. Since $\mu_k=(\mathrm{dd}^c u_k)^n$ is absolutely continuous with respect to λ_{2n} with a smooth density function, the critical exponents for μ_k are the same as those for λ_{2n}, that is, the integral $\int e^{-\varphi}\,d\mu_k$ converges around a if $\nu(\varphi;a)<2$, and it diverges if $\nu(\varphi;a)>2n$. Notice that, for $k>\frac{n}{2}$, the function u_k is Hölder continuous with an exponent strictly smaller than 1. I would like to thank S. Dinew for bringing this example to my attention.

EXAMPLE 4.10 (Lebesgue measure on the unit sphere). Consider the function $u(z)=\log^+\|z\|$ in $\mathbb{C}^n$. It is a Lipschitz p.s.h. function whose associated Monge–Ampère measure $\mu=(\mathrm{dd}^c u)^n$ is the Lebesgue measure on the unit sphere $S^{2n-1}=\{\|z\|=1\}$.

Consider the function $\psi(z)=c\log|z_1|=c\log\mathrm{dist}(z,H)$, where $H=\{z_1=0\}$. We claim that $e^{-\psi}$ is integrable in a neighborhood of $q=(0,\dots,0,1)$ if and only if $\nu(\psi;q)=c<2$. Notice that $\Sigma=H\cap S^{2n-1}$ is a sphere of dimension $2n-3$ passing through q. The restriction of the function $\mathrm{dist}(\cdot,H)$ to a neighborhood of q in S^{2n-1} is comparable to $\mathrm{dist}(\cdot,\Sigma)$, where the later is the distance with respect to the standard Riemannian metric on S^{2n-1}. Choosing local (real) coordinates in S^{2n-1} around q, we reduce the problem to study the integrability of the function $\rho=1/\mathrm{dist}(\cdot,V)^c$ in a neighborhood of the origin in $\mathbb{R}^{2n-1}$, where V is a linear subspace of dimension $2n-3$. It is not hard to check that such a ρ is integrable if and only if $c<2$. This proves the claim.

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