Michigan Math. J. 66 (2017), 611–623

Chern–Ricci Invariance Along G-Geodesics

Nefton Pali

Abstract. Over a compact oriented manifold, the space of Riemannian metrics and normalized positive volume forms admits a natural pseudo-Riemannian metric G, which is useful for the study of Perelman's $\mathcal{W}$ functional. We show that if the initial speed of a G-geodesic is G-orthogonal to the tangent space to the orbit of the initial point under the action of the diffeomorphism group, then this property is preserved along all points of the G-geodesic. We show also that this property implies preservation of the Chern–Ricci form along such G-geodesics under the extra assumption of complex antiinvariant initial metric variation and vanishing of the Nijenhuis tensor along the G-geodesic.

1. Statement of the Invariance Result

We consider the space $\mathcal{M}$ of smooth Riemannian metrics over a compact oriented manifold X of dimension m. We denote by $\mathcal{V}_1$ the space of positive smooth volume forms with integral one. Notice that the tangent space of $\mathcal{M} \times \mathcal{V}_1$ is

$$T_{\mathcal{M}\times\mathcal{V}_1} = C^\infty(X, S^2T_X^*) \oplus C^\infty(X, \Lambda^m T_X^*)_0,$$

where $C^\infty(X, \Lambda^m T_X^*)_0 := \{V \in C^\infty(X, \Lambda^m T_X^*) \mid \int_X V = 0\}$. We denote by $\operatorname{End}_g(T_X)$ the bundle of g-symmetric endomorphisms of T_X and by $C_\Omega^\infty(X, \mathbb{R})_0$ the space of smooth functions with zero integral with respect to Ω. We will use the fact that, for any $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$, the tangent space $T_{\mathcal{M}\times\mathcal{V}_1,(g,\Omega)}$ identifies with $C^\infty(X, \operatorname{End}_g(T_X)) \oplus C_\Omega^\infty(X, \mathbb{R})_0$ via the isomorphism

$$(v, V) \longmapsto (v_g^*, V_\Omega^*) := (g^{-1}v, V/\Omega).$$

In [Pal4], we consider the pseudo-Riemannian metric G over $\mathcal{M} \times \mathcal{V}_1$, defined over any point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ by the formula

$$G_{g,\Omega}(u, U; v, V) = \int_X [\langle u, v\rangle_g - 2U_\Omega^* V_\Omega^*]\Omega$$

for all $(u, U), (v, V) \in T_{\mathcal{M}\times\mathcal{V}_1}$. The gradient flow of Perelman's $\mathcal{W}$-functional [Per] with respect to the structure G is a modification of the Ricci flow with relevant properties (see [Pal4; Pal5]). The G-geodesics exists only for short time intervals $(-\varepsilon, \varepsilon)$. This is because the G-geodesics are uniquely determined by the evolution of the volume forms and the latter degenerate in finite time (see Section 2). In [Pal4], we show that the space G-orthogonal to the tangent of the

Received May 2, 2016. Revision received January 26, 2017.

orbit of a point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ under the action of the identity component of the diffeomorphism group is

$$\mathbb{F}_{g,\Omega} := \{(v, V) \in T_{\mathcal{M}\times\mathcal{V}_1} \mid \nabla_g^{*\Omega} v_g^* + \nabla_g V_\Omega^* = 0\},$$

where $\nabla_g^{*\Omega}$ denotes the adjoint of the Levi–Civita connection with respect to the volume form Ω. In this paper, we show the following conservative property.

PROPOSITION 1. *Let $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{M} \times \mathcal{V}_1$ be a G-geodesic such that $(\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_{g_0,\Omega_0}$. Then $(\dot{g}_t, \dot{\Omega}_t) \in \mathbb{F}_{g_t,\Omega_t}$ for all $t \in (-\varepsilon, \varepsilon)$.*

We consider now a compact symplectic manifold (X, ω), and we denote by $\mathcal{J}_\omega^{\mathrm{ac}}$ the space of smooth almost complex structures compatible with the symplectic form ω. We notice that the variations inside the space of metrics $\mathcal{M}_\omega^{\mathrm{ac}} := -\omega \cdot \mathcal{J}_\omega^{\mathrm{ac}} \subset \mathcal{M}$, at a point $g = -\omega J$, are J-antiinvariant. Thus, in this setup, it is natural to consider the subspace

$$\mathbb{F}_{g,\Omega}^J := \{(v, V) \in \mathbb{F}_{g,\Omega} \mid v = -J^* v J\}.$$

With these notations, we state the following result.

THEOREM 1 (Main result. The invariance of the Chern–Ricci form). *Let (X, J_0, g_0) be a compact almost-Kähler manifold with symplectic form $\omega := g_0 J_0$. Then, for any G-geodesic $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{M} \times \mathcal{V}_1$ with initial speed $(\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}^{J_0}_{g_0,\Omega_0}$, we have that $J_t := -\omega^{-1} g_t \in \mathcal{J}_\omega^{\mathrm{ac}}$ and $(\dot{g}_t, \dot{\Omega}_t) \in \mathbb{F}^{J_t}_{g_t,\Omega_t}$. Moreover, if the Nijenhuis tensor vanishes identically along the G-geodesic, then* $\mathrm{Ric}_{J_t}(\Omega_t) = \mathrm{Ric}_{J_0}(\Omega_0)$ *for all $t \in (-\varepsilon, \varepsilon)$.*

Our unique interest in this result concerns the Fano case $\omega = \mathrm{Ric}_{J_0}(\Omega_0)$. In this case the space of ω-compatible complex (integrable) structures $\mathcal{J}_\omega$ embeds naturally inside $\mathcal{M} \times \mathcal{V}_1$ via the Chern–Ricci form. (This is possible thanks to the $\partial\bar{\partial}$-lemma). The image of this embedding is

$$\mathcal{S}_\omega := \{(g, \Omega) \in \mathcal{M}_\omega \times \mathcal{V}_1 \mid \omega = \mathrm{Ric}_J(\Omega),\ J = -\omega^{-1} g\}$$

with $\mathcal{M}_\omega := -\omega \cdot \mathcal{J}_\omega \subset \mathcal{M}$. It is well known that the J-antilinear endomorphism sections associated with the metric-variations in $\mathcal{M}_\omega$ at a point $g = -\omega J$ are $\overline{\partial}_{T_{X,J}}$-closed. Thus, in the integrable setup, it is natural to consider the subspace

$$\mathbb{F}_{g,\Omega}^J[0] := \{(v, V) \in \mathbb{F}_{g,\Omega}^J \mid \overline{\partial}_{T_{X,J}} v_g^* = 0\}.$$

It has been showed in [Pal4] that this is the space G-orthogonal to the tangent to the orbit of the point $(g, \Omega) \in \mathcal{S}_\omega$ under the action of the identity component of the ω-symplectomorphism group. (See the identity 1.14 in [Pal4].) Furthermore, the product $G_{g,\Omega}$ is positive over $\mathbb{F}_{g,\Omega}^J[0]$ thanks to a result in [Pal4]. We conjecture the following slice-type result.

CONJECTURE 1. *Let (X, J) be a Fano manifold, and let $\omega \in 2\pi c_1(X)$ be a Kähler form. Then the distribution $(g, \Omega) \in \mathcal{S}_\omega \longmapsto \mathbb{F}_{g,\Omega}^J[0]$ with $J := -\omega^{-1} g$ is* *integrable over the space $\mathcal{S}_\omega$, with leave at the point (g, Ω) given locally, in a neighborhood of this point, by* $\Sigma^{\omega}_{g,\Omega} := \{(\gamma, \mu) \in \mathrm{Exp}_G(\mathbb{F}^J_{g,\Omega}) \mid \nabla_\gamma \omega = 0\}$.

We invite the reader to compare with [F-S] for other approaches concerning slice-type problems in the space of compatible complex structures. In view of the results in Section 9 of [Pal4], the solution of this conjecture is crucial for the proof of the dynamical stability of the Soliton–Kähler–Ricci flow [Pal7].

An important ingredient for the proof of the main Theorem 1 is the general variation formula (4.1). Particular cases of this variation formula have been intensively studied. See [Fu; Do; Mo; Ga; Pal3]. These formulas allow us to establish an important moment map picture in Kähler-geometry. See [Fu] for the integrable case and [Do] for the almost complex case. In the last section, we provide a formula relating the Bakry–Emery–Ricci tensor with the Chern–Ricci form.

2. Pure Evolving Volume Nature of the G-Geodesic Equation

We recall that the equation of a G-geodesic $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)}$ (see [Pal4]) rewrites in the form

$$(S)\begin{cases} \frac{d}{dt}\dot{g}^*_t + \dot{\Omega}^*_t \dot{g}^*_t = 0, \\ \ddot{\Omega}_t + \frac{1}{4}\{|\dot{g}_t|^2_{g_t} - 2(\dot{\Omega}^*_t)^2 - \int_X [|\dot{g}_t|^2_{g_t} - 2(\dot{\Omega}^*_t)^2]\Omega_t\}\Omega_t = 0. \end{cases}$$

The invariance of the scalar product of the speed of geodesics implies

$$G_t := G_{g_t,\Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) \equiv G_{g_0,\Omega_0}(\dot{g}_0, \dot{\Omega}_0; \dot{g}_0, \dot{\Omega}_0).$$

Therefore a solution of system (S) also satisfies

$$(S_1)\begin{cases} \frac{d}{dt}\dot{g}^*_t + \dot{\Omega}^*_t \dot{g}^*_t = 0, \\ \ddot{\Omega}_t + \frac{1}{4}[|\dot{g}_t|^2_{g_t} - 2(\dot{\Omega}^*_t)^2 - G_0]\Omega_t = 0. \end{cases}$$

The first equation in system (S_1) rewrites as

$$\dot{g}^*_t = \frac{\Omega_0}{\Omega_t}\dot{g}^*_0,$$

which provides the expression

$$g_t = g_0 \exp\left(\dot{g}^*_0 \int_0^t \frac{\Omega_0}{\Omega_s}\, ds\right).$$

We set $u_t := \Omega_t / \Omega_0$ and observe the trivial identities

$$|\dot{g}_t|^2_{g_t} = \mathrm{Tr}_{\mathbb{R}}(\dot{g}^*_t)^2 = u_t^{-2}|\dot{g}_0|^2_{g_0},$$
$$\dot{\Omega}^*_t = \dot{u}_t / u_t.$$

We deduce that system (S_1) is equivalent to the system

$$\begin{cases} g_t = g_0 \exp(\dot{g}_0^* \int_0^t u_s^{-1}\, ds), \\ \Omega_t = u_t \Omega_0, \\ 4\ddot{u}_t + \frac{|\dot{g}_0|^2_{g_0} - 2\dot{u}_t^2}{u_t} - u_t G_0 = 0, \\ u_0 = 1, \\ \int_X \dot{u}_0 \Omega_0 = 0. \end{cases}$$

The solution u is given by the explicit formula

$$\begin{aligned} u_t &= 1 + \dot{u}_0 \sum_{k\geq 0} \frac{(G_0/2)^k}{(2k+1)!} t^{2k+1} - \frac{1}{4}\underline{N_0} \sum_{k\geq 1} \frac{(G_0/2)^{k-1}}{(2k)!} t^{2k}, \\ \underline{N_0} &:= N_0 - G_0, \\ N_0 &:= |\dot{g}_0|^2_{g_0} - 2(\dot{\Omega}_0^*)^2. \end{aligned}$$

Thus the solution $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)}$ of system (S_1) satisfies $\int_X \Omega_t \equiv 1$. This implies $G_t \equiv G_0$. We infer that system (S_1) is equivalent to system (S).

In the case $G_0 > 0$, the previous formula for u_t reduces to the expression

$$u_t = 1 + \dot{\Omega}_0^* \gamma_0^{-1} \sinh(\gamma_0 t) - \underline{N_0}(2\gamma_0)^{-2}[\cosh(\gamma_0 t) - 1]$$

with $\gamma_0 := (G_0/2)^{1/2}$.

3. Conservative Properties Along G-Geodesics

In this section, we show Proposition 1.

Proof of Proposition 1. We first recall the fundamental variation formula

$$2[(D_{g,\Omega}\nabla^*_{\bullet}{}^{\bullet})(v, V)]v_g^* = \frac{1}{2}\nabla_g |v|^2_g - 2v_g^* \cdot (\nabla_g^{*\Omega} v_g^* + \nabla_g V_\Omega^*), \tag{3.1}$$

obtained in [Pal6] (see the formula (19) in [Pal6]). Using (3.1), we develop the derivative

$$\begin{aligned} 2\frac{d}{dt}(\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*) &= -2\dot{g}_t^* \cdot (\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*) + \frac{1}{2}\nabla_{g_t} |\dot{g}_t|^2_{g_t} \\ &\quad + 2\nabla_{g_t}^{*\Omega_t} \frac{d}{dt}\dot{g}_t^* + 2\nabla_{g_t} \frac{d}{dt}\dot{\Omega}_t^* - 2\dot{g}_t^* \cdot \nabla_{g_t} \dot{\Omega}_t^*. \end{aligned}$$

Writing the equations defining the G-geodesic $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)}$ in the form

$$\begin{cases} \frac{d}{dt}\dot{g}_t^* + \dot{\Omega}_t^* \dot{g}_t^* = 0, \\ 2\frac{d}{dt}\dot{\Omega}_t^* + (\dot{\Omega}_t^*)^2 + \frac{1}{2}|\dot{g}_t|^2_{g_t} - \frac{1}{2}G_{g_t,\Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) = 0, \end{cases}$$

we infer

$$\begin{aligned} 2\frac{d}{dt}(\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*) &= -2\dot{g}_t^* \cdot (\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*) \\ &\quad - 2\nabla_{g_t}^{*\Omega_t}(\dot{\Omega}_t^* \dot{g}_t^*) - \nabla_{g_t}(\dot{\Omega}_t^*)^2 - 2\dot{g}_t^* \cdot \nabla_{g_t} \dot{\Omega}_t^*, \end{aligned}$$

and thus

$$2\frac{d}{dt}(\nabla_{g_t}^{*\Omega_t}\dot{g}_t^* + \nabla_{g_t}\dot{\Omega}_t^*) = -2(\dot{g}_t^* + \dot{\Omega}_t^*\mathbb{I}) \cdot (\nabla_{g_t}^{*\Omega_t}\dot{g}_t^* + \nabla_{g_t}\dot{\Omega}_t^*).$$

Then the conclusion follows by Cauchy's uniqueness. □

Let $\mathcal{J} \subset C^\infty(X, \mathrm{End}_{\mathbb{R}}(T_X))$ be the set of smooth almost complex structures over X. For any nondegenerate closed 2-form ω over a symplectic manifold, we define the space $\mathcal{J}_\omega^{\mathrm{ac}}$ of ω-compatible almost complex structures as

$$\mathcal{J}_\omega^{\mathrm{ac}} := \{J \in \mathcal{J} \mid -\omega J \in \mathcal{M}\}.$$

With these notations, we have the following result.

Lemma 1. *Let $J_0 \in \mathcal{J}_\omega^{\mathrm{ac}}$, and let $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{M} \times \mathcal{V}_1$ be a G-geodesic such that $g_0 = -\omega J_0$ and $\dot{g}_0^* J_0 = -J_0\dot{g}_0^*$. Then $J_t := -\omega^{-1}g_t \in \mathcal{J}_\omega^{\mathrm{ac}}$ for all $t \in (-\varepsilon, \varepsilon)$.*

Proof. Using the identity $\dot{J}_t = J_t\dot{g}_t^*$ and the G-geodesic equation

$$\frac{d}{dt}\dot{g}_t^* + \dot{\Omega}_t^*\dot{g}_t^* = 0,$$

we obtain the variation formula

$$\frac{d}{dt}(J_t\dot{g}_t^* + \dot{g}_t^* J_t) = (J_t\dot{g}_t^* + \dot{g}_t^* J_t) \cdot (\dot{g}_t^* - \dot{\Omega}_t^*\mathbb{I}).$$

This implies $\dot{g}_t^* J_t = -J_t\dot{g}_t^*$ for all $t \in (-\varepsilon, \varepsilon)$, by Cauchy's uniqueness. We deduce in particular the evolution identity $2\dot{J}_t = [J_t, \dot{g}_t^*]$. Then $J_t^2 = -\mathbb{I}_{T_X}$, thanks to Lemma 4 in [Pal2]. We infer the required conclusion. □

4. The First Variation of the Ω-Chern–Ricci Form

Let (X, J) be an almost complex manifold. Any volume form $\Omega > 0$ induces a Hermitian metric h_Ω over the canonical bundle $K_{X,J} := \Lambda_J^{n,0}T_X^*$, which is given by the formula

$$h_\Omega(\alpha, \beta) := \frac{n!i^{n^2}\alpha \wedge \overline{\beta}}{\Omega}.$$

We define the Ω-Chern–Ricci form

$$\mathrm{Ric}_J(\Omega) := -i\mathcal{C}_{h_\Omega}(K_{X,J}),$$

where $\mathcal{C}_h(F)$ denotes the Chern curvature of a Hermitian vector bundle $(F, \overline{\partial}_F, h)$ equipped with a $(0, 1)$-type connection. Consider also a J-invariant Hermitian metric ω over X. We recall that the ω-Chern-=Ricci form is defined by the formula

$$\mathrm{Ric}_J(\omega) := \mathrm{Tr}_{\mathbb{C}}[J\mathcal{C}_\omega(T_{X,J})].$$

The fact that the metric h_{ω^n} over $K_{X,J}$ is induced by the metric ω over $T_{X,J}$ implies, by natural functorial properties, the identity $\mathrm{Ric}_J(\omega) = \mathrm{Ric}_J(\omega^n)$. Let now

$$\mathcal{KS} := \{(J, g) \in \mathcal{J} \times \mathcal{M} \mid g = J^*gJ, \nabla_g J = 0\}$$

be the space of Kähler structures over a compact manifold X. We recall that if $A \in \mathrm{End}_{\mathbb{R}}(T_X)$, then its transposed A_g^T with respect to g is given by $A_g^T = g^{-1}A^*g$. We observe that the compatibility condition $g = J^*gJ$ is equivalent to the condition $J_g^T = -J$. We define also the space of almost Kähler structures as

$$\mathcal{AKS} := \{(J, g) \in \mathcal{J} \times \mathcal{M} \mid g = J^*gJ, d(gJ) = 0\}.$$

With these notations, we have the following first variation formula for the Ω-Chern–Ricci form. (Compare with [Fu; Do; Mo; Ga; Pal3].)

PROPOSITION 2. *Let $(J_t, g_t)_t \subset \mathcal{AKS}$ and $(\Omega_t)_t \subset \mathcal{V}$ be two smooth paths such that $\dot{J}_t = (\dot{J}_t)_{g_t}^T$. Then we have the first variation formula*

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t) = L_{J_t\nabla_{g_t}^{*\Omega_t}\dot{J}_t - \nabla_{g_t}\dot{\Omega}_t^*}\,\omega_t \tag{4.1}$$

with $\omega_t = g_tJ_t$.

Proof. Step I. Local expressions. We first consider the case of constant volume form Ω. We recall a general basic identity. Let $(L, \overline{\partial}_L, h)$ be a Hermitian line bundle equipped with a (0, 1)-type connection over an almost complex manifold (X, J), and let $D_{L,h} = \partial_{L,h} + \overline{\partial}_L$ be the induced Chern connection. In explicit terms, $\partial_{L,h} := h^{-1} \cdot \partial_{\overline{L^*}} \cdot h$. We observe that, for any local nonvanishing section $\sigma \in C^\infty(U, L \smallsetminus 0)$ over an open set $U \subset X$, we have the identity

$$\begin{aligned}\sigma^{-1}\partial_{L,h}\sigma(\eta) &= |\sigma|_h^{-2}h(\partial_{L,h}\sigma(\eta), \sigma)\\ &= |\sigma|_h^{-2}[\eta_J^{1,0}.|\sigma|_h^2 - h(\sigma, \overline{\partial}_L\sigma(\overline{\eta}))]\\ &= \eta_J^{1,0}.\log|\sigma|_h^2 - \overline{\sigma^{-1}\overline{\partial}_L\sigma(\overline{\eta})}\end{aligned}$$

for all $\eta \in T_X \otimes_{\mathbb{R}} \mathbb{C}$. We infer the formula

$$i\sigma^{-1}D_{L,h}\sigma = i\partial_J\log|\sigma|_h^2 + 2\Re e(i\sigma^{-1}\overline{\partial}_L\sigma).$$

In the case $L = K_{X,J_t} := \Lambda_{J_t}^{n,0}T_X^*$ and $h \equiv h_\Omega$, for all

$$\beta_t = \beta_{1,t}^{1,0} \wedge \cdots \wedge \beta_{n,t}^{1,0} \in C^\infty(U, K_{X,J_t} \smallsetminus 0)$$

with $\beta_{r,t}^{1,0} := \beta_{r,J_t}^{1,0}$, $\beta_r \in C^\infty(U, T_X^* \otimes_{\mathbb{R}} \mathbb{C})$, $r = 1, \ldots, n$, we get the formula for the 1-form α_t:

$$\alpha_t := i\beta_t^{-1}D_{K_{X,J_t},h_\Omega}\beta_t = i\partial_{J_t}\log\frac{i^{n^2}\beta_t \wedge \bar{\beta}_t}{\Omega} + 2\Re e(i\beta_t^{-1}\overline{\partial}_{K_{X,J_t}}\beta_t).$$

We also notice the local expression $\mathrm{Ric}_{J_t}(\Omega) = -i\mathcal{C}_{h_\Omega}(K_{X,J_t}) = -d\alpha_t$. To expand the time derivative of the expression

$$\begin{aligned}\alpha_t(\eta) = i\eta_{J_t}^{1,0}.\log\frac{i^{n^2}\beta_t \wedge \bar{\beta}_t}{\Omega}\\ + 2\Re e\left[i\beta_t^{-1}\sum_{r=1}^n \beta_{1,t}^{1,0} \wedge \cdots \wedge (\eta_{J_t}^{0,1}\neg\overline{\partial}_{J_t}\beta_{r,t}^{1,0}) \wedge \cdots \wedge \beta_{n,t}^{1,0}\right],\end{aligned}$$

we first observe the formula

$$2\frac{d}{dt}(\overline{\partial}_{J_t}\beta^{1,0}_{J_t}) = J_t\dot{J}_t\neg[(d-2\overline{\partial}_{J_t})\beta^{1,0}_{J_t}] - i[d(\beta\cdot\dot{J}_t)]^{1,1}_{J_t}. \tag{4.2}$$

We notice indeed that, for bidegree reasons, we have the identity

$$2\overline{\partial}_{J_t}\beta^{1,0}_{J_t} = 2(d\beta^{1,0}_{J_t})^{1,1}_{J_t} = d\beta^{1,0}_{J_t} + J_t^* d\beta^{1,0}_{J_t} J_t.$$

Then, time deriving the latter, we infer the required formula (4.2).

Step II. Local choices. We fix an arbitrary time τ. We want to compute the time derivative $\dot{\alpha}_\tau(\eta)$. We take a relatively compact open set $U \subset X$. Then, for a sufficiently small $\varepsilon > 0$, the bundle map

$$\begin{aligned}\varphi_t := \det_{\mathbb{C}}\pi^{1,0}_{J_t} : \Lambda^{n,0}_{J_\tau}T^*_U &\longrightarrow \Lambda^{n,0}_{J_t}T^*_U,\\ \beta_1\wedge\cdots\wedge\beta_n &\longmapsto \beta_t := \beta^{1,0}_{1,t}\wedge\cdots\wedge\beta^{1,0}_{n,t}\end{aligned}$$

is an isomorphism for all $t \in (\tau-\varepsilon, \tau+\varepsilon)$. For notational simplicity, we set $D_t := D_{K_{X,J_t},h_\Omega}$. We also consider the connection $D_{\varphi_t} := \varphi_t^* D_t$ over the bundle $\Lambda^{n,0}_{J_\tau}T^*_U$. Explicitly, $D_{\varphi_t}\beta = \varphi_t^{-1}D_t\beta_t$. Then the expression $D_t\beta_t = \alpha_t \otimes \beta_t$ implies $D_{\varphi_t}\beta = \alpha_t \otimes \beta$. We deduce that

$$\dot{\alpha}_t = \frac{d}{dt}D_{\varphi_t}$$

is independent of the choice of β. We want to compute $\dot{\alpha}_\tau$ at an arbitrary point $p \in U$.

Step IIa. The Kähler case. (We consider this case first since it is drastically simpler.) Let ∇_{g_τ} be the Levi–Civita connection of g_τ. Using parallel transport and the Kähler assumption $\nabla_{g_\tau}J_\tau = 0$, we can construct (up to shrinking U around p), a frame $(\beta_r)^n_{r=1} \subset C^\infty(U, \Lambda^{1,0}_{J_\tau}T^*_U)$ satisfying $\nabla_{g_\tau}\beta_r(p) = 0$ for all $r = 1, \ldots, n$, and the identity

$$\omega_\tau = \frac{i}{2}\sum_{r=1}^n \beta_r \wedge \bar{\beta}_r$$

over U. Then $dV_{g_\tau} = 2^{-n}i^{n^2}\beta_\tau \wedge \bar{\beta}_\tau$. We now set $f_\tau := \log\frac{dV_{g_\tau}}{\Omega}$. The identity $d\beta_r = \mathrm{Alt}\,\nabla_{g_\tau}\beta_r$ implies $d\beta_r(p) = 0$. Then formula (4.2) implies the following identity at the point p:

$$\begin{aligned}2\frac{d}{dt}_{|t=\tau}(\overline{\partial}_{J_t}\beta^{1,0}_{r,t}) &= -i\beta_r(\nabla_{T_X,g_\tau}\dot{J}_\tau)^{1,1}_{J_\tau}\\ &= -i\beta_r\,\mathrm{Alt}(\nabla^{1,0}_{g_\tau,J_\tau}\dot{J}_\tau).\end{aligned}$$

(The last equality follows from the Kähler assumption.) We deduce

$$\eta^{0,1}_{J_\tau}\neg 2\frac{d}{dt}_{|t=\tau}(\overline{\partial}_{J_t}\beta^{1,0}_{r,t}) = i\beta_r\nabla^{1,0}_{g_\tau,J_\tau}\dot{J}_\tau\cdot\eta^{0,1}_{J_\tau} = i\beta_r\nabla^{1,0}_{g_\tau,J_\tau}\dot{J}_\tau\cdot\eta$$

at the point p. (The last equality follows also from the Kähler assumption.) Using this last identity, we obtain the following expression at the point p:

$$\begin{aligned}\dot{\alpha}_\tau(\eta) &= \frac{1}{2}\dot{J}_\tau\eta.f_\tau + \eta^{1,0}_{J_\tau}.\Re e\bigg(\beta_\tau^{-1}\sum_{r=1}^{n}\beta_1\wedge\cdots\wedge(\beta_r\dot{J}_\tau)^{1,0}_{J_\tau}\wedge\cdots\wedge\beta_n\bigg)\\ &\quad - \Re e\bigg(\beta_\tau^{-1}\sum_{r=1}^{n}\beta_1\wedge\cdots\wedge\beta_r\nabla^{1,0}_{g_\tau,J_\tau}\dot{J}_\tau\cdot\eta\wedge\cdots\wedge\beta_n\bigg)\\ &= -\frac{1}{2}(\mathrm{Tr}_{\mathbb{R}}\nabla_{g_\tau}\dot{J}_\tau - df_\tau\cdot\dot{J}_\tau)(\eta),\end{aligned}$$

thanks to the elementary identities $(\beta_r\dot{J}_\tau)^{1,0}_{J_\tau}=0$, $\mathrm{Tr}_{\mathbb{C}}A=\mathrm{Tr}_{\mathbb{C}}A^*$, and $\mathrm{Tr}_{\mathbb{R}}B = 2\Re e(\mathrm{Tr}_{\mathbb{C}}B^{1,0}_J)$ for all $B\in\mathrm{End}_{\mathbb{R}}(T_X)$. Using now the symmetry identities $\dot{J}_\tau = (\dot{J}_\tau)^T_{g_\tau}$ and $\nabla_{g_\tau,\xi}\dot{J}_\tau = (\nabla_{g_\tau,\xi}\dot{J}_\tau)^T_{g_\tau}$, we obtain

$$2\dot{\alpha}_\tau = \nabla^{*\Omega}_{g_\tau}\dot{J}_\tau\neg g_\tau$$

over U. We conclude, thanks to the Kähler condition and Cartan's identity, the required formula for arbitrary time t in the case of constant volume form.

Step IIb. The almost Kähler case. We first recall that, in this case, we have the classical identity

$$g(\nabla_{g,\xi}J\cdot\eta,\mu) = -2g(J\xi, N_J(\eta,\mu)), \tag{4.3}$$

where N_J is the Nijenhuis tensor defined by the formula

$$4N_J(\xi,\eta) := [\xi,\eta] + J[\xi,J\eta] + J[J\xi,\eta] - [J\xi,J\eta].$$

Identity (4.3), combined with the identity $N_J(J\eta,\mu) = -JN_J(\eta,\mu)$, implies

$$\nabla_{g,J\xi}J = -J\nabla_{g,\xi}J. \tag{4.4}$$

We also consider the Chern connection $D^{\omega}_{\Lambda^{1,0}_J T^*_X}$ of the complex vector bundle $\Lambda^{1,0}_J T^*_X$ with respect to the Hermitian product

$$\langle\alpha,\beta\rangle_\omega := \frac{1}{2}\mathrm{Tr}_\omega(i\alpha\wedge\bar{\beta}).$$

This connection is obviously the dual of the Chern connection $D^{\omega_\tau}_{T^{1,0}_{X,J}}$ of the Hermitian vector bundle $(T^{1,0}_{X,J},\omega)$. We denote by $D^{\omega}_{T_{X,J}}$ the Chern connection of $(T_{X,J},\omega)$. By abuse of notation, we denote with the same symbol its complex linear extension over $T_X\otimes_{\mathbb{R}}\mathbb{C}$. The latter satisfies the formula

$$D^{\omega}_{T_{X,J}}\xi = D^{\omega}_{T^{1,0}_{X,J}}\xi^{1,0}_J + \overline{D^{\omega}_{T^{1,0}_{X,J}}\overline{\xi^{0,1}_J}},\quad \forall\xi\in C^\infty(X,T_X\otimes_{\mathbb{R}}\mathbb{C}).$$

In the almost Kähler case, $D^{\omega}_{T_{X,J}}$ is related to the Levi–Civita connection ∇_g (see [Pal1] and use identity (4.3)) via the formula

$$D^{\omega}_{T_{X,J},\xi}\eta = \nabla_{g,\xi}\eta - \frac{1}{2}J\nabla_{g,\xi}J\cdot\eta \tag{4.5}$$

for all $\xi, \eta \in C^\infty(X, T_X \otimes_{\mathbb{R}} \mathbb{C})$. Thus

$$D^{\omega}_{T^{1,0}_{X,J}} \xi^{1,0}_J = \nabla_g \xi^{1,0}_J - \frac{1}{2} J \nabla_g J \cdot \xi^{1,0}_J$$

and

$$\begin{aligned} D^{\omega}_{\Lambda^{1,0}_J T^*_X} \beta^{1,0}_J \cdot \xi^{1,0}_J &= \nabla_g \beta^{1,0}_J \cdot \xi^{1,0}_J + \frac{1}{2} i \beta^{1,0}_J \cdot \nabla_g J \cdot \xi^{1,0}_J \\ &= \nabla_g \beta^{1,0}_J \cdot \xi^{1,0}_J \end{aligned}$$

since $\nabla_g J \cdot J = -J \nabla_g J$. We now apply these considerations to the almost Kähler structure (J_τ, g_τ). Using parallel transport, we can construct a complex frame $(\beta_r)_{r=1}^n \subset C^\infty(U, \Lambda^{1,0}_{J_\tau} T^*_U)$ satisfying $D^{\omega_\tau}_{\Lambda^{1,0}_{J_\tau} T^*_X} \beta_r(p) = 0$ for all $r = 1, \ldots, n$ and the identity

$$\omega_\tau = \frac{i}{2} \sum_{r=1}^n \beta_r \wedge \bar{\beta}_r$$

over U. Then, as before, we have the identity $dV_{g_\tau} = 2^{-n} i^{n^2} \beta_\tau \wedge \bar{\beta}_\tau$. We infer

$$\overline{\partial}_{J_\tau} \beta_r(p) = 0, \tag{4.6}$$

$$(\nabla_{g_\tau} \beta_r \cdot \xi^{1,0}_{J_\tau})(p) = 0, \tag{4.7}$$

$$\partial_{J_\tau} \beta_r(p) = 0 \tag{4.8}$$

for all $r = 1, \ldots, n$. The last equality follows from the elementary identities

$$\partial_{J_\tau} \beta_r(\xi, \eta) = d\beta_r(\xi, \eta) = \nabla_{g_\tau, \xi} \beta_r \cdot \eta - \nabla_{g_\tau, \eta} \beta_r \cdot \xi$$

for all $\xi, \eta \in C^\infty(X, T^{1,0}_{X,J_\tau})$. We observe now that formula (4.2) writes as

$$2 \frac{d}{dt} (\overline{\partial}_{J_t} \beta^{1,0}_{J_t}) = J_t \dot{J}_t \neg [(\partial_{J_t} - \overline{\partial}_{J_t}) \beta^{1,0}_{J_t} - \beta^{1,0}_{J_t} \cdot N_{J_t}] - i [d(\beta \cdot \dot{J}_t)]^{1,1}_{J_t}.$$

Thus, at the point p, we have the identity

$$\begin{aligned} 2 \frac{d}{dt}_{|t=\tau} (\overline{\partial}_{J_t} \beta^{1,0}_{r,t})(\eta^{0,1}_{J_\tau}, \mu) &= -\beta_r \cdot N_{J_\tau}(\eta^{0,1}_{J_\tau}, J_\tau \dot{J}_\tau \mu) \\ &\quad - i [d(\beta_r \cdot \dot{J}_\tau)]^{1,1}_{J_\tau} (\eta^{0,1}_{J_\tau}, \mu) \\ &= i \beta_r \cdot N_{J_\tau}(\eta, \dot{J}_\tau \mu) \\ &\quad - i\, d(\beta_r \cdot \dot{J}_\tau)(\eta^{0,1}_{J_\tau}, \mu^{1,0}_{J_\tau}). \end{aligned}$$

For notational simplicity, we set $\eta^{0,1}_\tau := \eta^{0,1}_{J_\tau}$, $\mu^{1,0}_\tau := \mu^{1,0}_{J_\tau}$, and we observe the expansion

$$\begin{aligned} d(\beta_r \cdot \dot{J}_\tau)(\eta^{0,1}_\tau, \mu^{1,0}_\tau) &= \nabla_{g_\tau, \eta^{0,1}_\tau} \beta_r \cdot \dot{J}_\tau \mu^{1,0}_\tau + \beta_r \cdot \nabla_{g_\tau, \eta^{0,1}_\tau} \dot{J}_\tau \cdot \mu^{1,0}_\tau \\ &\quad - \nabla_{g_\tau, \mu^{1,0}_\tau} \beta_r \cdot \dot{J}_\tau \eta^{0,1}_\tau - \beta_r \cdot \nabla_{g_\tau, \mu^{1,0}_\tau} \dot{J}_\tau \cdot \eta^{0,1}_\tau. \end{aligned}$$

We also notice the trivial identity $\beta_r \cdot \dot{J}_\tau \mu^{1,0}_\tau = \beta_r \cdot (\dot{J}_\tau \mu)^{0,1}_{J_\tau} = 0$ over U. Taking a covariant derivative of this, we infer

$$0 = \nabla_{g_\tau, \eta^{0,1}_\tau} \beta_r \cdot \dot{J}_\tau \mu^{1,0}_\tau + \beta_r \cdot \nabla_{g_\tau, \eta^{0,1}_\tau} \dot{J}_\tau \cdot \mu^{1,0}_\tau + \beta_r \cdot \dot{J}_\tau \nabla_{g_\tau, \eta^{0,1}_\tau} \mu^{1,0}_\tau.$$

Identity (4.4) implies

$$\nabla_{g_\tau,\eta_\tau^{0,1}}\mu_\tau^{1,0}=\Big(\nabla_{g_\tau,\eta_\tau^{0,1}}\mu-\frac{i}{2}\nabla_{g_\tau,\eta}J\cdot\mu\Big)^{1,0}_{J_\tau}.$$

Thus

$$\beta_r\cdot\dot J_\tau\nabla_{g_\tau,\eta_\tau^{0,1}}\mu_\tau=\beta_r\cdot\Big[\dot J_\tau\Big(\nabla_{g_\tau,\eta_\tau^{0,1}}\mu-\frac{i}{2}\nabla_{g_\tau,\eta}J\cdot\mu\Big)\Big]^{0,1}_{J_\tau}=0$$

and

$$d(\beta_r\cdot\dot J_\tau)(\eta_\tau^{0,1},\mu_\tau^{1,0})=-\beta_r\cdot\nabla_{g_\tau,\mu_\tau^{1,0}}\dot J_\tau\cdot\eta_\tau^{0,1}\tag{4.9}$$

at the point p since

$$\nabla_{g_\tau,\mu_\tau^{1,0}}\beta_r\cdot\dot J_\tau\eta_\tau^{0,1}=\nabla_{g_\tau,\mu_\tau^{1,0}}\beta_r\cdot(\dot J_\tau\eta)^{1,0}_{J_\tau}=0$$

at p thanks to (4.7). Taking a covariant derivative of the identity

$$\dot J_\tau J_\tau+J_\tau\dot J_\tau=0,$$

we obtain

$$\nabla_{g_\tau}\dot J_\tau J_\tau+\dot J_\tau\nabla_{g_\tau}J_\tau+\nabla_{g_\tau}J_\tau\dot J_\tau+J_\tau\nabla_{g_\tau}\dot J_\tau=0,$$

and thus

$$\begin{aligned}2\beta_r\cdot\nabla_{g_\tau,\mu_\tau^{1,0}}\dot J_\tau\cdot\eta_\tau^{0,1}&=2\beta_r\cdot(\nabla_{g_\tau,\mu_\tau^{1,0}}\dot J_\tau\cdot\eta)^{1,0}_{J_\tau}\\&\qquad-i\beta_r\cdot(\dot J_\tau\nabla_{g_\tau,\mu_\tau^{1,0}}J_\tau+\nabla_{g_\tau,\mu_\tau^{1,0}}J_\tau\dot J_\tau)\eta\\&=2\beta_r\cdot\nabla_{g_\tau,\mu_\tau^{1,0}}\dot J_\tau\cdot\eta-i\beta_r\cdot\dot J_\tau\nabla_{g_\tau,\mu}J_\tau\cdot\eta\end{aligned}$$

thanks to (4.4) and the fact that β_r is of type $(1,0)$ with respect to J_τ. We deduce

$$-i\,d(\beta_r\cdot\dot J_\tau)(\eta_\tau^{0,1},\mu_\tau^{1,0})=i\beta_r\cdot\Big(\nabla^{1,0}_{g_\tau,J_\tau,\mu}\dot J_\tau-\frac{1}{2}J_\tau\dot J_\tau\nabla_{g_\tau,\mu}J_\tau\Big)\eta$$

thanks to (4.9), and thus

$$\begin{aligned}\eta^{0,1}_{J_\tau}\neg 2\frac{d}{dt}_{|t=\tau}(\bar\partial_{J_t}\beta^{1,0}_{r,t})&=i\beta_r\cdot N_{J_\tau}(\eta,\dot J_\tau\bullet)\\&\qquad+i\beta_r\cdot\Big(\nabla^{1,0}_{g_\tau,J_\tau}\dot J_\tau-\frac{1}{2}J_\tau\dot J_\tau\nabla_{g_\tau}J_\tau\Big)\eta.\end{aligned}\tag{4.10}$$

Using (4.6) and (4.10), we obtain

$$\begin{aligned}2\dot\alpha_\tau(\eta)&=\dot J_\tau\eta.f_\tau\\&\qquad+2\Re e\Big\{i\beta_\tau^{-1}\sum_{l=1}^n\beta_1\wedge\cdots\wedge\Big[\eta^{0,1}_{J_\tau}\neg 2\frac{d}{dt}_{|t=\tau}(\bar\partial_{J_t}\beta^{1,0}_{l,t})\Big]\wedge\cdots\wedge\beta_n\Big\}\\&=df_\tau\cdot\dot J_\tau\eta\\&\qquad-2\Re e\,\mathrm{Tr}_{\mathbb C}\Big[N_{J_\tau}(\eta,\dot J_\tau\bullet)+\Big(\nabla^{1,0}_{g_\tau,J_\tau}\dot J_\tau-\frac{1}{2}J_\tau\dot J_\tau\nabla_{g_\tau}J_\tau\Big)\eta\Big]\\&=df_\tau\cdot\dot J_\tau\eta-\mathrm{Tr}_{\mathbb R}\Big[N_{J_\tau}(\eta,\dot J_\tau\bullet)+\Big(\nabla_{g_\tau}\dot J_\tau-\frac{1}{2}J_\tau\dot J_\tau\nabla_{g_\tau}J_\tau\Big)\eta\Big].\end{aligned}$$

We now show the identity

$$2\,\mathrm{Tr}_{\mathbb{R}}[N_{J_\tau}(\eta,\dot{J}_\tau\bullet)]=\mathrm{Tr}_{\mathbb{R}}(J_\tau\dot{J}_\tau\nabla_{g_\tau}J_\tau\cdot\eta). \tag{4.11}$$

Indeed, let $(e_k)_{k=1}^{2n}\subset T_{X,p}$ be a $g_\tau(p)$-orthonormal basis. Using (4.3), (4.4), and the symmetry assumption $\dot{J}_\tau=(\dot{J}_\tau)^T_{g_\tau}$, we obtain

$$\begin{aligned}2g(e_k,N_{J_\tau}(\eta,\dot{J}_\tau e_k))&=g(\nabla_{g_\tau,J_\tau e_k}J_\tau\cdot\eta,\dot{J}_\tau e_k)\\&=-g(\dot{J}_\tau J_\tau\nabla_{g_\tau,e_k}J_\tau\cdot\eta,e_k)\\&=g(J_\tau\dot{J}_\tau\nabla_{g_\tau,e_k}J_\tau\cdot\eta,e_k)\end{aligned}$$

and thus the required identity (4.11). We infer the formula

$$2\dot{\alpha}_\tau(\eta)=-\,\mathrm{Tr}_{\mathbb{R}}(\nabla_{g_\tau}\dot{J}_\tau\cdot\eta)+df_\tau\cdot\dot{J}_\tau\eta$$

over U. Using the symmetry identities $\dot{J}_\tau=(\dot{J}_\tau)^T_{g_\tau}$ and $\nabla_{g_\tau,\xi}\dot{J}_\tau=(\nabla_{g_\tau,\xi}\dot{J}_\tau)^T_{g_\tau}$, we infer

$$\begin{aligned}2\dot{\alpha}_\tau&=\nabla^{*\Omega}_{g_\tau}\dot{J}_\tau\neg g_\tau\\&=-J_\tau\nabla^{*\Omega}_{g_\tau}\dot{J}_\tau\neg\omega_\tau\end{aligned}$$

over U. We conclude the required variation formula for arbitrary time t in the case of constant volume form. In the case of variable volume forms, we fix an arbitrary time τ, and we differentiate with respect to time at $t=\tau$ the decomposition

$$\mathrm{Ric}_{J_t}(\Omega_t)=\mathrm{Ric}_{J_t}(\Omega_\tau)-dd^c_{J_t}\log\frac{\Omega_t}{\Omega_\tau}.$$

We obtain

$$\frac{d}{dt}_{|t=\tau}\mathrm{Ric}_{J_t}(\Omega_t)=\frac{d}{dt}_{|t=\tau}\mathrm{Ric}_{J_t}(\Omega_\tau)-dd^c_{J_\tau}\dot{\Omega}^*_\tau,$$

and thus

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t)=d[(J_t\nabla^{*\Omega_t}_{g_t}\dot{J}_t-\nabla_{g_t}\dot{\Omega}^*_t)\neg\omega_t]$$

thanks to the variation formula for the fixed volume form case. The conclusion follows from Cartan's identity for the Lie derivative of differential forms. □

We infer the following corollary.

Corollary 1. *Let ω be a symplectic form, and let $(J_t,\Omega_t)_t\subset\mathcal{J}^{\mathrm{ac}}_\omega\times\mathcal{V}$ be an arbitrary smooth family. Then we have the variation formulas*

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t)=-L_{\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+\nabla_{g_t}\dot{\Omega}^*_t}\,\omega-2\,d\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg N_{J_t})\dot{g}^*_t] \tag{4.12}$$

and

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t)=-L_{\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+\nabla_{g_t}\dot{\Omega}^*_t}\,\omega+d\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg\overline{\partial}_{T_{X,J_t}}\dot{g}^*_t)] \tag{4.13}$$

with $g_t:=-\omega J_t$.

Proof. We have $\dot g_t^* = -J_t \dot J_t$ and thus the property $\dot J_t = (\dot J_t)^T_{g_t}$, which allows us to apply (4.1). We notice now the equality

$$\nabla^{*\Omega}_{g_t}(J_t \dot J_t) = -\nabla_{g_t,e_k} J_t \cdot \dot J_t e_k + J_t \nabla^{*\Omega}_{g_t} \dot J_t$$

with respect to a g_t-orthonormal local frame $(e_k)_{k=1}^{2n}$ of T_X. We deduce

$$2\dot\alpha_t = -[\nabla^{*\Omega}_{g_t}(J_t \dot J_t) + \nabla_{g_t,e_k} J_t \cdot \dot J_t e_k]\neg\omega.$$

Identity (4.3) implies

$$\begin{aligned}\omega(\nabla_{g_t,e_k} J_t \cdot \dot J_t e_k, \xi) &= 2\omega(e_k, N_{J_t}(\dot J_t e_k, J_t\xi))\\ &= 2\omega(e_k, N_{J_t}(J_t \dot J_t e_k, \xi))\\ &= -2\,\mathrm{Tr}_{g_t}[\omega(\xi\neg N_{J_t})\dot g_t^*].\end{aligned}$$

We infer the variation formula (4.12). To show (4.13), we first notice the identity $\omega(e_k, N_{J_t}(g_t^{-1}e_k^*, \xi)) \equiv 0$ for arbitrary real local frame $(e_k)_k$ of T_X. Time differentiating this, we obtain

$$\omega(e_k, N_{J_t}(\dot g_t^* e_k, \xi)) = \omega(e_k, \dot N_{J_t}(e_k, \xi))$$

with respect to our g_t-orthonormal local frame. Then the general formula

$$2\frac{d}{dt} N_{J_t} = \overline\partial_{T_{X,J_t}}(J_t \dot J_t) + J_t \dot J_t N_{J_t} - (J_t \dot J_t)\neg N_{J_t}$$

(see the proof of Lemma 7 in [Pal2]) implies

$$\begin{aligned}2\omega(e_k, N_{J_t}(\dot g_t^* e_k, \xi)) &= \omega(e_k, N_{J_t}(\dot g_t^* e_k, \xi) + N_{J_t}(e_k, \dot g_t^*\xi) - \dot g_t^* N_{J_t}(e_k, \xi))\\ &\quad - \omega(e_k, \overline\partial_{T_{X,J_t}} \dot g_t^*(e_k, \xi))\\ &= \omega(e_k, N_{J_t}(\dot g_t^* e_k, \xi)) - \omega(\dot g_t^* e_k, N_{J_t}(e_k, \xi))\\ &\quad - \omega(\overline\partial_{T_{X,J_t}} \dot g_t^*(\xi, e_k), e_k).\end{aligned}$$

Assuming for simplicity that the g_t-orthonormal local frame $(e_k)_{k=1}^{2n}$ diagonalizes $\dot g_t^*$, we deduce the identity

$$2\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg N_{J_t})\dot g_t^*] = -\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg\overline\partial_{T_{X,J_t}} \dot g_t^*)],$$

which implies the variation formula (4.13). □

Combining Lemma 1, Proposition 1, and Corollary 1, we deduce the following result.

THEOREM 2. *Let (X, J_0, g_0) be a compact almost-Kähler manifold with symplectic form $\omega := g_0 J_0$. Then, for any G-geodesic $(g_t, \Omega_t)_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{M} \times \mathcal{V}_1$ with initial speed $(\dot g_0, \dot\Omega_0) \in \mathbb{F}^{J_0}_{g_0,\Omega_0}$, we have the properties $J_t := -\omega^{-1} g_t \in \mathcal{J}^{\mathrm{ac}}_\omega$ and $(\dot g_t, \dot\Omega_t) \in \mathbb{F}^{J_t}_{g_t,\Omega_t}$ and the variation formulas*

$$\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t) = -d\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg N_{J_t})\dot g_t^*],$$

$$2\frac{d}{dt}\mathrm{Ric}_{J_t}(\Omega_t) = d\,\mathrm{Tr}_{g_t}[\omega(\bullet\neg\overline\partial_{T_{X,J_t}} \dot g_t^*)].$$

The main Theorem 1 follows directly from the above statement since we assume vanishing of the Nijenhuis tensor along the G-geodesic.

ACKNOWLEDGMENT. I warmly thank the referee for pointing out a few inaccuracies in the original version of this manuscript.

References

[Do] S. K. Donalsdon, *Remarks on gauge theory, complex geometry and 4-manifold topology,* Fields medallists' lectures, World Sci. Ser. 20th Century Math., 5, pp. 384–403, World Sci. Publ., River Edge, NJ, 1997.

[Fu] A. Fujiki, *Moduli space of polarized algebraic manifolds and Kähler metrics* (translation of Sugaku 42 (1990), no. 3, 231–243), Sugaku Expositions 5 (1992), no. 2, 173–191. Publ. RIMS, Kyoto Univ. 24 (1988), 141–168.

[F-S] A. Fujiki and G. Schumacher, *The moduli space of Kähler structures on a real compact symplectic manifold,* Publ. RIMS, Kyoto Univ. 24 (1988), 141–168.

[Ga] P. Gauduchon, *Calabi's extremal metrics: An elementary introduction,* book available on the web.

[Mo] O. Modsen, *Symplectomorhismes hamiltoniens et métriques kähleriennes,* Mémoire de DEA, Univ. Paris 7, 2003.

[Pal1] N. Pali, *The Chern connection of the tangent bundle of almost complex manifolds,* New York J. Math. 11 (2005), 597–634.

[Pal2] ———, *The total second variation of Perelman's $\mathcal{W}$-functional,* Calc. Var. Partial Differential Equations 50 (2014), no. 1–2, 115–144, (2012), arXiv:1201.0969v1.

[Pal3] ———, *Variation formulas for the complex components of the Bakry–Emery–Ricci endomorphism,* Complex Var. Elliptic Equ. 60 (2015), no. 5, 635–667, (2014), arXiv:1406.0805.

[Pal4] ———, *The Soliton–Ricci flow with variable volume forms,* Complex Manifolds 3 (2016), no. 1, 41–144.

[Pal5] ———, *Variational stability of Kähler–Ricci solitons,* Adv. Math. 290 (2016), 15–35.

[Pal6] ———, *On complex deformations of Kähler–Ricci solitons,* 2015, Kodai Math. J., (to appear).

[Pal7] ———, *The stability of the Soliton–Kähler–Ricci flow,* in preparation.

[Per] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* arXiv:math/0211159.

Département de Mathématiques
Université Paris Sud
Bâtiment 425 F91405 Orsay
France

nefton.pali@math.u-psud.fr