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# Weak Amenability of the Central Beurling Algebras on $[FC]^-$ Groups

Varvara Shepelska & Yong Zhang

Abstract. We study weak amenability of central Beurling algebras $ZL^1(G,\omega)$. The investigation is a natural extension of the known work on the commutative Beurling algebra $L^1(G,\omega)$. For $[FC]^-$ groups, we establish a necessary condition, and for $[FD]^-$ groups, we give sufficient conditions for the weak amenability of $ZL^1(G,\omega)$. For a compactly generated $[FC]^-$ group with polynomial weight $\omega_\alpha(x) = (1+|x|)^\alpha$, we prove that $ZL^1(G,\omega_\alpha)$ is weakly amenable if and only if $\alpha < 1/2$.

## 1. Introduction

Let $G$ be a locally compact group. As it is customary, two functions equal to each other almost everywhere on $G$ with respect to the Haar measure will be regarded as the same. We denote the integral of a function $f$ on a (Borel-)measurable subset $K$ of $G$ against a fixed left Haar measure by $\int_K f\,dx$. The space of all complex-valued Haar-integrable functions on $G$ is denoted by $L^1(G)$. A weight on $G$ is a Borel-measurable function $\omega : G \to \mathbb{R}^+$ satisfying

$$\omega(xy) \le \omega(x)\omega(y) \quad (x, y \in G).$$

Given a weight $\omega$ on $G$, we consider the space $L^1(G,\omega)$ of all complex-valued Haar-measurable functions $f$ on $G$ that satisfy

$$\|f\|_\omega = \int |f(x)|\omega(x)\,dx < \infty.$$

With the convolution product $*$ and the norm $\|\cdot\|_\omega$, $L^1(G,\omega)$ is a Banach algebra, called a Beurling algebra on $G$. When $\omega = 1$, this is simply the group algebra $L^1(G)$. Let $ZL^1(G,\omega)$ be the closed subalgebra of $L^1(G,\omega)$ consisting of all $f \in L^1(G,\omega)$ such that $f^g = f$ for all $g \in G$, where $f^g(x) = f(g^{-1}xg)$ $(x \in G)$. Then $ZL^1(G,\omega)$ is a commutative Banach algebra, called a *central Beurling algebra* [19]. Indeed, $ZL^1(G,\omega)$ is the center of $L^1(G,\omega)$. It is well known that $ZL^1(G,\omega)$ is nontrivial if and only if $G$ is an [IN] group [22].

From [8, Rem. 8.8], a measurable weight $\omega$ on $G$ is always equivalent to a continuous weight $\tilde{\omega}$ on $G$, where the equivalence means that there are constants $c_1, c_2 > 0$ such that $c_1\omega(x) \le \tilde{\omega}(x) \le c_2\omega(x)$ for almost all $x \in G$. The equivalence implies that the respective Beurling algebras $L^1(G,\omega)$ and $L^1(G,\tilde{\omega})$ are

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isomorphic as Banach algebras. So are the central Beurling algebras $ZL^1(G,\omega)$ and $ZL^1(G,\tilde{\omega})$. For this reason, in our investigation, we will most of the time assume the weight $\omega$ to be a continuous function.

The study of $L^1(G,\omega)$ goes back to Beurling [4], who considered $G=\mathbb{R}$. We may find a good account of elementary theory concerning the general weighted group algebra in [26]. The structure of the center of group algebras and the central Beurling algebras were substantially studied in [19; 20].

Amenable Beurling algebras are essentially isomorphic to amenable group algebras [11; 30]. This is no longer true for the weak amenability. The weak amenability of Beurling algebras for commutative groups has been extensively investigated and is well characterized [3; 12; 27; 31], whereas for noncommutative groups, we hardly see a nontrivial example of a weakly amenable Beurling algebra. Recent investigations are in [5; 28; 29]. For approximate amenability of Beurling algebras we refer to [9]

We are concerned with weak amenability of central Beurling algebras $ZL^1(G,\omega)$. When $G$ is commutative, $ZL^1(G,\omega)$ coincides with $L^1(G,\omega)$. So investigation of weak amenability for $ZL^1(G,\omega)$ is a natural extension of the study for the commutative groups. We notice that some investigations on amenability and weak amenability of $ZL^1(G)$ (the case with trivial weight $\omega=1$) have been made recently in [2] and [1]. Even for amenability, all theses studies have answers only for particular cases of locally compact groups; in particular, those in [2] and [1] are only on compact and some discrete groups.

We will mainly focus on $[\mathrm{FC}]^-$ groups (groups with precompact conjugacy classes). We note that for the trivial weight $\omega=1$, it has been shown in [2] that $ZL^1(G)$ is always weakly amenable for an $[\mathrm{FC}]^-$ group $G$. When $G$ is compact, the result has a simple direct proof. In fact, $ZL^1(G)$ is generated by its minimal idempotents if $G$ is compact. So by a simple observation (see [17, Sect. 7]), $ZL^1(G,\omega)\simeq ZL^1(G)$ is always weakly amenable for compact $G$.

We recall here some standard notions concerning a locally compact group. More details can be found in [13; 14; 25].

For a locally compact group $G$,

(1) $G$ is an [IN] group if there is a compact neighborhood of the identity that is invariant under all inner automorphisms;
(2) $G$ is a [SIN] group if there is a compact neighborhood basis of the identity such that each member of the basis is invariant under inner automorphisms;
(3) $G$ is an $[\mathrm{FC}]^-$ group if the conjugacy class $\{gxg^{-1} : g\in G\}$ for each $x\in G$ has a compact closure in $G$;
(4) $G$ is an $[\mathrm{FD}]^-$ group if the closure of the commutator subgroup $G'$ of $G$ is compact in $G$ (where the commutator subgroup of $G$ is the subgroup generated by all elements of the form $xyx^{-1}y^{-1}$, $x,y\in G$).

Obviously, $G$ is an $[\mathrm{FD}]^-$ group if and only if there is a compact normal subgroup $K$ such that $G/K$ is abelian. It is also obvious that $[\mathrm{FD}]^-\subseteq[\mathrm{FC}]^-$. The less obvious inclusion $[\mathrm{FC}]^-\subseteq[\mathrm{IN}]$ was shown in [21, Prop. 3.1].

Section 2 is devoted to considering $[\mathrm{FC}]^-$ groups. We show that the projective tensor product $ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2)$ is weakly amenable if and only if both $ZL^1(G_1, \omega_1)$ and $ZL^1(G_2, \omega_2)$ are weakly amenable and, under some conditions, $ZL^1(G_1 \times G_2, \omega_1 \times \omega_2) \simeq ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2)$. We also show, among others, that a condition characterizing the weak amenability of the Beurling algebra on a commutative group remains necessary for central Beurling algebras on $[\mathrm{FC}]^-$ groups. In Section 3, we focus on the case for $[\mathrm{FD}]^-$ groups, establishing some sufficient conditions for $ZL^1(G, \omega)$ to be weakly amenable for this sort of groups $G$. For a compactly generated $[\mathrm{FC}]^-$ group $G$, we consider the polynomial weight $\omega_\alpha(x) = (1 + |x|)^\alpha$. We will show that $ZL^1(G, \omega_\alpha)$ is weakly amenable if and only if $0 \le \alpha < 1/2$. This last result notably generalizes [3, Thm. 2.4(iii)–(iv)].

## 2. Central Beurling Algebras on $[\mathrm{FC}]^-$ Groups

Let $G$ be a locally compact group, and let $\mathrm{Aut}(G)$ be the set of all topological automorphisms of $G$ onto itself. For any compact set $F$ of $G$ and any open neighborhood $U$ of $e$ in $G$ we denote

$$B(F, U) = \{\tau \in \mathrm{Aut}(G) : \tau(x) \in Ux, \tau^{-1}(x) \in Ux \text{ for all } x \in F\}.$$

The family of all subsets of the form $B(F, U)$ forms an open neighborhood basis at the identity $\iota$ of $\mathrm{Aut}(G)$. This defines a Hausdorff topology on $\mathrm{Aut}(G)$, called the open compact topology on $\mathrm{Aut}(G)$. It is well known that this topology is completely regular [15, Thms. 4.8 and 8.4]. With this topology, $\mathrm{Aut}(G)$ is a topological group (but it may not be locally compact) [15, Thms. 26.5 and 26.6]. All inner automorphisms of $G$ form a (completely regular) topological subgroup of $\mathrm{Aut}(G)$, denoted by $I(G)$. For $x \in G$, let $\beta_x$ be the inner automorphism of $G$ implemented by $x$, that is,

$$\beta_x(g) = xgx^{-1} \quad (g \in G).$$

Clearly, the natural mapping $x \mapsto \beta_x$ is a continuous group antihomomorphism from $G$ onto $I(G)$, so that $x \mapsto (\beta_x)^{-1} = \beta_{x^{-1}}$ is a continuous group homomorphism from $G$ onto $I(G)$.

Let $S$ be a semitopological semigroup, that is, $S$ is a semigroup with a Hausdorff topology such that the product $s \cdot t$ is separately continuous. Obviously, a topological group is a semitopological semigroup; in particular, $I(G)$ belongs to this class. The space $C_b(S)$ of all bounded complex-valued continuous functions on $S$ forms a Banach space with the supremum norm

$$\|f\|_{\sup} = \sup_{s \in S} |f(s)| \quad (f \in C_b(S)).$$

Indeed, with the pointwise product $C_b(S)$ is a unital commutative C*-algebra whose identity is the constant function 1. Let $s \in S$ and $f \in C_b(S)$. The *left translate* $\ell_s f$ of $f$ by $s$ is the function $\ell_s f(x) = f(sx)$ $(x \in S)$. The *right translate* $r_s f$ by $s$ is defined similarly. Clearly, for each $s \in S$, $\ell_s$ and $r_s$ define bounded operators on $C_b(S)$. A positive linear functional $m \in C_b(S)^*$ is called a *left invariant* *mean* on $C_b(S)$ if $\|m\| = m(1) = 1$ (i.e. $m$ is a mean) and $m(\ell_s f) = m(f)$ (i.e. $m$ is left invariant) for all $f \in C_b(S)$ and all $s \in S$. Similarly, a *right invariant mean* on $C_b(S)$ is a a mean $m \in C_b(S)^*$ satisfying $m(r_s f) = m(f)$ for all $f \in C_b(S)$ and all $s \in S$. For a locally compact group $G$, it is well known that $G$ is amenable if and only if $C_b(G)$ has an invariant mean, a mean on $C_b(G)$ that is both left invariant and right invariant [10].

For two semitopological semigroups $S$ and $H$, it is readily seen that if $C_b(S)$ has a left invariant mean and if there is a continuous semigroup homomorphism from $S$ onto $H$, then $C_b(H)$ has a left invariant mean. It is also readily seen that if there is a continuous antihomomorphism from $S$ onto $H$, then the existence of a right invariant mean on $C_b(S)$ implies the existence of a left invariant mean on $C_b(H)$.

Let $G$ be a locally compact group, and let $f \in C_b(G)$. Then, for each $x \in G$, we have $\hat{f}_x \in C_b(I(G))$, where $\hat{f}_x(\beta) = f(\beta^{-1}(x))$ $(\beta \in I(G))$. Suppose further that $G$ is an $[\mathrm{FC}]^-$ group. Then, as is well known, $G$ is amenable (see [25] or [18]). The above discussion shows that $C_b(I(G))$ has a left invariant mean, say $\mu \in C_b(I(G))^*$. Note that $C_b(I(G))^* = M(\delta I(G))$, where $\delta I(G)$ is the Stone–Cěch compactification of $I(G)$ [6, Cor. V.6.4]. Let $f \in C_{00}(G)$ and $K = \mathrm{supp}(f)$. Then

$$C_K = cl\{\beta(x) : \beta \in I(G), x \in K\}$$

is a compact subset of $G$, and the function $(\beta, x) \mapsto f(\beta^{-1}(x))$ is a continuous function on $I(G) \times G$ whose support sits in $I(G) \times C_K$. As explained before, we may regard the left invariant mean $\mu$ as in $M(\delta I(G))$. Restricting $\mu$ to $I(G)$, we obtain a positive finite Borel (probability) measure space $(I(G), \mu)$. Note that $C_K$ is of finite Haar measure as a compact subset of $G$. We then may apply Fubini's Theorem to the function $(\beta, x) \mapsto f(\beta^{-1}(x))$ on $I(G) \times C_K$ and define $P_\mu(f)$ by

$$P_\mu(f)(x) = \mu(\hat{f}_x) \quad (x \in G). \tag{1}$$

Clearly, $\mathrm{supp}(P_\mu(f)) \subset C_K$ and $P_\mu(f) \in L^1(G)$. By the left invariance of $\mu$ it is readily seen that $P_\mu(f)(\beta^{-1}(x)) = P_\mu(f)(x)$ ($\beta \in I(G)$ and $x \in G$). Whence $P_\mu(f) \in ZL^1(G)$. Moreover, the Fubini theorem ([15, Thm. 13.8]) asserts that

$$\|P_\mu(f)\|_1 \le \|f\|_1.$$

Since $C_{00}(G)$ is dense in $L^1(G)$, $P_\mu$ extends to a bounded linear operator from $L^1(G)$ into $ZL^1(G)$, still denoted by $P_\mu$. It is also easily seen that $P_\mu(f) = f$ when $f \in ZL^1(G)$. Therefore, $P_\mu \colon L^1(G) \to ZL^1(G)$ is a Banach space contractive projection. Although $P_\mu$ is usually not a Banach algebra homomorphism, it is a $ZL^1(G)$-bimodule morphism if we view both $L^1(G)$ and $ZL^1(G)$ as natural $ZL^1(G)$-bimodules. But we will not use this feature.

If $\mu_i$ is a left invariant mean on $C_b(I(G_i))$ $(i = 1, 2)$, then $\mu_1 \times \mu_2$ is a left invariant mean on $C_b(I(G_1) \times I(G_2))$. Note that $I(G_1) \times I(G_2) = I(G_1 \times G_2)$. This generates a contractive projection $P_{\mu_1 \times \mu_2}$ from $L^1(G_1 \times G_2)$ onto $ZL^1(G_1 \times G_2)$. On the other hand, the mapping $f \otimes g \mapsto f \times g$ ($f \in L^1(G_1), g \in L^1(G_2)$) defines a Banach algebra isometry $T \colon L^1(G_1) \widehat{\otimes} L^1(G_2) \to L^1(G_1 \times$

$G_2)$, which maps $ZL^1(G_1) \otimes ZL^1(G_2)$ into $ZL^1(G_1 \times G_2)$, where $f \times g$ denotes the function

$$f \times g(x, y) = f(x)g(y) \quad (x \in G_1, y \in G_2).$$

Since $ZL^1(G_i)$ is complemented in $L^1(G_i)$, the inclusion mappings

$$\iota_i : ZL^1(G_1) \to L^1(G_i) \quad (i = 1, 2)$$

induce a norm-preserving Banach algebra embedding:

$$\iota_1 \otimes \iota_2 : ZL^1(G_1) \widehat{\otimes} ZL^1(G_2) \to L^1(G_1) \widehat{\otimes} L^1(G_2).$$

We warn here that, in general, for closed subspaces $B_i$ of Banach spaces $A_i$ and embeddings $\iota_i$: $B_i \to A_i$ $(i = 1, 2)$, $\iota_1 \otimes \iota_2$: $B_1 \widehat{\otimes} B_2 \to A_1 \widehat{\otimes} A_2$ is not necessarily an embedding; it may not be even injective (see [32]). Denote the inclusion mapping $ZL^1(G_1 \times G_2) \to L^1(G_1 \times G_2)$ by $\iota$. Then we can easily verify that

$$\iota \circ P_{\mu_1 \times \mu_2} \circ T = T \circ \iota_1 \otimes \iota_2 \circ P_{\mu_1} \times P_{\mu_2}.$$

LEMMA 2.1. *Let $G_1$ and $G_2$ be locally compact* $[\mathrm{FC}]^-$ *groups. Then, as Banach algebras,*

$$ZL^1(G_1 \times G_2) \simeq ZL^1(G_1) \widehat{\otimes} ZL^1(G_2).$$

*Proof.* Consider the following chain:

$$ZL^1(G_1) \widehat{\otimes} ZL^1(G_2) \xrightarrow{\iota_1 \otimes \iota_2} L^1(G_1) \widehat{\otimes} L^1(G_2) \xrightarrow{T} L^1(G_1 \times G_2)$$
$$\xrightarrow{P_{\mu_1 \times \mu_2}} ZL^1(G_1 \times G_2).$$

From our discussion, the composition of the chain provides a Banach algebra isomorphism (in fact, isometric isomorphism) from $ZL^1(G_1) \widehat{\otimes} ZL^1(G_2)$ onto $ZL^1(G_1 \times G_2)$. □

Here we note that Lemma 2.1 generalizes the known results for compact and discrete cases in [2] and [1].

Now we consider the weighted case. If $\omega$ is a continuous weight on the $[\mathrm{FC}]^-$ group $G$, then for each $x \in G$, there is a constant $c_x \geq 1$ such that

$$\omega(\beta(x)) \leq c_x \omega(x) \quad (\beta \in I(G)).$$

Assume that there is an upper bound for all $c_x$. We then have the following:

PROPOSITION 2.2. *Let $G_1$, $G_2$ be* $[\mathrm{FC}]^-$ *groups, and $\omega_i$ be a weight on $G_i$ satisfying $\omega_i(gxg^{-1}) \leq c\omega_i(x)$ $(x, g \in G_i)$ $(i = 1, 2)$, where $c > 0$ is a constant. Then, as Banach algebras,*

$$ZL^1(G_1 \times G_2, \omega_1 \times \omega_2) \simeq ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2). \tag{2}$$

*Proof.* Up to equivalence, we may assume that $\omega_1$ and $\omega_2$ to be continuous. If $\omega$ is a continuous weight on an $[\mathrm{FC}]^-$ group $G$ such that $\omega(gxg^{-1}) \leq c\omega(x)$

($x, g \in G$), then we can still consider the mapping $P_\mu$ on $C_{00}(G)$ defined by (1). Let $f \in C_{00}(G)$. Then we have

$$|P_\mu(f)(x)\omega(x)| \le cP_\mu(|f|\omega)(x),$$
$$|P_\mu(|f|\omega)(x)| \le cP_\mu(|f|)(x)\omega(x) \quad (x \in G).$$

By the Fubini theorem we obtain

$$\|P_\mu(f)\|_\omega \le c\|f\|_\omega.$$

These are true for all $f \in C_{00}(G)$, which is dense in $L^1(G, \omega)$. So $P_\mu$ extends to a bounded linear mapping from $L^1(G, \omega)$ to $L^1(G, \omega)$. Similarly to the non-weighted case, we have $P_\mu(f) \in ZL^1(G, \omega)$. So $P_\mu$ is a continuous projection from $L^1(G, \omega)$ onto $ZL^1(G, \omega)$, and $\|P_\mu\| \le c$. Then we follow the same argument for Lemma 2.1 to get the isomorphic relation (2). □

As is well known, [FC]$^-$ groups are amenable [IN] groups [25]. For general amenable [IN] groups we have the following result.

PROPOSITION 2.3. *Let $G_1$ and $G_2$ be amenable* [IN] *groups, and let $\omega_1$ and $\omega_2$ be weights on them, respectively. Then $ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2)$ is weakly amenable if and only if both $ZL^1(G_1, \omega_1)$ and $ZL^1(G_2, \omega_2)$ are weakly amenable.*

*Proof.* Again, we may assume that the weights are continuous.

Since $ZL^1(G_1, \omega_1)$ and $ZL^1(G_2, \omega_2)$ are commutative, The sufficiency follows from [7, Prop. 2.8.71].

For the converse, we first note that if $G$ is an amenable [SIN] group and $\omega$ is a weight on $G$, then there is a character (namely, a bounded multiplicative linear functional) $\varphi > 0$ on $L^1(G, \omega)$ by [30, Lemma 1]. Restricting to $ZL^1(G, \omega)$, $\varphi$ is clearly nontrivial (note that $L^1(G, \omega)$ has a central bounded approximate identity). Now let $G$ be an amenable [IN] group. Then it is well known that there is a compact normal subgroup $K$ of $G$ such that $G/K \in$ [SIN] (see [16, Thm. 1]) and $G/K$ is still amenable. Define a weight $\hat{\omega}$ on $G/K$ by

$$\hat{\omega}(xK) = \inf_{t \in K} \omega(xt).$$

Then there is a standard Banach algebra homomorphism $T$ from $L^1(G, \omega)$ onto $L^1(G/K, \hat{\omega})$ ([26, Thm. 3.7.13]). In fact, $T$ is precisely formulated by

$$T(f)(xK) = \int_K f(xt)\,dt \quad (f \in L^1(G, \omega), x \in G).$$

Clearly, $T$ maps $ZL^1(G, \omega)$ onto $ZL^1(G/K, \hat{\omega})$. As we have shown, there is a character $\varphi$ on $L^1(G/K, \hat{\omega})$ that is nontrivial on $ZL^1(G/K, \hat{\omega})$. Then the composition $\phi = \varphi \circ T$ gives a character on $L^1(G, \omega)$ that is nontrivial on $ZL^1(G, \omega)$. Apply this to $ZL^1(G_2, \omega_2)$ and assume that $ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2)$ is weakly amenable. Then the mapping

$$f \otimes g \mapsto \phi(g)f \quad (f \in ZL^1(G_1, \omega_1), g \in ZL^1(G_2, \omega_2))$$

generates a Banach algebra homomorphism from $ZL^1(G_1, \omega_1) \widehat{\otimes} ZL^1(G_2, \omega_2)$ onto $ZL^1(G_1, \omega_1)$. Whence $ZL^1(G_1, \omega_1)$ is weakly amenable by [7, Prop. 2.8.64]. Similarly, $ZL^1(G_2, \omega_2)$ is weakly amenable. □

Consider the particular case $G = H \times K$, where $H$ is an $[\mathrm{FC}]^-$ group, and $K$ is a compact group. Let $\omega$ be a continuous weight on $G$. Define

$$\omega_H(x) = \omega(x, e_K) \quad (x \in H),$$

where $e_K$ is the unit of $K$. Then $\omega_H$ is a weight on $H$, and $\omega$ is equivalent to the weight $\omega_H \times 1$ on $H \times K$. Therefore, $ZL^1(G, \omega)$ is a Banach algebra isomorphic to $ZL^1(G, \omega_H \times 1)$. Now assume that $\omega$ satisfies $\omega(hxh^{-1}, e_K) \le c\omega(x, e_K)$ $(x, h \in H)$ for some constant $c$. Since $ZL^1(K)$ is weakly amenable (see [31, Prop. 5.1]), by Propositions 2.2 and 2.3 we see $ZL^1(G, \omega)$ is weakly amenable if and only if $ZL^1(H, \omega_H)$ is weakly amenable. This, in particular, leads us to the following extension of [31, Thm. 3.1], where $ZL^1(H, \omega_H) = L^1(H, \omega_H)$ since $H$ is abelian.

PROPOSITION 2.4. *Suppose that $G = H \times K$, $H$ is an Abelian group and $K$ is a compact group. Let $\omega$ be a weight on $G$. Then $ZL^1(G, \omega)$ is weakly amenable if and only if there is no nontrivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} < \infty. \tag{3}$$

*Proof.* We only need to note that there is a nontrivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that (3) holds if and only if there is a nontrivial continuous group homomorphism $\Phi : H \to \mathbb{C}$ such that

$$\sup_{h \in H} \frac{|\Phi(h)|}{\omega_H(h)\omega_H(h^{-1})} < \infty.$$

So the conclusion follows from [31, Thm. 3.1]. □

REMARK 2.5. According to [14, Thm. 4.3], if $G$ is a connected [SIN] group, then $G = V \times K$ for some (Abelian) vector group $V$ and a compact group $K$. So Proposition 2.4 is valid in particular for this kind of group $G$.

In fact, the necessity part of Proposition 2.4 remains true for general $[\mathrm{FC}]^-$ groups. To prove this, we first consider a general [IN] group.

LEMMA 2.6. *Let $G$ be an* [IN] *group, $\omega$ be a weight on $G$, and $U$ be an open set of $G$ with a compact closure and invariant under inner automorphisms of $G$. Suppose that there exists a continuous group homomorphism $\Phi : G \to \mathbb{C}$ nontrivial on $U$ and such that*

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

*Then there is a nontrivial continuous derivation from $ZL^1(G, \omega)$ into $L^\infty(G, \frac{1}{\omega})$. Consequently, $ZL^1(G, \omega)$ is not weakly amenable.*

*Proof.* Since $ZL^1(G,\omega)$ is the center of $L^1(G,\omega)$, $L^\infty(G,\frac{1}{\omega})$ is a symmetric Banach $ZL^1(G,\omega)$-bimodule. We construct a nontrivial continuous derivation $D : ZL^1(G,\omega) \to L^\infty(G,\frac{1}{\omega})$. When this is done, it follows from the definition of weak amenability for a commutative Banach algebra given in [3] that $ZL^1(G,\omega)$ is not weakly amenable.

We define $D$ as follows:

$$D(h)(t) = \int_U \Phi(t^{-1}\xi)h(t^{-1}\xi)\,d\xi \quad (t \in G, h \in ZL^1(G,\omega)). \tag{4}$$

First, we note that $D$ is nontrivial. To see this, we consider the function $h_\Phi = \chi_U \overline{\Phi}$, where $\chi_U$ is the characteristic function of $U$, and $\overline{\Phi}$ is the conjugate of $\Phi$. Since $\Phi$ is a group homomorphism and $U$ is invariant under $I(G)$, we have that $h_\Phi \in ZL^1(G,\omega)$. Moreover,

$$\begin{aligned} D(h_\Phi)(t) &= \int_U \Phi(t^{-1}\xi)h_\Phi(t^{-1}\xi)\,d\xi = \int_{U\cap tU} |\Phi(t^{-1}\xi)|^2\,d\xi \\ &= \int_{t^{-1}U\cap U} |\Phi(\xi)|^2\,d\xi. \end{aligned}$$

This shows that $D(h_\Phi)(t) > 0$ for $t$ in a neighborhood of the identity $e$ of $G$ because $|\Phi|^2 > 0$ on some open subset of $t^{-1}U \cap U$ when $t$ is near $e$. Hence, $D$ is nontrivial. Using the method of [29, Thm. 2.2], we can show that formula (4) defines a bounded derivation even from the whole $L^1(G,\omega)$ into $L^\infty(G,\frac{1}{\omega})$. So, it also defines a (nontrivial) continuous derivation from $ZL^1(G,\omega)$ into $L^\infty(G,\frac{1}{\omega})$. □

We need some elementary property of an $[\mathrm{FC}]^-$ group, which we state as follows.

LEMMA 2.7. *Let $G$ be an $[\mathrm{FC}]^-$ group. Then, for every $x \in G$, there exists an open precompact neighborhood $K_x$ of $x$ in $G$ that is invariant under inner automorphisms of $G$.*

*Proof.* It is known that an $[\mathrm{FC}]^-$ group $G$ belongs to [IN]. Let $B$ be a precompact open invariant neighborhood of $e$, and let $C_x$ be the conjugacy class of $x$ (which is also precompact and invariant). Then $K_x = BC_x$ satisfies our requirement. □

PROPOSITION 2.8. *Let $G$ be a locally compact $[\mathrm{FC}]^-$ group, and $\omega$ be a weight on $G$. Suppose that there exists a nontrivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{t\in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty. \tag{5}$$

*Then $ZL^1(G,\omega)$ is not weakly amenable.*

*Proof.* Since $\Phi$ is nontrivial, there exists $x \in G$ such that $\Phi(x) \neq 0$. Applying Lemma 2.7, we get an open neighborhood $U = K_x$ of $x$ that is invariant under inner automorphisms and has compact closure. Therefore Lemma 2.6 applies. □

We wonder whether the converse of Proposition 2.8 remains true as in the commutative case. We raise it here as an open problem. We note that, in many cases, $ZL^1(G,\omega)$ is isomorphic to a weighted commutative hypergroup algebra. So our question links to the general open problem of characterizing weak amenability of (commutative) hypergroup algebras. In particular, it would be of great interest to characterize weak amenability of $Z^B L^1(G,\omega)$ for $G \in [\mathrm{FIA}]_B^-$ and $B$ being a closed subgroup of $\mathrm{Aut}(G)$ with $I(G) \subseteq B$ (see [19; 23] for definitions).

## 3. Central Beurling Algebras on [FD]$^-$ Groups

In this section, we consider [FD]$^-$ groups and aim to establish some sufficient conditions for $ZL^1(G,\omega)$ to be weakly amenable. We first recall that $G$ is an [FD]$^-$ group if and only if there exists a compact normal subgroup $K$ of $G$ such that the quotient $G/K$ is abelian.

The following structural result, which is [24, Lemma 1] for $B = I(G)$, is crucial in the sequel.

Lemma 3.1. *Let $G$ be an* [FD]$^-$ *group, and $K$ a compact normal subgroup of $G$ such that $G/K$ is abelian. Let $\omega \geq 1$ be a weight on $G$ satisfying*

$$\lim_{n\to\infty} (\omega(x^n))^{1/n} = 1$$

*for all $x \in G$, and let $\hat{\omega}$ be the induced weight on $G/K$ defined by*

$$\hat{\omega}(xK) = \inf_{k\in K} \omega(xk) \quad (x \in G).$$

*Then $ZL^1(G,\omega)$ may be written as the closure of the linear span of a family of complemented ideals, each of which is isomorphic to a Beurling algebra of the form $L^1(S/K,\hat{\omega})$ for some open normal subgroup $S$ of $G$.*

We need also the following well-known result.

Lemma 3.2. *Let $A$ be a commutative Banach algebra, and $\{A_\gamma\}_{\gamma\in\Gamma}$ be a family of closed subalgebras of $A$ such that $A = \overline{\mathrm{lin}}\{A_\gamma\}_{\gamma\in\Gamma}$. If each $A_\gamma$ is weakly amenable, then so is $A$.*

We note that $L^1(S/K,\hat{\omega})$ in Lemma 3.1 is a commutative Beurling algebra. So [31, Thm. 3.1] applies for the weak amenability of it. This leads to the following result.

Theorem 3.3. *Let $G$ be an* [FD]$^-$ *group, and $\omega \geq 1$ be a continuous weight on $G$ satisfying*

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G). \tag{6}$$

*Then $ZL^1(G,\omega)$ is weakly amenable.*

*Proof.* First, we show that (6) implies that $\lim_{n\to\infty}(\omega(x^n))^{1/n} = 1$ for every $x \in G$. Since $\omega \geq 1$, it suffices to prove that

$$\limsup_{n\to\infty}(\omega(x^n))^{1/n} \leq 1 \quad (x \in G).$$

Fix $x \in G$ and let $\varepsilon > 0$ be arbitrary. Because $\lim_{n\to\infty} n^{1/n} = 1$, there exists $N_\varepsilon \in \mathbb{N}$ such that $n^{1/n} \leq (1+\varepsilon)$ for every $n \geq N_\varepsilon$. Using assumption (6) and the inequality $\omega \geq 1$, we can find $n_\varepsilon > N_\varepsilon$ such that

$$\omega(x^{n_\varepsilon}) \leq \omega(x^{n_\varepsilon})\omega(x^{-n_\varepsilon}) \leq n_\varepsilon = (n_\varepsilon^{1/n_\varepsilon})^{n_\varepsilon} \leq (1+\varepsilon)^{n_\varepsilon}.$$

For any $m \in \mathbb{N}$, there exist $k \in \mathbb{N} \cup \{0\}$ and $0 \leq l < n_\varepsilon$ such that $m = kn_\varepsilon + l$. Using the weight inequality for $\omega$, we can make the following estimates:

$$\begin{aligned}\omega(x^m) = \omega(x^{kn_\varepsilon+l}) &\leq (\omega(x^{n_\varepsilon}))^k\omega(x^l) \leq (1+\varepsilon)^{kn_\varepsilon}\omega(x^l)\\ &= \frac{(1+\varepsilon)^m\omega(x^l)}{(1+\varepsilon)^l} \leq c_\varepsilon(1+\varepsilon)^m,\end{aligned}$$

where

$$c_\varepsilon = \sup_{0\leq l<n_\varepsilon} \frac{\omega(x^l)}{(1+\varepsilon)^l}$$

is a constant that does not depend on $m$. It follows that

$$\limsup_{n\to\infty}(\omega(x^n))^{1/n} \leq \limsup_{n\to\infty}(c_\varepsilon(1+\varepsilon)^n)^{1/n} = 1+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain that $\limsup_{n\to\infty}(\omega(x^n))^{1/n} \leq 1$, as desired.

So, the condition of Lemma 3.1 is satisfied. Then there exists a family of complemented ideals $\{J_\gamma\}_{\gamma\in\Gamma}$ of $ZL^1(G,\omega)$ such that $\overline{\text{lin}}\{J_\gamma\}_{\gamma\in\Gamma} = ZL^1(G,\omega)$, and for each $\gamma \in \Gamma$, there exists an open subgroup $S_\gamma \supset K$ of $G$ for which $J_\gamma \simeq L^1(S_\gamma/K, \hat{\omega})$. Let $\Phi : S_\gamma/K \to \mathbb{C}$ be a nontrivial continuous group homomorphism. Choose $t_\gamma \in S_\gamma/K$ so that $\Phi(t_\gamma) \neq 0$. Then

$$\sup_{t\in S_\gamma/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{|\Phi(t_\gamma^n)|}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})} = \sup_{n\in\mathbb{N}} \frac{n|\Phi(t_\gamma)|}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})}. \tag{7}$$

Let $x_\gamma \in S_\gamma$ be a representative of $t_\gamma$, that is, $x_\gamma K = t_\gamma$. We note that, for each $x \in G$,

$$\hat{\omega}(xK) = \inf_{k\in K} \omega(xk) \leq \omega(x).$$

In particular, $\hat{\omega}(t_\gamma^n) \leq \omega(x_\gamma^n)$ and $\hat{\omega}(t_\gamma^{-n}) \leq \omega(x_\gamma^{-n})$ $(n \in \mathbb{N})$. Combining this with conditions (6) and (7), we obtain

$$\sup_{t\in S_\gamma/K} \frac{|\Phi(t)|}{\hat{\omega}(t)\hat{\omega}(t^{-1})} \geq \sup_{n\in\mathbb{N}} \frac{n|\Phi(t_\gamma)|}{\hat{\omega}(t_\gamma^n)\hat{\omega}(t_\gamma^{-n})} \geq \sup_{n\in\mathbb{N}} \frac{n|\Phi(t_\gamma)|}{\omega(x_\gamma^n)\omega(x_\gamma^{-n})} = \infty.$$

According to [31, Thm. 3.1], this implies that $J_\gamma \simeq L^1(S_\gamma/K, \hat{\omega})$ is weakly amenable. Then Lemma 3.2 applies. □

We now apply Theorem 3.3 to compactly generated $[\mathrm{FC}]^-$ groups, which are, in fact, $[\mathrm{FD}]^-$ groups according to [14, Thm. 3.20].

Let $G$ be a compactly generated locally compact group. Then there is an open symmetric neighborhood $U$ of the identity in $G$ with compact closure and satisfying $G = \bigcup_{n=1}^{\infty} U^n$. Following [24], we consider the length function $|\cdot| : G \to \mathbb{N}$ defined by

$$|x| = \min\{n \in \mathbb{N} : x \in U^n\} \quad (x \in G).$$

It is readily checked that $|x| \geq 1$ $(x \in G)$, and for every $\alpha \geq 0$, the corresponding polynomial weight $\omega_\alpha(x) = (1 + |x|)^\alpha$ $(x \in G)$ is, indeed, an upper semicontinuous weight on $G$. As addressed in the Introduction, $\omega_\alpha$ is equivalent to a continuous weight.

Theorem 3.4. *Let $G$ be a compactly generated noncompact* $[\mathrm{FC}]^-$ *group, and $\omega_\alpha$ be the weight on $G$ defined as before. Then $ZL^1(G, \omega_\alpha)$ is weakly amenable if and only if* $0 \leq \alpha < 1/2$.

*Proof.* From the definition of the length function we have $|x^{-1}| = |x|$ and $|x^n| \leq n|x|$ $(x \in G, n \in \mathbb{N})$. If $0 \leq \alpha < 1/2$, then

$$\sup_{n\in\mathbb{N}} \frac{n}{\omega_\alpha(x^n)\omega_\alpha(x^{-n})} = \sup_{n\in\mathbb{N}} \frac{n}{(1 + |x^n|)^\alpha (1 + |x^{-n}|)^\alpha} \geq \sup_{n\in\mathbb{N}} \frac{n}{(1 + n|x|)^{2\alpha}} = \infty \quad (x \in G).$$

This is still true if $\omega_\alpha$ is replaced by a continuous equivalent weight. Therefore, $ZL^1(G, \omega_\alpha)$ is weakly amenable by Theorem 3.3.

To prove the converse, we let $K$ be a compact subgroup of $G$ such that $G/K$ is abelian. The quotient group $H = G/K$ is clearly still compactly generated. By the structure theorem for compactly generated locally compact Abelian groups [15, Thm. II.9.8], $H$ is topologically isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n \times F$ for some integers $m$ and $n$ and some compact Abelian group $F$. Since $G$ is not compact, neither is $H$. Then either $\mathbb{R}$ or $\mathbb{Z}$ is a quotient group of $H$. Thus there is a nontrivial continuous group homomorphism $\phi\colon H \to \mathbb{R}$. Then $\Phi = \phi \circ q\colon G \to \mathbb{R}$ is a nontrivial continuous group homomorphism, where $q\colon G \to H$ is the quotient map. If $\alpha \geq 1/2$, this $\Phi$ satisfies inequality (5) with $\omega = \omega_\alpha$. In fact, for $x \in G$, there is a smallest $k \in \mathbb{N}$ such that $x \in U^k$. We have $|x| = k$ and

$$|\Phi(x)| \leq c_0 k = c_0|x|,$$

where $c_0 = \sup_{g\in U} |\Phi(g)|$, which is finite since $\overline{U}$ is compact. This leads to inequality (5) for $\omega = \omega_\alpha$ (and also for any continuous $\omega$ equivalent to $\omega_\alpha$) since $\alpha \geq 1/2$. Hence, $L^1(G, \omega_\alpha)$ is not weakly amenable due to Proposition 2.8. □

Remark 3.5. Consider again the general $[\mathrm{FD}]^-$ group $G$. Let $K$ be a compact normal subgroup of it such that $G/K$ is commutative. If there is a continuous nontrivial group homomorphism $\Phi\colon G \to \mathbb{C}$ such that (5) holds, then $ZL^1(G, \omega)$ is not weakly amenable from Proposition 2.8. If there is no such $\Phi$, then there is no such $\Phi$ for $G/K$ with weight $\hat{\omega}$. Then $L^1(G/K, \hat{\omega})$ is weakly amenable due to [31, Thm. 3.1]. We want to know whether $L^1(S/K, \hat{\omega})$ is weakly amenable for any open subgroup $S$ of $G$ containing $K$. If this is true, we will then obtain a characterization for the weak amenability of Beurling algebras on an [FD]$^-$ group. We note that $S/K$ is an open subgroup of $G/K$. However, in general, weak amenability of a Beurling algebra does not pass to the Beurling algebra on a subgroup (see Section 5 of [29]).

The situation is simple when $G/K$ is isomorphic to $\mathbb{R}$ or $\mathbb{Z}$.

PROPOSITION 3.6. *Suppose that $G$ is a locally compact group and has a compact normal subgroup $K$ such that $G/K \simeq \mathbb{R}$ or $G/K \simeq \mathbb{Z}$. Let $\omega \geq 1$ be a weight on $G$. Then $ZL^1(G, \omega)$ is weakly amenable if and only if there is no nontrivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ such that*

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty. \tag{8}$$

*Proof.* It suffices to show the sufficiency. If there is no nontrivial continuous group homomorphism $\Phi : G \to \mathbb{C}$ for which (8) holds, then, as is known, $L^1(G/K, \hat{\omega})$ is weakly amenable. This in turn implies that

$$\sup_{n \in \mathbb{N}} \frac{n}{\hat{\omega}(t^n)\hat{\omega}(t^{-n})} = \infty \quad (t \in G/K)$$

due to [31, Cor. 3.7]. Since $\omega(x) \leq c\hat{\omega}(xK)$ for $x \in G$, where

$$c = \max\{\omega(k) : k \in K\},$$

the last condition leads to

$$\sup_{n \in \mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G).$$

Then, applying Theorem 3.3, we conclude that $ZL^1(G, \omega)$ is weakly amenable. □

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## References

[1] M. Alaghmandan, Y. Choi, and E. Samei, *ZL-amenability and characters for the restricted direct products of finite groups,* J. Math. Anal. Appl. 411 (2014), 314–328.

[2] A. Azimifard, E. Samei, and N. Spronk, *Amenability properties of the centres of group algebras,* J. Funct. Anal. 256 (2009), 1544–1564.

[3] W. G. Bade, P. C. Curtis Jr., and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras,* Proc. Lond. Math. Soc. 55 (1987), 359–377.

[4] A. Beurling, *Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle,* 9th Congrés des Mathématiciens Scandinaves 1938, Helsinki, pp. 345–366, 1939.

[5] C. R. Borwick, *Johnson–Hochschild cohomology of weighted group algebras and augmentation ideals,* Ph.D. thesis, University of Newcastle upon Tyne, 2003.

[6] H. B. Conway, *A course in functional analysis,* Springer-Verlag, New York, 1985.
[7] H. G. Dales, *Banach algebras and automatic continuity,* London Math. Soc. Monogr. Ser., 24, The Clarendon Press, Oxford University Press, New York, 2000, Oxford Science Publications.
[8] F. Ghahramani, R. J. Loy, and Y. Zhang, *Generalized notions of amenability, II,* J. Funct. Anal. 254 (2008), 1776–1810.
[9] F. Ghahramani, E. Samei, and Y. Zhang, *Generalized amenability properties of the Beurling algebras,* J. Aust. Math. Soc. 89 (2010), 359–376.
[10] F. P. Greenleaf, *Invariant means on topological groups and their applications,* Van Nostrand Math. Stud., 16, Van Nostrand Reinhold Co., New York–Toronto, Ont.–London, 1969.
[11] N. Gronbaek, *Amenability of weighted convolution algebras on locally compact groups,* Trans. Amer. Math. Soc. 319 (1990), 765–775.
[12] ———, *A characterization of weakly amenable Banach algebras,* Studia Math. 94 (1989), 149–162.
[13] S. Grosser and M. Moskowitz, *On central topological groups,* Trans. Amer. Math. Soc. 127 (1967), 317–340.
[14] ———, *Compactness conditions in topological groups,* J. Reine Angew. Math. 246 (1971), 1–40.
[15] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I,* Springer-Verlag, New York, 1963.
[16] K. Iwasawa, *Topological groups with invariant compact neighborhoods of the identity,* Ann. of Math. (2) 54 (1951), 345–348.
[17] B. E. Johnson, *Non-amenability of the Fourier algebra of a compact group,* J. Lond. Math. Soc. (2) 50 (1994), 361–374.
[18] H. Leptin, *Zur harmonischen Analyse klassenkompakter Gruppen,* Invent. Math. 5 (1968), 249–254.
[19] J. Liukkonen and R. Mosak, *Harmonic analysis and centers of Beurling algebras,* Comment. Math. Helv. 57 (1977), 297–315.
[20] ———, *Harmonic analysis and centers of group algebras,* Trans. Amer. Math. Soc. 195 (1974), 147–163.
[21] J. R. Liukkonen, *Dual spaces of groups with precompact conjugacy classes,* Trans. Amer. Math. Soc. 180 (1973), 85–108.
[22] R. D. Mosak, *Central functions in group algebras,* Proc. Amer. Math. Soc. 29 (1971), 613–616.
[23] ———, *The $L^1$- and $C^*$-algebras of $[FIA]_B^-$ groups, and their representations,* Trans. Amer. Math. Soc. 163 (1972), 277–310.
[24] ———, *Ditkin's condition and primary ideals in central Beurling algebras,* Monatsh. Math. 85 (1978), 115–124.
[25] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras, Vol. 2. *-algebras,* Encyclopedia Math. Appl., 79, Cambridge University Press, Cambridge, 2001.
[26] H. Reiter and J. D. Stegeman, *Classical harmonic analysis and locally compact groups,* 2nd edition, London Math. Soc. Monogr. Ser., 22, The Clarendon Press, Oxford University Press, New York, 2000.
[27] E. Samei, *Weak amenability and 2-weak amenability of Beurling algebras,* J. Math. Anal. Appl. 346 (2008), 451–467.
[28] V. Shepelska, *Weak amenability of weighted group algebras on some discrete groups,* Studia Math. 230 (2015), 189–214.

[29] V. Shepelska and Y. Zhang, *Non-weakly amenable Beurling algebras,* Indiana Univ. Math. J., to appear.
[30] M. C. White, *Characters on weighted amenable groups,* Bull. Lond. Math. Soc. 23 (1991), 375–380.
[31] Y. Zhang, *Weak amenability of commutative Beurling algebras,* Proc. Amer. Math. Soc. 142 (2014), 1649–1661.
[32] ______, *Nilpotent ideals in a class of Banach algebras,* Proc. Amer. Math. Soc. 127 (1999), 3237–3242.

V. Shepelska
Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 5E6 Canada

shepelska@gmail.com

Y. Zhang
Department of Mathematics
University of Manitoba
Winnipeg, Manitoba
R3T 2N2 Canada

zhangy@cc.umanitoba.ca