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On Generic Vanishing for Pluricanonical Bundles

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Abstract. We study cohomology support loci and higher direct images of (log) pluricanonical bundles of smooth projective varieties or log canonical pairs. We prove that the zeroth cohomology support loci of log pluricanonical bundles are finite unions of torsion translates of subtori, and we give a generalization of the generic vanishing theorem to log canonical pairs. We also construct an example of morphism from a smooth projective variety to an Abelian variety such that a higher direct image of a pluricanonical bundle to the Abelian variety is not a GV-sheaf.

1. Introduction

Throughout this paper, we always assume that all varieties are defined over the complex number field. Let X be a smooth projective variety, and Δ be a simple normal crossing divisor on X. In this paper, we prove some results of generic vanishing theory for $K_X + \Delta$ or, more generally, for $m(K_X + \Delta)$ for any positive integer m.

In Section 3, we prove some results about the structure of cohomology support loci for a log canonical pair (X, Δ) (for the definition of cohomology support loci, see Section 2.1). Originally, Simpson [Sim] proved the following theorem.

Theorem 1.1 (Simpson). *Let X be a smooth projective variety. Then the cohomology support locus*

$$S^i_j(K_X) = \{\xi \in \operatorname{Pic}^0(X) \mid h^i(X, \mathcal{O}_X(K_X) \otimes \xi) \geq j\}$$

is a finite union of torsion translates of Abelian subvarieties of $\operatorname{Pic}^0(X)$ *for any $i \geq 0$ and $j \geq 1$.*

In [ClHa], a generalization of Theorem 1.1 for Kawamata log terminal pairs is discussed (see [ClHa, Thm. 8.3]). We generalize Theorem 1.1 for log canonical pairs.

Theorem 1.2 (= Theorem 3.5). *Let X be a smooth projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X, that is, a $\mathbb{Q}$-divisor whose coefficients are in* $[0, 1]$, *with simple normal crossing support, $f : X \to A$ be a morphism to an Abelian variety, and D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Then the cohomology support locus*

$$S^i_j(D, f) = \{\xi \in \operatorname{Pic}^0(A) \mid h^i(X, \mathcal{O}_X(D) \otimes f^*\xi) \geq j\}$$

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is a finite union of torsion translates of Abelian subvarieties of $\mathrm{Pic}^0(A)$ *for any* $i \geq 0$ *and* $j \geq 1$.

We will reduce Theorem 1.2 to the particular case where Δ is a (reduced) simple normal crossing divisor. This case was treated by Kawamata [Kaw] and was used in [CKP] for the proof of Corollary 1.4. Budur [Bud] proved a result for the cohomology support loci of unitary local systems, which includes it as a particular case (see Theorems 3.3 and 3.4). We will use Budur's result for the proof of Theorem 1.2.

We can also prove a similar result for pluricanonical divisors.

THEOREM 1.3 (= Theorem 3.9). *Let* X *be a normal projective variety,* Δ *be a boundary* $\mathbb{Q}$*-divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier and* (X, Δ) *is log canonical,* $f : X \to A$ *be a morphism to an Abelian variety, and* D *be a Cartier divisor on* X *such that* $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ *for some positive integer* m. *Then the cohomology support locus*

$$S^0_j(D, f) = \{\xi \in \mathrm{Pic}^0(A) \mid h^0(X, \mathcal{O}_X(D) \otimes f^*\xi) \geq j\}$$

is a finite union of torsion translates of Abelian subvarieties of $\mathrm{Pic}^0(A)$ *for any* $j \geq 1$.

Theorem 1.3 is one of the main results of this paper. Note that Theorem 1.3 states the structure of cohomology support loci of only zeroth cohomology and that (X, Δ) is an arbitrary log canonical pair. The special case where $\Delta = 0$ was proved by Chen and Hacon [ChHa, Thm. 3.2]). As a corollary, Theorem 1.3 implies the following result by Campana, Koziarz, and Păun.

COROLLARY 1.4 ([CKP, Thm. 0.1]). *Let* X *be a normal projective variety,* Δ *be a boundary* $\mathbb{Q}$*-divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier and* (X, Δ) *is log canonical. Assume that there exists a numerically trivial Cartier divisor* ρ *on* X *such that* $H^0(X, m(K_X + \Delta) + \rho) \neq 0$ *for some positive integer* m. *Then* $h^0(X, m'(K_X + \Delta)) \geq h^0(X, m(K_X + \Delta) + \rho) > 0$ *for some suitable multiple* m' *of* m.

Therefore, we can see Theorem 1.3 as a generalization of [CKP, Thm. 0.1]. We will also give an alternative proof of [CKP, Cor. 3.2] as an application of Theorem 1.3 (see Proposition 3.10).

Section 4 treats the higher direct images of pluricanonical bundles to Abelian varieties. In this direction, Hacon [Hac] first proved the following result.

THEOREM 1.5 (Hacon). *Let* X *be a smooth projective variety, and* $f : X \to A$ *be a morphism to an Abelian variety. Then the higher direct images* $R^j f_*\omega_X$ *are GV-sheaves for any* j.

In the notation in Theorem 1.5, a coherent sheaf $\mathcal{F}$ on A is called a *GV-sheaf* on A if the cohomology support loci

$$S^i(\mathcal{F}) = \{\xi \in \mathrm{Pic}^0(A) \mid h^i(A, \mathcal{F} \otimes \xi) \neq 0\}$$

satisfy $\operatorname{codim}_{\mathrm{Pic}^0(A)} S^i(\mathcal{F}) \geq i$ for every $i > 0$ (see also Section 2.3). Theorem 1.5 recovers the original generic vanishing theorem of Green and Lazarsfeld [GrLa]. We will give some generalizations of Theorem 1.5 and the generic vanishing theorem of Green and Lazarsfeld to log canonical pairs. For the precise statements, see Proposition 4.2 and Theorem 3.7, respectively.

Popa and Schnell [PoSc] gave a similar result for pluricanonical divisors.

THEOREM 1.6 ([PoSc, Thm. 1.10]). *Let X be a smooth projective variety, and $f : X \to A$ be a morphism to an Abelian variety. Then the direct image $f_*(\omega_X^{\otimes m})$ is a GV-sheaf for any positive integer m.*

Theorem 1.6 is a consequence of [PoSc, Thm. 1.7]. For the proof of Theorem 1.6, the following particular case of [PoSc, Thm. 1.7] is sufficient.

THEOREM 1.7 ([PoSc, Cor. 2.9]). *Let X be a smooth projective variety, and $f : X \to Y$ be a morphism to a projective variety Y of dimension n. If L is an ample and globally generated line bundle on Y and m is a positive integer, then*

$$H^i(Y, f_*\omega_X^{\otimes m} \otimes L^{\otimes l}) = 0$$

for all $i > 0$ and $l \geq m(n+1) - n$.

However, their proof of [PoSc, Thm. 1.7] does not work for the higher direct images $R^j f_*\omega_X^{\otimes m}$. They posed the following question in [PoSc, p. 2293]:

QUESTION 1.8. Let X be a smooth projective variety, and $f : X \to A$ be a morphism to an Abelian variety. Then is $R^j f_*(\omega_X^{\otimes m})$ a GV-sheaf for any $j \geq 1$ and $m \geq 2$?

We will give an answer of the question by constructing an example that $R^j f_*(\omega_X^{\otimes m})$ is not a GV-sheaf for some $j \geq 1$ and $m \geq 2$ (see Example 4.5). The existence of such an example implies that, in the notation of Theorem 1.7, we cannot obtain any effective lower bounds for positive integers l that satisfy

$$H^i(Y, R^j f_*\omega_X^{\otimes m} \otimes L^{\otimes l}) = 0$$

for any $i > 0$ (for details, see Remark 4.7).

2. Preliminaries

In this section, we collect some basic definitions.

2.1. Cohomology Support Loci

Let X be a smooth projective variety, $f : X \to A$ be a morphism to an Abelian variety, and $\mathcal{F}$ be a coherent sheaf on X. Set $S^i_j(\mathcal{F}, f) = \{\xi \in \mathrm{Pic}^0(A) \mid h^i(X, \mathcal{F} \otimes f^*\xi) \geq j\}$. We call $S^i_j(\mathcal{F}, f)$ a *cohomology support locus* of $\mathcal{F}$. This is a Zariski closed subset of the Abelian variety $\mathrm{Pic}^0(A)$. If f is an Albanese morphism, then we denote $S^i_j(\mathcal{F}, f)$ by $S^i_j(\mathcal{F})$. We simply denote $S^i_1(\mathcal{F}, f)$ by $S^i(\mathcal{F}, f)$.

2.2. Fourier–Mukai Transforms

Let A be an Abelian variety, and $\hat{A}$ be the dual Abelian variety of A. We define the functor $\Phi_{\mathcal{P}_A} : \mathrm{Coh}(A) \to \mathrm{Coh}(\hat{A})$ as $\Phi_{\mathcal{P}_A}(\mathcal{F}) = p_{2*}(p_1^*\mathcal{F} \otimes \mathcal{P}_A)$, where $\mathcal{P}_A$ is the (normalized) Poincaré bundle on $A \times \hat{A}$, and $p_1 : A \times \hat{A} \to A$ and $p_2 : A \times \hat{A} \to \hat{A}$ are the natural projections. The right derived functor $R\Phi_{\mathcal{P}_A}$ of $\Phi_{\mathcal{P}_A}$ is called the *Fourier–Mukai transform.*

Let X be a smooth projective variety. Given a morphism $f : X \to A$, we put $\mathcal{P}_X = (f \times \mathrm{id}_{\hat{A}})^*\mathcal{P}_A$. We define the functor $\Phi_{\mathcal{P}_X} : \mathrm{Coh}(X) \to \mathrm{Coh}(\hat{A})$ as $\Phi_{\mathcal{P}_X}(\mathcal{F}) = \pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{P}_X)$, where $\pi_1 : X \times \hat{A} \to X$ and $\pi_2 : X \times \hat{A} \to \hat{A}$ are the natural projections.

2.3. GV-Sheaves

Let X be a smooth projective variety, $f : X \to A$ be a morphism to an Abelian variety, and $\mathcal{F}$ be a coherent sheaf on X. We say that $\mathcal{F}$ is a *GV-sheaf with respect to the morphism f* if $\mathrm{codim}_{\mathrm{Pic}^0(A)} S^i(\mathcal{F}, f) \geq i$ for every $i > 0$. This condition is equivalent to the condition that $\mathrm{codim}_{\mathrm{Pic}^0(A)} \mathrm{Supp}\, R^i\Phi_{\mathcal{P}_X}(\mathcal{F}) \geq i$ for every $i > 0$. If $f = \mathrm{id}_A$, then we simply call such a sheaf a GV-sheaf on A. For details, see [PaPo].

2.4. $\mathbb{Q}$-Divisors

Let X be a normal variety. A *$\mathbb{Q}$-divisor* on X is a formal sum of finitely many prime divisors on X with rational coefficients.

2.5. Operations on $\mathbb{Q}$-Divisors

Let $D = \sum d_j D_j$ be a $\mathbb{Q}$-divisor on a normal variety X, where d_j are rational numbers, and D_j are prime divisors on X. We define the *round-down* $\lfloor D \rfloor$ of D as $\lfloor D \rfloor = \sum \lfloor d_j \rfloor D_j$, where $\lfloor d_j \rfloor$ is the largest integer among all the integers that are not larger than d_j. The *fractional part* $\{D\}$ of D is defined as $\{D\} = D - \lfloor D \rfloor$. Finally, we define

$$D^+ = \sum_{d_j > 0} d_j D_j, \qquad D^- = -\sum_{d_j < 0} d_j D_j, \qquad D^{=1} = \sum_{d_j = 1} D_j.$$

Then we have $D = D^+ - D^-$.

2.6. Unitary Local Systems

Let X be a complex manifold. A *local system* (*of rank one*) on X is a locally constant sheaf of one-dimensional $\mathbb{C}$-vector spaces on X. Then there is a natural correspondence between local systems on X and characters of the fundamental group $\pi_1(X)$ (a *character* of $\pi_1(X)$ is a group homomorphism $\rho : \pi_1(X) \to \mathbb{C}^*$). A *unitary local system* on X is a local system that corresponds to a character whose image is contained in $U(1)$. We denote the set of all unitary local systems on X by $U_B(X)$.

Let $\mathrm{Pic}^\tau(X)$ be the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles whose first Chern classes are torsion elements. Take $L \in \mathrm{Pic}^\tau(X)$. L has the Chern connection $\nabla : L \to \Omega^1_X \otimes L$, and $\mathrm{Ker}\,\nabla$ is a unitary local system on X. The correspondence $L \mapsto \mathrm{Ker}\,\nabla$ gives an isomorphism $\mathrm{Pic}^\tau(X) \cong U_B(X)$. For details, see [Kob, Ch. 1, Sects. 2 and 4].

3. Cohomology Support Loci of Pluricanonical Bundles

First, we refer to Timmerscheidt's mixed Hodge theory for unitary local systems (see [Tim]). Let X be a smooth projective variety, Δ be a simple normal crossing divisor on X, and $U = X \setminus \Delta$. Take $\mathcal{V} \in U_B(U)$. Then we have Deligne's canonical extension (L, ∇) of $\mathcal{V}$, where L is a holomorphic line bundle on X, and ∇ is a logarithmic connection on L with poles along Δ such that $\mathrm{Ker}\,\nabla|_U = \mathcal{V}$ (for details, see [EsVi, (1.3)] and [Bud, p. 226]). We have the logarithmic de Rham complex

$$0 \to L \xrightarrow{\nabla} \Omega^1_X(\log \Delta) \otimes L \xrightarrow{\nabla} \Omega^2_X(\log \Delta) \otimes L \xrightarrow{\nabla} \cdots .$$

We give to the complex $(\Omega^\bullet_X(\log \Delta) \otimes L, \nabla)$ the following filtration F:

$$F^p(\Omega^q_X(\log \Delta) \otimes L) = \begin{cases} 0 & \text{for } q < p, \\ \Omega^p_X(\log \Delta) \otimes L & \text{for } q \geq p. \end{cases}$$

Let $\iota : U \to X$ be the inclusion. $R\iota_*\mathcal{V}$ is quasi-isomorphic to $\Omega^\bullet_X(\log \Delta) \otimes L$ (see [EsVi, (1.4)]). Therefore, the filtration F induces the spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log \Delta) \otimes L) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X(\log \Delta) \otimes L) = H^{p+q}(U, \mathcal{V}).$$

In general, we cannot expect a natural mixed Hodge structure on $H^{p+q}(U, \mathcal{V})$ when $\mathcal{V}$ is not defined over $\mathbb{R}$. However, Timmerscheidt proved the following:

THEOREM 3.1 ([Tim, Thm. 7.1]). *In the previous notation, the spectral sequence degenerates at* E_1.

REMARK 3.2. Theorem 3.1 holds for unitary local systems of arbitrary rank.

Next, we refer to a result of Budur. The following statement is a particular case of [Bud, Thm. 8.3].

THEOREM 3.3 ([Bud]). *Let X be a smooth projective variety, Δ be a simple normal crossing divisor on X, and $U = X \setminus \Delta$. Then the image of the set*

$$\{\mathcal{V} \in U_B(X) \mid \dim \mathrm{Gr}^p_F H^{p+q}(U, \mathcal{V}|_U) \geq j\}$$

in $\mathrm{Pic}^\tau(X)$ is a finite union of torsion translates of Abelian subvarieties for all p, q, and j.

Take a unitary local system $\mathcal{V} \in U_B(X)$ on X and let $L \in \mathrm{Pic}^\tau(X)$ be the corresponding line bundle. Then the canonical extension of $\mathcal{V}|_U$ is L with the Chern connection. So by Theorem 3.1 we can restate Theorem 3.3 as follows.

THEOREM 3.4. *Let X be a smooth projective variety, Δ be a simple normal crossing divisor on X, and $U = X \setminus \Delta$. Then the set*

$$\{L \in \mathrm{Pic}^\tau(X) \mid \dim H^q(X, \Omega_X^p(\log \Delta) \otimes L) \geq j\}$$

is a finite union of torsion translates of Abelian subvarieties for all p, q, and j.

By using Theorem 3.4 we can prove Theorem 1.2 in Section 1.

THEOREM 3.5 (= Theorem 1.2). *Let X be a smooth projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, $f : X \to A$ be a morphism to an Abelian variety, and D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Then $S^i_j(D, f)$ is a finite union of torsion translates of Abelian subvarieties of* $\mathrm{Pic}^0(A)$ *for all $i \geq 0$ and $j \geq 1$.*

Proof. We divide the proof into two steps.

Step 1. We reduce to the case where Δ is a (*reduced*) simple normal crossing divisor.

Set $C = D - K_X - \lfloor \Delta \rfloor$. Then $C \sim_{\mathbb{Q}} \{\Delta\}$, so we can take a positive integer N such that $NC \sim N\{\Delta\}$. Let $\pi : Y \to X$ be the normalization of the cyclic cover

$$\mathrm{Spec} \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC).$$

Then it follows that

$$\pi_* \mathcal{O}_Y = \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC + \lfloor k\{\Delta\} \rfloor)$$

(see [Kol, Prop. 9.8]). So

$$\pi_* \mathcal{O}_Y(-\pi^* \lfloor \Delta \rfloor) = \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC + \lfloor k\{\Delta\} \rfloor - \lfloor \Delta \rfloor)$$

contains $\mathcal{O}_X(-C - \lfloor \Delta \rfloor) = \mathcal{O}_X(-(D - K_X))$ as a direct summand. Then by the standard argument (cf. the proof of [Bud, Thm. 6.1]) and Serre duality it follows that $S^i_j(D, f)$ is a finite union of torsion translates of Abelian subvarieties for every i and j if $S^i_j(-\pi^* \lfloor \Delta \rfloor, f \circ \pi)$ is a finite union of torsion translates of Abelian subvarieties for all i and j.

So we consider $(Y, \pi^* \lfloor \Delta \rfloor)$. Note that $\pi^* \lfloor \Delta \rfloor$ is a reduced divisor. Take a divisor Δ_Y on Y as $K_Y + \Delta_Y = \pi^*(K_X + \Delta_{\mathrm{red}})$. Then, by Hurwitz's formula we see that Δ_Y is reduced and $\pi^* \lfloor \Delta \rfloor \leq \Delta_Y$. Moreover, (Y, Δ_Y) is a log canonical pair since $(X, \Delta_{\mathrm{red}})$ is log canonical and π is finite (see [KoMo, Prop. 5.20]). These arguments imply that $(Y, \pi^* \lfloor \Delta \rfloor)$ is a log canonical pair.

Take a log resolution $\mu : Y' \to Y$ of $(Y, \pi^*\lfloor\Delta\rfloor)$. Set

$$\Delta_{Y'} = \mu^*(K_Y + \pi^*\lfloor\Delta\rfloor) - K_{Y'}.$$

Then

$$\Delta_{Y'}^{=1} = \mu_*^{-1}(\pi^*\lfloor\Delta\rfloor) + E$$

for some reduced μ-exceptional divisor E. Since $-K_{Y'/Y} = -(K_{Y'} - \mu^* K_Y)$ has no irreducible components with coefficient 1, every component of E is contained in $\mu^*\pi^*\lfloor\Delta\rfloor$. Since E is μ-exceptional, E is in fact contained in $\mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor$. So $F = \mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E$ is an effective and μ-exceptional divisor on Y'. By the Fujino–Kovács vanishing theorem (see [Kov] and [Fuj2]),

$$R^i\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) = 0$$

for $i > 0$. Therefore,

$$\begin{aligned} R\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) &\cong \mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu^*\pi^*\lfloor\Delta\rfloor + F) \\ &\cong \mathcal{O}_Y(-\pi^*\lfloor\Delta\rfloor). \end{aligned}$$

Thus, we have $S^i_j(-\Delta_{Y'}^{=1}, f \circ \pi \circ \mu) = S^i_j(-\pi^*\lfloor\Delta\rfloor, f \circ \pi)$. Moreover, $\Delta_{Y'}^{=1}$ is a simple normal crossing divisor on Y'. So it is sufficient to give a proof in the case where Δ is a simple normal crossing divisor.

Step 2. Assume that Δ is a simple normal crossing divisor. Put $U = X \setminus \Delta$. By Theorem 3.4 the set

$$\{L \in \mathrm{Pic}^\tau(X) \mid h^q(X, \Omega_X^p(\log\Delta) \otimes L) \geq j\}$$

is a finite union of torsion translates of Abelian subvarieties for any p, q, and j. In particular, $S^i_j(K_X + \Delta) = S^i_j(K_X + \Delta, \mathrm{alb}_X)$ is a finite union of torsion translates of Abelian subvarieties for any i and j, where alb_X denotes the Albanese morphism of X. Considering the universality of the Albanese morphism, it follows from this that $S^i_j(K_X + \Delta, f)$ is also a finite union of torsion translates of Abelian subvarieties. □

REMARK 3.6. Note that the pair (X, Δ) in the statement of Theorem 3.5 is not an arbitrary log canonical pair. The technical reason why Theorem 3.5 for arbitrary log canonical pairs is not made is that the arguments for unitary local systems require log smoothness of (X, Δ) (i.e. the condition that X is smooth and Δ has simple normal crossing support), and the statement for arbitrary log canonical pairs cannot easily be reduced to the log smooth case. On the other hand, Theorem 3.9 below holds for arbitrary log canonical pairs.

We can also prove a generalization of the generic vanishing theorem for log canonical pairs.

THEOREM 3.7. *Let X be a smooth projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, $f : X \to A$ be a morphism to an Abelian variety, and D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Set $l = \max\{\dim V - \dim f(V) \mid V = X$ or V is a log canonical center of $(X, \Delta)\}$. Then*

$$\operatorname{codim}_{\operatorname{Pic}^0(A)} S^i(D, f) \geq i - l$$

for any i.

For the definition of log canonical centers, see [Fuj1, Def. 4.6].

Proof. Let $S = \lfloor \Delta \rfloor$ and Δ_i be an irreducible component of S. Consider the exact sequence

$$\cdots \to R^j f_* \mathcal{O}_X(D - \Delta_i) \to R^j f_* \mathcal{O}_X(D) \to R^j f_* \mathcal{O}_{\Delta_i}(D|_{\Delta_i}) \to \cdots.$$

Then it follows that $R^j f_* \mathcal{O}_X(D) = 0$ for $j > l$ by induction of both the dimension of X and the number of irreducible components of S.

Consider the Leray spectral sequence

$$E_2^{p,q}(\xi) = H^p(A, R^q f_* \mathcal{O}_X(D) \otimes \xi) \Rightarrow H^{p+q}(X, \mathcal{O}_X(D) \otimes f^* \xi),$$

where $\xi \in \operatorname{Pic}^0(A)$. Then it follows by the spectral sequence that

$$S^i(D, f) \subset \bigcup_{q=0}^{l} S^{i-q}(R^q f_* \mathcal{O}_X(D)).$$

Furthermore, $R^q f_* \mathcal{O}_X(D)$ are GV-sheaves on A for all q (see Proposition 4.2), so

$$\operatorname{codim} S^{i-q}(R^q f_* \mathcal{O}_X(D)) \geq i - q$$

for $0 \leq i \leq l$. Hence, $\operatorname{codim} S^i(D, f) \geq i - l$. □

REMARK 3.8. In the notation of Theorem 3.7, assume in addition that (X, Δ) is a Kawamata log terminal pair. Then $l = \dim X - \dim f(X)$. In particular, assume that $\Delta = 0$. Then Theorem 3.7 is the original generic vanishing theorem of Green and Lazarsfeld [GrLa].

Next, we consider the cohomology support loci of $m(K_X + \Delta)$.

THEOREM 3.9 (= Theorem 1.3). *Let X be a normal projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is log canonical, $f : X \to A$ be a morphism to an Abelian variety, and D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$ for some positive integer m. Then $S_j^0(D, f)$ is a finite union of torsion translates of Abelian subvarieties of $\operatorname{Pic}^0(A)$ for any $j \geq 1$.*

Proof. Fix an element $\xi \in S_j^0(D, f)$. It is sufficient to show that there is a torsion translate of an Abelian subvariety $T \subset \operatorname{Pic}^0(A)$ such that $\xi \in T \subset S_j^0(D, f)$. This is because, once we show that $S_j^0(D, f)$ is a union of (possibly infinitely many) torsion translates of Abelian subvarieties of $\operatorname{Pic}^0(A)$, then it follows that $S_j^0(D, f)$

is a union of finitely many of them since $S^0_j(D, f)$ and all torsion translates of Abelian subvarieties are Zariski closed subsets of $\mathrm{Pic}^0(A)$, and there are only countably many of torsion translates of Abelian subvarieties.

Step 1. Take a projective birational morphism $\mu : X' \to X$ satisfying the following conditions:

(i) X' is a smooth projective variety;
(ii) $K_{X'} + \Delta' = \mu(K_X + \Delta) + E$, where Δ' is a boundary $\mathbb{Q}$-divisor on X', and E is an effective and μ-exceptional Cartier divisor on X';
(iii) $\mu^*|D + f^*\xi| = |V| + F$, where $|V|$ is a free linear system, and F is an effective Cartier divisor on X';
(iv) $\mathrm{Exc}(\mu) \cup \mathrm{Supp}\,\Delta' \cup \mathrm{Supp}\,F$ is a simple normal crossing divisor on X'.

$D' = \mu^* D + mE$ is a Cartier divisor on X' satisfying

$$D' \sim_{\mathbb{Q}} m\mu^*(K_X + \Delta) + mE = m(K_{X'} + \Delta').$$

For any $\eta \in \mathrm{Pic}^0(A)$, $h^0(X', \mathcal{O}_{X'}(D' + \mu^* f^*\eta)) = h^0(X, \mathcal{O}_X(D + f^*\eta))$. Therefore, $S^0_j(D', f \circ \mu) = S^0_j(D, f)$. Furthermore,

$$\begin{aligned} |D' + \mu^* f^*\xi| &= |\mu^*(D + f^*\xi) + mE| \\ &= \mu^*|D + f^*\xi| + mE \\ &= |V| + F + mE. \end{aligned}$$

By replacing (X, Δ, D) with (X', Δ', D') we can assume that the following conditions hold: X is smooth, $|D + f^*\xi| = |V| + F$ for some free linear system $|V|$ and some effective Cartier divisor F, and $\mathrm{Supp}\,\Delta \cup \mathrm{Supp}\,F$ is a simple normal crossing divisor.

Step 2. Since $\xi \in S^0_j(D, f)$, $\dim|V| \geq j - 1$. Take a general smooth divisor $M \in |V|$ with no irreducible component contained in $\mathrm{Supp}\,\Delta \cup \mathrm{Supp}\,F$. We can take $\xi_0 \in \mathrm{Pic}^0(A)$ such that $\xi = m\xi_0$ in $\mathrm{Pic}^0(A)$. Then

$$\begin{aligned} m(K_X + \Delta + f^*\xi_0) &= m(K_X + \Delta) + f^*\xi \\ &\sim_{\mathbb{Q}} D + f^*\xi \\ &\sim M + F. \end{aligned}$$

Therefore,

$$K_X + \Delta + f^*\xi_0 \sim_{\mathbb{Q}} \frac{1}{m}(M + F).$$

Then we have

$$\begin{aligned} D - F + (m-1)f^*\xi_0 &\sim_{\mathbb{Q}} m(K_X + \Delta) - F + (m-1)f^*\xi_0 \\ &= K_X + \Delta - F + (m-1)(K_X + \Delta + f^*\xi_0) \\ &\sim_{\mathbb{Q}} K_X + \Delta - \frac{1}{m}F + \frac{m-1}{m}M. \end{aligned}$$

Set

$$G = \left\lceil \left(\Delta - \frac{1}{m}F\right)^{-} \right\rceil \geq 0$$

and

$$D_0 = D - F + (m-1) f^* \xi_0 + G.$$

Then

$$D_0 \sim_{\mathbb{Q}} K_X + \Delta - \frac{1}{m} F + \frac{m-1}{m} M + G,$$

and

$$\Delta - \frac{1}{m} F + \frac{m-1}{m} M + G$$

is a boundary $\mathbb{Q}$-divisor with simple normal crossing support. By Theorem 3.5, $S^0_j(D_0, f)$ is a finite union of torsion translates of Abelian subvarieties. Moreover,

$$\begin{aligned} h^0(X, \mathcal{O}_X(D_0 + f^*\xi_0)) &= h^0(X, \mathcal{O}_X(D - F + f^*\xi + G)) \\ &\geq h^0(X, \mathcal{O}_X(D - F + f^*\xi)) \\ &= \dim |V| + 1 = j. \end{aligned}$$

Therefore, we have $\xi_0 \in S^0_j(D_0, f)$.

Step 3. Here we will prove that

$$S^0_j(D_0, f) + (m-1)\xi_0 \subset S^0_j(D, f).$$

Take $\alpha \in S^0_j(D_0, f)$. Then $D + f^*(\alpha + (m-1)\xi_0) = D_0 + F - G + f^*\alpha$. $F - G$ is obviously effective since Δ is effective. So we have

$$h^0(X, \mathcal{O}_X(D_0 + F - G + f^*\alpha)) \geq h^0(X, \mathcal{O}_X(D_0 + f^*\alpha)) \geq j.$$

Thus, $\alpha + (m-1)\xi_0 \in S^0_j(D, f)$, so it follows that

$$S^0_j(D_0, f) + (m-1)\xi_0 \subset S^0_j(D, f).$$

Step 4. By Step 2 we can take an Abelian subvariety $B \subset \mathrm{Pic}^0(A)$ and a torsion point $q \in \mathrm{Pic}^0(A)$ such that $\xi_0 \in B + q \subset S^0_j(D_0, f)$. By Step 3, $B + q + (m-1)\xi_0 \subset S^0_j(D, f)$. Represent $\xi_0 \in B + q$ as $\xi_0 = t + q$ for some $t \in B$. Then $B + q + (m-1)\xi_0 = B + mq + (m-1)t = B + mq$. Therefore, $\xi = \xi_0 + (m-1)\xi_0 \in B + q + (m-1)\xi_0 = B + mq \subset S^0_j(D, f)$. □

Proof of Corollary 1.4. By Theorem 3.9 we may assume that ρ is a torsion element. We take a positive integer N such that $N \cdot \rho = 0$. Then $h^0(Nm(K_X + \Delta)) = h^0(N(m(K_X + \Delta) + \rho)) \geq h^0(m(K_X + \Delta) + \rho)$. □

As another application of Theorem 3.9, we give an alternative proof of the following proposition in [CKP].

PROPOSITION 3.10 ([CKP, Cor. 3.2]). *Let X be a normal projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is log canonical, and $f : X \to A$ be a morphism to an Abelian variety. Then $\kappa(K_X + \Delta) \geq \kappa(K_X + \Delta + f^*\xi)$ for every $\xi \in \mathrm{Pic}^0(A)$.*

In [CKP], this is proved as a corollary of [CKP, Thm. 0.1] (see Corollary 1.4).

First we prove the following lemma.

LEMMA 3.11. *Let A be an Abelian variety, and $\mathcal{T}$ be the set of all torsion translates of Abelian subvarieties of A. For $T \in \mathcal{T}$, set* $\mathrm{ord}(T) = \min\{\mathrm{ord}(t) \mid t \in T_{\mathrm{tor}}\}$, *where T_{tor} denotes the set of all torsion points in T.*

(i) $\mathrm{ord}(mT) \le \mathrm{ord}(T) \le m \cdot \mathrm{ord}(mT)$ *for any $T \in \mathcal{T}$ and $m \in \mathbb{Z}_{>0}$.*

(ii) *For any $p \in A$, the set* $\{\mathrm{ord}(T) \mid T \in \mathcal{T}$ *and* $mp \in T$ *for some* $m \in \mathbb{Z}_{>0}\}$ *is bounded.*

Proof. (i) By definition, $\mathrm{ord}(t_0) = \mathrm{ord}(T)$ for some $t_0 \in T$. Then

$$\mathrm{ord}(mT) \le \mathrm{ord}(mt_0) \le \mathrm{ord}(t_0) = \mathrm{ord}(T).$$

Take $t \in T_{\mathrm{tor}}$. Then $\mathrm{ord}(T) \le \mathrm{ord}(t) \le m\,\mathrm{ord}(mt)$ since

$$m\,\mathrm{ord}(mt) \cdot t = \mathrm{ord}(mt) \cdot mt = 0.$$

So $\mathrm{ord}(T) \le m\,\mathrm{ord}(mT)$.

(ii) Suppose that $\{\mathrm{ord}(T) \mid T \in \mathcal{T}$ and $mp \in T$ for some $m \in \mathbb{Z}_{>0}\}$ is an unbounded set. Consider a sequence $S_k = (m_1, \dots, m_k; T_1, \dots, T_k)$ satisfying

- $m_1, \dots, m_k \in \mathbb{Z}_{>0}$, $T_1, \dots, T_k \in \mathcal{T}$,
- $m_1 p \in T_1, m_1 m_2 p \in T_2, \dots, m_1 \cdots m_k p \in T_k$, and
- $\dim T_1 > \dim T_2 > \cdots > \dim T_k$.

By assumption we can take $m_{k+1} \in \mathbb{Z}_{>0}$ and $T''_{k+1} \in \mathcal{T}$ such that $m_{k+1} p \in T''_{k+1}$ and $\mathrm{ord}(T''_{k+1}) > m_1 \cdots m_k\,\mathrm{ord}(T_k)$. We define $T'_{k+1}, T'_k \in \mathcal{T}$ as $T'_{k+1} = m_1 \cdots m_k T''_{k+1}$ and $T'_k = m_{k+1} T_k$. Then, using (i), we have

$$\mathrm{ord}(T'_{k+1}) \ge \frac{\mathrm{ord}(T''_{k+1})}{m_1 \cdots m_k} > \frac{m_1 \cdots m_k\,\mathrm{ord}(T_k)}{m_1 \cdots m_k} = \mathrm{ord}(T_k) \ge \mathrm{ord}(T'_k).$$

Put $T_{k+1} = T'_{k+1} \cap T'_k \in \mathcal{T}$. Then $m_1 \cdots m_{k+1} p \in T_{k+1}$. Since $\mathrm{ord}(T'_{k+1}) > \mathrm{ord}(T'_k)$, $T'_k \not\subset T'_{k+1}$. Then it follows that $T_{k+1} \subsetneq T'_k$. Therefore, $\dim T_{k+1} < \dim T'_k = \dim T_k$.

By continuing this process we have $S_1, S_2, \dots, S_k, S_{k+1}, \dots$. But this process must stop because of the third condition of S_k. So we have a contradiction. □

Proof of Proposition 3.10. Let $\mathcal{T}$ be the set of all torsion translates of Abelian subvarieties of $\mathrm{Pic}^0(A)$. There exists a positive integer m_0 such that $D = m_0(K_X + \Delta)$ is Cartier. Then it reduces to show that $\kappa(D) \ge \kappa(D + f^*\xi)$ for any $\xi \in \mathrm{Pic}^0(A)$. Set $d(m) = h^0(mD)$ and $d'(m) = h^0(m(D + f^*\xi))$. Then $m\xi \in S^0_{d'(m)}(mD, f)$, so by Theorem 3.9 we can take $T_m \in \mathcal{T}$ such that $m\xi \in T_m \subset S^0_{d'(m)}(mD, f)$. Take $t_m \in T_m$ such that $\mathrm{ord}(t_m) = \mathrm{ord}(T_m)$. It follows by Lemma 3.11(ii) that there exists a positive integer N such that $\mathrm{ord}(T_m) \mid N$ for any $m \in \mathbb{Z}_{>0}$. Hence, $Nt_m = 0$ for any $m \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned} d'(m) &= h^0(m(D + f^*\xi)) \\ &\le h^0(mD + f^*t_m) \\ &\le h^0(N(mD + f^*t_m)) \\ &= h^0(NmD) = d(Nm). \end{aligned}$$

Therefore,

$$\begin{aligned}\kappa(D+f^*\xi) &= \limsup_m \frac{\log d'(m)}{\log m}\\ &\le \limsup_m \frac{\log d(Nm)}{\log m}\\ &= \limsup_m \frac{\log d(Nm)}{\log Nm} = \kappa(D).\end{aligned}$$

□

4. Higher Direct Images of Pluricanonical Bundles

In this section, we discuss higher direct images of pluricanonical bundles to Abelian varieties. Popa and Schnell [PoSc] proved the following theorem.

THEOREM 4.1 ([PoSc, Thm. 1.10]). *Let X be a smooth projective variety, and $f : X \to A$ be a morphism to an Abelian variety. Then the direct image $f_*(\omega_X^{\otimes m})$ is a GV-sheaf for any positive integer m.*

In addition, we can prove the following log version of Theorem 1.5.

PROPOSITION 4.2. *Let X be a smooth projective variety, Δ be a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, $f : X \to A$ be a morphism to an Abelian variety, and D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Then the higher direct images $R^j f_*\omega_X(\Delta)$ are GV-sheaves for any j.*

Here we use a characterization of GV-sheaves on Abelian varieties by Hacon. He proved the following criterion in [Hac].

THEOREM 4.3 (Hacon). *Let A be an Abelian variety, and $\mathcal{F}$ be a coherent sheaf on A. $\mathcal{F}$ is a GV-sheaf on A if and only if $H^k(B, \phi^*\mathcal{F} \otimes L) = 0$ for $k > 0$, where $\phi : B \to A$ is any isogeny, and L is any ample line bundle on B.*

Proof of Proposition 4.2. Let $\phi : B \to A$ be an isogeny, and L be an ample line bundle on B. Set $Y = X \times_A B$. Let $\psi : Y \to X$ and $g : Y \to B$ be the natural morphisms. Fix an integer $j \ge 0$. By base change,

$$\phi^* R^j f_*\omega_X(\Delta) \cong R^j g_*\psi^*\omega_X(\Delta) \cong R^j g_*\omega_Y(\psi^*\Delta).$$

Applying the Ambro–Fujino vanishing theorem (see [Amb, Thm. 3.2] and [Fuj1, Thm. 6.3]) to the log canonical pair $(Y, \psi^*\Delta)$, we have

$$H^k(B, \phi^* R^j f_*\omega_X(\Delta) \otimes L) \cong H^k(B, R^j g_*\omega_Y(\psi^*\Delta) \otimes L) = 0$$

for $k > 0$. By Theorem 4.3 it follows that $R^j f_*\omega_X(\Delta)$ is a GV-sheaf. □

Popa and Schnell [PoSc] also asked the question whether the higher direct images $R^j f_*(\omega_X^{\otimes m})$ are GV-sheaves or not. Note that $R^j f_*\omega_X$ is a GV-sheaf for every j (see Theorem 1.5 or Proposition 4.2). Hence, the question is whether $R^j f_*(\omega_X^{\otimes m})$ is a GV-sheaf or not in the case where $j \ge 1$ and $m \ge 2$.

Here we give an example of a higher direct image of a pluricanonical bundle that is not a GV-sheaf.

LEMMA 4.4. *There exists an irregular smooth projective variety with big anticanonical bundle.*

Proof. Let A be an Abelian variety. We take an ample line bundle L on A and define a vector bundle E as the direct sum of L^{-1} and $\mathcal{O}_A$. Let $\pi : X = \mathbb{P}_A(E) \to A$ be the projective bundle on A associated to E. Clearly, the irregularity $q(X)$ of X is positive. The canonical bundle ω_X is isomorphic to $\pi^*(\omega_A \otimes \det E) \otimes \mathcal{O}_X(-\operatorname{rank} E) = \pi^* L^{-1} \otimes \mathcal{O}_X(-2)$ (see [Laz, 7.3.A]).

We will see that ω_X^{-1} is big. Let ξ and l be the numerical classes of $\mathcal{O}_X(1)$ and L, respectively. Note that ξ is an effective class since $H^0(X, \mathcal{O}_X(1)) = H^0(A, E) \neq 0$. The numerical class of ω_X^{-1} is equal to

$$2\xi + \pi^* l = \frac{N-1}{N} 2\xi + \frac{1}{N} 2\xi + \pi^* l,$$

where N is a sufficiently large integer such that $(1/N)2\xi + \pi^* l$ is ample. So the numerical class of ω_X^{-1} is represented by the sum of an effective class and an ample class. Therefore, ω_X^{-1} is big. $\square$

EXAMPLE 4.5. Let X be an irregular smooth projective variety of dimension $n \geq 2$ with big anticanonical bundle, and let $f = \mathrm{alb}_X : X \to A$ be the Albanese morphism of X. Now we show that $R^j f_* \omega_X^{\otimes m}$ is not a GV-sheaf for some positive integers j and m.

Let $\mathcal{P}_A$ be the Poincaré bundle on $A \times \hat{A}$, and $\mathcal{P}_X = (f \times \mathrm{id}_{\hat{A}})^* \mathcal{P}_A$. Then we have the Fourier–Mukai transforms $\Phi_{\mathcal{P}_A} : \mathrm{Coh}(A) \to \mathrm{Coh}(\hat{A})$ and $\Phi_{\mathcal{P}_X} : \mathrm{Coh}(X) \to \mathrm{Coh}(\hat{A})$ (see Section 2.2). It immediately follows that $\Phi_{\mathcal{P}_X} \cong \Phi_{\mathcal{P}_A} \circ f_*$.

Now we prove the following lemma.

LEMMA 4.6. *Let X be a smooth projective variety of dimension n, and D be a big Cartier divisor on X. Then*

$$S^0(mD) = \{\xi \in \mathrm{Pic}^0(X) \mid H^0(X, \mathcal{O}_X(mD + \xi)) \neq 0\} = \mathrm{Pic}^0(X)$$

for any sufficiently large and divisible m.

Proof. Since D is big, there exist a positive integer m_0, a very ample Cartier divisor H, and an effective Cartier divisor E such that $m_0 D \sim H + E$. For any positive integer m, we have

$$S^0(mm_0 D) = S^0(mH + mE) \supset S^0(mH).$$

We can take a positive integer m_1 satisfying

$$H^i(X, \mathcal{O}_X(mH + \xi)) = 0$$

for every $\xi \in \mathrm{Pic}^0(X)$, $m \geq m_1$, and $i > 0$ (take m_1 such that $m_1 H - K_X$ is ample). According to the notion of the Castelnuovo–Mumford regularity, $mH + \xi$

is 0-regular for every $\xi \in \mathrm{Pic}^0(X)$ and $m \geq m_1 + n$, and so it is globally generated. In particular, $S^0(mH) = \mathrm{Pic}^0(X)$ for every $m \geq m_1 + n$. Therefore, $S^0(mm_0D) = \mathrm{Pic}^0(X)$ for every $m \geq m_1 + n$. □

By the above lemma, we can take a positive integer m such that $S^n(\omega_X^{\otimes m}) = -S^0(\omega_X^{\otimes(1-m)}) = \mathrm{Pic}^0(X)$. Then it follows that

$$\mathrm{Supp}\, R^n \Phi_{\mathcal{P}_X}(\omega_X^{\otimes m}) = \mathrm{Pic}^0(X).$$

Consider the Grothendieck spectral sequence

$$E_2^{i,j} = R^i \Phi_{\mathcal{P}_A} R^j f_*(\omega_X^{\otimes m}) \Rightarrow R^{i+j} \Phi_{\mathcal{P}_X}(\omega_X^{\otimes m}).$$

Then there exists an integer i such that $\mathrm{Supp}\, R^i \Phi_{\mathcal{P}_A} R^{n-i} f_*(\omega_X^{\otimes m}) = \mathrm{Pic}^0(X)$. Note that $\dim f(X) > 0$ since X is irregular. Therefore, $R^n f_*(\omega_X^{\otimes m}) = 0$, and so i must be positive. Note that a coherent sheaf $\mathcal{F}$ on A is a GV-sheaf if and only if $\mathrm{codim}\, \mathrm{Supp}\, R^j \Phi_{\mathcal{P}_A} \mathcal{F} \geq j$ for $j > 0$ (see Section 2.3). Hence, it follows that $R^{n-i} f_*(\omega_X^{\otimes m})$ is not a GV-sheaf.

REMARK 4.7. Let X be a smooth projective variety of dimension k, $f : X \to Y$ be a morphism to a projective variety Y of dimension n, L be an ample and globally generated line bundle on Y, and $m > 0$ and $j \geq 0$ be integers. Suppose that we can take a positive integer $N = N(k, n, m, j)$ depending only on k, n, m, and j such that

$$H^i(Y, R^j f_* \omega_X^{\otimes m} \otimes L^{\otimes l}) = 0$$

for any $i > 0$ and $l \geq N$. Then, by the same argument as the proof of Theorem 1.6 by using Theorem 1.7 (see the proof of [PoSc, Thm. 1.10]), it follows that $R^j f_* \omega_X^{\otimes m}$ is a GV-sheaf for any morphism $f : X \to A$ to an Abelian variety for any $j \geq 0$ and $m \geq 1$. But this contradicts Example 4.5. So we cannot take such an integer N.

Next, we consider the case where X is a surface. Example 4.5 shows that $R^j f_*(\omega_X^{\otimes m})$ is not always a GV-sheaf when $\kappa(X) = -\infty$. On the other hand, the following proposition holds.

PROPOSITION 4.8. *Let X be a smooth projective surface, and $f : X \to A$ be a morphism to an Abelian variety. Assume that $\kappa(X) \geq 0$. Then $R^j f_*(\omega_X^{\otimes m})$ is a GV-sheaf for any j and m.*

Proof. This obviously holds when $\dim f(X) = 0$. So we may assume that $\dim f(X) \geq 1$. Fix a positive integer m. $f_*(\omega_X^{\otimes m})$ is a GV-sheaf by Theorem 4.1. $R^2 f_*(\omega_X^{\otimes m}) = 0$ since all fibers of f have dimension less than 2. So it is sufficient to show that $R^1 f_*(\omega_X^{\otimes m})$ is a GV-sheaf.

Since $\kappa(X) \geq 0$, we can take a series of contractions of (-1)-curves $\varepsilon : X \to X'$ such that the canonical bundle $\omega_{X'}$ of X' is semiample. Then the exceptional curves of ε are also contracted by f. Hence, we obtain a morphism $f' : X' \to A$ such that $f' \circ \varepsilon = f$. Consider the spectral sequence

$$E_2^{i,j} = R^i f'_* R^j \varepsilon_*(\omega_X^{\otimes m}) \Rightarrow R^{i+j} f_*(\omega_X^{\otimes m}).$$

Then $E_2^{i,j}=0$ except that $(i,j)=(0,0),(1,0),(0,1)$. So we have

$$\begin{aligned}R^1f_*(\omega_X^{\otimes m}) &\cong R^1f'_*\varepsilon_*(\omega_X^{\otimes m})\oplus f'_*R^1\varepsilon_*(\omega_X^{\otimes m})\\ &\cong R^1f'_*(\omega_{X'}^{\otimes m})\oplus f'_*R^1\varepsilon_*(\omega_X^{\otimes m}).\end{aligned}$$

Take any ample line bundle L on A. We have

$$H^k(A,R^1f'_*(\omega_{X'}^{\otimes m})\otimes L)=0 \quad \text{for } k>0$$

by the Ambro–Fujino vanishing theorem since $\omega_{X'}$ is semiample (see [Amb, Thm. 3.2] and [Fuj1, Thm. 6.3]). It is obvious that

$$H^k(A,f'_*R^1\varepsilon_*(\omega_X^{\otimes m})\otimes L)=0 \quad \text{for } k>0$$

since $\dim\operatorname{Supp} R^1\varepsilon_*(\omega_X^{\otimes m})=0$. Thus, it follows that

$$H^k(A,R^1f_*(\omega_X^{\otimes m})\otimes L)=0 \quad \text{for } k>0.$$

We show by using Theorem 4.3 that $R^1f_*(\omega_X^{\otimes m})$ is a GV-sheaf. Let $\phi:B\to A$ be an isogeny, and L be an ample line bundle on B. Set $Y=X\times_A B$. Let $\psi:Y\to X$ and $g:Y\to B$ be the natural morphisms. Then ψ is étale, and $\kappa(Y)\geq 0$. By base change,

$$\phi^*R^1f_*(\omega_X^{\otimes m})\cong R^1g_*\psi^*(\omega_X^{\otimes m})\cong R^1g_*(\omega_Y^{\otimes m}).$$

Hence,

$$H^k(B,\phi^*R^1f_*(\omega_X^{\otimes m})\otimes L)\cong H^k(B,R^1g_*(\omega_Y^{\otimes m})\otimes L).$$

Here $H^k(B,R^1g_*(\omega_Y^{\otimes m})\otimes L)=0$ for $k>0$ by the previous argument. Therefore, $H^k(B,\phi^*R^1f_*(\omega_X^{\otimes m})\otimes L)=0$ for $k>0$. Then it follows by Theorem 4.3 that $R^1f_*(\omega_X^{\otimes m})$ is a GV-sheaf. □

Taking this proposition into account, we modify the question of Popa–Schnell as follows.

QUESTION 4.9. Let X be a smooth projective variety of dimension $n\geq 3$, and $f:X\to A$ be a morphism to an Abelian variety. Assume that $\kappa(X)\geq 0$. Then is $R^jf_*(\omega_X^{\otimes m})$ a GV-sheaf for any j and m?

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