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A Remark on FI-Module Homology

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Abstract. We show that the FI-homology of an FI-module can be computed via a Koszul complex. As an application, we prove that the Castelnuovo–Mumford regularity of a finitely generated torsion FI-module is equal to its degree.

1. Introduction

Let R be a commutative ring. Let FI be the category of finite sets and injective maps. An FI-module is a (covariant) functor from FI to the category of R-modules. In [3, (11)], Church, Ellenberg, Farb, and Nagpal defined, for any FI-module V, a complex $\widetilde{S}_{-*}V$ of FI-modules. The same complex was also considered independently by Putman [7, §4], who called it the central stability chain complex. The purpose of our present paper is to give a proof that the homology of the complex $\widetilde{S}_{-*}V$ is the FI-homology $H_*^{\mathrm{FI}}(V)$ of V, whose definition we now recall.

Let $\mathbb{N}$ be the set of nonnegative integers. For each $n \in \mathbb{N}$, let $[n]$ be the set $\{1, \dots, n\}$ with n elements; in particular, $[0] = \emptyset$. It is convenient to introduce the nonunital R-algebra

$$A := \bigoplus_{0 \le m \le n} A_{m,n},$$

where $A_{m,n}$ is the free R-module on the set $\mathrm{Hom}_{\mathrm{FI}}([m],[n])$. For any $\alpha \in \mathrm{Hom}_{\mathrm{FI}}([m],[n])$ and $\beta \in \mathrm{Hom}_{\mathrm{FI}}([r],[s])$, their product $\alpha\beta$ in A is defined by

$$\alpha\beta := \begin{cases} \alpha \circ \beta & \text{if } s = m, \\ 0 & \text{else.} \end{cases}$$

Define a two-sided ideal J of A by

$$J := \bigoplus_{0 \le m < n} A_{m,n}.$$

For each $n \in \mathbb{N}$, let $e_n \in A_{n,n}$ be the identity endomorphism of $[n]$. A left A-module M is said to be *graded* if $M = \bigoplus_{n \in \mathbb{N}} e_n M$. If V is an FI-module, then $\bigoplus_{n \in \mathbb{N}} V([n])$ has the natural structure of a graded left A-module. The functor $V \mapsto \bigoplus_{n \in \mathbb{N}} V([n])$ is an equivalence from the category of FI-modules to the category of graded left A-modules. Thus, we shall not distinguish between the notion

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of FI-modules and the notion of graded left A-modules. By abuse of notation, we shall also denote $\bigoplus_{n\in\mathbb{N}} V([n])$ by V.

Let $a \in \mathbb{N}$. Following [1] and [2, Definition 2.3.7], the FI-*homology* functor H_a^{FI} is, by definition, the ath left-derived functor of the right exact functor

$$H_0^{\mathrm{FI}} : V \mapsto V/JV \tag{1}$$

from the category of FI-modules to itself. In other words, $H_a^{\mathrm{FI}}(V) := \mathrm{Tor}_a^A(A/J, V)$.

The main result of this paper is the following.

THEOREM 1. *Let V be an* FI-*module. For each $a \in \mathbb{N}$, we have*

$$H_a^{\mathrm{FI}}(V) \cong H_a(\widetilde{S}_{-*}V).$$

Let us briefly describe our strategy for proving Theorem 1. For each $d \in \mathbb{N}$ and a finite set X, let $M(d)(X) := R\,\mathrm{Hom}_{\mathrm{FI}}([d], X)$, the free R-module on the set of injections from $[d]$ to X. This defines the FI-module $M(d)$. We shall prove that for each $d \in \mathbb{N}$, we have $H_a(\widetilde{S}_{-*}M(d)) = 0$ whenever $a \geq 1$. This key result implies that for $a \geq 1$, the functor $V \mapsto H_a(\widetilde{S}_{-*}V)$ is coeffaceable because every FI-module is a quotient of $\bigoplus_{i\in I} M(d_i)$ for some $d_i \in \mathbb{N}$. It is easy to see that the collection of functors $V \mapsto H_a(\widetilde{S}_{-*}V)$ for $a \geq 0$ form a homological δ-functor with $H_0(\widetilde{S}_{-*}V) \cong V/JV$, and hence Theorem 1 follows by a standard result in homological algebra.

We give an application of Theorem 1.

For any FI-module V, its *degree* is defined to be

$$\deg(V) := \sup\{n \in \mathbb{N} \mid V([n]) \neq 0\},$$

where by convention the supremum of an empty set is $-\infty$. The *Castelnuovo–Mumford regularity* $\mathrm{reg}(V)$ of V is defined to be the infimum of the set of all $c \in \mathbb{Z}$ such that

$$\deg(H_a^{\mathrm{FI}}(V)) \leq c + a \quad \text{for each } a \geq 1,$$

where by convention the infimum of an empty set is ∞.

THEOREM 2. *Let V be a finitely generated* FI-*module with* $\deg(V) < \infty$. *Then*

$$\mathrm{reg}(V) = \deg(V).$$

After recalling the construction of the complex $\widetilde{S}_{-*}V$ in the next section, we prove the key result we need in Section 3. The proof of Theorem 1 will be given in Section 4, and the proof of Theorem 2 will be given in Section 5.

REMARK 3. Theorem 1 was first stated without proof in [3, Remark 2.21]; after our present paper was written, we learnt that a different proof was given by Church and Ellenberg in [1, Proposition 4.9]. Theorem 2 was first conjectured in [6, Corollary 5.22].

2. Reminder on Koszul Complex of an FI-Module

The complex $\widetilde{S}_{-*}V$ defined in [3, (11)] is a Koszul complex [5, §12.4]. To state its definition, we need a few notations.

For any finite set I, let RI be the free R-module with basis I, and $\det(I)$ the free R-module $\bigwedge^{|I|} RI$ of rank one. In particular, if $I = \emptyset$, then $\det(I) = R$. If $I = \{i_1, \dots, i_a\}$, then $i_1 \wedge \cdots \wedge i_a$ is a basis for $\det(I)$.

Let V be an FI-module, and X a finite set. Suppose $T \subset X$ and $i \in X \setminus T$. For any $v \in V(T)$, we shall write $i(v)$ for the element $\iota(v) \in V(T \cup \{i\})$, where ι is the inclusion map from T to $T \cup \{i\}$. For any $a \in \mathbb{N}$, let

$$(\widetilde{S}_{-a}V)(X) := \bigoplus_{\substack{I \subset X \\ |I| = a}} V(X \setminus I) \otimes_R \det(I). \tag{2}$$

The differential $d : (\widetilde{S}_{-a}V)(X) \to (\widetilde{S}_{-a+1}V)(X)$ is defined on each direct summand by the formula

$$d(v \otimes i_1 \wedge \cdots \wedge i_a) := \sum_{p=1}^{a} (-1)^p i_p(v) \otimes i_1 \wedge \cdots \widehat{i_p} \cdots \wedge i_a,$$

where $v \in V(X \setminus I)$, $I = \{i_1, \dots, i_a\}$, and $\widehat{i_p}$ means that i_p is omitted in the wedge product. If X' is a finite set and $\alpha : X \to X'$ is an injective map, then $\alpha : (\widetilde{S}_{-a}V)(X) \to (\widetilde{S}_{-a}V)(X')$ is defined on each direct summand by the formula

$$\alpha(v \otimes i_1 \wedge \cdots \wedge i_a) := \alpha(v) \otimes \alpha(i_1) \wedge \cdots \wedge \alpha(i_a),$$

where $v \in V(X \setminus I)$, $\alpha(v) \in V(X' \setminus \alpha(I))$, and $I = \{i_1, \dots, i_a\}$. This defines, for each $a \in \mathbb{N}$, an FI-module $\widetilde{S}_{-a}V$. Thus, we obtain a complex $\widetilde{S}_{-*}V$ of FI-modules:

$$\cdots \longrightarrow \widetilde{S}_{-2}V \longrightarrow \widetilde{S}_{-1}V \longrightarrow \widetilde{S}_0 V \longrightarrow 0. \tag{3}$$

For each $a \in \mathbb{N}$, the homology $H_a(\widetilde{S}_{-*}V)$ of the complex (3) is an FI-module.

Observe that for each $n \in \mathbb{N}$, the image of $d : (\widetilde{S}_{-1}V)([n]) \to (\widetilde{S}_0 V)([n])$ is the R-submodule of $V([n])$ spanned by all elements of the form $\alpha(v)$ for some injective map $\alpha : [n-1] \to [n]$ and $v \in V([n-1])$, so

$$H_0(\widetilde{S}_{-*}V) \cong V/JV. \tag{4}$$

Remark 4. In [3], the complex $\widetilde{S}_{-*}V$ is constructed via another complex B_*V defined in [3, (10)]. We have given a direct construction of $\widetilde{S}_{-*}V$.

3. A Key Fact

Recall that for each $d \in \mathbb{N}$, the FI-module $M(d)$ is defined by $M(d)(X) := R\,\mathrm{Hom}_{\mathrm{FI}}([d], X)$ for each finite set X.

Lemma 5. *Let $d \in \mathbb{N}$. If $n \neq d$, then $H_0((\widetilde{S}_{-*}M(d))([n])) = 0$.*

Proof. The complex $(\widetilde{S}_{-*}M(d))([n])$ is

$$\cdots \longrightarrow \bigoplus_{\substack{T\subset[n]\\|T|=n-1}} R\,\mathrm{Hom}_{\mathrm{FI}}([d],T)\otimes_R \det([n]\setminus T) \longrightarrow R\,\mathrm{Hom}_{\mathrm{FI}}([d],[n]) \longrightarrow 0.$$

If $n < d$, then $\mathrm{Hom}_{\mathrm{FI}}(d,[n]) = \emptyset$. If $n > d$, then the image of any injective map from $[d]$ to $[n]$ lies in some subset $T \subset [n]$ with $|T| = n-1$. The lemma follows. □

The following result plays a crucial role in this paper.

PROPOSITION 6. *Let $d \in \mathbb{N}$. If $a \geq 1$, then $H_a(\widetilde{S}_{-*}M(d)) = 0$.*

Proof. We have to show that for each $n \in \mathbb{N}$, the homology $H_a((\widetilde{S}_{-*}M(d))([n]))$ is 0 for all $a \geq 1$. We do this by induction on n. If $a > n$, then for any FI-module V, we have $(\widetilde{S}_{-a}V)([n]) = 0$. Therefore, if $a > n$, then $(\widetilde{S}_{-a}M(d))([n])$ is 0 for all $d \in \mathbb{N}$. Thus, the $n = 0$ case is trivial. Now suppose that $n > 0$ and that $H_a((\widetilde{S}_{-*}M(d))([n-1])) = 0$ for all $a \geq 1$ and $d \in \mathbb{N}$.

For each $a \in \mathbb{N}$, let C_{-a} be the R-submodule of $(\widetilde{S}_{-a}M(d))([n])$ spanned by the elements $\alpha \otimes i_1 \wedge \cdots \wedge i_a$ where α is an injective map from $[d]$ to $[n] \setminus \{i_1,\dots,i_a\}$ and $n \notin \{i_1,\dots,i_a\}$; thus, C_{-a} is spanned by the direct summands of (2) for $V = M(d)$ and $X = [n]$ with $n \notin I$. (Note that the definition of C_{-a} depends on d and n, but we suppress them from our notation. Moreover, C_{-a} is not an S_n-submodule of $(\widetilde{S}_{-a}M(d))([n])$.) It is clear that C_{-*} is a subcomplex of $(\widetilde{S}_{-*}M(d))([n])$.

For each basis element $\alpha\otimes i_1\wedge\cdots\wedge i_a \in C_{-a}$, we have either $n \notin \alpha([d])$ or $n = \alpha(r)$ for exactly one $r \in [d]$; if $n \notin \alpha([d])$, then we can consider α as an injective map from $[d]$ to $[n-1]\setminus\{i_1,\dots,i_a\}$, and so $\alpha\otimes i_1\wedge\cdots\wedge i_a$ can be considered as an element of $(\widetilde{S}_{-a}M(d))([n-1])$, whereas if $r \in [d]$ and $n = \alpha(r)$, then $\alpha|_{[d]\setminus\{r\}}$ can be considered as an injective map from $[d-1]$ to $[n-1]\setminus\{i_1,\dots,i_a\}$ (by fixing an identification of $[d]\setminus\{r\}$ with $[d-1]$), and so $\alpha|_{[d]\setminus\{r\}}\otimes i_1\wedge\cdots\wedge i_a$ can be considered as an element of $(\widetilde{S}_{-a}M(d-1))([n-1])$. Hence, we can identify the complex C_{-*} with

$$(\widetilde{S}_{-*}M(d))([n-1]) \oplus \left(\bigoplus_{r=1}^{d}(\widetilde{S}_{-*}M(d-1))([n-1])\right).$$

The quotient complex of $(\widetilde{S}_{-*}M(d))([n])$ by C_{-*} can be identified with $\Sigma(\widetilde{S}_{-*}M(d))([n-1])$, where Σ is the shift functor on complexes such that $(\Sigma K)_{-a} = K_{-a+1}$ for any complex K; specifically, each basis element $\alpha \otimes i_1 \wedge \cdots \wedge i_{a-1}$ of $(\widetilde{S}_{-a+1}M(d))([n-1])$ (where α is an injective map from $[d]$ to $[n-1]\setminus\{i_1,\dots,i_{a-1}\}$) can be identified with the element $\alpha\otimes i_1\wedge\cdots\wedge i_{a-1}\wedge n \in (\widetilde{S}_{-a}M(d))([n])$.

Therefore, we have a short exact sequence of complexes

$$0 \longrightarrow C_{-*} \longrightarrow (\widetilde{S}_{-*}M(d))([n]) \longrightarrow \Sigma(\widetilde{S}_{-*}M(d))([n-1]) \longrightarrow 0.$$

This gives a long exact sequence in homology

$$\cdots \longrightarrow H_a(C_{-*}) \longrightarrow H_a((\widetilde{S}_{-*}M(d))([n])) \longrightarrow H_{a-1}((\widetilde{S}_{-*}M(d))([n-1])) \longrightarrow H_{a-1}(C_{-*}) \longrightarrow \cdots .$$

According to the induction hypothesis, if $a \geq 2$, then we have $H_a(C_{-*}) = 0$ and $H_{a-1}((\widetilde{S}_{-*}M(d))([n-1])) = 0$. Consequently $H_a((\widetilde{S}_{-*}M(d))([n])) = 0$. By the induction hypothesis we have $H_1(C_{-*}) = 0$ as well. Moreover, by Lemma 5 we have $H_0((\widetilde{S}_{-*}M(d))([n-1])) = 0$ whenever $d \neq n-1$. Hence, we have $H_1((\widetilde{S}_{-*}M(d))([n])) = 0$ whenever $d \neq n-1$, and if $d = n-1$, then the complex $(\widetilde{S}_{-*}M(d))([n])$ is

$$0 \longrightarrow \bigoplus_{\substack{T \subset [n] \\ |T| = n-1}} R\,\mathrm{Hom}_{\mathrm{FI}}([n-1], T) \otimes_R \det([n] \setminus T) \longrightarrow R\,\mathrm{Hom}_{\mathrm{FI}}([n-1], [n]) \longrightarrow 0.$$

Since the last sequence is exact, we have $H_1((\widetilde{S}_{-*}M(n-1))([n])) = 0$. □

4. Proof of Theorem 1

We will now prove Theorem 1.

Proof of Theorem 1. For any FI-module V and $a \in \mathbb{N}$, we shall write

$$\widetilde{H}_a(V) := H_a(\widetilde{S}_{-*}V);$$

our aim is to prove that $H_a^{\mathrm{FI}}(V) \cong \widetilde{H}_a(V)$.

Any short exact sequence $0 \to V' \to V \to V'' \to 0$ of FI-modules gives a short exact sequence $0 \to \widetilde{S}_{-*}V' \to \widetilde{S}_{-*}V \to \widetilde{S}_{-*}V'' \to 0$ of complexes. It follows that the system $\widetilde{H} = (\widetilde{H}_*)$ is a homological δ-functor. For any FI-module V, there is a surjective homomorphism $\bigoplus_{i \in I} M(d_i) \to V$ for some $d_i \in \mathbb{N}$. Hence, by Proposition 6 the functor $\widetilde{H}_a$ is coeffaceable for each $a \geq 1$. By [4, Proposition 2.2.1] it follows that $\widetilde{H}$ is a universal δ-functor. Since, by (1) and (4), we have

$$\widetilde{H}_0(V) \cong \frac{V}{JV} \cong H_0^{\mathrm{FI}}(V),$$

it follows that $\widetilde{H}_a(V) \cong H_a^{\mathrm{FI}}(V)$ for all a. □

The following corollary was first proved in [3, Proposition 2.25]; let us show that it is an immediate consequence of Theorem 1 and [3, Theorem A].

Corollary 7. *Suppose R is noetherian and V is a finitely generated* FI-*module. For each $a \in \mathbb{N}$, we have* $\deg(H_a(\widetilde{S}_{-*}V)) < \infty$.

Proof. By [3, Theorem A] there is a projective resolution $\cdots \to P_2 \to P_1 \to P_0$ of V, where each P_a is a finitely generated projective FI-module. Applying the functor (1) to the complex P_*, we obtain a complex P_*/JP_* where each P_a/JP_a is a finitely generated FI-module. But for any finitely generated FI-module W with $JW = 0$, we have $W([n]) = 0$ for all n sufficiently large; indeed, if W is generated by a finite set of elements $w_i \in W([n_i])$ for $i = 1, \dots, m$, then $W([n]) = 0$ for $n > \max\{n_1, \dots, n_m\}$ since $\alpha(w_i) = 0$ for every $\alpha \in \mathrm{Hom}_{\mathrm{FI}}([n_i], [n])$ with $n > n_i$. Therefore, we have $\deg(P_a/JP_a) < \infty$. By Theorem 1 we have $H_a(\widetilde{S}_{-*}V) \cong H_a^{\mathrm{FI}}(V) \cong H_a(P_*/JP_*)$. Consequently, $\deg(H_a(\widetilde{S}_{-*}V)) < \infty$. □

5. An Application

The goal of this section is to prove Theorem 2.

Proof of Theorem 2. The $V = 0$ case is trivial, so assume that $V \neq 0$. Set $p = \deg(V)$.

Let $a \in \mathbb{N}$. If $n > p + a$, then $V([n - a]) = 0$, so $(\widetilde{S}_{-a}V)([n]) = 0$, and hence by Theorem 1 we have $H_a^{\mathrm{FI}}(V)([n]) = 0$. We claim that $H_a^{\mathrm{FI}}(V)([p + a]) \neq 0$ for all a sufficiently large; this would imply the theorem.

We have:

$$\begin{aligned}
(\widetilde{S}_{-(a+1)}V)([p+a]) &\cong V([p-1])^{\oplus\binom{p+a}{a+1}}, \\
(\widetilde{S}_{-a}V)([p+a]) &\cong V([p])^{\oplus\binom{p+a}{a}}, \\
(\widetilde{S}_{-(a-1)}V)([p+a]) &\cong 0,
\end{aligned}$$

where $V([p-1])$ is 0 if $p = 0$. By Theorem 1, to prove our claim, it suffices to show the following statement: *for all a sufficiently large, there is no surjective R-linear map*

$$V([p-1])^{\oplus\binom{p+a}{a+1}} \longrightarrow V([p])^{\oplus\binom{p+a}{a}}.$$

If R is a field, then the statement is clear since $\dim(V([p-1])^{\oplus\binom{p+a}{a+1}})$ is a polynomial in a of degree at most $p - 1$, whereas $\dim(V([p])^{\oplus\binom{p+a}{a}})$ is a polynomial in a of degree p. Suppose now that R is any commutative ring and set $M = V([p])$; we can reduce the statement to the case where R is a field using the following observations:

- Since M is a nonzero R-module, there exists a maximal ideal $\mathfrak{q}$ of R such that the localization $M_{\mathfrak{q}}$ is nonzero; thus, we reduce to the case where R is a ring.
- If R is a local ring with unique maximal ideal $\mathfrak{q}$, then since M is a nonzero finitely generated R-module, it follows by Nakayama's lemma that $M/\mathfrak{q}M$ is nonzero; thus, we reduce to the case where R is a field.

This concludes the proof of the theorem. □

We believe that the conclusion of this theorem holds for torsion modules that might not be finitely generated. However, the technique used here (comparing dimensions) does not work in this general case. It would be interesting to find a proof valid for all torsion modules.

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