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On Coxeter Mapping Classes and Fibered Alternating Links

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Abstract. Alternating-sign Hopf plumbing along a tree yields fibered alternating links whose homological monodromy is, up to a sign, conjugate to some alternating-sign Coxeter transformation. Exploiting this tie, we obtain results about the location of zeros of the Alexander polynomial of the fibered link complement implying a strong case of Hoste's conjecture, the trapezoidal conjecture, bi-orderability of the link group, and a sharp lower bound for the homological dilatation of the monodromy of the fibration. The results extend to more general hyperbolic fibered 3-manifolds associated with alternating-sign Coxeter graphs.

1. Introduction

In this paper, we study mapping classes defined by bipartite Coxeter graphs with sign-labels on the vertices determined by the bipartite structure. If the graph is connected and has at least two vertices, then these *alternating-sign Coxeter mapping classes* are pseudo-Anosov, and if the Coxeter graph is a tree, then the associated mapping class is the monodromy of an alternating fibered knot or link, which we call an (*alternating*) *Coxeter link.*

There has long been interest in the location of roots of Alexander polynomials for alternating links. Murasugi [18] showed that the coefficients of the polynomials have alternating signs, and hence no real root can be negative. Hoste conjectured that the real part of all zeros must be bounded from below by -1. This and related conjectures were settled for some classes of alternating links in [15; 13; 27; 7].

Using properties of alternating-sign Coxeter transformations, we give a simple proof that the roots of the Alexander polynomials for alternating Coxeter links are real and positive. By a result of Perron and Rolfsen [20] this implies that the fundamental group of the complement of an alternating Coxeter link is bi-orderable. Applying an interlacing property for alternating-sign Coxeter graphs, we show that the homological dilatations are monotone under graph inclusion. Thus, the minimum homological dilatation achieved by an alternating Coxeter link is $\frac{3+\sqrt{5}}{2}$, the square of the golden ratio. Similar properties hold for the Alexander polynomial of the mapping torus of alternating-sign Coxeter mapping classes.

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REMARK 1. Hirasawa and Murasugi [7] similarly studied the roots of Alexander polynomials for quasi-rational knots and links, which include the Coxeter links discussed in this paper, and they also proved stability and interlacing properties of the Alexander polynomial for these examples. By applying the constructs of Coxeter graphs and Coxeter transformations in this paper, we simplify their proofs in this context and extend the results to more general mapping classes and mapping tori associated to alternating-sign Coxeter graphs.

1.1. Alexander Polynomials of Alternating Knots and Links

The Alexander polynomial $\Delta(t) \in \mathbf{Z}[t]$ is an invariant of a finitely presented group with a prescribed homomorphism onto $\mathbf{Z}$. Given a knot or link K in S^3, each oriented Seifert surface S defines a surjective homomorphism of $\pi_1(S^3 \setminus K)$ to $\mathbf{Z}$ by algebraic intersection of closed paths with S. Denote by $\Delta_S(t)$ the associated Alexander polynomial. If $M = S^3 \setminus K$ is fibered over the circle with fiber S and monodromy ϕ, then $\Delta_S(t)$ is the characteristic polynomial of the homological monodromy $\phi_{\text{hom}} : H_1(S; \mathbf{R}) \to H_1(S; \mathbf{R})$ (this can be deduced from either the Fox calculus or the Seifert algorithm for finding $\Delta_S(t)$; see, for example, [24]). Given any mapping class ϕ on a surface S, write $\Delta_{S,\phi}(t)$ for the characteristic polynomial of the homological monodromy. It follows that if K is a fibered link with monodromy (S, ϕ), and $\Delta_K(t)$ is the Alexander polynomial of K, then we have

$$\Delta_K(t) = \Delta_S(t) = \Delta_{S,\phi}(t).$$

There are few restrictions on the Alexander polynomial: any monic reciprocal polynomial can be realized as $\Delta_{S,\phi}(t)$ up to multiples of t and $(t-1)$, where (S, ϕ) is the monodromy of some fibered link [12]. The story is different when we confine ourselves to *alternating knots and links:* those that admit a planar projection such that over and under crossings are alternating. Murasugi [18] showed that if S is the Seifert surface defined by an alternating planar projection, then $\Delta(-t)$ has degree $2g$, and the coefficients for the powers t^k are all strictly positive or strictly negative for $0 \leq k \leq 2g$. This implies, for example, that any real root of $\Delta(t)$ must be positive.

In 2002, Hoste conjectured the following:

CONJECTURE 2 (Hoste). *For alternating knots, the real part of any zero of the Alexander polynomial is strictly greater than* -1.

A lower bound on the real part of roots of $\Delta(t)$ was found by Lyubich and Murasugi [15] for two-bridge links. The results were later improved by Koseleff and Pecker [13] and Stoimenow [27]. Hirasawa and Murasugi [7] showed that for a large class of alternating links, the roots of the Alexander polynomial are real and positive, a property of integer polynomials known as *real stability.*

Our first result is the following:

THEOREM 3. *If* (S,ϕ) *is an alternating-sign Coxeter mapping class, then* $\Delta_{(S,\phi)}(t)$ *has real stability. In particular, the Alexander polynomial of an alternating Coxeter link has real stability.*

Fox's *trapezoidal conjecture* concerns the coefficients of Alexander polynomials of alternating knots.

CONJECTURE 4 ([4]). *Let* $\Delta(t) = a_{2g}t^{2g} + \cdots + a_0$ *be the Alexander polynomial of an alternating knot. Then there exists an integer* k *satisfying* $0 \le k \le g$ *such that*

$$|a_0| < \cdots < |a_k| = \cdots = |a_{2g-k}| > \cdots > |a_{2g}|.$$

The trapezoidal conjecture has been verified for several classes of alternating knots, for example, for algebraic alternating knots by Murasugi [19] and alternating knots of genus two by Ozsváth and Szabó [22] and Jong [11].

Real stability implies the trapezoidal property for integer polynomials. The coefficient sequence of a polynomial $a_{2g}t^{2g} + \cdots + a_0 \in \mathbf{R}[t]$ with only positive real roots is strictly *log-concave,* that is,

$$a_i^2 > a_{i-1}a_{i+1}$$

for all $i = 2, \ldots, 2g-1$; see, for example, [29]. Thus, the trapezoidal property of Alexander polynomials of alternating Coxeter links follows from Theorem 3 (cf. [7]). More generally, we have the following:

COROLLARY 5. *If* (S,ϕ) *is an alternating-sign Coxeter mapping class, then* $\Delta_{(S,\phi)}(t)$ *is trapezoidal. In particular, alternating-sign Coxeter links have trapezoidal Alexander polynomials.*

1.2. Bi-orderable Groups

A second application of Theorem 3 is the bi-orderability of knot groups and fundamental groups of 3-manifolds.

A group G is *bi-orderable* if it admits a total order $<$ on G that is compatible with the group operation, that is,

$$a \le b \text{ and } c \le d \quad \text{implies} \quad ac \le bd.$$

Perron and Rolfsen showed that if all the eigenvalues of the homological action of a surface homeomorphism ϕ are real and positive, then the fundamental group of its mapping torus is bi-orderable [20; 21]. Thus, Theorem 3 has this immediate consequence.

COROLLARY 6. *The mapping torus of an alternating-sign Coxeter mapping class has bi-orderable fundamental group.*

1.3. Dilatations of Mapping Classes

A *mapping class* on an oriented compact surface S of finite type is a self-homeomorphism up to isotopy relative to the boundary. The *homological dilatation* λ_{hom} of a mapping class ϕ is the largest eigenvalue (in modulus) of the characteristic polynomial of the action of ϕ on first homology. By the Nielsen–Thurston classification theorem the mapping classes fall into three types: those that are periodic, nonperiodic but preserving the isotopy class of a simple closed multicurve, and *pseudo-Anosov.* The third type is the most general and has the property that for some pair of transverse measured singular foliations $(\mathcal{F}^{\pm}, \nu^{\pm})$, the mapping class stretches the measure ν^{-} by λ and ν^{+} by λ^{-1} for some $\lambda > 1$. The constant $\lambda_{\text{geo}} = \lambda$ is the (*geometric*) *dilatation* of the mapping class. The homological and geometric dilatations are related as follows:

$$\lambda_{\text{hom}}(\phi) \le \lambda_{\text{geo}}(\phi)$$

with equality if and only if ϕ is *orientable,* that is, its invariant foliations $\mathcal{F}^{\pm}$ are orientable (see, e.g., [3]).

The mapping torus of a mapping class (S, ϕ) is the three-dimensional manifold

$$M = M_{(S,\phi)} = S \times [0,1]/(x,1) \sim (\phi(x), 0).$$

By a theorem of Thurston [28], this manifold admits a hyperbolic structure if and only if ϕ is pseudo-Anosov. The associated fibration $M \to S^1$ defines a surjective homomorphism $\pi_1(M) \to \mathbf{Z}$ and a corresponding Alexander polynomial $\Delta_{(S,\phi)}(t)$.

We show that the dilatation of alternating-sign Coxeter mapping classes is monotonic with respect to graph inclusion. Thus, the minimum dilatation for alternating-sign Coxeter mapping classes is achieved by the alternating-sign A_2 graph, which in turn is geometrically realized by the figure eight knot.

THEOREM 7. *The minimum homological and geometric dilatation of alternating-sign Coxeter mapping classes is the square of the golden ratio* $\frac{3+\sqrt{5}}{2}$ *and is geometrically realized as the monodromy of the figure eight knot.*

REMARK 8. By a result of McMullen [17] the spectral radius of the classical Coxeter transformations is minimized by the E_{10} Coxeter graph, also known as the $(2,3,7)$ star-like graph [16]. The associated Coxeter link is the $(-2,3,7)$-pretzel link [8], and the dilatation of its monodromy is the conjectural smallest Salem number, known as Lehmer's number [14], which is smaller than the square of the golden ratio.

REMARK 9. By contrast to Theorem 7, when dropping the assumption of alternating signs, it is possible to find mixed-sign Coxeter graphs whose associated mapping classes have dilatation arbitrarily close to 1 (see [9]).

1.4. Organization

In Section 2 we recall some definitions and properties of classical Coxeter systems and generalize them to mixed-sign Coxeter systems. The analog of Alexander polynomials for Coxeter systems is the Coxeter polynomial, the characteristic polynomial of the Coxeter transformation. For bipartite alternating-sign Coxeter systems, we prove real stability for the Coxeter polynomial and the interlacing property. Section 3 discusses geometric realizations of alternating-sign Coxeter systems and contains proofs of Theorems 3 and 7.

2. Bipartite Coxeter Graphs

A *mixed-sign Coxeter graph* is a pair $(\Gamma, \mathfrak{s})$, where Γ is a finite connected graph without self- or double edges, and $\mathfrak{s}$ is an assignment of a sign $+$ or $-$ to every vertex v_i of Γ. Let $\mathbf{R}^{V_\Gamma}$ be the vector space of $\mathbf{R}$-labelings of the vertices of Γ. For $v \in V_\Gamma$, let $[v]$ be the corresponding element of $\mathbf{R}^{V_\Gamma}$ giving the label 1 on v and 0 on all other vertices of Γ. The real vector space $\mathbf{R}^{V_\Gamma}$ is equipped with a symmetric bilinear form B given by $B([v_i],[v_i]) = -2 \cdot \mathfrak{s}(v_i)$ and otherwise $B([v_i],[v_j]) = a_{ij}$, where $A = (a_{ij})$ is the adjacency matrix of Γ. With every vertex v_i, we associate a reflexion s_i about the hyperplane of $\mathbf{R}^{V_\Gamma}$ perpendicular to $[v_i]$, given by the formula

$$s_i([v_j]) = [v_j] - 2\frac{B([v_i],[v_j])}{B([v_i],[v_i])}[v_i].$$

The *Coxeter transformation* is the product $C = s_1 \cdots s_n$ of all these reflections. For trees, this product does not depend, up to conjugation, on the order of multiplication [26], but in general it does. For bipartite Coxeter graphs Γ, however, there is a distinguished conjugacy class, the *bipartite Coxeter transformation* C_{+-} given by $C_{+-} = C_+C_-$, where C_+ is any product of all the reflections corresponding to vertices in one part of the partition, and C_- is any product of all the reflections corresponding to vertices in the other part. This is well defined since all the reflections corresponding to vertices in one part of the partition commute pairwise.

If all signs $\mathfrak{s}$ of a bipartite Coxeter graph are positive, then theorems of A'Campo and McMullen state that the eigenvalues of the bipartite Coxeter transformation are on the unit circle or positive real and that the spectral radius is monotonic with respect to graph inclusion [1; 17]. We now prove analogs of these theorems for *alternating-sign Coxeter graphs,* the case where the bipartition of the graph Γ is defined by the signs $\mathfrak{s}$.

PROPOSITION 10. *Let* $(\Gamma, \mathfrak{s})$ *be an alternating-sign Coxeter graph. Then the eigenvalues of the bipartite Coxeter transformation* C_{+-} *are real and strictly negative.*

Proof. Let $(\Gamma, \mathfrak{s})$ be an alternating-sign Coxeter graph. Number the vertices of Γ starting with all the positive ones and then proceeding to the negative ones. With this vertex numbering, the adjacency matrix $A = A(\Gamma)$ of Γ becomes a 2×2-block matrix with zero blocks on the diagonal and blocks X and $X^\top$ in the upper right and lower left, respectively. Using the formula for the s_i, we have that the products C_+ and C_- corresponding to the partition are given by

$$C_+ = \begin{pmatrix} -I & X \\ 0 & I \end{pmatrix}, \qquad C_- = \begin{pmatrix} I & 0 \\ -X^\top & -I \end{pmatrix}.$$

Multiplication of C_+ and C_- shows that the bipartite Coxeter transformation $C_{+-} = C_+C_-$ is symmetric. Therefore, C_{+-} has only real eigenvalues. It is left to show that there are no positive eigenvalues. Note that $(C_+ + C_-)^2 = -A(\Gamma)^2$. Furthermore, by expanding we obtain

$$(C_+ + C_-)^2 = 2I + C_{+-} + C_{+-}^{-1},$$

and thus, for any eigenvalue $\lambda \in \mathbf{R}$ of C_{+-}, we have

$$2 + \lambda + \lambda^{-1} = -\alpha^2,$$

where α is some eigenvalue of the adjacency matrix $A(\Gamma)$. It follows that $2 + \lambda + \lambda^{-1}$ is a nonpositive real number since α is a real number. In particular, every eigenvalue λ of the alternating-sign Coxeter transformation C_{+-} is strictly negative. □

2.1. Interlacing Property

Let Γ and Γ' be alternating-sign Coxeter graphs such that Γ is a subgraph of Γ'. We say that Γ' is obtained from Γ by a *vertex extension* if the vertex set of Γ' contains one more element w than the vertex set of Γ and if the edges of Γ are precisely the edges of Γ' that do not have w as an endpoint.

PROPOSITION 11. *Let* $(\Gamma, \mathfrak{s})$ *and* $(\Gamma', \mathfrak{s}')$ *be two alternating-sign Coxeter graphs. If* Γ' *is a vertex extension of* Γ, *then the eigenvalues of the bipartite Coxeter transformations* C_{+-} *and* C'_{+-} *are interlaced, that is, if* $\alpha_1 \leq \cdots \leq \alpha_s$ *are the eigenvalues of* $C_{+-}(\Gamma)$, *and* $\beta_1 \leq \cdots \leq \beta_{s+1}$ *are the eigenvalues of* $C_{+-}(\Gamma')$, *then*

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \alpha_s \leq \beta_{s+1}.$$

Proof. Let $(\Gamma, \mathfrak{s})$ be an alternating-sign Coxeter graph with bipartite Coxeter transformation C_{+-}. From the proof of Proposition 10 we recall that the eigenvalues of C_{+-} are in one-to-one correspondence with the eigenvalues of the adjacency matrix $A(\Gamma)$. More precisely, the correspondence is given by

$$-\alpha^2 = 2 + \lambda + \lambda^{-1},$$

where λ and α are eigenvalues of C_{+-} and $A(\Gamma)$, respectively. Since Γ is bipartite, the eigenvalues of $A(\Gamma)$ are symmetric with respect to the origin [2]. Furthermore, since $\max(|\lambda|, |\lambda|^{-1})$ is monotonically increasing with respect to α^2, there exists a monotonic transformation of $\mathbf{R}$ taking the eigenvalues of $A(\Gamma)$ to the eigenvalues of C_{+-}. Now let $(\Gamma', \mathfrak{s}')$ be an alternating-sign Coxeter graph with bipartite Coxeter transformation C'_{+-} such that Γ' is a vertex-extension of Γ. Then the eigenvalues of $A(\Gamma)$ and $A(\Gamma')$ are interlaced [2], and therefore so are the eigenvalues of C_{+-} and C'_{+-}. □

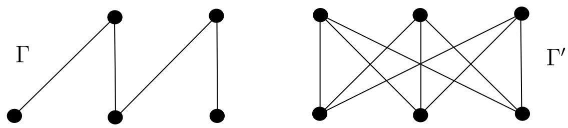

Figure 1

PROPOSITION 12. *The minimum spectral radius for an alternating-sign Coxeter transformation is realized by the alternating-sign A_2 Coxeter graph, and the spectral radius is the square of the golden mean.*

Proof. Noting that every nontrivial alternating-sign Coxeter graph is a (perhaps multiple) vertex extension of the alternating-sign A_2 graph, the statement follows from Proposition 11. □

REMARK 13. If a bipartite graph Γ is a subgraph of another bipartite graph Γ' with one more vertex but Γ' is not a vertex-extension of Γ, then the eigenvalues of the corresponding adjacency matrices need not be interlaced. Choosing Γ and Γ' as in Figure 1, the eigenvalues of the adjacency matrix of Γ are given by $\{-\sqrt{3}, -1, 0, 1, \sqrt{3}\}$, and the eigenvalues of the adjacency matrix of Γ' are given by $\{-3, 0, 0, 0, 0, 3\}$. In particular, these eigenvalues are not interlaced. However, focusing on the largest eigenvalue, it is still true that the spectral radius is monotonic under graph inclusion.

PROPOSITION 14. *Let* $(\Gamma, \mathfrak{s})$ *and* $(\Gamma', \mathfrak{s}')$ *be two alternating-sign Coxeter graphs. If* Γ *is a subgraph of* Γ', *then the spectral radius of* C_{+-} *is less than or equal to the spectral radius of* C'_{+-}.

Proof. The proof is basically the same as that of Proposition 11. However, instead of interlacing (which does not necessarily apply in the case of noninduced subgraphs), we use Perron–Frobenius theory and the fact that $A(\Gamma)$ is dominated by a submatrix of $A(\Gamma')$. □

REMARK 15. General Coxeter graphs are defined with arbitrary edge weights $m_{ij} \geq 3$. The corresponding entries a_{ij} of the adjacency matrix are then defined to be $a_{ij} = 2 \cdot \cos(2\pi/m_{ij})$. Although we formulated Propositions 10 and 14 for constant edge-weights $m_{ij} = 3$, they also hold in this generalized context. Proposition 10 holds without change of wording. For Proposition 14, we must add the assumption that when Γ is a subgraph of Γ', every edge-weight of Γ is less than or equal to the edge-weight of Γ'.

3. Geometric Realization

In this section, we associate fibered alternating links and more general mapping tori to alternating-sign Coxeter graphs $(\Gamma, \mathfrak{s})$.

3.1. Mapping Classes from Mixed-Sign Coxeter Systems

Mixed-sign Coxeter systems, defined by Coxeter graphs with ordered signed vertices, are useful for building examples of mapping classes.

As in the *classical* (or *positive-sign*) case, a mixed-sign Coxeter graph with n vertices defines a subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R})$ generated by reflections. In the classical case the reflections preserve an associated symmetric bilinear form $2I - A$, where A is the adjacency matrix of the Coxeter graph. For a mixed-sign Coxeter system, the bilinear form is given by $2I_{\mathfrak{s}} - A$, where $I_{\mathfrak{s}}$ is a diagonal matrix with ± 1 entries on the diagonal depending on the signs $\mathfrak{s}$ assigned to vertices of the Coxeter graph. For mixed-sign Coxeter graphs, just as for classical ones, we can explicitly construct mapping classes whose homological monodromy is conjugate to the Coxeter transformation up to sign [8; 9; 14; 28].

Classical bipartite Coxeter systems have been shown to have many useful properties. A'Campo showed that all eigenvalues of the Coxeter transformation are real or lie on the unit circle. This condition is sometimes called *bistability* [7]. Since the traces of the eigenvalues over the reals are related to the eigenvalues of the adjacency matrix of the Coxeter graph, the eigenvalues satisfy an interlacing theorem. McMullen [17] used this to prove monotonicity of the spectral radius of Coxeter transformations with respect to graph inclusion and found a sharp lower bound for the gap between 1 and the next smallest spectral radius of Coxeter transformations. It follows, in particular, that the classical Coxeter mapping classes associated to bipartite classical Coxeter graphs that are not spherical or affine have dilatation bounded from below by Lehmer's number, which is approximately 1.17628 [14].

REMARK 16. By contrast to Theorem 3, A'Campo [1] showed that for any classical bipartite Coxeter graph that is not spherical or affine, the roots of the corresponding Coxeter polynomials are either on the unit circle or positive real, with at least one root greater than 1. If Hoste's conjecture is true, then this gives a homological proof of the fact that the knots associated to classical bipartite Coxeter graphs that are not spherical or affine can never be alternating. This can also be proved independently: such a knot is positive, that is, it has a diagram with only positive crossings. For the signature $|\sigma|$ and genus g, we have $|\sigma| < 2g$ since $2g$ equals the number of vertices and $|\sigma|$ equals the signature of the bilinear form $2I - A$. But for knots that are both positive and alternating, we have $|\sigma| = 2g$, for example, by properties of Rasmussen's s-invariant [23].

Let $\mathcal{L}$ be an arrangement of line segments in the plane whose intersection graph equals Γ. That is, with each vertex v of Γ, there is an associated line segment ℓ_v in $\mathcal{L}$, and two line segments in $\mathcal{L}$ intersect if the corresponding vertices are connected by an edge of Γ. A *planar realization of* Γ is an embedding of $\mathcal{L}$ in $\mathbf{R}^2$ with coordinate axes x and y, so that if $\mathfrak{s}(v) = 1$, then ℓ_v is parallel to the y-axis, and if $\mathfrak{s}(v) = -1$, then ℓ_v is parallel to the x-axis.

If Γ has a planar realization, then we thicken the ℓ_v into rectangular strips $\ell_v \times [-1, 1]$ (resp., $[-1, 1] \times \ell_v$), so that each segment ℓ_v is identified with $\ell_v \times \{0\}$

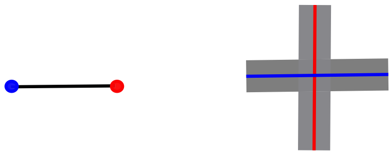

Figure 2

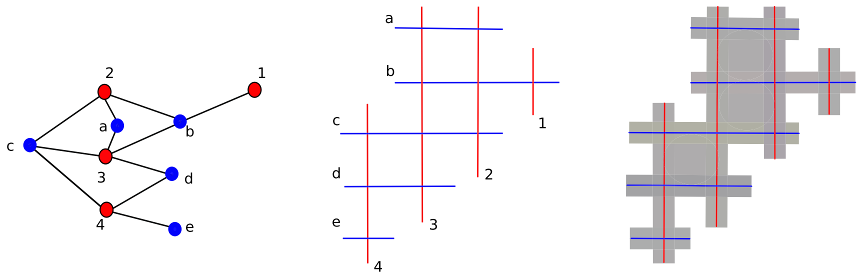

Figure 3

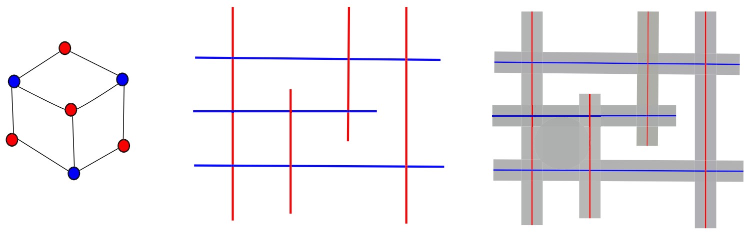

Figure 4

(resp., $\{0\} \times \ell_v$). If v and w are adjacent on Γ, then the rectangular strips ℓ_v and ℓ_w are glued together at right angles as in Figure 2. The thickenings and gluings can be made so that all rectangular strips in each bipartite partition are parallel to one another. A planar realization is *fillable* if it is possible to attach (possibly nonconvex) polygons to the planar graph along closed cycles, so that the interior of the polygon does not include any endpoint of a line segment. Figure 3 gives an example of a fillable planar realization, and Figure 4 gives an example of a nonfillable planar realization.

Think of the planar realization as being embedded in S^3. Let S be the filled planar realization after gluing together each end of the horizontal strips to its opposite with a single positive full twist, and the end of each vertical strip to its opposite by a single negative full twist. The boundary of S is a link $K \subset S^3$ with distinguished Seifert surface S. We call (K, S) a *Coxeter link* associated to Γ.

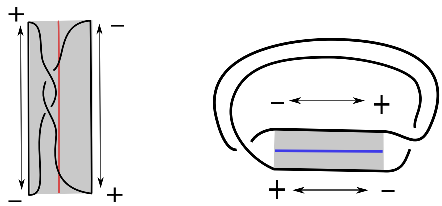

Figure 5

PROPOSITION 17. *If* Γ *is an alternating-sign Coxeter graph with a fillable planar realization, then any associated Coxeter link is alternating.*

Proof. The link K has an alternating planar diagram coming from drawing each vertical and horizontal Hopf band as in Figure 5. Here the shaded rectangle is the original neighborhood of the line segment associated with a vertex of Γ. The signs indicate over $(+)$ and under $(-)$ crossings. Thus, we can see that for each vertex $v \in V_\Gamma$, when proceeding along ℓ_v, there is always a $-$ sign on the right and a $+$ sign on the left, where $-$ indicates an upcoming underpass, and $+$ indicates an upcoming overpass. Since the signs are consistent on vertical and horizontal segments ($-$ appears on the right, and $+$ appears on the left, no matter from which direction you approach an endpoint of a segment), the link K is alternating. □

PROPOSITION 18. *The Coxeter link of an alternating-sign Coxeter graph is fibered, and the homological monodromy is conjugate to* $-C_{+-}$.

Proof. Since the surface S can be obtained from a disk by Hopf plumbings, the boundary of S is a fibered link K with fiber S. All the strips become annuli on S. The monodromy of the fibration is the product of right or left Dehn twists around core curves of the annuli, right or left being determined by whether the twist is positive or negative [5; 18; 25].

Let V_Γ be the set of vertices of Γ. For $v \in V_\Gamma$, let γ_v be the closed curve defined by ℓ_v. Then the homology classes $[\gamma_v]$ form a basis for $H_1(S; \mathbb{R})$, and the monodromy ϕ of S is the product of positive Dehn twists on γ_v for v such that $\mathfrak{s}(v) = 1$ composed with the product of negative Dehn twists on γ_v for v such that $\mathfrak{s}(v) = -1$. Let $\mathbf{R}^{V_\Gamma}$ be the vector space of $\mathbf{R}$-labelings of the vertices. For $v \in V_\Gamma$, let $[v]$ be the corresponding element of $\mathbf{R}^{V_\Gamma}$ giving the label 1 on v and 0 on all other vertices of Γ. There is a commutative diagram

$$\begin{array}{ccc} \mathbf{R}^{V_\Gamma} & \longrightarrow & H_1(S;\mathbf{R}) \\ {\scriptstyle -C_{+-}}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathbf{R}^{V_\Gamma} & \longrightarrow & H_1(S;\mathbf{R}) \end{array}$$

where the horizontal arrows taking $[v]$ to $[\gamma_v]$ are isomorphisms.

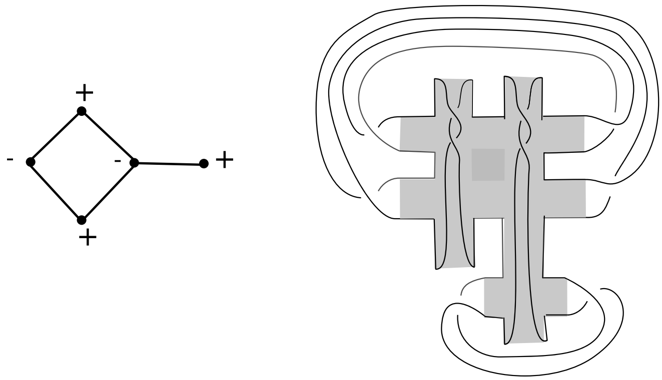

Figure 6

The Coxeter transformation decomposes as

$$C_{+-} = C_+ C_- = -M(M^T)^{-1},$$

where $M = -C_+$; see [10]. By construction, M is also the Seifert matrix for S in $S^3 \setminus K$ with respect to the generators for homology given by the core curves of the attached Hopf bands. Thus,

$$\phi_* = (M^T)^{-1} M$$

(see, e.g., [24]) and is conjugate to $-C_{+-}$. □

Corollary 19. *The Alexander polynomial $\Delta(t)$ satisfies*

$$\Delta(t) = c(-t),$$

where $c(t)$ is the characteristic polynomial of the Coxeter transformation C_{+-} of Γ.

Proof. The Alexander polynomial $\Delta_S(t)$ is the characteristic polynomial of $M(M^T)^{-1} = -C_{+-}$. □

Example 20. Figure 6 gives an example of an alternating-sign Coxeter graph and fillable planar realization.

Then

$$C_+ = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & -1 \end{bmatrix}.$$

Setting the orientation on the Seifert surface S so that the shaded area is oriented positively toward the viewer, we see that $-C_+$ is the Seifert matrix and

$$C_- = -(C_+^T)^{-1}.$$

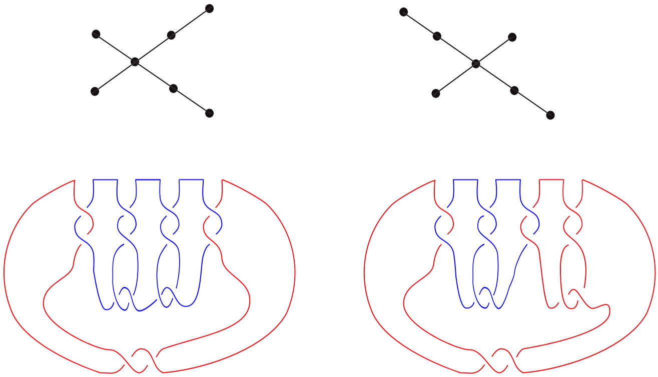

Figure 7

The Coxeter transformation is given by

$$C_{+-} = C_+ C_- = -\begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The associated Alexander and Coxeter polynomials are:

$$\Delta(t) = t^5 - 10t^4 + 27t^3 - 27t^2 + 10t - 1,$$
$$c(t) = t^5 + 10t^4 + 27t^3 + 27t^2 + 10t + 1.$$

REMARK 21. The link associated to a Coxeter graph is not uniquely determined by the combinatorics of the graph. Figure 7 shows two different planar embeddings of a Coxeter graph. The two links realizing these embeddings are distinct: one of them has an unknotted component, whereas the other does not. Although for a large class of classical Coxeter trees, two different planar embeddings always yield distinct but mutant links by a theorem of Gerber [6], we do not know whether the same holds in the alternating-sign case.

In general, even if Γ does not have a planar realization, it is possible to find a surface S and a system of simple closed curves $\{\gamma_v\}$ in one-to-one correspondence with V_Γ such that:

(1) the intersection matrix of the γ_v equals the adjacency matrix for V_Γ; and
(2) the complementary components of the union of γ_v are either disks or boundary parallel annuli

(see, e.g., [9]). Since Γ is bipartite, the system of curves partitions into two multicurves γ_+ and γ_- that intersect transversally. Let τ_+ and τ_- be the positive Dehn twist along γ_+ and the negative Dehn twist along γ_-, respectively. Let $\phi = \tau_+\tau_-$. We call (S, ϕ) a *geometric realization* of $(\Gamma, \mathfrak{s})$.

LEMMA 22. *Let E be the set of eigenvalues of $-C_{+-}$, and let F be the set of eigenvalues of the homological action of ϕ. Then*

$$F \setminus \{1\} \subset E \setminus \{1\}.$$

Proof. The proof follows along the same lines as the proof of Proposition 18, the only difference being that the horizontal arrows in the commutative diagram need not be one-to-one or onto. The cokernel is generated by boundary parallel curves whose homology classes are fixed by ϕ_*, and hence their homology classes are contained in the eigenspace for 1. □

Let (S, ϕ) be a geometric realization of an alternating-sign Coxeter graph $(\Gamma, \mathfrak{s})$. Then the eigenvalues of the homological action of ϕ are real and strictly positive by Proposition 10 and Lemma 22. This implies Theorem 3. Similarly, Theorem 7 follows directly from Proposition 12 and Lemma 22.

Combining Proposition 11 with Corollary 19, we also have the following interlacing result.

THEOREM 23. *If K' and K are alternating-sign Coxeter links associated with Γ' and Γ, respectively, where Γ' is a vertex extension of Γ, then the roots of the Alexander polynomial of K' and that of K are interlacing.*

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