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Cut Limits on Hyperbolic Extensions

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Abstract. Hyperbolic extensions were defined and studied in [4]. Cut limits of families of metrics were introduced in [5]. In this paper, we show that if a family of metrics $\{h_\lambda\}$ has cut limits, then the family of hyperbolic extensions $\{\mathcal{E}_k(h_\lambda)\}$ also has cut limits.

The results in this paper are used in the problem of smoothing Charney–Davis strict hyperbolizations [2; 3].

1. Introduction

This paper deals with the relationship between two concepts: "hyperbolic extensions", which were studied in [4], and "cut limits of families of metrics", which were defined in [5]. Before stating our main result, we first introduce these concepts here.

1.1. Hyperbolic Extensions

Recall that the hyperbolic n-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^k \times \mathbb{H}^{n-k}$ with warp product metric $(\cosh^2 r)\sigma_{\mathbb{H}^k} + \sigma_{\mathbb{H}^{n-k}}$, where $\sigma_{\mathbb{H}^l}$ denotes the hyperbolic metric of $\mathbb{H}^l$, and $r : \mathbb{H}^{n-k} \to [0, \infty)$ is the distance to a fixed point in $\mathbb{H}^{n-k}$. For instance, in the case $n = 2$, since $\mathbb{H}^1 = \mathbb{R}^1$, we have that $\mathbb{H}^2$ is isometric to $\mathbb{R}^2 = \{(u, v)\}$ with metric $\cosh^2 v\, du^2 + dv^2$. The concept of "hyperbolic extension" is a generalization of this construction; we explain this in the next paragraph.

Let (M^n, h) be a complete Riemannian manifold with *center* $o = o_M \in M$, that is, the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism. The warp product metric

$$f = (\cosh^2 r)\sigma_{\mathbb{H}^k} + h$$

on $\mathbb{H}^k \times M$ is the *hyperbolic extension* (*of dimension* k) of the metric h. Here r is the distance-to-o function on M. We write $\mathcal{E}_k(M) = (\mathbb{H}^k \times M, f)$ and $f = \mathcal{E}_k(h)$. We also say that $\mathcal{E}_k(M)$ is the *hyperbolic extension* (*of dimension* k) *of* (M, h) (or just of M). Hence, for instance, we have $\mathcal{E}_k(\mathbb{H}^l) = \mathbb{H}^{k+l}$. Also, write $\mathbb{H}^k = \mathbb{H}^k \times \{o_M\} \subset \mathcal{E}_k(M)$, and we have that any $p \in \mathbb{H}^k$ is a center of $\mathcal{E}_k(M)$ (see Remarks 2.3 (3)).

Remarks 1.1.

1. Let M^n have center o. Using a fixed orthonormal basis on T_oM and the exponential map, we can identify M with $\mathbb{R}^n$, and $M - \{o\}$ with $\mathbb{R}^n - \{0\} =$

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$\mathbb{S}^{n-1} \times (0, \infty)$. Hence, the spheres $\mathbb{S}^{n-1} \times \{r\} \subset \mathbb{S}^{n-1} \times (0, \infty)$ are geodesic spheres, and the rays $t \mapsto tv = (v, t) \in \mathbb{S}^{n-1} \times (0, \infty) = M - \{o\}$, are distance-minimizing geodesic rays emanating from the center.

2. Let g' be another metric on M. Suppose we can write $g' = g'_r + dr^2$ on $\mathbb{S}^{n-1} \times (0, \infty) = M - \{o\}$ (this last identification is done using g). Then the geodesic spheres around o and the geodesics emanating from o for g and g' coincide.

1.2. Cut Limits

Before we talk about "cut limits", we need some preliminary definitions and facts. Let (M^n, g) have center o. Then the metric g (outside the center) has the form $g = g_r + dr^2$. Here we are identifying (see Remarks 1.1) the space $M - \{o\}$ with $\mathbb{S}^{n-1} \times (0, \infty)$, and thus each g_r is a metric on the sphere $\mathbb{S}^{n-1}$.

EXAMPLES.

1. The Euclidean metric $\sigma_{\mathbb{R}^n}$ on $\mathbb{R}^n$ can be written as $\sigma_{\mathbb{R}^n} = g_r + dr^2$ with $g_r = r^2\sigma_{\mathbb{S}^{n-1}}$, where $\sigma_{\mathbb{S}^{n-1}}$ is the round metric on the sphere $\mathbb{S}^{n-1}$.

2. The hyperbolic metric $\sigma_{\mathbb{H}^n}$ on $\mathbb{R}^n$ can be written as $\sigma_{\mathbb{H}^n} = g_r + dr^2$ with $g_r = \sinh^2(r)\sigma_{\mathbb{S}^{n-1}}$.

Let (M, g) have center o and write $g = g_r + dr^2$. Let $r_0 > 0$. We can think of the metric g_{r_0} as being obtained from $g = g_r + dr^2$ by "cutting" g along the sphere of radius r_0, so we call the metric g_{r_0} on $\mathbb{S}^{n-1}$ the *spherical cut of g at r_0*. Let

$$\hat{g}_{r_0} = \left(\frac{1}{\sinh^2(r_0)}\right) g_{r_0}. \tag{1.1}$$

We call the metric $\hat{g}_{r_0}$ on $\mathbb{S}^{n-1}$ given by (1.1) the *normalized spherical cut of g at r_0*. In the particular case that $g = g_r + dr^2$ is a warped-by-sinh metric, we have $g_r = \sinh^2(r)g'$ for some fixed g' independent of r. In this case, the spherical cut of $g = \sinh^2(r)g' + dt^2$ at r_0 is $\sinh^2(r_0)g'$, and the normalized spherical cut at r_0 is $\hat{g}_{r_0} = g'$.

EXAMPLE. If $g = \sigma_{\mathbb{H}^n} = \sinh^2(r)\sigma_{\mathbb{S}^{n-1}} + dr^2$, then the normalized spherical cut at r_0 is $(\widehat{\sigma_{\mathbb{H}^n}})_{r_0} = \sigma_{\mathbb{S}^{n-1}}$, and the spherical cut at r_0 is $\sinh^2(r_0)\sigma_{\mathbb{S}^{n-1}}$.

Let (M^n, g) have center o. We now consider families of metrics $\{g_\lambda\}_{\lambda > \lambda_0}$ on M of the form $g_\lambda = (g_\lambda)_r + dr^2$. Here $\lambda_0 > 0$, and the identification $M - \{o\} = \mathbb{S}^{n-1} \times (0, \infty)$ is done using g; see Remarks 1.1. We call such a family an $\odot$-*family of metrics on* (M, g). (We use the symbol $\odot$ to evoke the idea that all metrics g_λ have a common center and spheres.) The reason we are interested in these families is that they are key ingredients in Riemannian hyperbolization [3] (also see [5]). Moreover, Main Theorem in this paper is used in [3].

Let $b \in \mathbb{R}$. By cutting each g_λ at $b + \lambda$ we obtain a one-parameter family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ of metrics on the sphere $\mathbb{S}^{n-1}$. (The metric $\widehat{(g_\lambda)}_{\lambda+b}$ is the normalized spherical cut of g_λ at $\lambda + b$.) Here $\lambda > \max\{\lambda_0, -b\}$, so that the definition makes sense. We say that the $\{g_\lambda\}$ has *cut limit at b* if this family C^2-converges as $\lambda \to \infty$. That is, there is a C^2 metric $\hat{g}^b_\infty$ on $\mathbb{S}^{n-1}$ such that

$$|\widehat{(g_\lambda)}_{\lambda+b} - \hat{g}^b_\infty|_{C^2(\mathbb{S}^{n-1})} \longrightarrow 0 \quad \text{as } \lambda \to \infty. \tag{1.2}$$

Here the arrow means convergence in the C^2-norm on the space of C^2 metrics on $\mathbb{S}^{n-1}$.

REMARK 1.2. The C^2 norm is taken with respect to a fixed locally finite atlas with extendable charts, that is, charts that can be extended to the (compact) closure of their domains.

Let $I \subset \mathbb{R}$ be an interval (compact or noncompact). We say that the $\odot$-family $\{g_\lambda\}$ has *cut limits on I* if the convergence in (1.2) is uniform with compact supports in the variable in $b \in I$. Explicitly this means: for every $\varepsilon > 0$ and compact $K \subset I$, there is λ_* such that $|\widehat{(g_\lambda)}_{\lambda+b'} - \hat{g}_{\infty+b'}|_{C^2(\mathbb{S}^{n-1})} < \varepsilon$ for $\lambda > \lambda_*$ and $b' \in K$.

REMARK 1.3. Equivalently, the $\odot$-family $\{g_\lambda\}$ has *cut limits on I* if for every $\varepsilon > 0$ and $b \in I$, there are λ_* and a neighborhood U of b in I such that $|\widehat{(g_\lambda)}_{\lambda+b'} - \hat{g}_{\infty+b'}|_{C^2(\mathbb{S}^{n-1})} < \varepsilon$ for $\lambda > \lambda_*$ and $b' \in U$.

If $\{g_\lambda\}$ has cut limits on I, then it has a cut limit at every $b \in I$. Finally, we say that $\{g_\lambda\}$ has *cut limits* if $\{g_\lambda\}$ has cut limits on $\mathbb{R}$.

REMARK 1.4. If $\{g_\lambda\}_\lambda$ is a family of metrics, then $\{g_{\lambda(\lambda')}\}_{\lambda'}$ is a *reparameterization* of $\{g_\lambda\}_\lambda$, where $\lambda' \mapsto \lambda(\lambda')$ is a change of variables. For instance, if we use translations, then the following holds: $\{g_\lambda\}_\lambda$ has cut limits at b if and only if $\{g_{\lambda'+a}\}_{\lambda'}$ has cut limits at $b + a$; here the change of variables is $\lambda = \lambda' + a$.

1.3. Statement of Main Result

Here is a natural question:

QUESTION. If $\{h_\lambda\}_\lambda$ has cut limits, does $\{\mathcal{E}_k(h_\lambda)\}_\lambda$ have cut limits?

REMARK. More generally, we can ask whether $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ has cut limits, where $\lambda = \lambda(\lambda')$. Of course, the answer would depend on the change of variables $\lambda = \lambda(\lambda')$.

Our main result gives an affirmative answer to this question, provided that the family $\{h_\lambda\}$ is, in some sense, nice near the origin. Explicitly, we say that $\{h_\lambda\}_{\lambda>\lambda_0}$ is *hyperbolic around the origin* if there is $B \in \mathbb{R}$ such that

$$\widehat{(h_\lambda)}_{\lambda+b} = \sigma_{\mathbb{S}^{n-1}}$$

for every $b \le B$ and every $\lambda > \max\{\lambda_0, -b\}$. Note that this implies that each h_λ is canonically hyperbolic on the ball of radius $\lambda + B$, that is, $h_\lambda = \sinh^2(r)\sigma_{\mathbb{S}^{n-1}} + dr^2$ on the ball of radius $\lambda + B$. Examples of $\odot$-families that are hyperbolic around the origin are families obtained using hyperbolic forcing [5].

As mentioned before, our main result answers affirmatively the question posed. Moreover, it also says that some reparameterized families $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ have cut limits as well for certain change of variables $\lambda = \lambda(\lambda')$. Write $\lambda = \lambda(\lambda', \theta) = \sinh^{-1}(\sinh(\lambda') \sin\theta)$ for a fixed θ. We say that $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ is the θ-*reparameterization of* $\{\mathcal{E}_k(h_\lambda)\}_\lambda$. Note that if we consider a hyperbolic right triangle with one angle equal to θ and side (opposite to θ) of length λ, then λ' is the length of the hypotenuse of the triangle. All θ-reparameterizations, in the limit $\lambda' \to \infty$, differ just by translations; that is, a simple calculation shows that $\lim_{\lambda' \to \infty} \lambda(\lambda') - \lambda' = \ln \sin\theta$. We are now ready to state our main result.

MAIN THEOREM. *Let M have center o. Let $\{h_\lambda\}_\lambda$ be an $\odot$-family of metrics on M. If $\{h_\lambda\}_\lambda$ is hyperbolic around the origin and has cut limits, then, for every $\theta \in (0, \pi/2]$, the θ-reparameterization $\{\mathcal{E}_k(h_\lambda)\}_{\lambda'}$ has cut limits.*

Note that $\theta = \pi/2$ gives $\lambda = \lambda'$, answering the question posed. The paper is structured as follows. In Section 2, we review some facts about hyperbolic extensions. In Section 3, we introduce useful coordinates on the spheres of a hyperbolic extension. In Section 4, we study normalized spherical cuts on hyperbolic extensions. Finally, in Section 5, we deal with cut limits in a bit more detail and prove Main Theorem.

2. Hyperbolic Extensions

Notational convention: we will denote all fixed centers on manifolds by the same letter "o". If the manifold M needs to be specified, then we will write $o = o_M$, which means that o is a center in M.

Note that $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$ (see [1, p. 23]). Let η be a complete geodesic in M passing though o, and let η^+ be one of its two geodesic rays (beginning at o). Then η is a totally geodesic subspace of M, and η^+ is convex (see [4]). Also, let γ be a complete geodesic in $\mathbb{H}^k$. The following two results are proved in Section 3 of [4].

LEMMA 2.1. *The subspace $\gamma \times \eta^+$ is convex in $\mathcal{E}_k(M)$, and $\gamma \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.*

COROLLARY 2.2. *The subspaces $\mathbb{H}^k \times \eta^+$ and $\gamma \times M$ are convex in $\mathcal{E}_k(M)$. Also, $\mathbb{H}^k \times \eta$ is totally geodesic in $\mathcal{E}_k(M)$.*

REMARKS 2.3.

1. By convexity we mean the following: a set A is convex if given two points in A, any distance minimizing geodesic joining these points lies in A.

2. As pointed out in Section 3 of [4], the proof of Lemma 2.1 (which is Lemma 3.1 in [4]) can easily be adapted to show that $\{y\} \times M$ are convex in $\mathcal{E}_k(M)$. Alternatively, it is not hard to prove that $\{y\} \times M$ is convex in $\gamma \times M$; this, together with Corollary 2.2, implies that $\{y\} \times M$ are convex in $\mathcal{E}_k(M)$.

3. Note that $\mathbb{H}^k \times \eta$ (with metric induced by $\mathcal{E}_k(M)$) is isometric to $\mathbb{H}^k \times \mathbb{R}$ with warp product metric $\cosh^2 v\sigma_{\mathbb{H}^k} + dv^2$, which is just the hyperbolic $(k+1)$-space $\mathbb{H}^{k+1}$. Also, $\gamma \times \eta$ is isometric to $\mathbb{R} \times \mathbb{R}$ with warp product metric $\cosh^2 v\, du^2 + dv^2$, which is just the hyperbolic 2-space $\mathbb{H}^2$. In particular, every point in $\mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset \mathcal{E}_k(M)$ is a center point.

As before, we use h to identify $M - \{o\}$ with $\mathbb{S}^{n-1} \times \mathbb{R}^+$. Sometimes, we will denote a point $v = (u, r) \in \mathbb{S}^{n-1} \times \mathbb{R}^+ = M - \{o\}$ by $v = ru$. Fix a center $o \in \mathbb{H}^k \in \mathcal{E}_k(M)$. Then, for $y \in \mathbb{H}^k - \{o\}$, we can also write $y = tw$, $(w, t) \in \mathbb{S}^{k-1} \times \mathbb{R}^+$. Similarly, using the exponential map, we can identify $\mathcal{E}_k(M) - \{o\}$ with $\mathbb{S}^{k+n-1} \times \mathbb{R}^+$, and for $p \in \mathcal{E}_k(M) - \{o\}$, we can write $p = sx$, $(x, s) \in \mathbb{S}^{k+n-1} \times \mathbb{R}^+$.

We denote the metric on $\mathcal{E}_k(M)$ by f, and we can write $f = f_s + ds^2$. Since $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$, we can write $\mathbb{H}^k - \{o\} = \mathbb{S}^{k-1} \times \mathbb{R}^+ \subset \mathbb{S}^{k+n-1} \times \mathbb{R}^+$ and $\mathbb{S}^{k-1} \subset \mathbb{S}^{k+n-1}$.

A point $p \in \mathcal{E}_k(M) - \mathbb{H}^k$ has two sets of coordinates: the *polar coordinates* $(x, s) = (x(p), s(p)) \in \mathbb{S}^{k+n-1} \times \mathbb{R}^+$ and the *hyperbolic extension coordinates* $(y, v) = (y(p), v(p)) \in \mathbb{H}^k \times M$. Write $M_o = \{o\} \times M$. Therefore, we have the following functions:

the distance to o function:
$$s : \mathcal{E}_k(M) \to [0, \infty), \qquad s(p) = d_{\mathcal{E}_k(M)}(p, o);$$
the direction of p function:
$$x : \mathcal{E}_k(M) - \{o\} \to \mathbb{S}^{n+k-1}, \qquad p = s(p)x(p);$$
the distance to $\mathbb{H}^k$ function:
$$r : \mathcal{E}_k(M) \to [0, \infty), \qquad r(p) = d_{\mathcal{E}_k(M)}(p, \mathbb{H}^k);$$
the projection on $\mathbb{H}^k$ function:
$$y : \mathcal{E}_k(M) \to \mathbb{H}^k;$$
the projection on M function:
$$v : \mathcal{E}_k(M) \to M;$$
the projection on $\mathbb{S}^{n-1}$ function:
$$u : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{S}^{n-1}, \qquad v(p) = r(p)u(p);$$
the length of y function:
$$t : \mathcal{E}_k(M) \to [0, \infty), \qquad t(p) = d_{\mathbb{H}^k}(y(p), o);$$
the direction of y function:
$$w : \mathcal{E}_k(M) - M_o \to \mathbb{S}^{k-1}, \qquad y(p) = t(p)w(p).$$

Note that $r = d_M(v, o)$. Note also that, by Lemma 2.1, the functions w and u are constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is, $w(sx) = w(x)$ and $u(sx) = u(x)$.

Let ∂_r and ∂_s be the gradient vector fields of r and s, respectively. Since the M-fibers $M_y = \{y\} \times M$ are convex, the vectors ∂_r are the velocity vectors of the speed one geodesics of the form $a \mapsto (y, au)$, $u \in \mathbb{S}^{n-1} \subset M$. These geodesics emanate from (and orthogonally to) $\mathbb{H}^k \subset \mathcal{E}_k(M)$. Also, the vectors ∂_s are the velocity vectors of the speed one geodesics emanating from $o \in \mathcal{E}_k(M)$. For $p \in \mathcal{E}_k(M)$, denote by $\triangle = \triangle(p)$ the right triangle with vertices o, $y = y(p)$, p and sides the geodesic segments $[o, p] \in \mathcal{E}_k(M)$, $[o, y] \in \mathbb{H}^k$, $[p, y] \in \{y\} \times M \subset \mathcal{E}_k(M)$. (These geodesic segments are unique and well defined because: (1) $\mathbb{H}^k$ is convex in $\mathcal{E}_k(M)$, (2) $(y, o) = o_{\{y\}\times M}$ and o are centers in $\{y\} \times M$ and $\mathbb{H}^k \subset \mathcal{E}_k(M)$, respectively.)

Let $\alpha : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{R}$ be the angle between ∂_s and ∂_r (in that order); thus, $\cos\alpha = f(\partial_r, \partial_s)$, $\alpha \in [0, \pi]$. Then $\alpha = \alpha(p)$ is the interior angle at $p = (y, v)$ of the right triangle $\triangle = \triangle(p)$. We call $\beta(p)$ the interior angle of this triangle at o, that is, $\beta(p) = \beta(x)$ is the spherical distance between $x \in \mathbb{S}^{k+n-1}$ and the totally geodesic subsphere $\mathbb{S}^{k-1}$. Alternatively, β is the angle between the geodesic segment $[o, p] \subset \mathcal{E}_k(M)$ and the convex submanifold $\mathbb{H}^k$. Therefore, β is constant on each geodesic emanating from $o \in \mathcal{E}_k(M)$, that is, $\beta(sx) = \beta(x)$. The following corollary follows from Lemma 2.1 (see 3.1 in [4]).

COROLLARY 2.4. *Let η^+ (or η) be a geodesic ray (line) in M through o containing $v = v(p)$, and γ a geodesic line in $\mathbb{H}^k$ through o containing $y = y(p)$. Then $\triangle(p) \subset \gamma \times \eta^+ \subset \gamma \times \eta$.*

REMARK 2.5. Note that the right geodesic triangle $\triangle(p)$ has sides of lengths $r = r(p)$, $t = t(p)$, and $s = s(p)$. By Lemma 2.1 and Remarks 2.3 (3) we can consider $\triangle$ as contained in a totally geodesic copy of the hyperbolic 2-space $\mathbb{H}^2(p)$. The plane $\mathbb{H}^2(p)$ is well defined for p outside $\mathbb{H}^k \cup (\{o\} \times M)$. We will write $\mathbb{H}^2(p) = \gamma_w \times \eta_u$, where $p = (y, v) \in \mathbb{H}^k \times M$, $y = tw$, $v = ru$.

Hence, by Remark 2.5, using hyperbolic trigonometric identities, we can find relations between r, t, s, α, and β. For instance, using the hyperbolic law of sines, we get

$$\sinh(r) = \sin(\beta)\sinh(s). \tag{2.1}$$

In Section 4, we will need the following result.

PROPOSITION 2.6. *The following identity holds outside $\mathbb{H}^k \cup (\{o\} \times M)$:*

$$(\sinh^2(s))\, d\beta^2 + ds^2 = \cosh^2(r)\, dt^2 + dr^2.$$

Proof. First, a particular case. Take $M = \mathbb{R}$ and $k = 1$; hence, $\mathcal{E}_k(M) = \mathcal{E}_1(\mathbb{R}) = \mathbb{H}^2$. In this case, the left-hand side of the identity is the expression of the metric of $\mathbb{H}^2$ in polar coordinates (β, s), and right-hand side of the equation is the expression of the same metric in the hyperbolic extension coordinates $(r, t) = (v, y)$. (Here r and t are "signed" distances.) Hence, the equation holds in this particular case. This particular case, together with the fact that $\mathbb{H}^2(p)$ is isometric to $\mathbb{H}^2$, and the following claim prove the proposition.

CLAIM. *The functionals $d\beta, ds, dt, dr$, at $p \in \mathbb{H}^k \cup (\{o\} \times M)$, are zero on vectors perpendicular to $\mathbb{H}^2(p)$.*

Proof. To prove the claim, let u be a vector perpendicular to $\mathbb{H}^2(p)$ at p. Since the ray $s \mapsto sx(p)$ is contained in $\mathbb{H}^2(p)$, we have that u is tangent to the sphere of radius $s(p)$ centered at o. Therefore, $ds(u) = 0$.

Next, we prove that $dr(u) = 0$ and $dt(u) = 0$. Note that u is a linear combination of vectors perpendicular to $\mathbb{H}^2(p)$ that are either tangent to $\{y\} \times M$ or $\mathbb{H}^k \times \{v\}$, where $y = y(p)$ and $v = v(p)$. Therefore, it is enough to assume that u is tangent to $\{y\} \times M$ or $\mathbb{H}^k \times \{v\}$.

First, assume that u is perpendicular to $\mathbb{H}^2(p)$ and tangent to $\{y\} \times M$. Since u is tangent to $\{y\} \times M$, we get that $dt(u)$. Since u is perpendicular to the ray $r \mapsto rv$ in $\{y\} \times M$ (because this ray is contained in $\mathbb{H}^2(p)$), we get that $dr(u) = 0$.

Next, assume that u is perpendicular to $\mathbb{H}^2(p)$ and tangent to $\mathbb{H}^k \times \{v\}$. Then $dr(u) = 0$, and since u is perpendicular to the ray $t \mapsto ty$ in $\mathbb{H}^k \times \{v\}$ (because this ray is contained in $\mathbb{H}^2(p)$), we get that $dt(u) = 0$.

Finally, the equation $d\beta(u) = 0$ follows from $ds(u) = 0$, $dt(u) = 0$, $dr(u) = 0$, the fact that β is a function of s, t, r, and the chain rule. This proves the claim and concludes the proof of Proposition 2.6. $\square$

$\square$

3. Coordinates on the Spheres $S_s(\mathcal{E}_k(M))$

Let N^n have center o. The geodesic sphere of radius r centered at o will be denoted by $\mathbb{S}_r = \mathbb{S}_r(N)$, and we can identify $\mathbb{S}_r$ with $\mathbb{S}^{n-1} \times \{r\}$.

Let M have center o and metric h. Consider the hyperbolic extension $\mathcal{E}_k(M)$ of M with center $o \in \mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset \mathcal{E}_k(M)$ and metric f. Since $\mathbb{H}^k \subset \mathcal{E}_k(M)$ is convex, we can write $\mathbb{S}_s(\mathcal{E}_k(M)) \cap \mathbb{H}^k = \mathbb{S}_s(\mathbb{H}^k)$. Equivalently, $(\mathbb{S}^{k+n-1} \times \{s\}) \cap \mathbb{H}^k = \mathbb{S}^{k-1} \times \{s\}$. Write $M_o = \{o\} \times M$. Also write

$$E_k(M) = \mathcal{E}_k(M) - (\mathbb{H}^k \sqcup M_o)$$

and

$$S_s(\mathcal{E}_k(M)) = \mathbb{S}_s(\mathcal{E}_k(M)) \cap E_k(M) = \mathbb{S}_s(\mathcal{E}_k(M)) - (\mathbb{H}^k \sqcup M_o).$$

Note that the functions α and β are well defined and smooth on $E_k(M)$ and that $0 < \beta(p) < \pi/2$. Moreover, by Remark 2.5 the plane $\mathbb{H}^2(p) = \gamma_w \times \eta_u$ is well defined for $p \in E_k(M)$. As in Remark 2.5, here $p = (y, v) \in \mathbb{H}^k \times M$, $y = tw$, $v = ru$. Recall that $\triangle(p) \subset \mathbb{H}^2(p)$ (see Corollary 2.4 and Remark 2.5).

By the identification between $\mathbb{S}^{n+k-1} \times \{s\}$ with $\mathbb{S}_s(\mathcal{E}_k(M))$ and Lemma 2.1 we have that $\mathbb{H}^2(p) \cap \mathbb{S}_s(\mathcal{E}_k(M))$ gets identified with a geodesic circle $\mathbb{S}^1(p) \subset \mathbb{S}^{n+k-1}$. Moreover, since $\mathbb{H}^2(p)$ and $\mathbb{H}^k$ intersect orthogonally on γ_w, we have that the spherical geodesic segment $[x(p), w(p)]_{\mathbb{S}^{n+k-1}}$ intersects $\mathbb{S}^{k-1} \subset \mathbb{S}^{n+k-1}$ orthogonally at w. This, together with the fact that $\beta < \pi/2$, implies that $[x(p), w(p)]_{\mathbb{S}^{n+k-1}}$ is a length-minimizing spherical geodesic in $\mathbb{S}^{k+n-1}$ joining x to w. Consequently, $\beta = \beta(p)$ is the length of $[x(p), w(p)]_{\mathbb{S}^{n+k-1}}$.

We now give a set of coordinates on $S_s(\mathcal{E}_k(M))$. For $p \in S_s(\mathcal{E}_k(M))$, define

$$\Xi(p) = \Xi_s(p) = (w, u, \beta) \in \mathbb{S}^{k-1} \times \mathbb{S}^{n-1} \times (0, \pi/2),$$

where $w = w(p)$, $u = u(p)$, $\beta = \beta(p)$. Note that Ξ is constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is, $\Xi(sx) = \Xi(x)$.

Using hyperbolic trigonometric identities (e.g., identity (2.1)) we can find well-defined and smooth functions $r = r(s, \beta)$ and $t = t(s, \beta)$ such that r, s, t are the lengths of the sides of a right geodesic triangle on $\mathbb{H}^2$ with angle β opposite the side with length r. With these functions, we can construct explicitly a smooth inverse to Ξ.

REMARKS 3.1.

1. For $(w, u) \in \mathbb{S}^{k-1} \times \mathbb{S}^{n-1}$, we have

$$\Xi((\gamma_w \times \eta_u) \cap S_s(\mathcal{E}_k(M))) = \{\pm w\} \times \{\pm u\} \times (0, \pi/2).$$

By Lemma 2.1 the paths $a \mapsto (\pm w, \pm u, a)$ are four spherical (open) geodesic segments emanating orthogonally from $\mathbb{S}^{k-1}$.

2. For $w \in \mathbb{S}^{k-1}$, we have

$$\Xi((\gamma_w \times M) \cap S_s(\mathcal{E}_k(M))) = \{\pm w\} \times \mathbb{S}^{n-1} \times (0, \pi/2).$$

By Corollary 2.2 we have that this set is a spherical geodesic ball of radius $\pi/2$ and of dimension n (with its center deleted) intersecting $\mathbb{S}^{k-1}$ orthogonally at w. Note that the geodesic segments on this ball emanating from w are the spherical geodesic segments of item 1 for all $u \in \mathbb{S}^{n-1}$.

3. For $w \in \mathbb{S}^{k-1}$ and r with $0 < r < s$, we have

$$\Xi((\gamma_w \times \mathbb{S}_r(M)) \cap S_s(\mathcal{E}_k(M))) = \{w\} \times \mathbb{S}^{n-1} \times \beta(r),$$

where $\beta(r)$ is the angle of the right geodesic hyperbolic triangle with sides of length s (opposite to the right angle) and r, opposite to β. By identity (2.1) we have $\beta = \sin^{-1}(\sinh(r)/\sinh(s))$.

4. Since the M-fibers $\{y\} \times M$ are orthogonal in $\mathcal{E}_k(M)$ to the $\mathbb{H}^k$-fibers $\mathbb{H}^k \times \{v\}$, items 1, 2, and 3 imply that the $\mathbb{S}^{k-1}$-fibers, the $\mathbb{S}^{n-1}$-fibers, and $(0, \pi/2)$-fibers are mutually orthogonal in $\mathbb{S}^{k-1} \times \mathbb{S}^{n-1} \times (0, \pi/2)$ with the metric $\Xi_* f$.

5. The map

$$\Xi' = (\Xi, s) : E_k(M) \to \mathbb{S}^{k-1} \times \mathbb{S}^{n-1} \times (0, \pi/2) \times \mathbb{R}^+$$

gives coordinates on $E_k(M)$.

4. Spherical Cuts on Hyperbolic Extensions

Let (N^m, g) have center o. Recall from Introduction that the metric g_r on $\mathbb{S}_r$ is called the *spherical cut of g at r* and that the metric $\hat{g}_r = (1/\sinh^2(r)) g_r$ is the *normalized spherical cut of g at r*.

Now let (M^n, h) have center o. Thus, we can write $h = h_r + dr^2$, where each h_r is a metric on $\mathbb{S}^{n-1}$. As before, we denote by $f = \mathcal{E}_k(h)$ the hyperbolic extension of h, and we write $f = f_s + ds^2$ on $\mathcal{E}_k(M) - \{o\}$; each f_s is a metric on $\mathbb{S}^{n+k-1}$. We use the map $\Xi = \Xi_s$ of Section 3, which gives coordinates on $S_s(\mathcal{E}_k(M))$. Note that the metric $\Xi_* f_s$ is a metric on $\mathbb{S}^{k-1} \times \mathbb{S}^{n-1} \times (0, \pi/2)$, and it is the expression of f_s in the Ξ-coordinates.

PROPOSITION 4.1. *The expression of f_s in the Ξ-coordinates is given by*

$$\Xi_* f_s = (\sinh^2(s)\cos^2(\beta))\sigma_{\mathbb{S}^{k-1}} + h_r + (\sinh^2(s))\, d\beta^2,$$

where $r = \sinh^{-1}(\sinh(s)\sin(\beta))$ (*see identity* (2.1)).

REMARK 4.2. Note that the function $r = r(s, \beta)$ is the same function used in Introduction for the θ-reparameterizations $\lambda = \lambda(\lambda', \theta)$.

Proof of Proposition 4.1. By Remarks 3.1 (4) we have that $\Xi_* f_s$ has the form $A + B + C$, where $A(u, \beta)$ is a metric on $\mathbb{S}^{k-1} \times \{u\} \times \{\beta\}$, $B(w, \beta)$ is a metric on $\{w\} \times \mathbb{S}^{n-1} \times \{\beta\}$, and $C(u, \beta)$ is a metric on $\{w\} \times \{u\} \times (0, \pi/2)$, that is, $C = f(w, u, \beta)\, d\beta^2$ for some positive function f.

Now, by definition we have

$$f = \cosh^2(r)\sigma_{\mathbb{H}^k} + h_r + dr^2 = \cosh^2(r)(\sinh^2(t)\sigma_{\mathbb{S}^{k-1}} + dt^2) + h_r + dr^2.$$

By Proposition 2.6 and the identity $\cosh(r)\sinh(t) = \sinh(s)\cos(\beta)$ (which follows from the law of sines and the second law of cosines; also see identity (2.1)) we can write

$$f_s + ds^2 = f = (\sinh^2(s)\cos^2(\beta))\sigma_{\mathbb{S}^{k-1}} + h_r + (\sinh^2(s))\, d\beta^2 + ds^2.$$

This proves the proposition. □

Hence, Proposition 4.1 gives the expression of the spherical cut at s of the metric $f = \mathcal{E}_k(h)$ in the Ξ-coordinates. The next corollary does the same for the normalized spherical cut $\hat{f}$ of f at s.

COROLLARY 4.3. *The expression of $\hat{f}_s$ in the Ξ-coordinates is given by*

$$\Xi_*(\hat{f}_s) = \cos^2(\beta)\sigma_{\mathbb{S}^{k-1}} + \sin^2(\beta)\hat{h}_r + d\beta^2,$$

where r is as in Proposition 4.1.

Proof. Since $\sinh^2(r)\hat{h}_r = h_r$ and $\sinh^2(s)\hat{f}_s = f_s$, the corollary follows from Proposition 4.1 and identity (2.1). □

5. Cut Limits and Proof of Main Theorem

First, a bit of notation. Let (N^m, g) have center o. Recall that we can write the metric on $N - \{o\} = \mathbb{S}^{m-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where r is the distance to o. Let $A \subset \mathbb{S}^{m-1}$ be open and denote by $\mathrm{C}\,A$ the open cone $A \times \mathbb{R}^+ \subset \mathbb{S}^{m-1} \times \mathbb{R}^+ \subset M$. We write $A_r = \mathrm{C}\,A \cap \mathbb{S}_r(M) = A \times \{r\}$. We say that $\{g_\lambda\}_\lambda$ is an $\odot$-*family of metrics over* A if each g_λ is a metric defined on $\mathrm{C}\,A$ and g_λ and if it can be written in the form $g_\lambda = (g_\lambda)_r + dr^2$ on $\mathrm{C}\,A$. We say that the $\{g_\lambda\}$ has a *cut limit over A at b* if there is a C^2 metric $\hat{g}^b_\infty$ on A such that (1.2) holds, where the arrow in (1.2) now means uniform convergence in the $C^2(A)$-norm on the space of C^2 metrics on $A \subset \mathbb{S}^{m-1}$. Also, *cut limits over A on I* and *cut limits over A* are defined similarly.

Let M^n have metric h and center o. As always, we identify $M - \{o\}$ with $\mathbb{S}^{n-1} \times \mathbb{R}^+$ and M with $\mathbb{R}^n$. Choose a center $o \in \mathbb{H}^k \subset \mathcal{E}_k(M)$. Let $\{h_\lambda\}_\lambda$ be an $\odot$-family of metrics on M; thus, o is a center for all h_λ. Denote by $f_\lambda = \mathcal{E}_k(h_\lambda)$ the hyperbolic extension of h_λ. We have that $\{f_\lambda\}_\lambda$ is an $\odot$-family on $\mathcal{E}_k(M)$. From now on we assume $\theta \in (0, \pi/2]$ fixed. Next θ-reparameterize $\{f_\lambda\}_\lambda$, that is, we use the change of variables $\lambda = \lambda(\lambda') = \sinh^{-1}(\sinh(\lambda')\sin\theta)$. (Note that λ' plays the role of the variable s in identity (2.1), and λ plays the role of r.) We obtain in this way the $\odot$-family $\{f_{\lambda(\lambda')}\}_{\lambda'}$. Write $S = \mathbb{S}^{n+k-1} - \{\mathbb{S}^{k-1} \sqcup \mathbb{S}^{n-1}\}$, where $\mathbb{S}^{k-1} \subset \mathbb{H}^k \times \{o\}$ and $\mathbb{S}^{n-1} \subset \{o\} \times M$.

PROPOSITION 5.1. *Assume that $\{h_\lambda\}$ has cut limits on the interval $J_c = (-\infty, c]$ and that it is hyperbolic around the origin. Then, for each $c' < c + \ln\sin(\theta)$, the family $\{f_{\lambda(\lambda')}\}_{\lambda'}$ has cut limits on $J_{c'}$ over S.*

Proof. By hypothesis $\{h_\lambda\}$ is hyperbolic around the origin. Hence, there is B such that

$$\widehat{(h_\lambda)}_{\lambda+b} = \sigma_{\mathbb{S}^{n-1}} \quad \text{for all } b \leq B. \tag{5.1}$$

Hence, the metrics h_λ are canonically hyperbolic on the ball of radius $\lambda + B$. Also, since we are assuming that $\{h_\lambda\}$ has cut limits on J_c, we have that

$$b \in J_c \quad \Longrightarrow \quad \widehat{(h_\lambda)}_{\lambda+b} \xrightarrow{C^2} \hat{h}^b_\infty \quad \text{as } \lambda \to \infty \tag{5.2}$$

uniformly on $\mathbb{S}^{n-1}$ and uniformly with compact supports in the variable $b \in J_c$.

As mentioned before, we can write $f_\lambda = (f_\lambda)_s + ds^2$. We have to compute the limit of $\widehat{(f_{\lambda(\lambda')})}_{\lambda'+b}$ as $\lambda' \to \infty$. Let the Ξ-coordinates be as defined in Section 3 for the space $(\mathcal{E}_k(M), f)$. From Corollary 4.3 we can express $(\hat{f}_\lambda)_s$ in Ξ-coordinates:

$$\Xi_*(\widehat{(f_{\lambda(\lambda')})}_{\lambda'+b}) = \cos^2(\beta)\sigma_{\mathbb{S}^{k-1}} + \sin^2(\beta)\widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)} + d\beta^2,$$

where $r = r(s, \beta)$ is given by identity (2.1) (see also Proposition 4.1 and Remark 4.2). Therefore, we want to find the limit of $\widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)}$ as $\lambda' \to \infty$. To do this, taking the inverse of $\lambda = \lambda(\lambda')$, we get $\lambda' = \lambda'(\lambda) = \sinh^{-1}(\sinh(\lambda)/\sin(\theta))$. Hence,

$$\lim_{\lambda'\to\infty} \widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)} = \lim_{\lambda\to\infty} \widehat{(h_\lambda)}_{\vartheta(\lambda,\beta,b)}, \tag{5.3}$$

where

$$\vartheta(\lambda, \beta, b) = r(\lambda'(\lambda) + b, \beta) = \sinh^{-1}\left(\sinh\left\{b + \sinh^{-1}\left(\frac{\sinh(\lambda)}{\sin(\theta)}\right)\right\}\sin(\beta)\right),$$

and a straightforward calculation shows

$$\lim_{\lambda\to\infty} (\vartheta(\lambda, \beta, b) - \lambda) = b + \ln\left(\frac{\sin(\beta)}{\sin(\theta)}\right). \tag{5.4}$$

This convergence is uniform with compact supports in the $C^2(S)$-topology (see caveat below). Choose $c' \in \mathbb{R}$ such that $c' < c - \ln(\sin(\pi/2)/\sin(\theta)) = c +$

$\ln\sin(\theta)$. Since $\beta \in (0, \pi/2)$, we get

$$b \in J_{c'} \quad \Longrightarrow \quad \left(b + \ln\left(\frac{\sin(\beta)}{\sin(\theta)}\right)\right) \in J_c. \tag{5.5}$$

Hence, from (5.2), (5.3), (5.4), and (5.5) we get

$$\lim_{\lambda' \to \infty} \widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)} = \hat{h}_\infty^{b+\ln(\sin\beta/\sin\theta)}. \tag{5.6}$$

CAVEAT. The limit (5.3) (hence also in (5.6)) is uniform *with compact supports* in the β direction, but not uniform in the β direction. The problem occurs when $\beta \to 0$.

We next deal with the problem mentioned in the caveat, that is, we have to show that the limit in (5.6) is uniform in the variable $\beta \in (0, \pi/2)$ (not just uniform with compact supports). The convergence in (5.4) (hence in (5.6)) is uniform for β near $\pi/2$, but the convergence in (5.4) is certainly not uniform near 0. Here is where we will need the extra condition of the family being hyperbolic near the origin. We will need the following claim.

CLAIM. *Let $c, B, \theta \in \mathbb{R}$. Choose c' with $c' < c + \ln\sin\theta$. Then there is $\beta_1 > 0$ such that $r(\lambda' + c', \beta_1) \leq \lambda(\lambda') + B$ for every λ' sufficiently large.*

Proof. A calculation shows that taking $\beta_1 = \sin^{-1}(e^{2(B-c-1})$ works. (Find the limit $\lambda' \to \infty$ of both terms in the inequality and use the fact that $c' < c + \ln\sin\theta$.) This proves the claim. □

Since the function $r = r(s, \beta)$ is increasing in both variables, the claim implies that $r(\lambda' + b, \beta) \leq \lambda(\lambda') + B$ for all $b \leq c'$, $\beta \leq \beta_1$, and λ' sufficiently large (how large not depending on b, nor on β). This, together with (5.1), implies that for all $b \leq c'$, $\beta \leq \beta_1$, and λ' sufficiently large, we have

$$\widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)} = \sigma_{\mathbb{S}^{n-1}}.$$

Hence, for all $b \in J_{c'}$ and $\beta \leq \beta_1$, we have

$$\lim_{\lambda' \to \infty} \widehat{(h_{\lambda(\lambda')})}_{r(\lambda'+b,\beta)} = \sigma_{\mathbb{S}^{n-1}}.$$

Since $\beta_1 > 0$, the problem mentioned in the caveat (i.e., when $\beta \to 0$) has been removed. This proves the proposition. □

Taking $c \to \infty$ in Proposition 5.1 gives the following corollary.

COROLLARY 5.2. *Assume that $\{h_\lambda\}$ has cut limits and that it is hyperbolic around the origin. Then $\{f_{\lambda(\lambda')}\}_{\lambda'}$ has cut limits over S.*

Proof of Main Theorem. Note that the only difference between Corollary 5.2 and Main Theorem is that in the corollary the cut limits exist *over* $S \subset \mathbb{S}^{n+k-1}$. Hence, we have to show that the existence of cut limits over S implies the existence of cut limits on the whole of $\mathbb{S}^{n+k-1}$. Corollary 5.2 and (1.2) in Introduction imply

$$|\widehat{((f_\lambda)|S)}_{\lambda'+b} - \hat{f}^b_\infty|_{C^2(S)} \longrightarrow 0 \quad \text{as } \lambda' \to \infty,$$

where $\hat{f}^b_\infty$ is a metric on S. In particular, for every b, the one-parameter family $\widehat{((f_\lambda)|S)}_{\lambda'+b}$ is Cauchy, that is,

$$|\widehat{((f_{\lambda(\lambda'_1)})|S)}_{\lambda'_1+b} - \widehat{((f_{\lambda(\lambda'_2)})|S)}_{\lambda'_2+b}|_{C^2(S)} \longrightarrow 0 \tag{5.7}$$

uniformly on S as $\lambda'_1, \lambda'_2 \to \infty$. But since S is dense in $\mathbb{S}^{n+k-1}$, we get that $|g|_S|_{C^2(S)} = |g|_{C^2(\mathbb{S}^{n+k-1})}$ for any C^2 (pointwise) bilinear form g on $\mathbb{S}^{n+k-1}$. Therefore, we can drop the restriction "$|_S$" in (5.7) to get

$$|\widehat{(f_{\lambda(\lambda'_1)})}_{\lambda'_1+b} - \widehat{(f_{\lambda(\lambda'_2)})}_{\lambda'_2+b}|_{C^2(\mathbb{S}^{n+k-1})} \longrightarrow 0 \quad \text{as } \lambda' \to 0.$$

This implies that the family $\widehat{(f_\lambda)}_{\lambda'+b}$ is Cauchy. Since the space of C^2 metrics on $\mathbb{S}^{n+k-1}$ with the C^2 norm is a complete metric space, this Cauchy sequence converges to some $\hat{f}^b_\infty$. Note that $\hat{f}^b_\infty$ is a symmetric bilinear form on $\mathbb{S}^{n+k-1}$, and it is positive definite on S. It remains to prove that $\hat{f}^b_\infty$ is also positive definite outside S. Recall that $S = \mathbb{S}^{n+k-1} - (\mathbb{S}^{k-1} \sqcup \mathbb{S}^{n-1})$. It is straightforward to verify that we have $\hat{f}^b_\infty|_{\mathbb{S}^{k-1}} = \sigma_{\mathbb{S}^{k-1}} + \sigma_{\mathbb{H}^n}$. On the other hand, on $\mathbb{S}^{n-1}$ we have $\beta = \pi/2$, and hence $\lambda = \lambda'$. Also, by definition we have $f_\lambda = \cosh^2(r)\sigma_{\mathbb{H}^k} + h_\lambda$. But on M_o we get $r = s$. Therefore,

$$\begin{aligned}\widehat{((f_\lambda)|_{\mathbb{S}^{n-1}})}_{\lambda'+b} &= \widehat{((f_\lambda)|_{\mathbb{S}^{n-1}})}_{\lambda+b} = \operatorname{cotanh}^2(\lambda+b)\sigma_{\mathbb{H}^k} + \widehat{(h_\lambda)}_{\lambda+b}\\ &\longrightarrow \operatorname{cotanh}^2(\lambda+b)\sigma_{\mathbb{H}^k} + \hat{h}^b_\infty.\end{aligned}$$

Consequently, $\hat{f}_{b+\infty}$ is positive definite on $\mathbb{S}^{n-1}$. Thus, it is positive definite outside S. This proves Main Theorem. □

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