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Differentiability Inside Sets with Minkowski Dimension One

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Abstract. We investigate Minkowski, or box-counting, dimension of universal differentiability sets of Lipschitz functions. Whilst existing results concern the Lebesgue measure and Hausdorff dimension of these fractal sets, the Minkowski dimension is stronger than Hausdorff, and we demonstrate that the lower bound one on Minkowski dimension is tight for any Euclidean space. Spaces other than the real line allow for a further refinement of the bound: the 1-Hausdorff measure of such sets must be infinite.

1. Introduction

Background and Overview of Main Results

In the present paper, we answer a natural question pointed out by Olsen in 2009, whether there is a universal differentiability set of Minkowski dimension one. Our answer is affirmative: a compact universal differentiability set with upper and lower Minkowski dimension one in $\mathbb{R}^d$, for all d, is constructed explicitly. Namely, we prove a stronger statement:

Theorem (Theorem 5.6(1)). *For every $d \geq 1$, there exists a compact subset $S \subseteq \mathbb{R}^d$ of Minkowski dimension one such that for any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, the set of points $x \in S$ such that g is Fréchet differentiable at x is a dense subset of S.*

Recall that Lipschitz functions on Banach spaces have rather strong differentiability properties. The classical Rademacher theorem says that Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$ are differentiable almost everywhere with respect to the Lebesgue measure. For $d = 1$, the converse statement also holds: Each subset N of $\mathbb{R}$ with Lebesgue measure zero admits a Lipschitz function nowhere differentiable on N; see [14; 7]. However, Preiss [11] proved that all Euclidean spaces of dimension higher than one contain Lebesgue null sets that capture a point of differentiability of every Lipschitz function on the space.

Sets containing a point of differentiability of every Lipschitz function are said to have the universal differentiability property and are called universal differentiability sets (UDS). The result of [11] has sparked a modern investigation into the nature of such sets. Clearly, the set S in Theorem 5.6 quoted is a UDS.

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One of the central questions about universal differentiability sets is how small they can be. To detect smaller universal differentiability sets, we must appeal to various other notions of size in addition to the Lebesgue measure. Doré and Maleva [2; 4] prove that every Euclidean space contains compact universal differentiability sets of Hausdorff dimension one. This result is also shown to be optimal: any set of Hausdorff dimension smaller than one fails to be a UDS. See also [3; 13] for further investigation of the universal differentiability property in infinite-dimensional spaces and for mappings with higher-dimensional codomain.

The Minkowski dimension is a much finer tool for distinguishing between small sets than the Hausdorff dimension. Indeed, $\dim_H(A) \le \underline{\dim}_M(A) \le \overline{\dim}_M(A)$ for any set A. Sets where the lower Minkowski dimension $\underline{\dim}_M(A)$ and upper Minkowski dimension $\overline{\dim}_M(A)$ coincide with value p are said to have Minkowski dimension p. Universal differentiability sets constructed in [11] are dense in $\mathbb{R}^d$ and have Hausdorff dimension 1 and Minkowski dimension d. In [4; 3], the UDS are compact but still have Hausdorff dimension 1 and Minkowski dimension d. The methods employed in [11; 4; 3] fail to achieve a breakthrough on Minkowski dimension. This motivates a new way to construct fractal sets developed in the present paper.

In addition to uncovering universal differentiability sets, which are, in the sense of the Minkowski dimension, smaller than all previous known examples, we also establish a new restriction on the minimal possible size of UDS. We prove in Theorem 2.2 and Corollary 2.3 that any universal differentiability set must have infinite one-dimensional Hausdorff measure.

THEOREM (Theorem 2.2). *Let $S \subseteq \mathbb{R}^d$, where $d \ge 2$, be an $\mathcal{H}^1$-measurable set of finite one-dimensional Hausdorff measure*

$$\mathcal{H}^1(S) = \liminf_{\varepsilon \to 0+} \left\{ \sum \operatorname{diam}(S_i) : S \subseteq \bigcup_{i=1}^{\infty} S_i \text{ and } \operatorname{diam}(S_i) \le \varepsilon \right\}.$$

Then S is a nonuniversal differentiability set.

This indicates that our main result is optimal in the following sense: Denoting, for a set $S \subseteq \mathbb{R}^d$, the minimal number of ε-cubes (defined in (4.1)) needed to cover S by $N_\varepsilon(S)$, the universal differentiability set U that we construct satisfies $\limsup_{\varepsilon \to 0} N_\varepsilon(U)\varepsilon^p = 0$ whenever $p > 1$. In contrast, any universal differentiability set E must satisfy $\liminf_{\varepsilon \to 0} N_\varepsilon(E)\varepsilon = \infty$.

This naturally leads to a question: describe all exact dimension functions $f(x)$, asymptotically (as $x \to 0$) between x and x^p for all $p > 1$, that necessarily determine a nonuniversal differentiability set. See [10] for more information on exact dimension functions.

The Idea of Construction

To get a universal differentiability set of Minkowski dimension one, it is necessary to control the number N_δ as $\delta \to 0$. The set we construct will be defined by an inductive procedure. The final set is the intersection of the sets described in the nth step, over all $n \geq 1$.

Let us explain first how to get the lower Minkowski dimension, in other words, to only control N_δ for a specific sequence $\delta = \delta_n \searrow 0$. Assume that $p > 1$ is a fixed number and we want to make sure that the set to be constructed has lower Minkowski dimension less than p. Imagine that we have reached the nth step of the construction where we require $N_{\delta_n} \delta_n^p < 1$. The idea for the next step is to divide each δ_n-cube by a $K_n \times \cdots \times K_n$ grid into smaller $\delta_{n+1} = \delta_n / K_n$-cubes. If K_n is big enough, then since $\delta_n^p / \delta_{n+1}^p = K_n^p$, we are free to choose inside the given δ_n-cube any number of δ_{n+1}-cubes up to K_n^p. We then have that the product $N_{\delta_{n+1}} \delta_{n+1}^p$ is bounded by 1 from above as well. Since this is satisfied for all n, we conclude that $\underline{\dim}_M(S) \leq p$. Since this is true for every $p > 1$, we obtain a set of lower Minkowski dimension 1.

Getting the inequality for the upper dimension $\overline{\dim}_M(S) \leq 1$ is more intricate. As n grows, the sequence K_n must tend to infinity. Otherwise, we would get many points of porosity inside S (see below for the definition and discussion of porosity). In order to prove that $\overline{\dim}_M(S) \leq p$, we should be able to show that there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, the set S can be covered by a controlled number N_δ of δ-cubes. In other words, $N_\delta \delta^p$ should stay bounded for all δ below a certain threshold. Choosing n such that $\delta_{n+1} < \delta \leq \delta_n$ gives $N_\delta \delta^p \leq N_{\delta_{n+1}} \delta_n^p = N_{\delta_{n+1}} \delta_{n+1}^p K_n^p$, and the factor $K_n^p \to \infty$ makes it impossible to have a constant upper estimate for $N_\delta \delta^p$. The idea here is that we need to leave a "gap" for an unbounded sequence in the upper estimate for $N_{\delta_{n+1}} \delta_{n+1}^p$ and to make sure that K_n^p fits inside that gap. The realization of that gap is inequality (4.18).

The success of the construction of course depends on being able to show that the set obtained has universal differentiability property. This is achieved by making sure that the set satisfies the "layering property", proved to be sufficient for universal differentiability in Section 3. The latter requires that the set is closed and that arbitrarily close to each of its points the set contains, at all scales below a certain threshold, line segments of length proportional to their distance to the point. We achieve this by carefully positioning such line segments at each step of the construction and defining the final set as an intersection of closed tubular neighborhoods of these. The difficulty of the construction arises due to the necessity of including tubes of length bounded from below whilst controlling upper estimates for N_δ.

To conclude, let us briefly explain why we should be concerned about porosity points. A set W is called porous if there is $\lambda \in (0, 1)$ such that for all $x \in W$ and $\varepsilon > 0$, there is y such that $0 < \|y\| < \varepsilon$ and $B(x + y, \lambda\|y\|) \cap W = \varnothing$. If W is porous, then the distance to W, $f(\cdot) = \operatorname{dist}(\cdot, W)$, is a 1-Lipschitz function not differentiable at every $x \in W$. Since our aim is to construct a universal differentiability set, we try to avoid as much as possible constructions that lead to a set with many porosity points. More information about porous and σ-porous sets (countable unions of porous sets) can be found in the survey [15], and a further discussion of relations between problems about differentiability of Lipschitz functions and the theory of porous and σ-porous sets is presented in the recent book [8].

Structure of the Paper

We begin, in Section 2, by proving Theorem 2.2, which implies that any universal differentiability set in $\mathbb{R}^d$, with $d \geq 2$, has infinite one-dimensional Hausdorff measure. From this we show that our next main result, the existence of a universal differentiability set with Minkowski dimension one, is optimal in many respects. In Section 3, we establish a sufficient condition for the universal differentiability property, which we use later in Section 5. Section 4 is devoted to the construction of a family of nested closed sets of Minkowski dimension one for which we later verify the universal differentiability property. Finally, in Section 5, we apply the result established in Section 3 to describe a compact universal differentiability set of Minkowski dimension 1 and obtain our main result, Theorem 5.6, which guarantees the existence of a universal differentiability set S of Minkowski dimension 1 in which any Lipschitz function is differentiable on a dense subset. We moreover obtain the following quantitative estimate on the set S we construct:

THEOREM (Theorem 5.6(2)). *For any pair of integer sequences s_k, P_k satisfying $s_k, P_k \to \infty$, a universal differentiability set $S \subseteq \mathbb{R}^d$, satisfying Theorem 5.6(1), can be constructed so that for each $n \geq 1$, the set S may be covered by $\frac{1}{\delta}\prod_{k=1}^{n} s_k^{P_k}$ boxes with side $\delta = Q^{-(s_1+\cdots+s_n)}$, where $Q \in (1,2]$ is fixed.*

2. Optimality

We begin by defining the key notion of Lipschitz condition and of differentiability of a real-valued function f on a Banach space X.

A function $f : X \to \mathbb{R}$ is said to be *Fréchet differentiable* at a point $x \in X$ if the limit

$$f'(x,e) = \lim_{t\to 0} \frac{f(x+te) - f(x)}{t}$$

exists uniformly in $e \in \overline{B}(0,1)$ and is a bounded linear map.

A function $f : X \to \mathbb{R}$ is called *Lipschitz* if there exists $L > 0$ such that $|f(y) - f(x)| \leq L\|y - x\|_X$ for all $x, y \in X$ with $y \neq x$. If $f : X \to \mathbb{R}$ is a Lipschitz function, then the number

$$\operatorname{Lip}(f) = \sup\left\{\frac{|f(y) - f(x)|}{\|y - x\|_X} : x, y \in X, y \neq x\right\}$$

is finite and is called the Lipschitz constant of f.

An analytic set S in a separable space X is called a *universal differentiability set* if for every Lipschitz function $f : X \to \mathbb{R}$, there exists $x \in S$ such that f is Fréchet differentiable at x. A set $S \subseteq X$ for which one can find a Lipschitz function not Fréchet differentiable at any $x \in S$ is referred to as a nonuniversal differentiability set.

Before turning our attention to verifying the existence of a compact universal differentiability set of Minkowski dimension one in $\mathbb{R}^d$, let us first demonstrate that, in many ways, this result is the best possible.

Firstly, we emphasise that there are no universal differentiability sets that have Minkowski dimension, or even Hausdorff dimension, smaller than one; from [4, Lemma 1.2] we have that any universal differentiability set $S \subseteq \mathbb{R}^d$ satisfies $\underline{\dim}_M(S) \geq \dim_H(S) \geq 1$.

Supposing that there exists a universal differentiability set S with Minkowksi dimension equal to one, we have that $\limsup_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon^p = 0$ whenever $p > 1$. It is then natural to ask whether we can do better: Can we find S with $\limsup_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon = 0$ or even $\liminf_{\varepsilon\to 0} N_\varepsilon(S)\varepsilon < \infty$? In the present section, we prove that when $d \geq 2$, these stronger conditions are impossible to achieve and that any universal differentiability set in $\mathbb{R}^d$ must have infinite one-dimensional Hausdorff measure. In other words, if $f(x) = x$ is an exact dimension function (see [10]) for a set E, then E must be a nonuniversal differentiability set.

The following lemma is a general statement about universal differentiability sets.

Lemma 2.1. *If X is a Banach space and $A, B \subseteq X$ are such that A is a nonuniversal differentiability set and there is a nonzero continuous linear mapping $P : X \to \mathbb{R}$ such that the Lebesgue measure of $P(B)$ is zero, then the union $S = A \cup B$ is a nonuniversal differentiability set.*

Proof. Since A is a nonuniversal differentiability set, there exists a (nonzero) Lipschitz function $f : X \to \mathbb{R}$ that is not Fréchet differentiable at any $x \in A$.

Since $C = P(B) \subseteq \mathbb{R}$ has measure zero, there exists a G_δ set $C' \supseteq C$ of measure zero. By [7, Thm. 1][1] there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere outside C' and for every $x \in C'$,

$$g'_+(t) = \limsup_{s\to t} \frac{g(s) - g(t)}{s - t} = 1 \quad \text{and} \quad g'_-(t) = \liminf_{s\to t} \frac{g(s) - g(t)}{s - t} = -1.$$

Let $e \in X$ be such that $Pe = 1$. Define the Lipschitz function $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) = \frac{1}{2\|e\| \operatorname{Lip}(f)} f(x) + g(P(x)).$$

Note that if $x \in S$ and $P(x) \in C'$, then $\widetilde{f}'_+(x, e) - \widetilde{f}'_-(x, e) \geq 1$, where $\widetilde{f}'_\pm(x, e)$ denote directional upper/lower derivatives of $\widetilde{f}$. Thus, $\widetilde{f}$ is not Fréchet differentiable at x.

If $x \in S$ and $P(x) \notin C'$, then $x \in A$, which implies that f is not Fréchet differentiable at x. However, $P(x) \notin C'$ means that $g(P(\cdot))$ is differentiable at x, so that $\widetilde{f}$ is not Fréchet differentiable at x.

[1]Paper [7] gives a new proof of the characterisation of sets of nondifferentiability points of Lipschitz functions on $\mathbb{R}$. This characterisation was first given by Zahorski [14]. The existence of the function g follows from the proof of [14, Lemma 8].

This implies that the Lipschitz function $\widetilde{f}$ is not Fréchet differentiable at any $x \in S$, and hence S is a nonuniversal differentiability set. $\square$

THEOREM 2.2. *Let $S \subseteq \mathbb{R}^d$, where $d \geq 2$, be an $\mathcal{H}^1$-measurable set of finite one-dimensional Hausdorff measure*

$$\mathcal{H}^1(S) = \liminf_{\varepsilon \to 0+} \left\{ \sum \operatorname{diam}(S_i) \colon S \subseteq \bigcup_{i=1}^{\infty} S_i \text{ and } \operatorname{diam}(S_i) \leq \varepsilon \right\}.$$

Then S is a nonuniversal differentiability set.

Proof. Since $\mathcal{H}^1(S) < \infty$, by Federer's structure theorem [6, 3.3.13] S can be decomposed into a union $S = A' \cup B'$, where A' is $\mathcal{H}^1$-rectifiable, and B' has projection of one-dimensional Lebesgue measure zero for almost all one-dimensional subspaces of $\mathbb{R}^d$.

The fact that A' is $\mathcal{H}^1$-rectifiable means that there exists a countable collection of one-dimensional Lipschitz curves $\gamma_i : [0,1] \to \mathbb{R}^d$ such that $\mathcal{H}^1(A' \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$, where $\Gamma_i = \gamma_i([0,1])$. Note that the union of curves $A = \bigcup_{i=1}^{\infty} \Gamma_i$ is a σ-porous set (in fact, a countable union of closed porous sets) since each Γ_i is porous (and closed). By [1, Thm. 6.48] (see also [12]) we can conclude that A is a nonuniversal differentiability set.

Define now $B = B' \cup (A' \setminus A)$. Fix any line L such that the projection of B' onto L, $\operatorname{proj}_L(B')$, has one-dimensional Lebesgue measure zero. Since $\mathcal{H}^1(A' \setminus A) = 0$, we conclude that $\operatorname{proj}_L(B)$ has one-dimensional Lebesgue measure zero too.

It remains to apply Lemma 2.1 to $A, B \subseteq \mathbb{R}^d$ and $P = \operatorname{proj}_L$ and to note that $S \subseteq A \cup B$ to get that S is a nonuniversal differentiability set. $\square$

COROLLARY 2.3. *Let $S \subseteq \mathbb{R}^d$, where $d \geq 2$, be a universal differentiability set.*

Then $\liminf_{\varepsilon \to 0} N_\varepsilon(S)\varepsilon = \infty$, where $N_\varepsilon(S)$ is defined according to Definition 4.1.

Proof. We see that $\liminf_{\varepsilon \to 0} N_\varepsilon(S)\varepsilon \geq \frac{1}{2}\mathcal{H}^1(S)$, and the latter must be infinite for a universal differentiability set by Theorem 2.2. $\square$

REMARK. The proof of Corollary 2.3 works if $N_\varepsilon(S)$ is the minimal number of Euclidean balls of radius ε needed to cover the set S. We will later switch to covering the set by ε-cubes (see Definition 4.1) that are rotated ℓ_∞-balls of radius ε and prove that for the compact universal differentiability set, we construct, $\lim_{\varepsilon \to 0} N_\varepsilon^{\text{cubes}}(S)\varepsilon^p = 0$ for every $p > 1$. Note that since

$$N_\varepsilon^{\text{Eucl. balls}} \geq N_\varepsilon^{\text{cubes}} \geq N_{\varepsilon\sqrt{d}}^{\text{Eucl. balls}},$$

we get $\liminf_{\varepsilon \to 0} N_\varepsilon^{\text{cubes}}(S)\varepsilon = \infty$ for any universal differentiability set in $\mathbb{R}^d$, $d \geq 2$. However, for the compact universal differentiability set, we construct we have $\lim_{\varepsilon \to 0} N_\varepsilon^{\text{Eucl. balls}}(S)\varepsilon^p = 0$ for every $p > 1$.

3. Differentiability

In this section, we prove a sufficient condition for a set to have the universal differentiability property. Proposition 3.2 is based on [3, Lemma 3.5] and says that the universal differentiability property is satisfied for all sets S that can be decomposed into layers with the geometric property that every point in S can be approximated, in a special way, by line segments contained in a nearby layer of S. We prove this theorem in any Banach space with separable dual and use it for $X = \mathbb{R}^d$ in Section 5 to show that the sets (4.17) constructed in Section 4 are in fact closed universal differentiability sets of Minkowski dimension one.

Let $(M, \|\cdot\|)$ be a normed space. We call the set $\mathcal{W}_M := M^3$ of triples from M the *wedge space* of M, and we define the metric on $\mathcal{W}_M$ by

$$d(t', t) = \max_{1\le i\le 3} \|t'_i - t_i\|$$

for $t = (t_1, t_2, t_3)$ and $t' = (t'_1, t'_2, t'_3)$. Of course, the distance d depends on the norm chosen on M.

Given $t \in \mathcal{W}_M$, we call the union of segments $W(t) = [t_1, t_2] \cup [t_2, t_3]$ a *wedge*. Note that triples (t_1, t_2, t_3) and (t_3, t_2, t_1) correspond to the same wedge for any $t_1, t_2, t_3 \in M$ although the distance between them is not zero in general.

For $\alpha > 0$ and subsets $S_1, S_2 \subseteq M$, we say that S_1 is an *α-wedge approximation* for S_2 if for any $t \in \mathcal{W}_M$ with $W(t) \subseteq S_2$, there exists $t' \in \mathcal{W}_M$ with $W(t') \subseteq S_1$ and $d(t', t) \le \alpha$.

Lemma 3.1 is a restatement of [3, Lemma 3.5].

LEMMA 3.1. *Let X be a Banach space with separable dual, and $(\mathcal{W}, d) = (\mathcal{W}_X, d)$ be the wedge space equipped with the standard wedge distance. Suppose that the nested collection $(T_\lambda)_{0\le\lambda\le 1}$ of nonempty closed subsets of X satisfies the condition that for any $\eta > 0$, $\lambda \in (0, 1]$, and $x \in \bigcup_{0\le\lambda'<\lambda} T_{\lambda'}$, there is a $\delta_1 = \delta_1(\eta, \lambda, x) > 0$ such that for all $\delta \in (0, \delta_1)$ the set T_λ is an $\eta\delta$-wedge approximation for $B_\delta(x)$.*

Then, for each $\lambda \in (0, 1]$, the set T_λ is a closed universal differentiability set. Furthermore, for every Lipschitz function $g : X \to \mathbb{R}$, the set $D_{g,\lambda}$ of points $x \in T_\lambda$ where g is Fréchet differentiable is dense in T_λ. Moreover, for any $0 \le \lambda' < \lambda \le 1$, $x \in T_{\lambda'}$, $r > 0$, and any nonzero continuous linear map $P\colon X \to \mathbb{R}$, there exists a bounded open interval I containing Px such that the set $I \setminus P(D_{g,\lambda} \cap B_r(x))$ has Lebesgue measure 0.

PROPOSITION 3.2. *Let X be a Banach space with separable dual. Suppose that $(U_\lambda)_{\lambda\in[0,1]}$ is a family of closed subsets of X satisfying $U_{\lambda_1} \subseteq U_{\lambda_2}$ whenever $0 \le \lambda_1 \le \lambda_2 \le 1$. Suppose further that for any $\eta \in (0, 1)$, $\lambda \in [0, 1)$, and $\psi \in (0, 1 - \lambda)$, there exists*

$$\Delta_1 = \Delta_1(\eta, \lambda, \psi) > 0$$

such that whenever $x \in U_\lambda$, $\delta \in (0, \Delta_1)$, and v_1, v_2, v_3 are in the open unit ball in X, there exist $v'_1, v'_2, v'_3 \in X$ such that $\|v'_i - v_i\| \le \eta$ and $[x + \delta v'_1, x + \delta v'_3] \cup [x + \delta v'_3, x + \delta v'_2] \subseteq U_{\lambda+\psi}$. Then, for each $\lambda \in (0, 1]$, the set U_λ is a universal *differentiability set, and for every Lipschitz function $g : X \to \mathbb{R}$, the set $D_{g,\lambda}$ of points $x \in U_\lambda$ where g is Fréchet differentiable is dense in U_λ.*

Proof. Define $T_\lambda := U_\lambda$, a nested collection of nonempty closed subsets of X. Let $\eta > 0$, $\lambda \in (0,1]$, and $x \in T_{\lambda'} = U_{\lambda'}$ for some $\lambda' \in [0,\lambda)$ be fixed; let $\psi = \lambda - \lambda'$ and $\delta_1 = \Delta_1(\eta, \lambda, \psi)$. We show that for every $\delta \in (0, \delta_1)$, the set T_λ is an $\eta\delta$-wedge approximation of $B_\delta(x)$. Indeed, take any wedge $W(t) \subseteq B_\delta(x)$ and let $v_i = (t_i - x)/\delta$. Since $\delta \in (0, \Delta_1)$ and $\|v_i\| < 1$, there exist $v_1', v_2', v_3' \in X$ such that $\|v_i' - v_i\| \le \eta$ and $[x + \delta v_1', x + \delta v_3'] \cup [x + \delta v_3', x + \delta v_2'] \subseteq U_{\lambda'+\psi} = U_\lambda = T_\lambda$. Denoting $t_i' = x + \delta v_i'$, we get $W(t') \subseteq T_\lambda$ and $d(t', t) = \delta \sup \|v_i' - v_i\| \le \eta\delta$. Hence, Lemma 3.1 proves the statement. □

REMARK 3.3. The "moreover" property from Lemma 3.1 is also satisfied for U_λ.

4. The Set

We let $d \ge 2$ and construct a universal differentiability set of upper Minkowski dimension one in $\mathbb{R}^d$. There are many equivalent ways of defining the (upper and lower) Minkowski dimension of a bounded subset of $\mathbb{R}^d$; several examples can be found in [9, p. 41–45]. The equivalent definition given further will be most convenient for our use. We let S^{d-1} denote the unit sphere in $\mathbb{R}^d$. By an *ε-cube* with center $x \in \mathbb{R}^d$, parallel to $e \in S^{d-1}$, we mean any subset of $\mathbb{R}^d$ of the form

$$C(x, \varepsilon, e) = \left\{ x + \sum_{i=1}^{d} t_i e_i : e_1 = e, t_i \in [-\varepsilon, \varepsilon] \right\}, \tag{4.1}$$

where $e_2, \ldots, e_d \in S^{d-1}$ and $\langle e_i, e_j \rangle = 0$ whenever $1 \le i \ne j \le d$.

DEFINITION 4.1. Given a bounded subset A of $\mathbb{R}^d$ and $\varepsilon > 0$, we denote by $N_\varepsilon(A)$ the minimum number of (closed) ε-cubes required to cover A. That is, $N_\varepsilon(A)$ is the smallest integer n for which there exist ε-cubes $C_1, C_2, \ldots, C_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} C_i.$$

We define the *upper Minkowski dimension* (respectively *lower Minkowski dimension*) of A by

$$\overline{\dim}_M(A) = \inf\left\{ s > 0 \colon \limsup_{\varepsilon \to 0+} N_\varepsilon(A)\varepsilon^s = 0 \right\}$$

$$\left(\text{respectively } \underline{\dim}_M(A) = \inf\left\{ s > 0 \colon \liminf_{\varepsilon \to 0+} N_\varepsilon(A)\varepsilon^s = 0 \right\} \right).$$

For a point $x \in \mathbb{R}^d$ and $w > 0$, we shall write $\overline{B}_w(x)$ for the closed ball with center x and radius w with respect to the Euclidean norm. For a bounded subset V of $\mathbb{R}^d$, we let $\overline{B}_w(V) = \bigcup_{x \in V} \overline{B}_w(x)$. The cardinality of a finite set F is denoted by $|F|$. Given a real number α, we write $[\alpha]$ for the integer part of α.

Fix two sequences of positive integers (s_k) and (M_k) such that the following conditions are satisfied:

$$4 \leq M_k + 1 \leq s_k, \quad M_k, s_k \to \infty, \frac{M_k \log s_k}{s_k} \to 0, \tag{4.2}$$

and there exists a sequence $\tilde{s}_k \geq s_k$ such that

$$\frac{\tilde{s}_k - \tilde{s}_{k-1}}{s_k} \to 0. \tag{4.3}$$

REMARK 4.2. Before we explain how sequences s_k and M_k satisfying (4.2) and (4.3) can be chosen, we note that in order to prove that the set U_λ as in (4.17) is a universal differentiability set, we only use that

$$s_k, M_k \to \infty \quad \text{and} \quad M_k/s_k \to 0.$$

This can be seen from the proof of Lemma 5.4. The rest of conditions in (4.2) and (4.3) are needed to prove that the Minkowski dimension of the set U_λ is equal to 1.

Note that if there exists a sequence $\tilde{s}_k \geq s_k$ such that the sequence $(\tilde{s}_k - \tilde{s}_{k-1})_{k\geq 2}$ is bounded and $s_k \to \infty$, then (4.3) is satisfied. Hence, an example of sequences (s_k), $(\tilde{s}_k)$ satisfying (4.2) and (4.3) is $\tilde{s}_k = ak + b$ with $a > 0$ and any integer sequence $s_k \to \infty$ such that $3 \leq s_k \leq \tilde{s}_k$.

We also remark that if $s_k \to \infty$ is such that

$$\frac{s_k}{s_{k+1}} \to 1, \tag{4.4}$$

then (4.3) is satisfied with $\tilde{s}_k = s_k$. Indeed, in such case, $(\tilde{s}_k - \tilde{s}_{k-1})/s_k = 1 - s_{k-1}/s_k \to 0$.

An example of an integer sequence $s_k \to \infty$ satisfying condition (4.4) is $s_k = \max\{3, [F(k)]\}$, where $F(x)$ has the form $F(x) = \sum_{\lambda\in\Lambda} a_\lambda x^\lambda$ where Λ is a finite subset of $\mathbb{R}$, $a_\lambda \in \mathbb{R}$, and both $\max\Lambda > 0$ and $a_{\max\Lambda} > 0$. Also, whenever $s_k \to \infty$ satisfies condition (4.4), the sequence $s'_k = [\log s_k]$ also satisfies this condition and tends to infinity.

Once (s_k) is defined, there is much freedom to choose (M_k). For example, we may take $M_k = \max\{3, [s_k^\alpha]\}$ with $\alpha \in (0, 1)$ or $M_k = \max\{3, [\log s_k]\}$, et cetera.

Having defined the sequences (s_k) and (M_k), we fix a number $Q \in (1, 2]$ and introduce the sequence (w_k) defined by

$$w_1 = Q^{-s_1}, \qquad w_k = Q^{-s_k} w_{k-1}, \quad k \geq 2. \tag{4.5}$$

We further fix two integer sequences (A_k) and (B_k) such that $1 \leq A_k,\ B_k \leq s_k$, $A_k, B_k \to \infty$, and $M_k/B_k \to 0$.

For each $k \geq 1$, let $\mathcal{E}_k$ be a maximal $1/A_k$-separated subset of S^{d-1}. We note the following two properties of the sets $\mathcal{E}_k$:

$$|\mathcal{E}_k| \leq A_k^{2d} \quad \text{and} \quad \forall e \in S^{d-1}\ \exists e' \in \mathcal{E}_k \quad \text{s.t. } \|e - e'\| \leq \frac{1}{A_k}. \tag{4.6}$$

DEFINITION 4.3. Given a line segment $l = x + [a, b]e \subseteq \mathbb{R}^d$ and $0 < w < \text{length}(l)/2$, define $\mathcal{F}_w(l)$ to be a finite collection of w-cubes of the form $C(x_i, w, e)$, defined by (4.1), with $x_i \in l$, such that

$$\overline{B}_w(l) \subseteq \bigcup_{C \in \mathcal{F}_w(l)} C \quad \text{and} \quad |\mathcal{F}_w(l)| < \frac{\text{length}(l)}{w}. \tag{4.7}$$

Let l_1 be a line segment in $\mathbb{R}^d$ of length greater than $2w_1$ and set $\mathcal{L}_1 = \{l_1\}$. We refer to the collection $\mathcal{L}_1$ as "the lines of level 1".

Suppose that $k \geq 2$ and that we have defined the collections $\mathcal{L}_r$ of lines of level r for integers $r = 1, 2, \ldots, k-1$. Before we describe how to construct the lines of the kth level, let us first explain roughly how these line segments take part in the construction of our final set. Our final set is defined as the intersection of countably many layers, where the kth layer can be thought of as the union of w_k-neighborhoods of the lines in the collection $\mathcal{L}_k$. To calculate the Minkowski dimension of this intersection, we compute, for each k, the number of w_k-cubes needed to cover the kth layer.

The collection $\mathcal{L}_k$ will be partitioned into exactly $M_k + 1$ classes. Having constructed the lines of class m, we construct the lines of class $m + 1$ with the intention of providing "good" approximations of wedges in a w_{k-1}-neighborhood of each line of class m, in the spirit of Section 3. The precise meaning of "good" here depends on the level k of the construction: At level k, we add line segments that provide $\alpha_k w_k$-wedge approximations, and we ensure that $\alpha_k \searrow 0$ as $k \to \infty$. In the context of Lemma 3.1, the sequence $w_k \searrow 0$ will be used to approximate δ whilst the sequence $\alpha_k \searrow 0$ corresponds to η.

Each class of line segments in the collection $\mathcal{L}_k$ will be further partitioned into categories according to the length of the lines. This will allow for the control and the calculation of the Minkowski dimension of our set. Each category consists of lines of equal length, and the length of these line segments governs the density with which they should occur: the wedge approximation property of Section 3 can be achieved if line segments of length δ occur with density proportional to δ. Thus, it is natural to group the line segments based on their lengths, and the partition of $\mathcal{L}_k$ into categories enables the efficient computation of the number of w_k-cubes needed to cover the kth layer.

We first define the collections of lines of level k, class 0, by

$$\mathcal{L}_{(k,0)} = \mathcal{L}_{k-1}. \tag{4.8}$$

We will say that all lines of level k, class 0, have the empty category.

DEFINITION 4.4. Given a bounded line segment $l \subseteq \mathbb{R}^d$, an integer $j \geq 1$ with $\text{length}(l) \geq Q^j w_k / B_k$, and a direction $e \in S^{d-1}$, we define a collection of line segments $\mathcal{R}_{l,k}(j, e)$ as follows: Let $\Phi \subseteq l$ be a maximal $Q^j w_k / B_k$-separated set and define

$$\mathcal{R}_{l,k}(j, e) = \{\phi_x : x \in \Phi\},$$

where ϕ_x is the line given by

$$\phi_x = x + [-1, 1]Q^j w_k e. \tag{4.9}$$

We note for future reference that

$$|\mathcal{R}_{l,k}(j, e)| \le \frac{2B_k \operatorname{length}(l)}{Q^j w_k}. \tag{4.10}$$

For $j \in \{1, 2, \dots, s_k\}$, we define the collection of lines of level k, class 1, category (j), by

$$\mathcal{L}_{(k,1)}^{(j)} = \bigcup_{l \in \mathcal{L}_{(k,0)}} \bigcup_{e \in \mathcal{E}_k} \mathcal{R}_{l,k}(j, e). \tag{4.11}$$

We emphasise that all the lines in $\mathcal{L}_{(k,1)}^{(j)}$ have the same length. Indeed, from Definition 4.4 we get

$$\operatorname{length}(l) = 2Q^j w_k \quad \text{for all lines } l \in \mathcal{L}_{(k,1)}^{(j)}.$$

The collection $\mathcal{L}_{(k,1)}$ is now defined by

$$\mathcal{L}_{(k,1)} = \bigcup_{1 \le j \le s_k} \mathcal{L}_{(k,1)}^{(j)}.$$

Suppose that $1 \le m < M_k$ and that we have defined the collection $\mathcal{L}_{(k,m)}$. Assume that this collection is partitioned into categories

$$\mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}$$

where the j_i are integers satisfying

$$1 \le j_{i+1} \le j_i \le s_k \quad \text{for all } i. \tag{4.12}$$

For an integer sequence $(j_1, \dots, j_m, j_{m+1})$ satisfying (4.12), we define the collection of lines of level k, class $(m+1)$, category $(j_1, \dots, j_{m+1})$, by

$$\mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})} = \bigcup_{l \in \mathcal{L}_{(k,m)}^{(j_1,\dots,j_m)}} \Bigg(\bigcup_{e \in \mathcal{E}_k} \mathcal{R}_{l,k}(j_{m+1}, e) \Bigg), \tag{4.13}$$

and we set

$$\mathcal{L}_{(k,m)} = \bigcup_{(j_1,\dots,j_{m+1})} \mathcal{L}_{(k,m+1)}^{(j_1,\dots,j_{m+1})}.$$

Finally, the collection of lines of level k is defined by

$$\mathcal{L}_k = \bigcup_{0 \le m \le M_k} \mathcal{L}_{(k,m)}.$$

This completes the construction of lines of all levels, classes, and categories.

For $k \ge 1$, we let

$$\mathcal{C}_k = \bigcup_{l \in \mathcal{L}_k} \mathcal{F}_{w_k}(l).$$

It is immediate from Definition 4.3 that $\mathcal{C}_k$ is a cover of the set

$$\bigcup_{l\in\mathcal{L}_k} \overline{B}_{w_k}(l).$$

LEMMA 4.5. *For all $k \geq 2$, $0 \leq m \leq M_k$, and $(j_1, \ldots, j_m)$, the collection $\mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}$ satisfies*

$$\sum_{l\in\mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}} \mathrm{length}(l) \leq 2|\mathcal{C}_{k-1}|(4s_k|\mathcal{E}_k|)^m Q^{s_k} w_k.$$

Proof. Using the definition of the collections $\mathcal{L}_{(k,0)}$ and $\mathcal{C}_{k-1}$, we may conclude that every line $l \in \mathcal{L}_{(k,0)}$ is covered by cubes in $\mathcal{F}_{w_{k-1}}(l) \subseteq \mathcal{C}_{k-1}$. Each cube in $\mathcal{F}_{w_{k-1}}(l)$ intersects l in a line segment of length at most $2w_{k-1}$. Therefore, we have that

$$\sum_{l\in\mathcal{L}_{(k,0)}} \mathrm{length}(l) \leq 2|\mathcal{C}_{k-1}|w_{k-1} = 2|\mathcal{C}_{k-1}|Q^{s_k} w_k.$$

Hence, the statement of the lemma holds for $m = 0$.

Suppose that $m \geq 0$ and that the statement of the lemma holds for m. Using (4.13) and (4.10), we deduce

$$\begin{aligned} |\mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_{m+1})}| &\leq 2B_k|\mathcal{E}_k|w_k^{-1}Q^{-j_{m+1}} \sum_{l\in\mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}} \mathrm{length}(l) \\ &\leq |\mathcal{C}_{k-1}|(4B_k|\mathcal{E}_k|)^{m+1} Q^{s_k-j_{m+1}}. \end{aligned} \tag{4.14}$$

Each line in the collection $\mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_{m+1})}$ has length $2Q^{j_{m+1}}w_k$. Therefore, the total length may be estimated by

$$\begin{aligned} \sum_{l\in\mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_{m+1})}} \mathrm{length}(l) &\leq |\mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_{m+1})}|2Q^{j_{m+1}} \\ &\leq 2|\mathcal{C}_{k-1}|(4B_k|\mathcal{E}_k|)^{m+1} Q^{s_k} w_k. \end{aligned}$$

□

In the next lemma, we establish an upper bound on the size of the collection $\mathcal{C}_k$.

LEMMA 4.6. *For each $k \geq 2$, the collection $\mathcal{C}_k$ satisfies*

$$|\mathcal{C}_k| \leq 2(M_k+1)|\mathcal{C}_{k-1}|(4s_k A_k^{2d} B_k)^{M_k} Q^{s_k}. \tag{4.15}$$

Proof. Using (4.7) and Lemma 4.5, we may write

$$\begin{aligned} |\mathcal{C}_k| &\leq \sum_{0\leq m\leq M_k} \sum_{(j_1,\ldots,j_m)} \sum_{l\in\mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}} |\mathcal{F}_{w_k}(l)| \\ &\leq \frac{1}{w_k} \sum_{0\leq m\leq M_k} \sum_{(j_1,\ldots,j_m)} \left(\sum_{l\in\mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}} \mathrm{length}(l) \right) \end{aligned}$$

$$\leq \frac{1}{w_k} \sum_{0 \leq m \leq M_k} \sum_{(j_1,\ldots,j_m)} (2|\mathcal{C}_{k-1}|(4B_k|\mathcal{E}_k|)^m Q^{s_k} w_k)$$
$$\leq 2(M_k+1)|\mathcal{C}_{k-1}|(4s_k B_k|\mathcal{E}_k|)^{M_k} Q^{s_k}.$$

It remains to apply (4.6) to get the final estimate. □

REMARK 4.7. Since $A_k, B_k \leq s_k$ and $4 \leq M_k + 1 \leq s_k$, we have that

$$\frac{|\mathcal{C}_k|}{|\mathcal{C}_{k-1}|} \leq s_k^{(3+2d)M_k+2} Q^{s_k}. \tag{4.16}$$

In fact, Lemma 4.6 proves a much better estimate since A_k, B_k may be chosen to grow substantially slower than s_k.

We now define a collection of closed sets $(U_\lambda)_{\lambda\in[0,1]}$. Eventually, we will show that each U_λ with $\lambda \in (0, 1]$ is a compact universal differentiability set of Minkowski dimension one.

DEFINITION 4.8. For $\lambda \in [0, 1]$, we let

$$U_\lambda = \bigcap_{k=1}^{\infty} \left(\bigcup_{0\leq m_k\leq\lambda M_k} \left(\bigcup_{l\in\mathcal{L}_{(k,m_k)}} \overline{B}_{\lambda w_k}(l) \right)\right). \tag{4.17}$$

We emphasise that the single line segment l_1 of level 1 is contained in the set U_λ for every $\lambda \in [0, 1]$. Hence, every U_λ is nonempty. Note also that $U_{\lambda_1} \subseteq U_{\lambda_2}$ whenever $0 \leq \lambda_1 \leq \lambda_2 \leq 1$. Finally, since the unions in (4.17) are finite, it is clear that for each $0 \leq \lambda \leq 1$, the set U_λ is closed.

LEMMA 4.9. *For $\lambda \in [0, 1]$, the set U_λ has Minkowski dimension one.*

Proof. For any $\lambda \in [0, 1]$, we have that U_λ contains a line segment. Hence, each of the sets U_λ has lower Minkowski dimension at least one. We also have $U_\lambda \subseteq U_1$ for all $\lambda \in [0, 1]$. Therefore, to complete the proof, it suffices to show that the set U_1 has upper Minkowski dimension one.

To show $\overline{\dim}_M(U_1) \leq 1$, it suffices to argue that $\overline{\dim}_M(U_1) \leq p$ for all $p > 1$. Fix an arbitrary $p \in (1, 2)$.

Using Definition 4.8, we see that, for each $k \geq 1$,

$$U_1 \subseteq \bigcup_{l\in\mathcal{L}_k} \overline{B}_{w_k}(l),$$

whilst the latter set is covered by the cubes in the collection $\mathcal{C}_k$. Therefore, we have that $\mathcal{C}_k$ is also a cover of U_1. By Definition 4.1 this means

$$N_{w_k}(U_1) \leq |\mathcal{C}_k| \quad \text{for all } k \geq 1.$$

We claim that the sequence $|\mathcal{C}_k| {w_k}^p Q^{ps_k}$ is bounded, that is, there exists $H > 0$ such that

$$|\mathcal{C}_k| {w_k}^p Q^{ps_k} \leq H \quad \forall k \geq 1. \tag{4.18}$$

Assume that the claim is valid. Fix an arbitrary $w \in (0, w_1)$. There exists an integer $k \geq 1$ such that $w_{k+1} \leq w < w_k$. This implies $N_w(U_1) \leq N_{w_{k+1}}(U_1)$, so that

$$N_w(U_1)w^p \leq N_{w_{k+1}}(U_1)w_k^p = N_{w_{k+1}}(U_1)w_{k+1}^p Q^{ps_{k+1}} \leq H. \tag{4.19}$$

Hence, the sequence $N_w(U_1)w^p$ is uniformly bounded from above by a fixed constant H. Since this is true for any arbitrarily small $w \in (0, w_1)$, we conclude that $\overline{\dim}_M(U_1) \leq p$.

It only remains to establish the claim (4.18). We prove a more general statement, namely, that the sequence $|\mathcal{C}_k|w_k^p Q^{p\tilde{s}_k}$ tends to zero for any sequence $\tilde{s}_k \geq s_k$ satisfying condition (4.3).

Indeed, using (4.16), we obtain

$$\begin{aligned}\frac{|\mathcal{C}_k|w_k^p Q^{p\tilde{s}_k}}{|\mathcal{C}_{k-1}|w_{k-1}^p Q^{p\tilde{s}_{k-1}}} &\leq s_k^{(3+2d)M_k+2} Q^{-(p-1)s_k} Q^{p(\tilde{s}_k-\tilde{s}_{k-1})} \\ &\leq Q^{(p-1)s_k/2} Q^{-(p-1)s_k} Q^{p(\tilde{s}_k-\tilde{s}_{k-1})}\end{aligned} \tag{4.20}$$

for k sufficiently large. The latter inequality follows from

$$\frac{((3+2d)M_k+2)\log s_k}{s_k} < \frac{(p-1)\log Q}{2},$$

which is true by (4.2) for k sufficiently large. We then see that the product of the three terms in (4.20) tends to zero as $k \to \infty$ since (4.3) implies that

$$p(\tilde{s}_k - \tilde{s}_{k-1}) < \frac{(p-1)s_k}{4}$$

for k sufficiently large. □

5. Main Result

The objective of this section is to prove Theorem 5.6, which guarantees, in every finite-dimensional space, the existence of a compact universal differentiability set S of Minkowski dimension one. In Section 2, we established that this result is optimal. Note that we will always assume that $d \geq 2$ since the case $d = 1$ is trivial (we can simply take $S = [0, 1]$).

We first establish several lemmas. The statements we prove typically concern a line l of level k, class m, category $(j_1, \ldots, j_m)$, where $0 \leq m \leq M_k$. When $m = 0$, we interpret the category $(j_1, \ldots, j_m)$ as the empty category and assume that $j \leq j_m$ for all integers j.

LEMMA 5.1. *Let $k \geq 2$, $0 \leq m < M_k$, and $l \in \mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}$. Let $e \in \mathcal{E}_k$ and $1 \leq j_{m+1} \leq j_m \leq s_k$. If $x \in l$, then there exists $x' \in l$ such that $\|x' - x\| \leq Q^{j_{m+1}} w_k / B_k$ and*

$$l' = x' + [-1, 1]Q^{j_{m+1}} w_k e \in \mathcal{L}_{(k,m+1)}^{(j_1,\ldots,j_m,j_{m+1})}.$$

Proof. By definition the collection $\mathcal{R}_{l,k}(j_{m+1}, e)$ has an element l' satisfying the conclusions of this lemma. □

LEMMA 5.2. *Let $k \geq 2$ and suppose $1 \leq m \leq M_k$. Let $x \in l \in \mathcal{L}_{(k,m)}^{(j_1,\ldots,j_m)}$, and i_m be an integer with $j_m < i_m \leq s_k$. Then there exist an integer sequence $s_k \geq i_1 \geq \cdots \geq i_{m-1} \geq i_m$ and a line $l' \in \mathcal{L}_{(k,m)}^{(i_1,\ldots,i_m)}$ such that l' is parallel to l and there exists a point $x' \in l'$ with $\|x' - x\| \leq m Q^{i_m} w_k / B_k$.*

Proof. Suppose that either

(i) $n = 1$, or
(ii) $2 \leq n \leq M_k$ and the statement of Lemma 5.2 holds for $m = 1, \ldots, n-1$.

We prove that in both cases, the statement of Lemma 5.2 holds for $m = n$. The proof will then be complete by induction.

Let the line l, integers $j_1, \ldots, j_n, i_n$, and point $x \in l$ be given by the hypothesis of Lemma 5.2 when we set $m = n$. Let $e \in \mathcal{E}_k$ be the direction of l. By (4.11) in case (i), or (4.13) in case (ii), there exists a line $l^{(n-1)}$ of level k, class $n-1$, category $(j_1, \ldots, j_{n-1})$, such that the line l belongs to the collection $\mathcal{R}_{l^{(n-1)},k}(j_n, e)$.

By Definition 4.4 the line l has the form

$$l = z + [-1, 1] Q^{j_n} w_k e,$$

where $z \in l^{(n-1)}$. Therefore, we may write

$$z = x + \beta e, \tag{5.1}$$

where

$$|\beta| \leq Q^{j_n} w_k. \tag{5.2}$$

We now distinguish between two cases. First, suppose that $i_n \leq j_{n-1}$. Note that this is certainly the case if $n = 1$. Setting $i_a = j_a$ for $a = 1, \ldots, n-1$, we get that $s_k \geq i_1 \geq \cdots \geq i_{n-2} \geq i_{n-1} \geq i_n$. The line $l^{(n-1)} \in \mathcal{L}_{(k,n-1)}^{(i_1,\ldots,i_{n-1})}$, the direction $e \in \mathcal{E}_k$, the integer i_n, and the point $z \in l^{(n-1)}$ now satisfy the conditions of Lemma 5.1. Hence, there is a line l' of level k, class n, category $(i_1, \ldots, i_n)$, and a point z' with

$$\|z' - z\| \leq \frac{Q^{i_n} w_k}{B_k} \tag{5.3}$$

such that the line segment l' is given by

$$l' = z' + [-1, 1] Q^{i_n} w_k e.$$

Finally, set

$$x' = z' - \beta e,$$

so that $x' \in l'$, using (5.2). We deduce, using (5.3) and (5.1), that $\|x' - x\| \leq Q^{i_n} w_k / B_k \leq n Q^{i_n} w_k / B_k$. This completes the proof for the case $i_n \leq j_{n-1}$.

Now suppose that $i_n > j_{n-1}$. In this situation, we must be in case (ii). We set $i_{n-1} = i_n > j_{n-1}$. The conditions of Lemma 5.2 are now readily verified for $z \in l^{(n-1)} \in \mathcal{L}_{(k,n-1)}^{(j_1,\ldots,j_{n-1})}$ and the integer i_{n-1}. Therefore, by (ii) and Lemma 5.2 there exist an integer sequence $s_k \geq i_1 \geq \cdots \geq i_{n-2} \geq i_{n-1}$ and a line $l'' \in \mathcal{L}_{(k,n-1)}^{(i_1,\dots,i_{n-1})}$ such that l'' is parallel to $l^{(n-1)}$, and there exists a point $y'' \in l''$ such that

$$\|y'' - z\| \leq \frac{(n-1)Q^{i_{n-1}} w_k}{B_k}. \tag{5.4}$$

The conditions of Lemma 5.1 are now readily verified for the line $l'' \in \mathcal{L}_{(k,n-1)}^{(i_1,\dots,i_{n-1})}$, the direction $e \in \mathcal{E}_k$, the integer i_n, and the point $y'' \in l''$. Hence, there exist a line $l' \in \mathcal{L}_{(k,n)}^{(i_1,\dots,i_n)}$ and a point $y' \in l'$ such that

$$\|y' - y''\| \leq \frac{Q^{i_n} w_k}{B_k}, \tag{5.5}$$

and the line l' is given by

$$l' = y' + [-1, 1]Q^{i_n} w_k e.$$

We set

$$x' = y' - \beta e.$$

Using (5.2) and $i_n > j_n$, we get that $x' \in l'$. Moreover, using (5.1), (5.4), and (5.5), we obtain $\|x' - x\| \leq n Q^{i_n} w_k / B_k$. □

LEMMA 5.3. *Let $\lambda \in [0, 1)$, $\psi \in (0, 1 - \lambda)$, and suppose that $x \in U_\lambda$. Suppose that the integers $n \geq 1$, $t \in \{0, 1, \dots, s_n - 1\}$, and a number $\delta > 0$ satisfy*

$$\psi Q^{t-1} w_n < \delta \leq \psi Q^t w_n \quad \textit{and} \quad \frac{Q}{B_n} < \psi. \tag{5.6}$$

Let $f \in \mathcal{E}_n$ and suppose that $y \in l \in \mathcal{L}_{(n,r)}^{(h_1,\dots,h_r)}$, where

$$r \leq (\lambda + \psi)M_n - 2, \quad h_r = t + 1. \tag{5.7}$$

Then there exist a line $l' \in \mathcal{L}_{(n,1+r)}^{(h_1,\dots,h_r,t+1)}$ and a point $y' \in l$ such that

$$\|y' - y\| \leq \frac{Q^2}{\psi B_n}\delta, \tag{5.8}$$

$$l' = y' + [-1, 1]Q^{t+1} w_n f, \tag{5.9}$$

and $y' + [-1, 1]\tau f \subseteq U_{\lambda+\psi} \cap l'$ whenever

$$0 \leq \tau \leq \left(Q - \frac{Q^2}{\psi B_n}\right)\delta - \|y - x\|. \tag{5.10}$$

Proof. Choose a sequence of integers $(m_k)_{k\geq 1}$ with $0 \leq m_k \leq \lambda M_k$ and a sequence $(l_k)_{k\geq 1}$ of line segments such that $l_k \in \mathcal{L}_{(k,m_k)}$ is a line of level k, class m_k, and

$$x \in \bigcap_{k=1}^{\infty} \overline{B}_{\lambda w_k}(l_k). \tag{5.11}$$

Note that $Q\delta < \psi w_n Q^{t+1} \le \psi w_n Q^{s_n} = \psi w_{n-1} \le \psi w_k$ for all $k \le n-1$. This, together with (5.11), implies that

$$\overline{B}_{Q\delta}(x) \subseteq \overline{B}_{(\lambda+\psi)w_k}(l_k) \quad \text{for } 1 \le k \le n-1. \tag{5.12}$$

Now, the line $l \in \mathcal{L}_{(n,r)}^{(h_1,\dots,h_r)}$, the direction $f \in \mathcal{E}_n$, the integer $t+1$, and the point $y \in l$ satisfy the conditions of Lemma 5.1. Therefore, there exist a line l' of level n, class $1+r$, category $(h_1,\dots,h_r,t+1)$, and a point $y' \in l'$ such that (5.9) holds and

$$\|y'-y\| \le \frac{Q^{t+1}w_n}{B_n} = \frac{Q^2}{\psi B_n}\psi Q^{t-1}w_n \le \frac{Q^2}{\psi B_n}\delta. \tag{5.13}$$

Recall that l' is a line of level n. Hence, from (4.8) we have that l' is a line of level k, class 0, for all $k \ge n+1$. We now set

$$l'_k = l' \quad \text{for all } k \ge n \quad \text{and} \quad l'_k = l_k \quad \text{for } 1 \le k \le n-1. \tag{5.14}$$

Then for each $k \ge 1$, we have that l'_k is a line of level k, class m'_k, where

$$m'_k = \begin{cases} m_k & \text{if } 1 \le k \le n-1, \\ 1+r & \text{if } k = n, \\ 0 & \text{if } k \ge n+1. \end{cases}$$

From $m_k \le \lambda M_k$ and (5.7) we have that $0 \le m'_k \le (\lambda+\psi)M_k$ for all k. Hence, by Definition 4.8,

$$\bigcap_{k=1}^{\infty} \overline{B}_{(\lambda+\psi)w_k}(l'_k) \subseteq U_{\lambda+\psi}. \tag{5.15}$$

Suppose τ is a real number satisfying (5.10) (note that by (5.6) we have that $\psi - Q/B_n$ is nonnegative). Since $\psi < 1$,

$$0 \le \tau \le Q^{t+1}\left(\psi - \frac{Q}{B_n}\right)w_n \le Q^{t+1}w_n.$$

Hence, $y' + [-1,1]\tau f \subseteq l'$ by (5.9).

From (5.12), (5.13), and (5.10) we have that, for all $1 \le k \le n-1$,

$$y' + [-1,1]\tau f \subseteq l' \cap \overline{B}_{Q\delta}(x) \subseteq l' \cap \overline{B}_{(\lambda+\psi)w_k}(l_k).$$

Putting this together with (5.14) and (5.15), we conclude that

$$y' + [-1,1]\tau f \subseteq U_{\lambda+\psi} \cap l'$$

since $l' \subseteq \overline{B}_{(\lambda+\psi)w_k}(l') = \overline{B}_{(\lambda+\psi)w_k}(l_k)$ for all $k \ge n$. □

The next lemma represents the crucial step toward our main result, Theorem 5.6.

LEMMA 5.4. *Let* $\lambda \in (0,1)$, $\psi \in (0, 1-\lambda)$, *and* $\eta \in (0,1/2)$. *Then there exists a real number*

$$\delta_0 = \delta_0(\lambda,\psi,\eta) > 0 \tag{5.16}$$

such that for any $x \in U_\lambda$, $e \in S^{d-1}$, *and* $\delta \in (0,\delta_0)$, *there exist* $e' \in S^{d-1}$, *integers* $n \ge 1$, $t \in \{0,1,\dots,s_n-1\}$, *and a pair* (x', l'), *consisting of a point and a straight line segment, with* $x' \in l' \in \mathcal{L}_{(n,r)}^{(h_1,\dots,h_r)}$, *satisfying the following properties*:

(i) *Condition* (5.6) *of Lemma* 5.3 *is satisfied*;

(ii) *The condition*

$$r \leq (\lambda + \psi)M_n - 4, \qquad h_r = t + 1 \tag{5.17}$$

is satisfied (*a stronger version of* (5.7));

(iii) $\|x' - x\| \leq \eta\delta$, $\|e' - e\| \leq \eta$, *and*

$$x' + [-1, 1]\delta e' \subseteq U_{\lambda+\psi} \cap l'. \tag{5.18}$$

Moreover, δ_0 *can be chosen to be independent of* $Q \in (1, 2]$.

Proof. We will find $\delta_0' = \delta_0'(\lambda, \psi, \eta)$ such that for any $x \in U_\lambda$, $e \in S^{d-1}$, and $\delta \in (0, \delta_0')$, conclusions (i), (ii), and (iii) of Lemma 5.4 are valid when (5.18) is replaced by the weaker statement

$$x' + [-1, 1]\frac{\delta}{2}e' \subseteq U_{\lambda+\psi} \cap l'. \tag{5.19}$$

Then, defining $\delta_0 = \frac{1}{2}\delta_0'(\lambda, \psi, \eta/2)$, we will get that the conclusion of this lemma, including (5.18), is satisfied.

Since $(w_k)_{k\geq 1}$ is strictly decreasing, and the sequences (A_k), (B_k), and (M_k) satisfy $A_k, B_k, M_k \to \infty$, $M_k/B_k \to 0$, we may choose $\delta_0' \in (0, \frac{\psi}{2}w_1)$ small enough so that whenever $\psi w_k \leq 2\delta_0'$, we have

$$\frac{1}{A_k} \leq \frac{\eta\psi}{8}, \qquad \frac{1}{B_k} \leq \frac{\eta\psi}{8(M_k + 3)}, \qquad \psi M_k \geq 6.$$

Since $Q \in (1, 2]$, this implies that whenever $\psi w_k \leq Q\delta_0'$, we have

$$\frac{1}{A_k} \leq \frac{\eta\psi}{2Q^2}, \qquad \frac{1}{B_k} \leq \frac{\eta\psi}{2Q^2(M_k + 3)}, \qquad \psi M_k \geq 6. \tag{5.20}$$

Let $x \in U_\lambda$ and fix $\delta \in (0, \delta_0')$. Choose a sequence of integers $(m_k)_{k\geq 1}$ with $0 \leq m_k \leq \lambda M_k$ and a sequence $(l_k)_{k\geq 1}$ of line segments such that $l_k \in \mathcal{L}_{(k,m_k)}$ is a line of level k, class m_k, and

$$x \in \bigcap_{k=1}^{\infty} \overline{B}_{\lambda w_k}(l_k).$$

Note that $Q\delta < \psi w_1$ since $Q \leq 2$. Since $w_k \to 0$, there is a unique natural number $n \geq 2$ satisfying

$$\psi w_n \leq Q\delta < \psi w_{n-1}. \tag{5.21}$$

We remark for further reference that from (5.20), $\delta \in (0, \delta_0')$, and (5.21) we can deduce that

$$\frac{Q^2}{\psi A_n}\delta \leq \frac{\eta}{2}\delta \leq \frac{\delta}{4} \tag{5.22}$$

and

$$(M_n + 3)\frac{Q^2}{\psi B_n}\delta \leq \frac{\eta}{2}\delta \leq \frac{\delta}{4}. \tag{5.23}$$

Since $w_{n-1} = Q^{s_n} w_n$, by (5.21) there exists $t \in \{0, 1, \ldots, s_n - 1\}$ satisfying

$$\psi Q^t w_n \leq Q\delta < \psi Q^{t+1} w_n.$$

Further, from (5.20), $\delta \in (0, \delta_0)$, and (5.21) we have that $Q/s_n \leq \psi$. Hence, δ, n, and t satisfy (5.6). By (4.6) there exists a direction $e' \in \mathcal{E}_n$ such that $\|e' - e\| \leq 1/A_n$, whilst $1/A_n \leq \eta$ follows from (5.20), $\delta \in (0, \delta_0')$, and (5.21). Hence, we have $\|e' - e\| \leq \eta$, as required.

Note that $\overline{B}_{\lambda w_n}(l_n)$ is a tube of level n, class m_n, containing the point x. Let the line l_n have category $(j_1, \dots, j_{m_n})$. We can write $x = z + \alpha g$ where $z \in l_n$, $g \in S^{d-1}$, and $\alpha \in [0, \lambda w_n]$. Next, using (4.6), pick $g' \in \mathcal{E}_n$ such that $\|g' - g\| \leq 1/A_n$. Apply now Lemma 5.1 to $z \in l_n$ to find a line

$$l''' = z' + [-1, 1]Q w_n g' \in \mathcal{L}_{(n, 1+m_n)}^{(j_1, \dots, j_{m_n}, 1)},$$

where $z' \in l_n$ and $\|z' - z\| \leq Q w_n / B_n$. Let $x''' = z' + \alpha g'$; then, using (5.21), we have

$$\begin{aligned} \|x''' - x\| &\leq \|z' - z\| + \alpha \|g' - g\| \\ &\leq Q w_n \left(\frac{1}{A_n} + \frac{1}{B_n} \right) \\ &\leq \frac{Q^2}{\psi} \left(\frac{1}{A_n} + \frac{1}{B_n} \right) \delta. \end{aligned} \tag{5.24}$$

From (5.20), $\delta \in (0, \delta_0')$, and (5.21) we have $\psi M_n \geq 6$. In particular,

$$m_n + 2 \leq \lambda M_n + 2 \leq (\lambda + \psi) M_n - 4,$$

and (5.17) is satisfied when $r = m_n + 2$ and $h_r = t + 1$.

We will now show that there exist a line l' of level n, class $2 + m_n$, category $(j_1, \dots, j_{1+m_n}, t+1)$, and a point $x' \in l'$ such that

$$\|x' - x'''\| \leq \frac{(m_n + 2) Q^2}{\psi B_n} \delta \quad \text{and} \quad x' + [-1, 1] \frac{\delta}{2} e' \subseteq U_{\lambda + \psi} \cap l'. \tag{5.25}$$

Once (5.25) is established, the proof is completed by combining (5.25) and (5.24) with (5.22) and (5.23) to get

$$\|x' - x\| \leq \frac{Q^2}{\psi} \delta \left(\frac{m_n + 3}{B_n} + \frac{1}{A_n} \right) \leq \eta \delta. \tag{5.26}$$

Thus, it only remains to verify (5.25). We distinguish two cases, $t = 0$ and $t \geq 1$.

If $t = 0$, then the conditions of Lemma 5.3 are satisfied for λ, ψ, x, δ, t, n, $f = e'$, $l = l'''$, $r = 1 + m_n$, $(h_1, \dots, h_r) = (j_1, \dots, j_{1+m_n})$, and $y = x''' \in l'''$. Therefore, by Lemma 5.3 there exist a line l' of level n, class $2 + m_n$, category $(j_1, \dots, j_{1+m_n}, 1)$, and point $x' \in l'$ such that

$$\|x' - x'''\| \leq \frac{Q^2}{\psi B_n} \delta \quad \text{and} \tag{5.27}$$

$$x' + [-1, 1] \tau e' \subseteq U_{\lambda + \psi} \cap l' \tag{5.28}$$

$$\text{whenever } 0 \leq \tau \leq \left(Q - \frac{Q^2}{\psi B_n} \right) \delta - \|x''' - x\|.$$

Therefore, using (5.23) and (5.24), we get

$$\left(Q - \frac{Q^2}{\psi s_n}\right)\delta - \|x''' - x\| \geq \left(Q - \frac{1}{4}\right)\delta > \frac{\delta}{2}.$$

Hence, by (5.28) we have $x' + [-1, 1]\frac{\delta}{2}e' \subseteq U_{\lambda+\psi} \cap l'$, and we obtain (5.25).

Now assume that we are in the remaining case $t \geq 1$. Set $i_{1+m_n} = t + 1$, so that $j_{1+m_n} = 1 < i_{1+m_n} \leq s_n$. Observe that the line $l''' \in \mathcal{L}^{(j_1,\ldots,j_{1+m_n})}_{(n,1+m_n)}$, the integer $i_{1+m_n} > j_{1+m_n}$, and the point $x''' \in l'''$ satisfy the conditions of Lemma 5.2. Therefore, by Lemma 5.2 there exists an integer sequence $s_k \geq i_1 \geq \cdots \geq i_{1+m_n} \geq 1$ together with a line l'' of level n, class $1 + m_n$, category $(i_1, \ldots, i_{1+m_n})$, such that l'' is parallel to l''', and there exists a point $x'' \in l''$ with

$$\|x'' - x'''\| \leq \frac{(1 + m_n)Q^{t+1}w_n}{s_n} \leq (1 + m_n)\frac{Q^2}{\psi B_n}\delta. \tag{5.29}$$

Set $i_{2+m_n} = t + 1$, so that $i_{2+m_n} = i_{1+m_n}$. Note that the conditions of Lemma 5.3 are satisfied for λ, ψ, x, δ, t, n, $f = e'$, $l = l''$, $r = 1 + m_n$, $(h_1, \ldots, h_r) = (i_1, \ldots, i_{1+m_n})$, and $y = x'' \in l''$. Hence, by Lemma 5.3 there exist a line segment l' of level n, class $2 + m_n$, category $(i_1, \ldots, i_{1+m_n}, t + 1)$, and a point $x' \in l'$ with

$$\|x' - x''\| \leq \frac{Q^2}{\psi B_n}\delta \quad \text{and} \tag{5.30}$$

$$x' + [-1, 1]\tau e' \subseteq U_{\lambda+\psi} \cap l' \tag{5.31}$$

$$\text{whenever } 0 \leq \tau \leq \left(Q - \frac{Q^2}{\psi B_n}\right) - \|x'' - x\|.$$

We observe that

$$\|x' - x'''\| \leq \frac{(m_n + 2)Q^2}{\psi B_n}\delta,$$

using (5.30) and (5.29). Moreover, combining (5.29) with (5.24) yields

$$\|x'' - x\| \leq \frac{Q^2}{\psi}\delta\left(\frac{1}{A_n} + \frac{m_n + 2}{B_n}\right).$$

Therefore, by (5.22) and (5.23),

$$\left(Q - \frac{Q^2}{\psi B_n}\right)\delta - \|x'' - x\| \geq \left(Q - \frac{1}{2}\right)\delta \geq \frac{\delta}{2}.$$

We conclude, using (5.31), that $x' + [-1, 1]\frac{\delta}{2}e' \subseteq U_{\lambda+\psi} \cap l'$. We have now verified (5.25). □

LEMMA 5.5. *Let* $\eta \in (0, 1)$, $\lambda \in [0, 1)$, *and* $\psi \in (0, 1 - \lambda)$. *Then there exists a number*

$$\Delta_1 = \Delta_1(\eta, \lambda, \psi) > 0$$

such that whenever $x \in U_\lambda$, $\delta \in (0, \Delta_1)$, and v_1, v_2, v_3 are in the open unit ball in $\mathbb{R}^d$, there exist $v_1', v_2', v_3' \in \mathbb{R}^d$ such that

$$\|v_i' - v_i\| \leq \eta \quad \textit{and} \tag{5.32}$$

$$[x + \delta v_1', x + \delta v_3'] \cup [x + \delta v_3', x + \delta v_2'] \subseteq U_{\lambda+\psi}. \tag{5.33}$$

Moreover, Δ_1 can be chosen to be independent of $Q \in (1, 2]$.

Proof. Fix positive numbers a, b, c such that

$$a + 2b + 3c < \frac{1}{2}. \tag{5.34}$$

Using the notation of Lemma 5.4, choose $0 < \Delta_1 \leq \delta_0(\lambda, \psi, a\eta)$ such that

$$\max\left\{\frac{1}{A_k}, \frac{4}{\psi B_k}\right\} \leq b\eta \quad \text{whenever } \psi w_k < 2\Delta_1,$$

implying that

$$\max\left\{\frac{1}{A_k}, \frac{Q^2}{\psi B_k}\right\} \leq b\eta \quad \text{whenever } \psi w_k < Q\Delta_1 \tag{5.35}$$

since $Q \in (1, 2]$.

Fix $x \in U_\lambda$, $\delta \in (0, \Delta_1)$, and v_1, v_2, v_3 in the open unit ball in $\mathbb{R}^d$. We may assume that

$$0 < \|v_i\| \leq c \quad \text{for each } i = 1, 2, 3 \tag{5.36}$$

and v_1, v_2, v_3 are distinct vectors.

Set $e_1 = v_1/\|v_1\|$. Since $\delta < \delta_0(\lambda, \psi, a\eta)$, Lemma 5.4 asserts that there exist $e_1' \in S^{d-1}$, integers n, t, and $x' \in l' \in \mathcal{L}_{(n,r)}^{(h_1,\ldots,h_r)}$ such that (5.6) and (5.17) are satisfied, together with

$$\begin{aligned} &\|x' - x\| \leq a\eta\delta, \\ &\|e_1' - e_1\| \leq a\eta, \quad \text{and} \quad x' + [-1, 1]\delta e_1' \subseteq U_{\lambda+\psi} \cap l'. \end{aligned} \tag{5.37}$$

Denote $l_1 := l'$ and set

$$x_1 = x' + \delta\|v_1\|e_1' \quad \text{and} \quad e_3 = \frac{v_3 - v_1}{\|v_3 - v_1\|}. \tag{5.38}$$

Let $e_3' \in \mathcal{E}_n$ be such that $\|e_3' - e_3\| \leq 1/A_n$. Note that (5.35) implies

$$\frac{Q}{B_n} \leq \frac{Q^2}{\psi B_n} \leq b\eta$$

since by (5.6) we have $\psi w_n \leq \psi Q^t w_n < Q\delta < Q\Delta_1$. This means we can now apply Lemma 5.3 to the point $x \in U_\lambda$, integers n, t found before, δ satisfying (5.6), $f := e_3'$, and $y := x_1 \in [x', x' + \delta e_1'] \subseteq l_1 \in \mathcal{L}_{(n,r)}^{(h_1,\ldots,h_r)}$. Let the point $y' \in l_1$ and the line $l_1' \in \mathcal{L}_{(n,1+r)}^{(h_1,\ldots,h_r,h_r)}$ be given by the conclusion of Lemma 5.3.

We now define $x_1' = y'$ and note that (5.8) and (5.35) imply

$$\|x_1' - x_1\| = \|y' - x_1\| \leq b\eta\delta,$$

so that using (5.36), we get $\|x_1' - x'\| \leq (b\eta + c)\delta$.

We claim that the straight line segment $[x_1' - \frac{1}{2}\delta e_3', x_1' + \frac{1}{2}\delta e_3']$ is inside $U_{\lambda+\psi}$. Indeed, we verify that $\tau = \delta/2$ satisfies (5.10). Using

$$\|x_1 - x\| \le \|x_1 - x'\| + \|x' - x\| \le (c + a\eta)\delta \tag{5.39}$$

and $Q > 1$, together with (5.34) and $0 < \eta < 1$, we get

$$\left(Q - \frac{Q^2}{\psi B_n}\right)\delta - \|x_1 - x\| \ge (1 - b\eta)\delta - (c + a\eta)\delta > \frac{\delta}{2}.$$

Let

$$x_3 = x_1' + \delta\|v_3 - v_1\|e_3'.$$

Denote $l_3 = l_1'$ and set $e_2 = (v_2 - v_3)/\|v_2 - v_3\|$. Find $e_2' \in \mathcal{E}_n$ with $\|e_2' - e_2\| \le 1/A_n$ and apply Lemma 5.3 to the point $x \in U_\lambda$, n, t, and δ satisfying (5.6) and found earlier, $f := e_2'$, and $y := x_3 \in [x_1', x_1' + \delta e_3'] \subseteq l_3 \in \mathcal{L}_{(n,1+r)}^{(h_1,\dots,h_r,h_r)}$. We note that condition (5.7) of Lemma 5.3 is satisfied for $r + 1$ instead of r because of (5.17). Let the point $y' \in l_3$ and the line $l_2 \in \mathcal{L}_{(n,2+r)}^{(h_1,\dots,h_r,h_r,h_r)}$ be given by the conclusion of Lemma 5.3.

Let $x_3' = y'$. We now verify that $[x_3' - \frac{1}{2}\delta e_2', x_3' + \frac{1}{2}\delta e_2'] \subseteq U_{\lambda+\psi}$. We again show that $\tau = \delta/2$ satisfies (5.10). Indeed, using (5.39), we get

$$\begin{aligned}\|x_3 - x\| &\le \|x_3 - x_1'\| + \|x_1' - x_1\| + \|x_1 - x\| \\ &\le 2c\delta + b\eta\delta + (c + a\eta)\delta = (a\eta + b\eta + 3c)\delta.\end{aligned}$$

Hence, using (5.34) and $0 < \eta < 1$, we conclude

$$\left(Q - \frac{Q^2}{\psi B_n}\right)\delta - \|x_3 - x\| \ge (1 - b\eta)\delta - (a\eta + b\eta + 3c)\delta > \frac{\delta}{2}.$$

Finally, define

$$x_2' = x_3' + \|v_2 - v_3\|\delta e_2'.$$

We are now left to see that v_i', $i = 1, 2, 3$, defined according to

$$x + \delta v_i' = x_i' \iff v_i' = \frac{x_i' - x}{\delta} \tag{5.40}$$

satisfy the conclusions of Lemma 5.5.

Indeed, let us verify $[x_1', x_3'] \cup [x_3', x_2'] \subseteq U_{\lambda+\psi}$. First, we see that $x_3' \in l_3$ and, by (5.34),

$$\|x_3' - x_1'\| \le \|x_3' - x_3\| + \|x_3 - x_1'\| \le b\eta\delta + 2c\delta < \frac{\delta}{2};$$

hence, $[x_1', x_3'] \subseteq [x_1' - \frac{1}{2}\delta e_3', x_1' + \frac{1}{2}\delta e_3'] \subseteq U_{\lambda+\psi}$. For the second straight line segment, we see that $\|x_2' - x_3'\| \le 2c\delta$ and $x_2' \in l_2$, so that $[x_3', x_2'] \subseteq [x_3' - \frac{1}{2}\delta e_2', x_3' + \frac{1}{2}\delta e_2'] \subseteq U_{\lambda+\psi}$.

By (5.40) we see that (5.32) is equivalent to

$$\|(x_i' - x_i) - \delta v_i\| \le \eta\delta \quad \text{for all } i = 1, 2, 3.$$

We note first that, using (5.34),

$$\begin{aligned}\|(x_1' - x) - \delta v_1\| &\le \|(x_1 - x') - \delta v_1\| + \|x_1' - x_1\| + \|x' - x\| \\ &\le c\delta\|e_1' - e_1\| + (a + b)\eta\delta \le (a + b + ac)\eta\delta < \eta\delta.\end{aligned}$$

Next,

$$\begin{aligned}\|(x_3' - x) - \delta v_3\| &\le \|x_3' - x_3\| + \|(x_3 - x_1') - \delta(v_3 - v_1)\| + \|(x_1' - x) - \delta v_1\| \\ &\le b\eta\delta + \delta\|v_3 - v_1\|\|e_3' - e_3\| + (a + b + ac)\eta\delta \\ &\le (a + 2b + ac + 2bc)\eta\delta < \eta\delta\end{aligned}$$

using $\|v_3 - v_1\| \le 2c$ and $\|e_3' - e_3\| \le 1/A_k \le b\eta$. Finally, using the definition of x_2', we get, in a similar way,

$$\begin{aligned}\|(x_2' - x) - \delta v_2\| &= \|(x_3' - x) + \delta\|v_2 - v_3\|e_2' - \delta v_2\| \\ &\le \|(x_3' - x) + \delta(v_2 - v_3) - \delta v_2\| + \delta\|v_2 - v_3\|\|e_2' - e_2\| \\ &= \|(x_3' - x) - \delta v_3\| + \delta\|v_2 - v_3\|\|e_2' - e_2\| \\ &\le (a + 2b + ac + 4bc)\eta\delta < \eta\delta\end{aligned}$$

since $a + 2b + ac + 4bc < 2(a + 2b + 3c) < 1$. □

We are now ready to prove our main result.

THEOREM 5.6. *For every $d \ge 1$, there exists a compact subset $S \subseteq \mathbb{R}^d$ of Minkowski dimension one with the universal differentiability property. Moreover,*

(1) *this set S can be constructed in such a way that for any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, the set of points $x \in S$ such that g is Fréchet differentiable at x is a dense subset of S;*
(2) *for any pair of integer sequences s_k, P_k satisfying $s_k, P_k \to \infty$, a universal differentiability set $S \subseteq \mathbb{R}^d$ satisfying* (1) *can be constructed so that, for each $n \ge 1$, the set S may be covered by $\frac{1}{\delta}\prod_{k=1}^{n} s_k^{P_k}$ boxes with side $\delta = Q^{-(s_1+\cdots+s_n)}$, where $Q \in (1, 2]$ is fixed.*

Proof. From Lemma 5.5 we have that the family of compact sets (U_λ), $\lambda \in [0, 1]$, satisfies the conditions of Proposition 3.2, where $X = \mathbb{R}^d$. Therefore, by Proposition 3.2 the set U_λ is a universal differentiability set with property (1) for each $\lambda \in (0, 1]$. By Lemma 4.9 these sets have Minkowski dimension one.

Let us now explain how the property described in (2) can be achieved. Given such sequences s_k, P_k, we choose a sequence M_k satisfying (4.2) and

$$(3 + 2d)M_k + 2 \le P_k \quad \forall k \ge 1.$$

Using (4.16), we see that, for each $\lambda \in (0, 1]$, the universal differentiability sets U_λ constructed in Section 4, with the sequences s_k and M_k, possess property (2). □

REMARK (added in proof). While the present paper was being prepared for publication, the first named author obtained a result [5] that shows that any universal differentiability set S contains a relatively closed subset $\ker(S)$ such that $\ker(S)$ is also a UDS and every Lipschitz function is differentiable on a dense subset of $\ker(S)$. This means that if S is a compact universal differentiability set of Minkowski dimension 1, then $\ker(S)$ is also a compact universal differentiability set. The Minkowski dimension of $\ker(S)$ must then be equal to 1 by [3, Lemma 2.1]. This provides an alternative way to deduce part (1) of Theorem 5.6 from the very fact that S is a UDS of Minkowski dimension 1.

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