A Construction of Slice Knots via Annulus Twists

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Abstract. We give a new construction of slice knots via annulus twists. The simplest slice knots obtained by our method are those constructed by Omae. In this paper, we introduce a sufficient condition for given slice knots to be ribbon and prove that all Omae's knots are ribbon.

1. Introduction

The annulus twist is a certain operation on knots along an annulus embedded in the 3-sphere S^3. Osoinach [Os] found that this operation is useful in the study of 3-manifolds. Using annulus twists, he gave the first example of a 3-manifold admitting infinitely many presentations by 0-framed knots. For more studies, see [AJOT; AJLO; BGL; K; Tak; Te; Om].

Recently, the first author, Jong, Omae, and Takeuchi [AJOT] constructed knots related to the slice-ribbon conjecture: Let $K \subset S^3$ be a slice knot admitting an annulus presentation (for the definition, see Section 2), and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. They proved that K_n bounds a smoothly embedded disk in a certain homotopy 4-ball $W(K_n)$ with $\partial W(K_n) \approx S^3$. A natural question is the following:

QUESTION. Is $W(K_n)$ diffeomorphic to the standard 4-ball B^4?

If $W(K_n)$ is not diffeomorphic to B^4, then the homotopy 4-sphere obtained by capping it off is a counterexample of the smooth four-dimensional Poincaré conjecture. For related studies, see [A1; A2; FGMW; G1; G2; N; NS; Tan]. Our first result is the following.

THEOREM 3.1. *Let K be a slice knot admitting an annulus presentation, and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. Then the homotopy 4-ball $W(K_n)$ associated to K_n is diffeomorphic to B^4, that is,*

$$W(K_n) \approx B^4.$$

In particular, K_n is a slice knot.

The slice knots constructed in Theorem 3.1 are relevant to the slice-ribbon conjecture. Recall that a knot K in $S^3 = \partial B^4$ is called *slice* if it bounds a smoothly embedded disk $D \subset B^4$, and the embedded disk $D \subset B^4$ is called a *slice disk* for K.

A knot in S^3 is called *ribbon* if it bounds a smoothly immersed disk $\Sigma \subset S^3$ with only ribbon singularities. It is well known that a ribbon knot bounds a special type of slice disk in B^4, called a *ribbon disk,* which is obtained from the immersed disk $\Sigma \subset S^3$. In particular, any ribbon knot is a slice knot. The *slice-ribbon conjecture* states the converse, that is, any slice knot is a ribbon knot. There are some affirmative results on the slice-ribbon conjecture; see [CD; GJ; Le; Li]. On the other hand, Gompf, Scharlemann, and Thompson [GST] demonstrated slice knots that might not be ribbon. Similarly, there is no apparent reason for the slice knots K_n in Theorem 3.1 to be ribbon. For more recent works, see [CP; LM; P].

Let $\mathcal{K}_n$ ($n \geq 0$) be the knot obtained from 8_{20} (with an appropriate annulus presentation) by the n-fold annulus twist. These are the simplest slice knots obtained by our method and were studied by Omae [Om] in a different viewpoint. We will prove that these slice knots are ribbon. To prove this, we introduce a sufficient condition for given slice knots to be ribbon.

LEMMA 5.1. *Let HD be a handle diagram of B^4 without 3-handles. Suppose that the handle diagram HD is changed into the empty handle diagram of B^4 by the following handle moves*:

(1) *2-handle slides over 1-handles or 2-handles, and*
(2) *1-handle slides over 1-handles, and*
(3) *adding or canceling 1/2-handle pairs, and*
(4) *isotopies.*

Then the belt sphere of any 2-handle of HD is a ribbon knot.

Our second result is the following.

THEOREM 5.4. *The slice knot $\mathcal{K}_n$ ($n \geq 0$) is ribbon.*

We outline the proof as follows. By the construction, $\mathcal{K}_n$ ($n \geq 0$) is isotopic to the belt-sphere of a 2-handle of a certain handle diagram *HD* of B^4 without 3-handles; see the proof of Lemma 2.7. By (rather long) handle calculus we prove that *HD* is changed into the empty handle diagram of B^4 by handle slides, canceling 1/2-handle pairs, and isotopies. By Lemma 5.1, $\mathcal{K}_n$ is ribbon.

In Section 6, we consider two conjectures. The first one is the following.

CONJECTURE 6.1. *Let HD be a handle diagram of B^4 without 3-handles. Then the belt-sphere of any 2-handle of HD is a ribbon knot.*

Note that if Conjecture 6.1 is true, then slice knots in Theorem 3.1 and Gompf, Scharlemann, and Thompson's slice knots in [GST] are ribbon. In this sense, to solve Conjecture 6.1 is the first step toward an affirmative answer to the slice-ribbon conjecture. For the details, see Section 6.

This paper is organized as follows. In Section 2, we recall some basic definitions and introduce some terminology. In Section 3, we prove the main result (Theorem 3.1). First, we give a picture of $W(K_n)$. After adding a canceling 2/3-handle pair to $W(K_n)$ suitably, we prove that $W(K_n) \approx B^4$. In Section 4, we
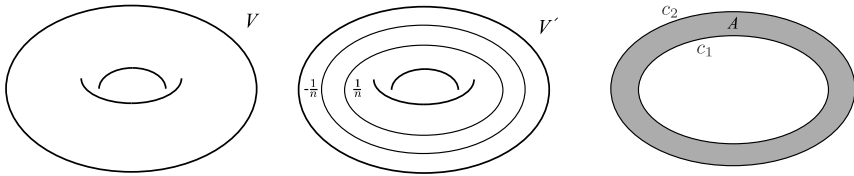
Figure 1 The definitions of V, V', and A, c_1, c_2

give an alternative proof of Theorem 3.1 in a special case by a log transformation. In Section 5, we give a sufficient condition for given slice knots to be ribbon (Lemma 5.1). As an application, we prove Theorem 5.4. In Section 6, we give two conjectures.

NOTATIONS. We denote by $M_K(n)$ the 3-manifold obtained from S^3 by n-surgery on a knot K in S^3 and by $X_K(n)$ the smooth 4-manifold obtained from B^4 by attaching a 2-handle along K with framing n. For a given 4-manifold X, we denote the boundary of X by ∂X.

The symbol $\approx$ stands for a diffeomorphism. In figures, we denote by $\sim$ an isotopy and by $\rightarrow$ a handle slide, a handle canceling, or a blow-up.

2. Preliminaries

In this section, we first define an annulus twist and an annulus presentation and introduce the dotted circle notation for the exteriors of slice disks. After that, we recall the knots constructed by Omae and describe the corresponding homotopy 4-balls. Finally, we give a remark on canceling pairs.

ANNULUS TWIST. Let V be the solid torus standardly embedded in S^3, and V' the 3-manifold as in Figure 1. Then the following is known.

LEMMA 2.1 (cf. Theorem 2.1 in [Os]). *There exists a* (*natural*) *diffeomorphism*

$$\phi_n : V' \longrightarrow V$$

such that $\phi_n|_{\partial V'} = id$.

REMARK 2.2. Osoinach [Os] considered the diffeomorphism ϕ_n^{-1}.

Let $A \subset \mathbb{R}^2 \cup \{\infty\} \subset S^3$ be an embedded annulus and set $\partial A = c_1 \cup c_2$ as in Figure 1. An *n-fold annulus twist along A* is the following operation:

(1) Regard c_1 as a $\frac{1}{n}$-framed knot and c_2 as a $-\frac{1}{n}$-framed knot for $n \in \mathbb{Z}$, and
(2) take a solid torus V' which is a neighborhood of A, and
(3) apply the diffeomorphism ϕ_n in Lemma 2.1.

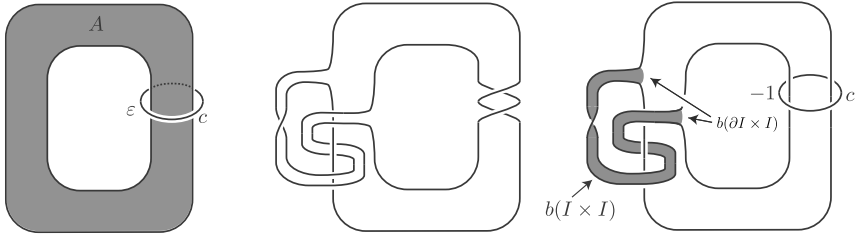A 1-fold annulus twist along A is called an *annulus twist along A*.

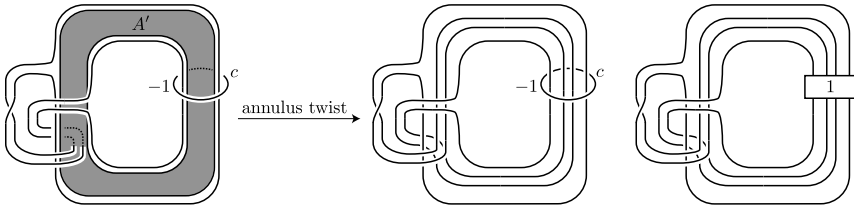**Figure 2** The knot 8_{20} depicted in the center admits an annulus presentation on the right

Figure 3 The associated annulus A' (left), an annulus twist along A', and the resulting knot (right)

ANNULUS PRESENTATION. The first author, Jong, Omae, and Takeuchi [AJOT] introduced the notion of an annulus presentation[1] of a knot for which we can associate an annulus.

We recall the definitions of an annulus presentation of a knot as follows. Let $A \subset \mathbb{R}^2 \cup \{\infty\} \subset S^3$ be a trivially embedded annulus with an ε-framed unknot c in S^3 as shown on the left of Figure 2, where $\varepsilon = \pm 1$. Take an embedding of a band $b: I \times I \to S^3$ such that

- $b(I \times I) \cap \partial A = b(\partial I \times I)$,
- $b(I \times I) \cap \text{int } A$ consists of ribbon singularities, and
- $b(I \times I) \cap c = \emptyset$,

where $I = [0, 1]$. Throughout this paper, we assume that $A \cup b(I \times I)$ is orientable. This means that we deal with only 0-framed knots; see [AJOT]. For simplicity, we also assume that $\varepsilon = -1$. If a knot $K \subset S^3$ is isotopic to the knot $(\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$ in $M_c(-1) \approx S^3$, then we say that K admits an *annulus presentation* (A, b, c). A typical example of an annulus presentation of a knot is given in Figure 2.

Let K be a knot admitting an annulus presentation (A, b, c). Shrinking the annulus A slightly, we obtain an annulus $A' \subset A$ as shown in Figure 3. We apply the n-fold ($n \in \mathbb{Z}$) annulus twist along A' and blow down the -1-framed unknot c. Figure 3 illustrates the case $n = 1$. We call the resulting knot *the knot obtained*

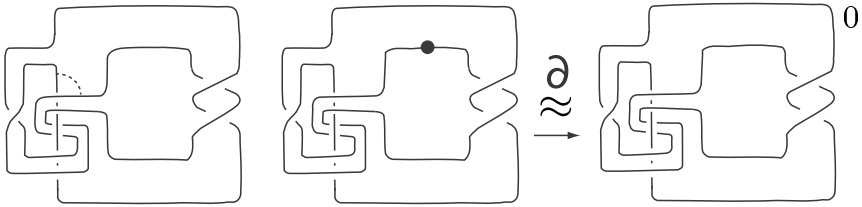[1]In [AJOT], it was called a band presentation.

Figure 4 The knot 8_{20}, and $B^4 \setminus N(D)$, and $\partial(B^4 \setminus N(D))$

from K by the n-fold annulus twist without mentioning A'. The first author, Jong, Omae, and Takeuchi proved the following.

LEMMA 2.3 ([AJOT]). *Let K be a knot admitting an annulus presentation, and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. Then*

$$M_K(0) \approx M_{K_n}(0).$$

If K is a slice knot, then K_n bounds a smoothly embedded disk in a homotopy 4-ball $W(K_n)$ such that $\partial W(K_n) \approx S^3$.

REMARK 2.4. Under the assumption of Lemma 2.3, we can also prove that $X_K(0) \approx X_{K_n}(0)$; see [AJOT].

CARVING A SLICE DISK. It is known that the dotted circle notation for the exteriors of unknotted disks (1-handles) can be generalized to the exteriors of ribbon disks (see [A, Subs. 1.4] or [GS, Subs. 6.2]). Now we further generalize the dotted circle notation for the exteriors of slice disks.

Let $K \subset S^3$ be a slice knot, and $D \subset B^4$ a slice disk for K. The *exterior* of D in B^4 is defined to be $B^4 \setminus N(D)$,[2] where $N(D)$ is an open tubular neighborhood of D. We describe $B^4 \setminus N(D)$ by K with a dot on K. The important thing is that we do not consider handle decompositions for $B^4 \setminus N(D)$. The advantage of this new notation is that the boundary of $B^4 \setminus N(D)$, which is diffeomorphic to the 0-surgery of K, is obtained by changing the dot to 0.

EXAMPLE 2.5. Let K be the knot 8_{20}. Then K is a ribbon knot since if we add a band along the dashed arc on the left of Figure 4, then we obtain the two-component unlink. In particular, K is a slice knot. Let D be an arbitrary slice disk for K (which is not necessarily a ribbon disk). Then $B^4 \setminus N(D)$ and $\partial(B^4 \setminus N(D))$ are represented by the pictures in Figure 4, respectively.

REMARK 2.6. The new notation for the exteriors of slice disks does not determine the 4-manifold uniquely since this notation does not indicate the choice of a slice disk.

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[2]In four-dimensional topology, it is usually called the complement of D. On the other hand, in knot theory, the complement of D implies $B^4 \setminus D$.
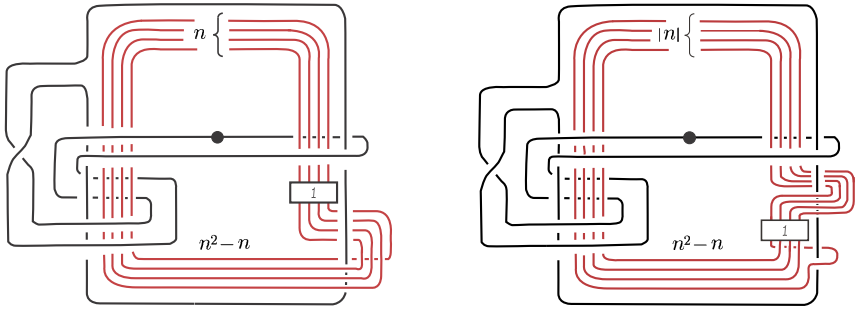
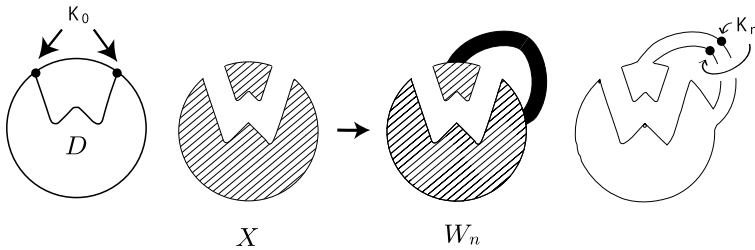**Figure 5**   Homotopy 4-balls W_n for the case $n \geq 0$ (left) and for the case $n < 0$ (right)

Figure 6   Schematic pictures

OMAE'S KNOTS AND HOMOTOPY 4-BALLS. Let D be a slice disk in B^4. In this paper, we consider homotopy 4-balls W such that $\partial W \approx S^3$ obtained from the exterior of D by attaching handles along the boundary of the exterior of D. Here we deal with the following particular case.

Recall that the knot 8_{20} admits an annulus presentation; see Figure 2. Let $\mathcal{K}_n$ be the knot obtained from 8_{20} by the n-fold annulus twist. In her master thesis [Om], Omae studies these knots $\mathcal{K}_n$ for $n \geq 0$. We can prove the following lemma.

LEMMA 2.7.   *The knot $\mathcal{K}_n$ bounds a smoothly embedded disk in a homotopy 4-ball W_n such that $\partial W_n \approx S^3$, which is represented by the picture in Figure 5.*

Here we outline the proof of Lemma 2.7. For the details, see [AJOT, Lemma 2.3].

Outline of the proof.   First, we consider the case $n \geq 0$. Recall that $\mathcal{K}_0 = 8_{20}$ is a slice knot; see Example 2.5. Let $D \subset B^4$ be an arbitrary slice disk for $\mathcal{K}_0$, and X the exterior of D. The homotopy 4-ball W_n is obtained from X by attaching a 2-handle h_n^2 along the meridian μ_n of $\mathcal{K}_n$ in $M_{\mathcal{K}_n}(0) \approx \partial X$ with framing 0. Schematic pictures are given in Figure 6. It is not difficult to see that the knot $\mathcal{K}_n$ is isotopic to the belt-sphere of the 2-handle h_n^2. Thus, $\mathcal{K}_n$ bounds the cocore disk of h_n^2, which is a smoothly embedded disk in W_n.
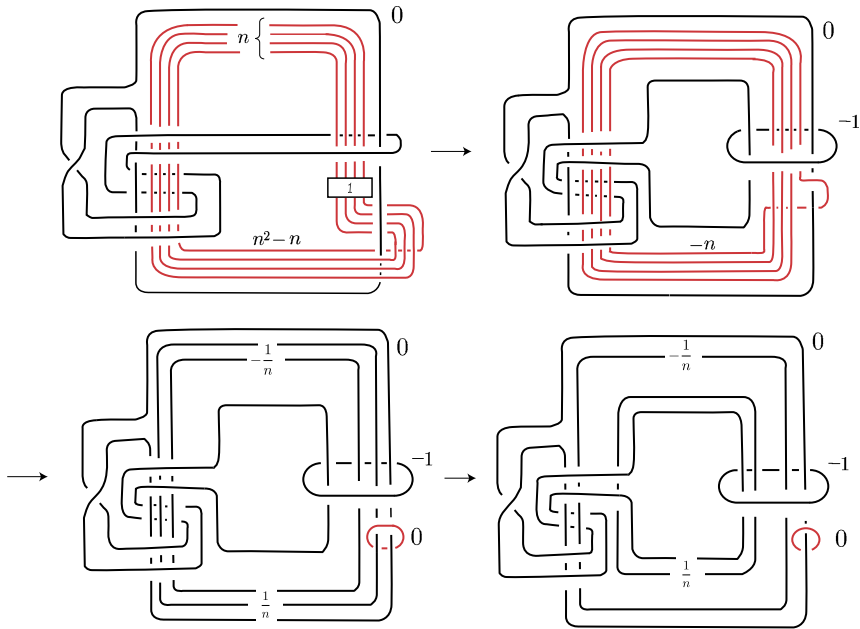
Figure 7 A diffeomorphism from $M_{\mathcal{K}_0}(0)$ to $M_{\mathcal{K}_n}(0)$. $M_{\mathcal{K}_0}(0)$ is represented by the first picture. The second picture is obtained by a blow up. The third picture is obtained by applying ϕ_n^{-1} in Lemma 2.1. The last picture is obtained by a handle slide. Then we obtain $M_{\mathcal{K}_n}(0)$ from the last picture by applying ϕ_n in Lemma 2.1 and a blow down

Finally, we draw a picture of the homotopy 4-ball $W_n = X \cup h_n^2$. To describe the attaching circle and its framing of the 2-handle h_n^2 precisely, we recall two diffeomorphisms. One of them is $f_n : M_{\mathcal{K}_0}(0) \to M_{\mathcal{K}_n}(0)$, which is given in Figure 7 (for a while, we ignore the framed knots colored red). The other is the diffeomorphism from ∂X to $M_{\mathcal{K}_0}(0)$, denoted by g, which is given by changing the dot to 0. By the definition of W_n, it is obtained from X by attaching a 2-handle along $(f_n \circ g)^{-1}(\mu_n)$ in ∂X with a suitable framing, which is 0-framing in $M_{\mathcal{K}_n}(0)$. By Figure 7 we can check that the framing is $n^2 - n$. Therefore, W_n is represented by the picture on the left of Figure 5.

Next, we consider the case $n < 0$. Set $n = -m$ for some positive integer m. Let $D \subset B^4$ be an arbitrary slice disk for $\mathcal{K}_0 = 8_{20}$, and X the exterior of D in B^4. The homotopy 4-ball W_{-m} is obtained from X by attaching a 2-handle along the meridian μ_{-m} of $\mathcal{K}_{-m}$ in $M_{\mathcal{K}_{-m}}(0) \approx \partial X$ with framing 0. Then the knot $\mathcal{K}_{-m}$ bounds a smoothly embedded disk in W_{-m} by the same argument.

Similarly, we draw a picture of the homotopy 4-ball $W_{-m} = X \cup h_{-m}^2$. To describe the attaching circle and its framing of the 2-handle h_{-m}^2 precisely, we recall two diffeomorphisms. One of them is $f_{-m} : M_{\mathcal{K}_0}(0) \to M_{\mathcal{K}_{-m}}(0)$, which is given in Figure 8 (for a while, we ignore the framed knots colored red). The
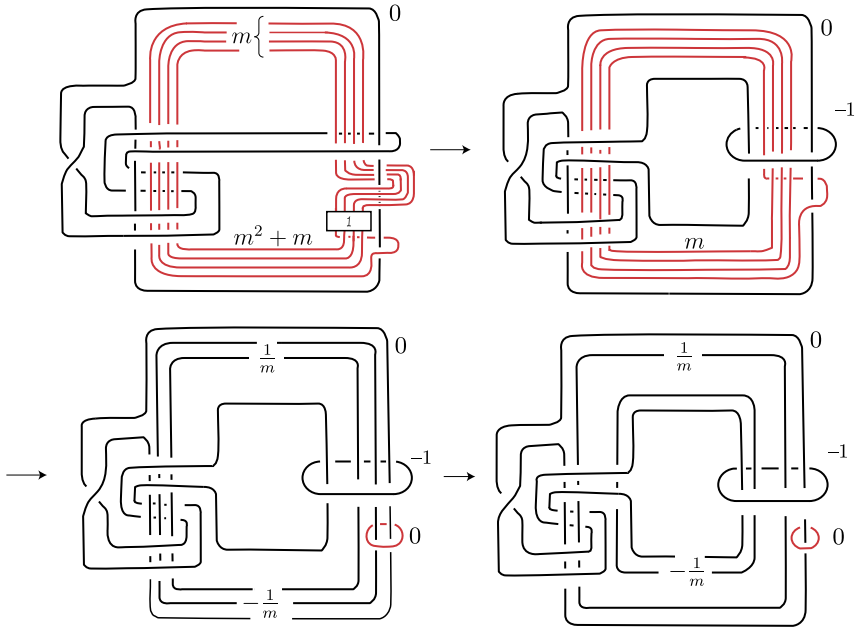
Figure 8 A diffeomorphism from $M_{\mathcal{K}_0}(0)$ to $M_{\mathcal{K}_{-m}}(0)$. $M_{\mathcal{K}_0}(0)$ is represented by the first picture. The second picture is obtained by a blow up. The third picture is obtained by applying ϕ_{-m}^{-1} in Lemma 2.1. The last picture is obtained by a handle slide. Then we obtain $M_{\mathcal{K}_{-m}}(0)$ from the last picture by applying ϕ_{-m} in Lemma 2.1 and a blow down

other is the diffeomorphism from ∂X to $M_{\mathcal{K}_0}(0)$, denoted by g, which is given by changing the dot to 0. By the definition of W_{-m}, it is obtained from X by attaching a 2-handle along $(f_{-m} \circ g)^{-1}(\mu_{-m})$ in ∂X with a suitable framing, which is 0-framing in $M_{\mathcal{K}_{-m}}(0)$. By Figure 8 the framing is $m^2 + m (= n^2 - n)$. Therefore, $W_{-m} = W_n$ is represented by the picture on the right of Figure 5. □

CANCELING PAIRS. Recall that, under the dotted circle notation for the exteriors of unknotted disks (1-handles), a 1/2-handle pair cancels if the attaching circle of the 2-handle intersects the spanning disk of the dotted circle in a single point. In this case, the 2-handle with any framing is allowed.

Under the dotted circle notation for the exteriors of slice disks, there exist another type of canceling pairs. A pair of a dotted slice knot ("1-handle") and a 2-handle cancels if the attaching circle of the 2-handle is the unknot in S^3 and its framing is zero, and the dotted slice knot intersects the spanning disk of the attaching circle of the 2-handle in a single point. For example, the picture in Figure 9 represents B^4. This canceling phenomena is justified by the following lemma.
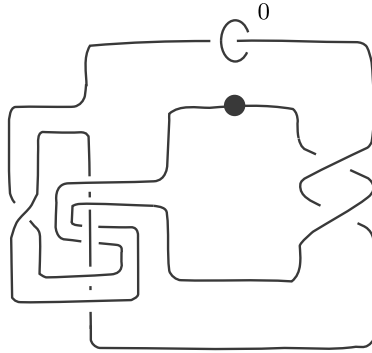
Figure 9   Another type of canceling pair

LEMMA 2.8. *Let K be a slice knot, and D a slice disk for K. Then B^4 is obtained from the exterior of D by attaching a 2-handle along the meridian of K with framing 0.*

Proof. Let $N(D)$ be an open tubular neighborhood of D, $\overline{N(D)}$ the closure of $N(D)$, and $f : \overline{N(D)} \to D^2 \times D^2$ a diffeomorphism such that $f^{-1}(D^2 \times \{0\})$ is D. It is easy to see that

$$(B^4 - N(D)) \cap \overline{N(D)} = f^{-1}(D^2 \times \partial D^2).$$

Therefore, we can regard $N(D)$ as a 2-handle h^2. Then its attaching region is $f^{-1}(D^2 \times \partial D^2)$. By the definition, the attaching circle of the 2-handle is isotopic to the meridian of K, and the framing is 0. □

3. A Construction of Slice Knots via Annulus Twists

In this section, we prove the following theorem by introducing a canceling 2/3-handle pair.

THEOREM 3.1. *Let K be a slice knot admitting an annulus presentation, and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. Then the homotopy 4-ball $W(K_n)$ associated to K_n is diffeomorphic to B^4, that is,*

$$W(K_n) \approx B^4.$$

In particular, K_n is a slice knot.

Proof. First we consider the case $K = 8_{20}$ with the annulus presentation as on the right of Figure 2 and $n \geq 0$. By Lemma 2.7, $K_n = \mathcal{K}_n$ bounds a smoothly embedded disk in the homotopy 4-ball W_n given by the picture on the left of Figure 5. We prove the following Claims 1 and 2.

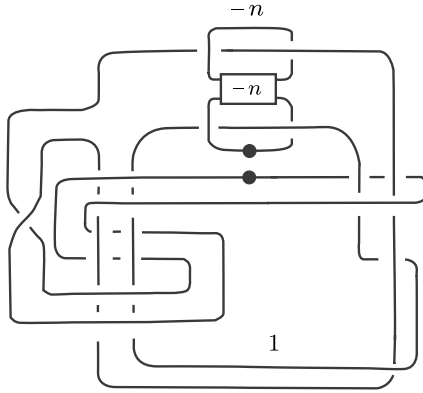CLAIM 1. *W_n ($n \geq 0$) is also represented by the picture in Figure 10.*

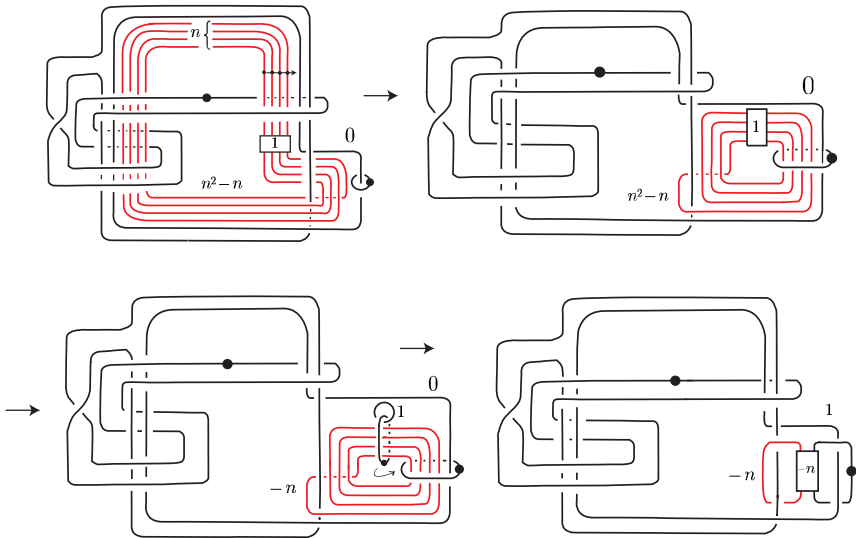**Figure 10** The homotopy 4-ball W_n

Figure 11 Pictures of the homotopy 4-balls W_n ($n \geq 0$)

Proof. Inserting a canceling 1/2-handle pair to W_n, we obtain the first picture in Figure 11. By handle slides we obtain the second picture. By inserting a canceling 1/2-handle pair to W_n and handle slides we obtain the third picture. After a 1-handle slide (and a 2-handle slide, annihilating a canceling 1/2-handle pair and isotopy), we obtain the last picture. Therefore, W_n is represented by the picture in Figure 10. □
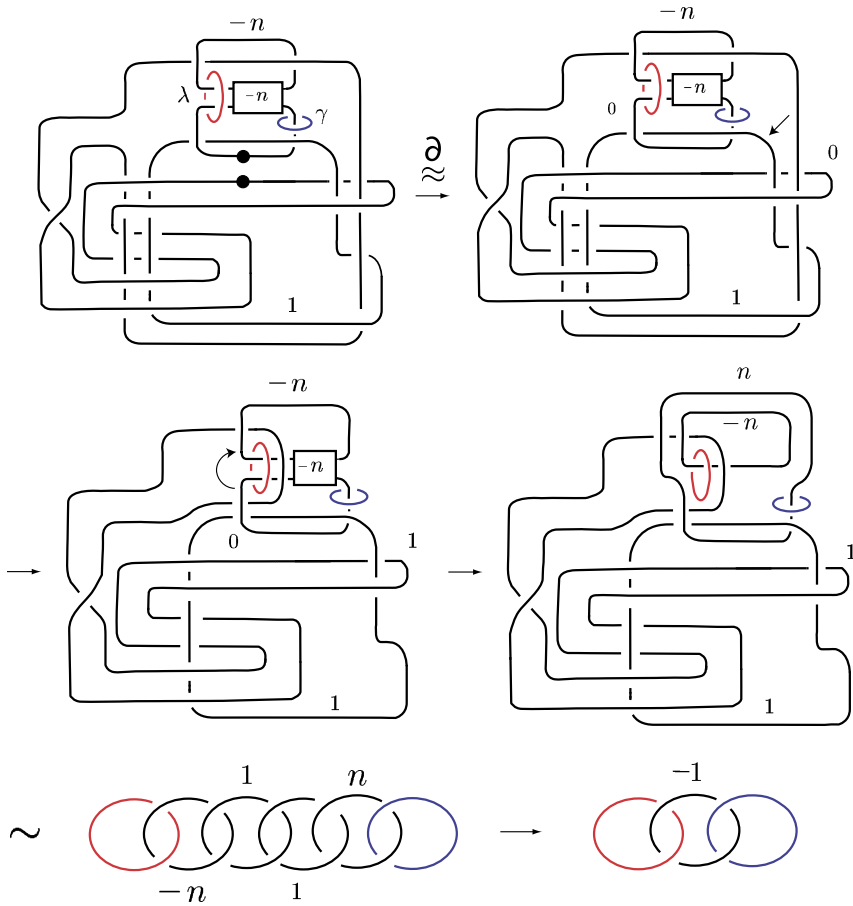
CLAIM 2. $W_n \approx W_{n-1}$.

Figure 12 A specific diffeomorphism identifying ∂W_n with S^3, which tells us that two curves $\gamma, \lambda \subset \partial W_n$ are isotopic

Proof. We show that two curves $\gamma, \lambda \subset \partial W_n$ described in Figure 12 are isotopic and each curve is the unknot in $\partial W_n = S^3$. By Claim 1, W_n is represented by the first picture in Figure 12. We replace the two dotted circles with the zero-framed circles. Then we obtain the second picture in Figure 12. Handle calculus in Figure 12 illustrates the diffeomorphism from ∂W_n to S^3.

Furthermore, if we regard γ (or λ) as a -1-framed knot, then it is isotopic to the 0-framed unknot in S^3. Now we insert a canceling 2/3-handle pair to W_n. Then W_n is diffeomorphic to the first picture in Figure 13. By a handle slide we obtain the second picture, which is diffeomorphic to W_{n-1}. □

By Claim 2, $W_n \approx W_{n-1} \approx \cdots \approx W_1 \approx W_0$. After canceling 1/2-handle pair, the picture of W_0 is that in Figure 9, which represents B^4. Therefore, $W_n \approx B^4$, and $K_n = \mathcal{K}_n$ is a slice knot.
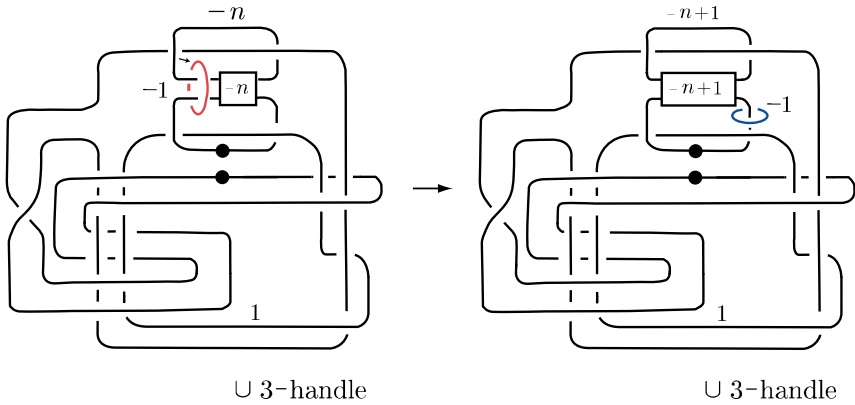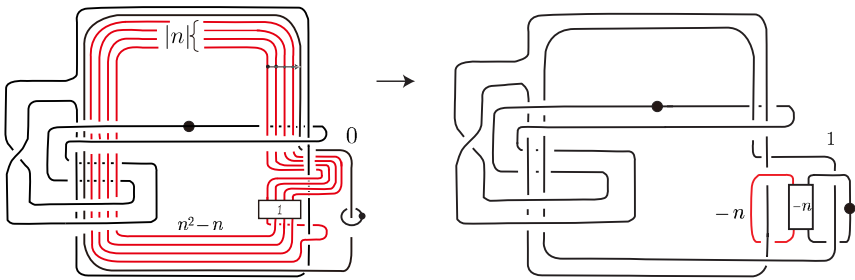
Figure 13   A handle slide

Figure 14   Pictures of the homotopy 4-balls W_n ($n < 0$)

Next we consider the case $K = 8_{20}$ with the annulus presentation and $n < 0$. Again by Lemma 2.7, $K_n = \mathcal{K}_n$ bounds a smoothly embedded disk in the homotopy 4-ball W_n given by the picture on the right of Figure 5. We prove the following claim.

CLAIM 3. W_n ($n < 0$) *is also represented by the picture in Figure 10.*

Proof. Inserting a canceling 1/2-handle pair to W_n, we obtain the first picture in Figure 14. By a similar handle calculus to that in Figure 11 we obtain the second picture. Therefore, W_n ($n < 0$) is represented by the picture in Figure 10 again.   ☐

By the same argument as that in Claim 2 we can prove that $W_n \approx B^4$ and $K_n = \mathcal{K}_n$ ($n < 0$) is a slice knot.

Now we consider the general case. First, suppose that $n \geq 0$. In this case, we can also associate a diffeomorphism $f_n \colon M_K(0) \to M_{K_n}(0)$ as described in Figure 7. Let μ_n be the meridian of K_n in $M_{K_n}(0)$. Then $f_n^{-1}(\mu_n)$ is as in the first picture in Figure 15 (after ignoring the framing). Then, as in the proof of Lemma 2.7,
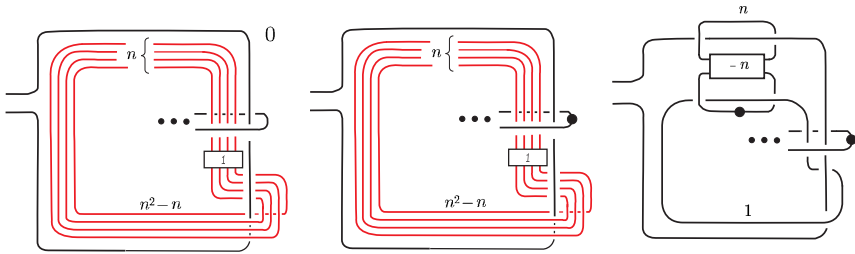
Figure 15 Pictures of the homotopy 4-ball $W(K_n)$.

we see that K_n bounds a smoothly embedded disk in a homotopy 4-ball $W(K_n)$, which is represented by the second picture in Figure 15. By the same manner it is proved that $W(K_n)$ is also represented by the third picture in Figure 15. Then we can prove that $W(K_n) \approx B^4$ by the same argument. Therefore, K_n is a slice knot.

For the case $n < 0$, by a similar argument to that in Claim 3, K_n bounds a smoothly embedded disk in a homotopy 4-ball $W(K_n)$, which is represented by the third picture in Figure 15 again. Then we can prove that $W(K_n) \approx B^4$ by the same argument. Therefore, K_n is a slice knot. □

4. Log Transformation and Fishtail Neighborhood

In this section, we give an alternative proof of Theorem 3.1 in the case $K = 8_{20}$ with the ribbon disk described in Example 2.5. More precisely, we prove that W_n and W_0 are related by a log transformation along a certain torus in W_n, where W_n is the homotopy 4-ball given by the picture in Figure 10. Lemma 4.1 in Gompf [G2] ensures that W_n and W_0 are diffeomorphic, which implies that $W_n \approx B^4$.

LOG TRANSFORMATION. Let X be an oriented 4-manifold, T an embedded torus with $T \cdot T = 0$, and $\phi \colon T^2 \times \partial D^2 \to \partial \nu(T)$ a diffeomorphism, where $\nu(T) (\approx T^2 \times D^2)$ is a closed neighborhood of T in X. Removing int $\nu(T)$ from X and attaching $T^2 \times D^2$ by ϕ, we obtain

$$(X - \mathrm{int}\,\nu(T)) \cup_\phi T^2 \times D^2.$$

Suppose that

$$\phi_*([\{\mathrm{pt.}\} \times \partial D^2]) = p[\{\mathrm{pt.}\} \times \partial D^2] + q[\gamma \times \{\mathrm{pt.}\}]$$

for some essential simple closed curve γ in T. Then we call this surgery a *logarithmic transformation with multiplicity p, direction γ, and auxiliary multiplicity q*. If $p = 1$, then we call this logarithmic transformation a *q-fold Dehn twist along T parallel to γ*.

FISHTAIL NEIGHBORHOOD. The fishtail neighborhood F is an elliptic fibration over D^2 with one fishtail singular fiber, which has the handle decomposition in Figure 16. It is well known that the -1-framed meridian in Figure 16 is isotopic to the vanishing cycle of F. Gompf [G2] proved the following assertion.
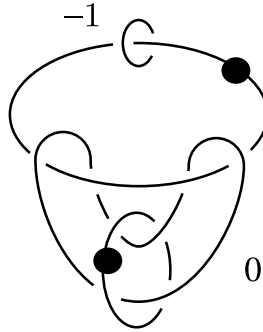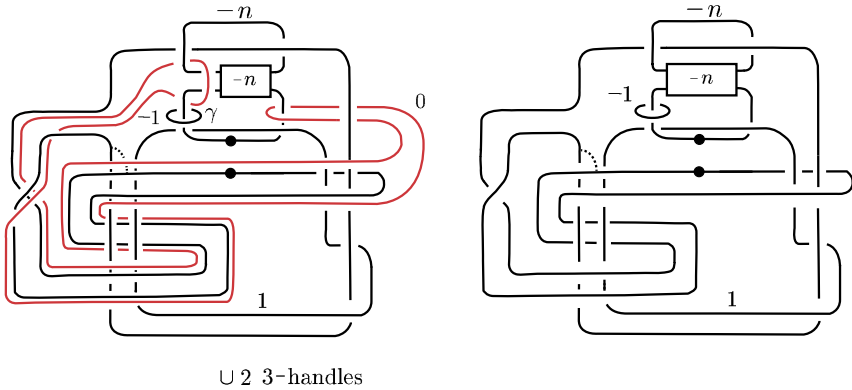
Figure 16   A handle decomposition of F

$\cup\, 2\ \ 3\text{-handles}$

Figure 17   A handle decomposition of W_n and the handlebody picture of $W_n + \gamma^{-1}$

LEMMA 4.1 ([G2]).   *Let X be a 4-manifold, and T be a regular fiber of a fishtail neighborhood F embedded in X. Then the q-fold Dehn twist along T parallel to the vanishing cycle of F does not change the diffeomorphism type of X.*

We prove the following.

LEMMA 4.2.   *The homotopy 4-ball W_n also has the handle decomposition given by the first picture in Figure 17.*

Proof.   We fix a diffeomorphism identifying ∂W_n with S^3. We use the diffeomorphism described in Figure 12 again. Recall that this diffeomorphism tells us that the -1-framed γ is isotopic to the 0-framed unknot in S^3 (for the details, see the proof of Theorem 3.1). Therefore, by inserting a canceling 2/3-handle pair to W_n we obtain

$$W_n \approx W_n + \gamma^{-1} \cup (3\text{-handle}),$$

where $W_n + \gamma^{-1}$ is the handlebody given by the second picture in Figure 17.
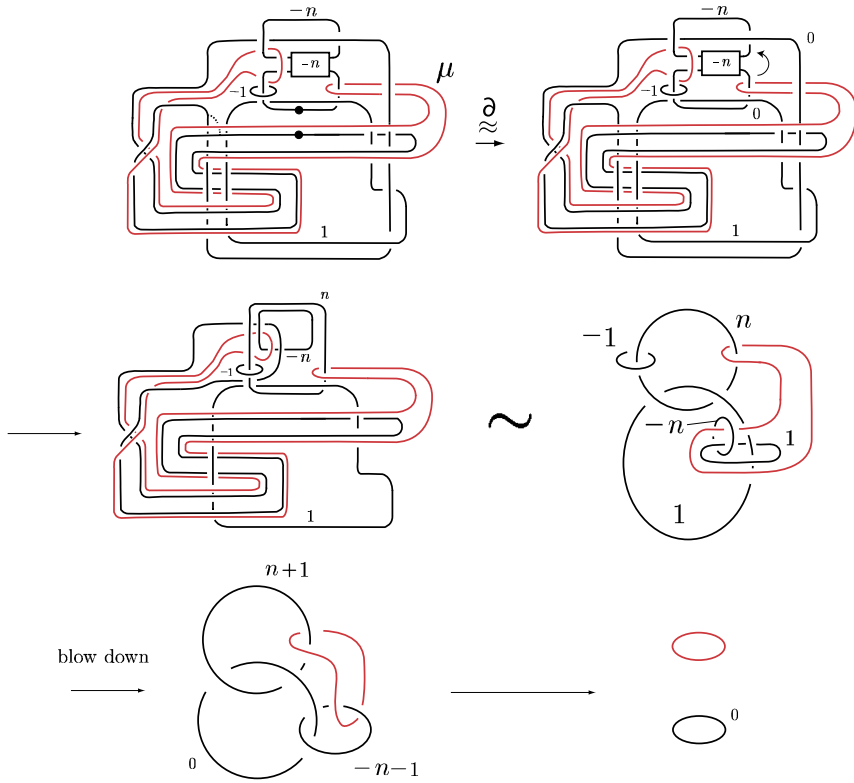
Figure 18 A diffeomorphism identifying $\partial(W_n + \gamma^{-1})$ with $S^1 \times S^2$, which tells us that the curve μ is the unknot in $S^1 \times S^2$

Next, we fix a diffeomorphism identifying $\partial(W_n + \gamma^{-1})$ with $S^1 \times S^2$ described in Figure 18 (for a while, we ignore the curve μ). This diffeomorphism tells us that $\mu \subset \partial(W_n + \gamma^{-1})$ is the unknot in $S^1 \times S^2$. Furthermore, if we regard μ as a 0-framed knot, then it is isotopic to the 0-framed unknot in $S^1 \times S^2$. Therefore, by inserting a canceling 2/3-handle pair to W_n we obtain the first picture in Figure 17. $\qquad\square$

Now we prove the main result in this section.

Proof of Theorem 3.1 in the case $K = 8_{20}$. The second picture of Figure 19 is a subhandlebody of W_n. By isotopy we see that it is diffeomorphic to $F \cup$ (1-handle), where F is the fishtail neighborhood. Therefore, by removing the 1-handle we can find F as a submanifold of W_n.

Let T be a regular fiber of F embedded in W_n. The 1-fold Dehn twist along T parallel to γ is 1-untwisting along γ. For the details, see [AY] or [GS]. Thus, the local deformation is as in Figure 20.
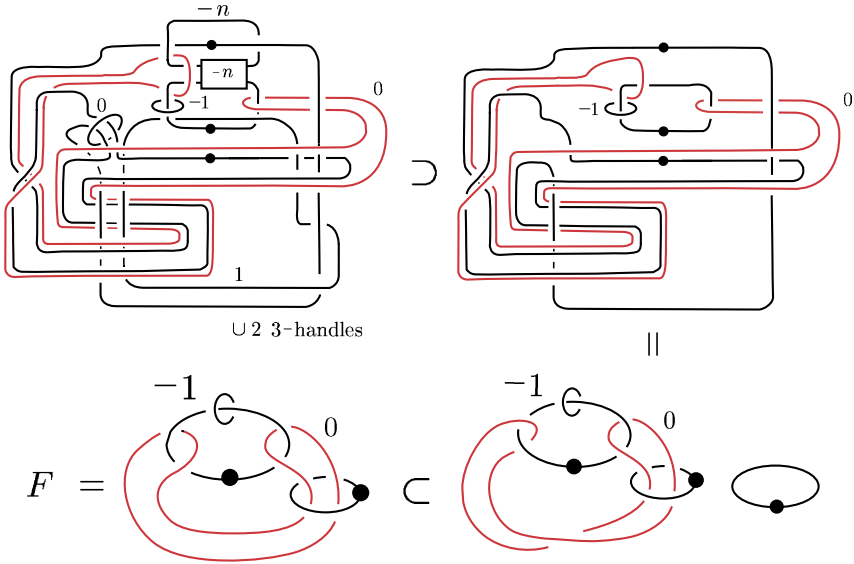
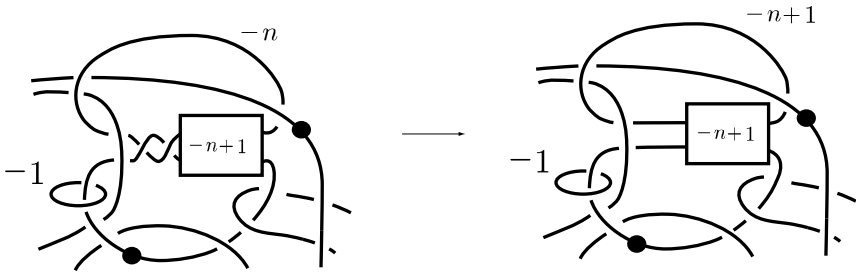**Figure 19**   An embedding of the fishtail neighborhood F

Figure 20   The 1-fold Dehn twist along T parallel to γ

As a result, performing the n-fold Dehn twist along T parallel to γ and removing the canceling 2/3-handle pairs, we obtain W_0 that is diffeomorphic to B^4. By Lemma 4.1, $W_n \approx W_0$. Therefore, $K_n = \mathcal{K}_n$ (obtained from 8_{20}) is a slice knot.   □

5. A Sufficient Condition to be Ribbon

In this section, we give a sufficient condition for a slice knot to be ribbon (Lemma 5.1) and prove that all the knots obtained from 8_{20} by annulus twists are ribbon (Theorem 5.4).
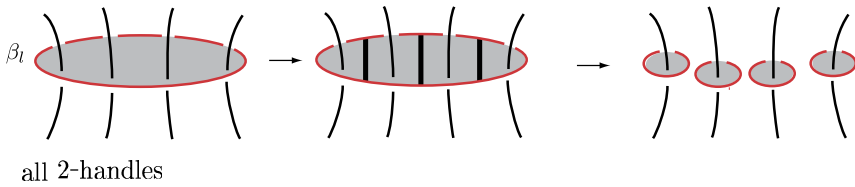
β_l

all 2-handles

Figure 21   Band surgeries along mutually disjoint bands ($m = 4$)

LEMMA 5.1. *Let HD be a handle diagram of B^4 without 3-handles. Suppose that the handle diagram HD is changed into the empty handle diagram of B^4 by the following handle moves*:

(1) *2-handle slides over 1-handles or 2-handles, and*
(2) *1-handle slides over 1-handles, and*
(3) *adding or canceling 1/2-handle pairs, and*
(4) *isotopies.*

Then the belt sphere of any 2-handle of HD is a ribbon knot.

Proof. Let

$$HD = HD_0 \to HD_1 \to \cdots \to HD_n = \text{(empty handle diagram)}$$

be a sequence of handle diagrams satisfying the condition of Lemma 5.1. By rearranging the sequence we can assume the following:

$HD_0 \to HD_1 \to \cdots \to HD_k$ (adding canceling 1/2-handle pairs),
$HD_k \to HD_{k+1} \to \cdots \to HD_l$ (1- and 2-handle slides and isotopies),
$HD_l \to HD_{l+1} \to \cdots \to HD_n$ (annihilating canceling 1/2-handle pairs).

Let β be the belt sphere of a 2-handle of HD, and β_i ($i = 1, 2, \ldots, l$) the corresponding knot in HD_i. Then β is the unknot in HD, and we see that β_l is also the unknot in HD_l. Furthermore, we can find a smoothly embedded disk D in HD_l such that $\partial D = \beta_l$, the disk D does not intersect any dotted 1-handles,[3] and D intersects transversely with some attaching spheres of 2-handles as the left in Figure 21. Let m be the number of intersections between D and the attaching spheres of 2-handles of HD_l. By band surgeries along mutually disjoint bands $B_1, B_2, \ldots, B_{m-1}$ as in the middle picture in Figure 21, we obtain an m-component link L such that each component is the meridian of the attaching sphere of a 2-handle of HD_l.

Finally, we consider the sequence $HD_l \to \cdots \to HD_n$. Let L' be the link in HD_n corresponding to L. Then it is the m-component unlink in S^3. In other words, the knot β is deformed into the m-component unlink by band surgeries along $m - 1$ bands. This means that β is a ribbon knot. □

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[3]We can choose D in this way since the link consisting of dotted circles (representing 1-handles) and β_l is the unlink.
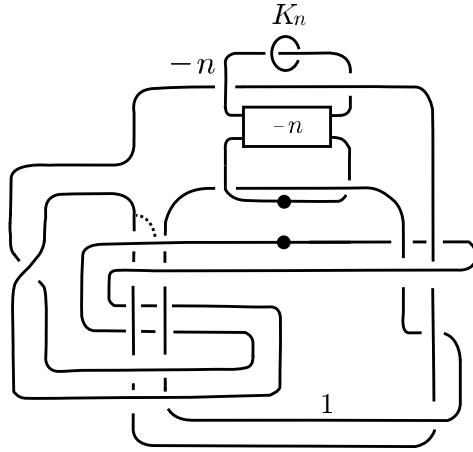
Figure 22   The slice knot $\mathcal{K}_n$ in ∂W_n

Let 8_{20} be the knot with the annulus presentation on the right of Figure 2, and $\mathcal{K}_n$ ($n \geq 0$) the knot obtained from 8_{20} by the n-fold annulus twist. By Theorem 3.1, $\mathcal{K}_n$ is a slice knot. There is no apparent reason for $\mathcal{K}_n$ to be ribbon. Our result is that, indeed, $\mathcal{K}_n$ is a ribbon knot. To prove this, we first observe the following.

LEMMA 5.2. *The slice knot $\mathcal{K}_n$ is located as in Figure 22.*

Proof. By the proofs of Lemma 2.7 and Theorem 3.1 we obtain this lemma immediately.   □

REMARK 5.3. Let K be any ribbon knot in ∂B^4. Then it is not difficult to see that B^4 admits a handle decomposition

$$h^0 \cup h_1^1 \cup \cdots \cup h_n^1 \cup h_1^2 \cup \cdots \cup h_n^2$$

such that the belt sphere of some 2-handle is isotopic to K, where h^0 is a 0-handle, h_i^1 ($i = 1, \ldots, n$) is a 1-handle, and h_j^2 ($j = 1, \ldots, n$) is a 2-handle. For the converse, see Conjecture 6.1.

Now we prove the following.

THEOREM 5.4. *The slice knot $\mathcal{K}_n$ ($n \geq 0$) is ribbon.*

Proof. Let *HD* be the handle diagram given by the picture in Figure 22. By Lemma 5.2, $\mathcal{K}_n$ is isotopic to the belt sphere of a 2-handle of *HD*. By Lemma 5.1, if *HD* is changed into the empty handle diagram by handle slides, adding or canceling 1/2-handle pairs, and isotopies, then $\mathcal{K}_n$ is a ribbon knot. Such operations are realized in Figures 23, 24, 25, and 26. As a result, $\mathcal{K}_n$ is a ribbon knot.   □
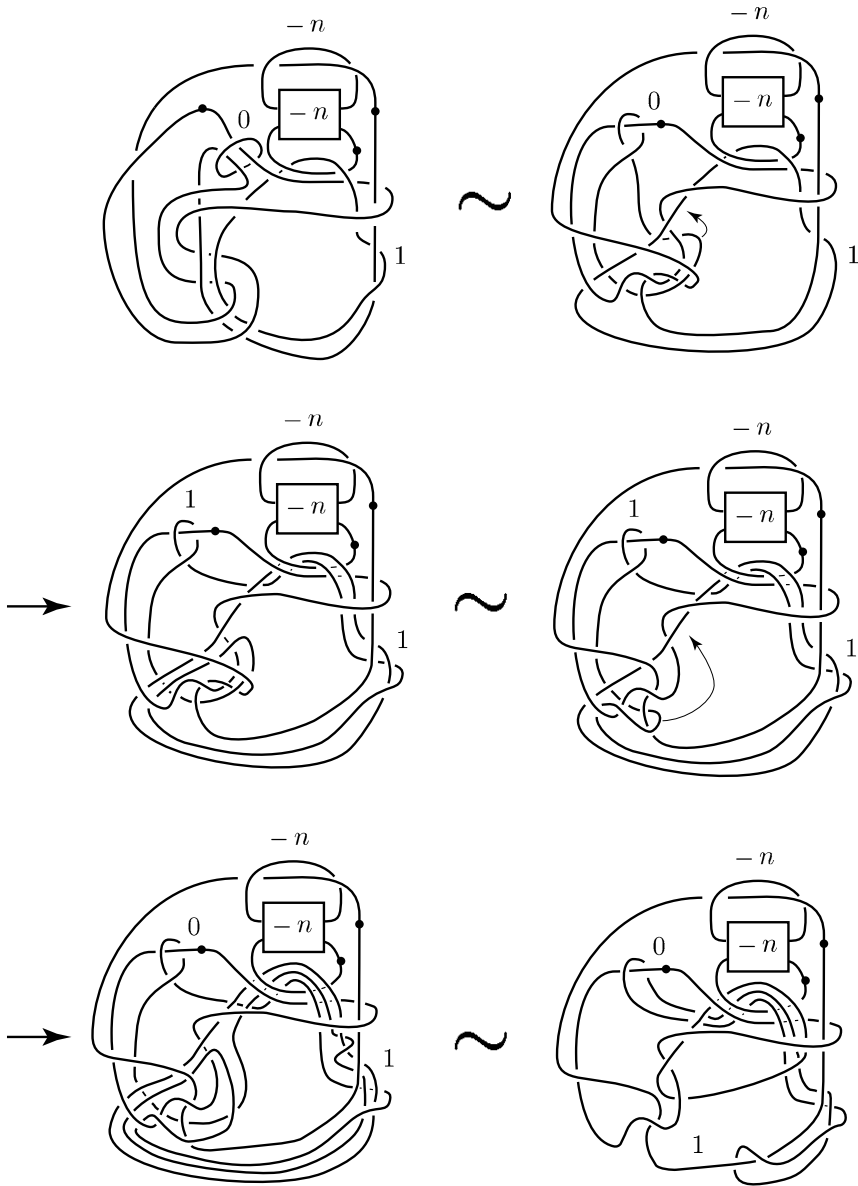
Figure 23 Handle calculus without adding canceling 2/3-handle pairs

Now we draw a ribbon presentation of $\mathcal{K}_n$. Keeping track of $\mathcal{K}_n$ through the handle calculus, though it is rather troublesome, we can obtain a ribbon presentation of $\mathcal{K}_n$ as in Figure 27.
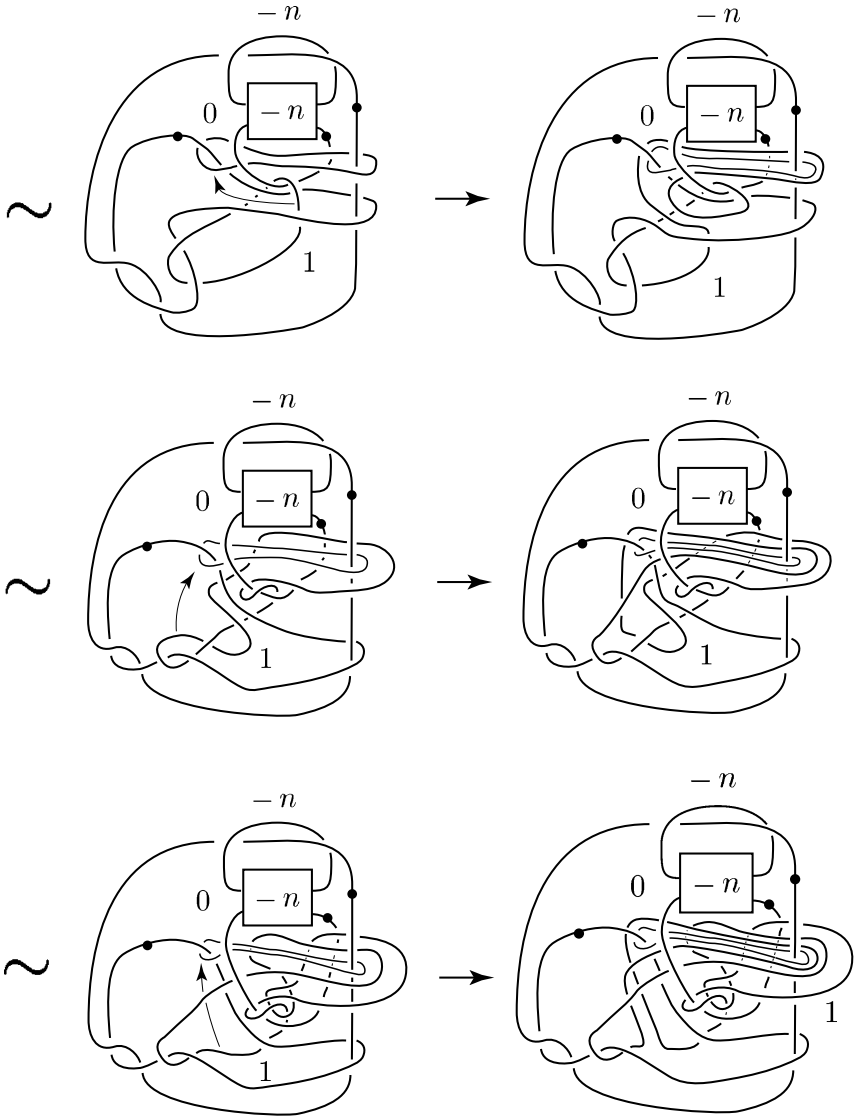
Figure 24   Handle calculus without adding canceling 2/3-handle pairs

6. Two Conjectures

In this section, we consider two conjectures.

Conjecture 6.1. *Let HD be a handle diagram of B^4 without 3-handles. Then the belt-sphere of any 2-handle of HD is a ribbon knot.*
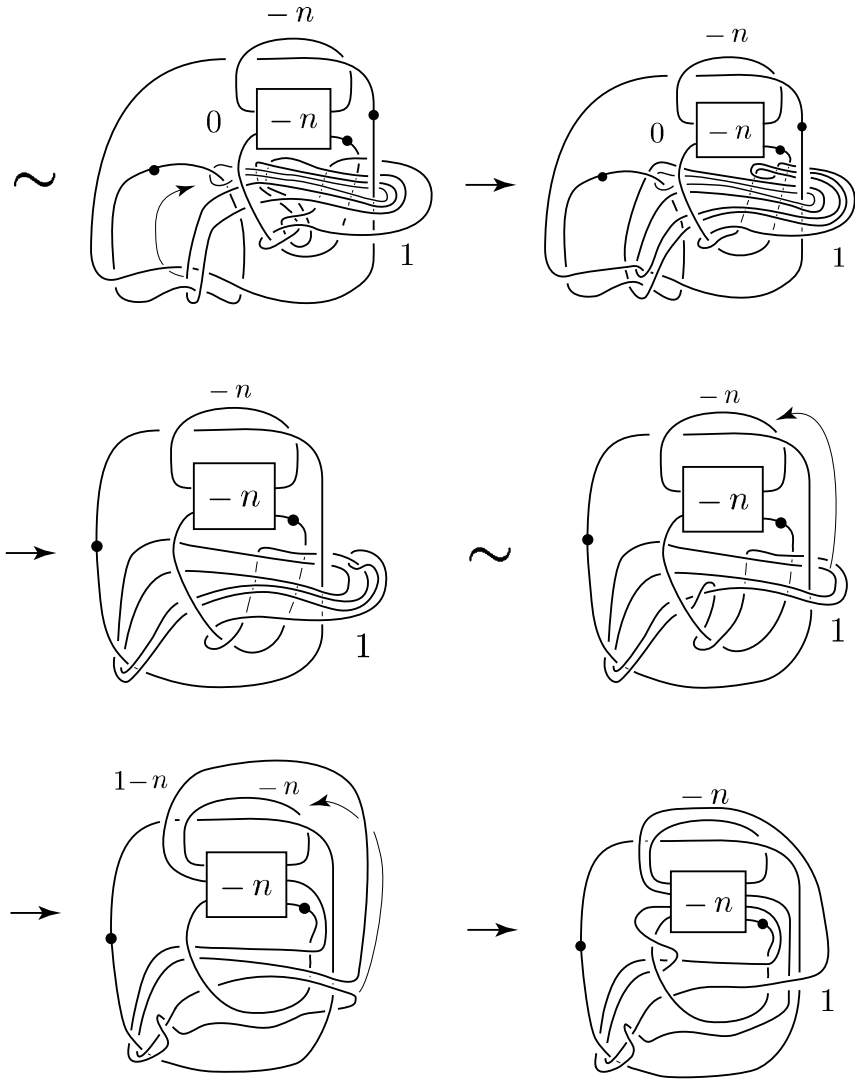
Figure 25 Handle calculus without adding canceling 2/3-handle pairs

Recall that each slice knot in Theorem 3.1 is isotopic to the belt-sphere of a 2-handle of a certain handle diagram of B^4 without 3-handles; see the proofs of Lemma 2.7 and Theorem 3.1. Therefore, if Conjecture 6.1 is true, then all slice knots in Theorem 3.1 are ribbon.

A partial answer to Conjecture 6.1 is Lemma 5.1. However, it is conjectured that some handle diagrams of B^4 without 3-handles do not satisfy the assumption of Lemma 5.1 as follows.
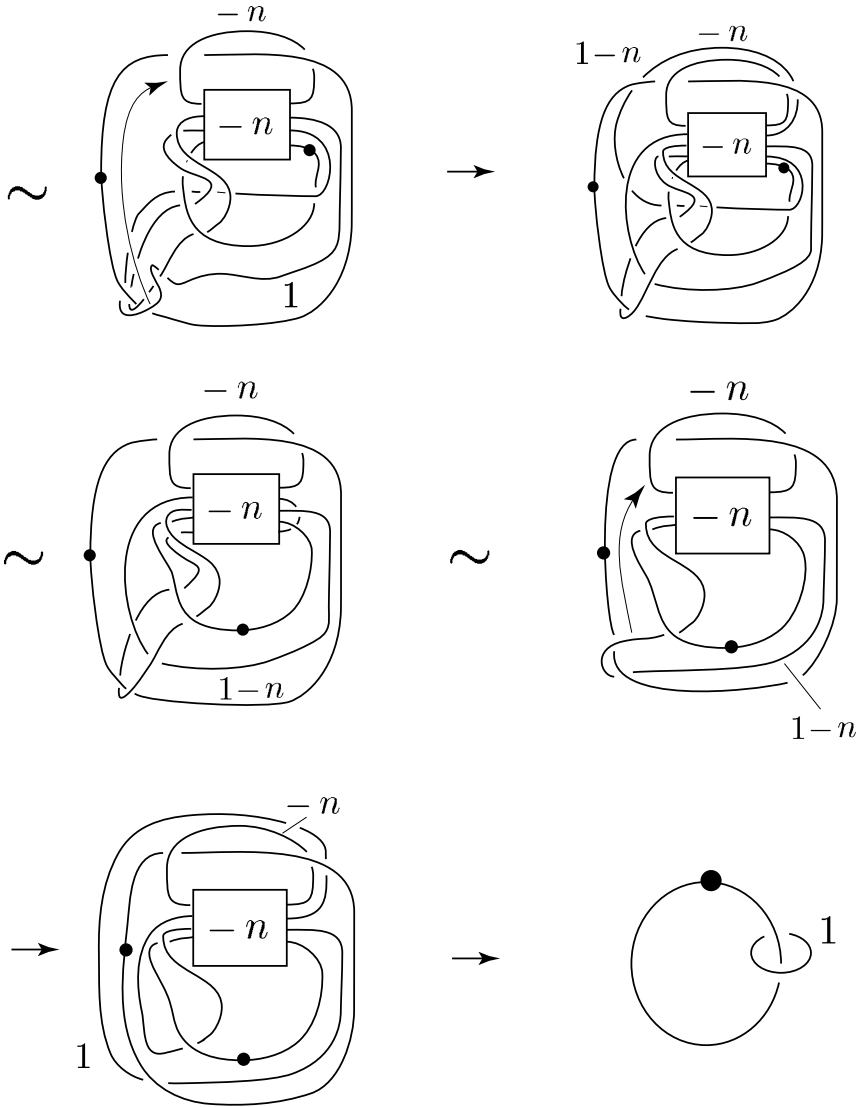
Figure 26 Handle calculus without adding canceling 2/3-handle pairs

CONJECTURE 6.2 ([G1, Conjecture B]). *For $n \geq 3$ and $k \neq 0$, the handle diagrams $H_{n,k}$ in Figure 28 cannot be deformed into the empty handle diagram without introducing a 3-handle.*

Let $L_{n,k}$ be the two-component link in S^3 that consists of the two belt-spheres of the two 2-handles of $H_{n,k}$; see the right half of Figure 28. By the definition, $L_{n,k}$ is a slice link, that is, it bounds two smoothly embedded disjoint disks in B^4. Gompf,
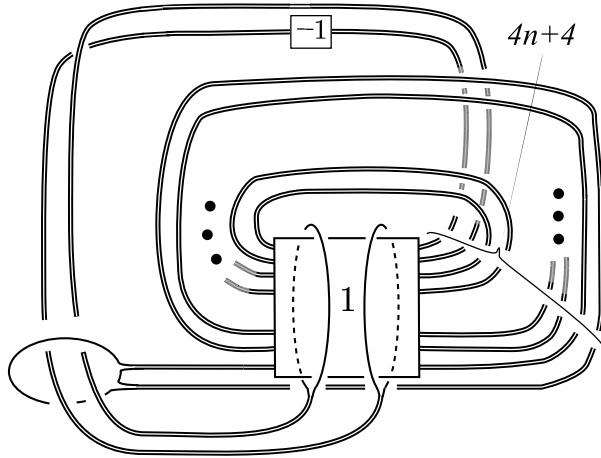
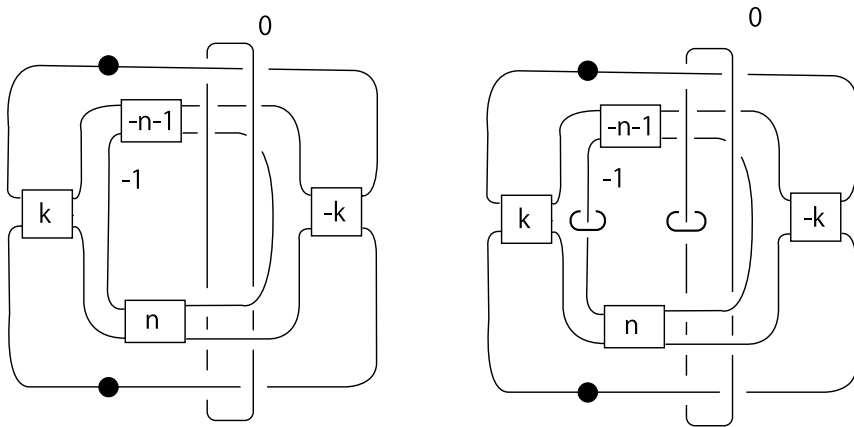**Figure 27** A ribbon presentation of $\mathcal{K}_n$ $(n \geq 1)$

Figure 28 The handle diagram $H_{n,k}$ of B^4 (left) and the 2-component link $L_{n,k}$ in $S^3 = \partial B^4$ (right)

Scharlemann, and Thompson [GST] considered a slice knot obtained from $L_{n,k}$ by attaching a certain band. After a single 2-handle slide (along the band), it turns out that the slice knot is isotopic to the belt-sphere of a 2-handle of a certain handle diagram of B^4 without 3-handles. Therefore, if Conjecture 6.1 is true, Gompf, Scharlemann, and Thompson's slice knots are also ribbon. In this sense, solving Conjecture 6.1 is the first step toward an affirmative answer to the slice-ribbon conjecture.

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