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A Remark on the Ueno–Campana's Threefold

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Dedicated to Fabrizio Catanese on his 65th birthday

Abstract. We show that the Ueno–Campana's threefold cannot be obtained as the blow-up of any smooth threefold along a smooth center, answering negatively a question raised by Oguiso and Truong.

1. Introduction

Let $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ be the complex elliptic curve of period τ. There exist exactly two elliptic curves with automorphism group bigger than $\{\pm 1\}$: these are defined respectively by the periods $\sqrt{-1}$ and the cubic root of unity $\omega := (-1+\sqrt{-3})/2$.

We consider the diagonal action of the cyclic group generated by $\sqrt{-1}$ (resp. $-\omega$) on the product

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega),$$

and we denote by X_4 (resp. X_6) the minimal resolution of their quotients

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}/\langle\sqrt{-1}\rangle \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega/\langle -\omega\rangle).$$

The minimal resolutions are obtained by a single blow-up at the maximal ideal of each singular point of the quotients.

The threefolds X_4 and X_6 have been extensively studied in the past. In particular, they admit an automorphism of positive entropy (e.g., see [Ogu15] for more details). The variety X_4 is now referred as the *Ueno–Campana's threefold.* It has been recently shown that X_4 and X_6 are rational. Indeed, Oguiso, and Truong [OT15] showed the rationality of X_6, and Colliot-Théléne [Col15] showed the rationality of X_4, after the work of Catanese, Oguiso, and Truong [COT14]. The unirationality of X_4 was conjectured by Ueno [Uen75], whilst Campana asked about the rationality of X_4 in [Cam11].

The aim of this note is to give a negative answer to the following question raised by Oguiso and Truong (see [Ogu15, Question 5.11] and [Tru15, Question 2]).

Question 1.1. Can X_4 or X_6 be obtained as the blow-up of $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$, or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along smooth centers?

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Our main result is the following:

Theorem 1.2. *Let A be an Abelian variety of dimension three, and let G be a finite group acting on A such that the quotient map*

$$\rho \colon A \to Z = A/G$$

is étale in codimension 2.

Assume that there exists a resolution $f \colon X \to Z$ given by the blow-up of the singular points of Z and such that the exceptional divisor at each singular point of Z is irreducible.

Then X cannot be obtained as the blow-up of a smooth threefold along a smooth centre.

Note that Theorem 1.2 provides a negative answer to Question 1.1. Very recently, using different methods, Lesieutre [Les15] announced that Question 1.1 admits a negative answer.

2. Preliminary Results

We use some of the methods introduced in [CT14]. Let X be a normal projective threefold with isolated quotient singularities. Given a basis $\gamma_1, \dots, \gamma_m$ of $H^2(X, \mathbb{C})$, the *cubic form associated to X* is the homogeneous polynomial of degree 3 defined by

$$F_X(x_1, \dots, x_m) = (x_1\gamma_1 + \cdots + x_m\gamma_m)^3 \in \mathbb{C}[x_1, \dots, x_m].$$

Note that, modulo the natural action of $\mathrm{GL}(m, \mathbb{C})$, the cubic F_X does not depend on the choice of the base, and it is a topological invariant of the underlying manifold X (see [OvdV95] for more details). In particular, if

$$\mathcal{H}_{F_X} = (\partial_{x_i}\partial_{x_j} F_X)_{i,j=1,\dots,m}$$

denotes the *Hessian matrix* associated to F_X and $p \in H^2(X, \mathbb{C})$, then the rank of $\mathcal{H}_{F_X}$ at p is well defined.

The following basic tool was used in [CT14] in a more general context. We provide a proof for the reader's convenience.

Lemma 2.1. *Let Y be a normal projective threefold with isolated quotient singularities, and let $f \colon X \to Y$ be the blow-up of Y along a point $q \in Y$ (resp. a curve $C \subseteq Y$). Assume that the exceptional divisor of f is irreducible and let E be its class in $H^2(X, \mathbb{C})$.*

Then the rank of the Hessian matrix $\mathcal{H}_{F_X}$ of F_X at E is one (resp. at most two).

Note that by [CT14, Lemmas 2.7 and 2.12] the rank of $\mathcal{H}_{F_X}$ is never zero.

Proof of Lemma 2.1. We have $H^2(X, \mathbb{C}) = \langle E, f^*(\gamma_1), \dots, f^*(\gamma_m)\rangle$ where $\gamma_1, \dots, \gamma_m$ is a basis of $H^2(Y, \mathbb{C})$.

Consider the cubic form F_X associated to X with respect to this basis:

$$F_X(x_0,\dots,x_m)=\left(x_0E+\sum_{i=1}^m x_i f^*(\gamma_i)\right)^3.$$

Since $f^*(\gamma_i)\cdot f^*(\gamma_j)\cdot E=0$ for all $i, j=1,\dots,m$, we have

$$F_X(x_0,\dots,x_m)=x_0^3E^3+3\sum_{i=1}^m x_0^2x_iE^2f^*(\gamma_i)+\left(\sum_{i=1}^m x_i f^*(\gamma_i)\right)^3.$$

Let $a=E^3$, and let $b_i=E^2f^*(\gamma_i)$ for $i=1,\dots,m$. Note that if f is the blow-up of a point $q\in Y$, then $b_1=\dots=b_m=0$.

Thus, we have

$$F_X(x_0,\dots,x_m)=ax_0^3+3\sum_{i=1}^m b_ix_0^2x_i+G(x_1,\dots,x_m),$$

where G is a homogeneous cubic polynomial in the variables $x_1,\dots,x_m$, that is, it does not depend on x_0. Let $p=y_0E+\sum_{i=1}^m y_if^*\gamma_i\in H^2(X,\mathbb{C})$ for some $y_0,\dots,y_m\in\mathbb{C}$, and let $p'=(y_1,\dots,y_m)$. After removing the first row and the first column, the Hessian matrix $\mathcal{H}_{F_X}(p)$ of F_X at p coincides with the Hessian matrix $\mathcal{H}_G(p')$ of G at p'.

In particular, if $p=E$, then $p'=(0,\dots,0)$, and $\mathcal{H}_G(p')$ is the zero matrix. Thus, the rank of the Hessian of F_X at p is at most two. In addition, if $b_1=\dots=b_m=0$, then the rank of $\mathcal{H}_F$ at p is exactly one. □

3. Proofs

LEMMA 3.1. *Let A be an Abelian variety of dimension 3, and let G be a finite group acting on A such that the quotient map $\rho\colon A\to Z=A/G$ is étale in codimension 2. Let F_Z be the cubic form associated to Z, and let $p\in H^2(Z,\mathbb{C})$ such that* $\operatorname{rk}\mathcal{H}_{F_Z}(p)\le 1$.

Then $p=0$.

Proof. The morphism ρ induces an immersion of vector spaces

$$\rho^*\colon H^2(Z,\mathbb{C})\to H^2(A,\mathbb{C}).$$

Thus, there exists a basis of $H^2(A,\mathbb{C})$ such that if F_A is the cubic associated to A with respect to this basis and d is the degree of ρ, then

$$F_Z(x_1,\dots,x_m)=d\cdot F_A(x_1,\dots,x_m,0,\dots,0).$$

It is enough to show that if $q\in H^2(A,\mathbb{C})$ is such that the rank of $\mathcal{H}_{F_A}$ at q is not greater than one, then $q=0$.

Write $A=\mathbb{C}^3/\Gamma$ and consider z_1,z_2,z_3 coordinates on $\mathbb{C}^3$. Then a basis of $H^2(A,\mathbb{C})$ is given by

$$\begin{aligned} z_{ij}&=dz_i\wedge dz_j, && 1\le i<j\le 3,\\ z_{i\bar{j}}&=dz_i\wedge d\bar{z}_j, && i,j\in\{1,2,3\},\\ z_{\bar{i}\bar{j}}&=d\bar{z}_i\wedge d\bar{z}_j, && 1\le i<j\le 3.\end{aligned}$$

For any $x \in H^2(A,\mathbb{C})$, let x_{ij}, $x_{i\bar{j}}$, and $x_{\bar{i}\bar{j}}$ be the coordinates of x with respect to the basis, and let F'_A be the cubic associated to this basis. It is enough to show that if $q \in H^2(A,\mathbb{C})$ is such that the rank of $\mathcal{H}_{F'_A}$ at q is not greater than one, then $q=0$. Let q_{ij}, $q_{i\bar{j}}$, and $q_{\bar{i}\bar{j}}$ be the coordinates of q.

The (2×2)-minor of $\mathcal{H}_{F'_A}$ at x defined by the rows corresponding to x_{12} and x_{13} and the columns corresponding to $x_{2\bar{1}}$ and $x_{3\bar{1}}$ is given by

$$\begin{pmatrix} 0 & 6x_{\bar{2}\bar{3}} \\ 6x_{\bar{2}\bar{3}} & 0 \end{pmatrix}.$$

It follows that $q_{\bar{2}\bar{3}}=0$. By choosing suitable (2×2)-minors it follows easily that each coordinate of q is zero. Thus, the claim follows. □

Proof of Theorem 1.2. Suppose not. Then there exists a smooth projective threefold Y such that X can be obtained as the blow-up $g\colon X\to Y$ at a smooth centre. Let E be the exceptional divisor of g. Let k be the number of singular points of Z, and let $E_1,\dots,E_k$ be the exceptional divisors on X corresponding to the singular points of Z.

We want to prove that $E=E_i$ for some $i=1,\dots,k$. Denote by p the class of E in $H^2(X,\mathbb{C})$. Lemma 2.1 implies that the rank of $\mathcal{H}_{F_X}$ at p is not greater than two.

Let $\gamma_1,\dots,\gamma_m\in H^2(Z,\mathbb{C})$ be a basis, and let F_Z be the associated cubic form. Then $f^*\gamma_1,\dots,f^*\gamma_m,[E_1],\dots,[E_k]$ is a basis of $H^2(X,\mathbb{C})$, and if F_X denotes the associated cubic form, then we have

$$F_X(x_1,\dots,x_m,y_1,\dots,y_k)=F_Z(x_1,\dots,x_m)+\sum_{i=1}^k a_i y_i^3,$$

where $a_i=E_i^3$ is a nonzero integer, for $i=1,\dots,k$.

Thus, the Hessian matrix of F_X is composed by two blocks: one is the Hessian matrix of F_Z, and the other one is a diagonal matrix whose only nonzero entries are $6a_i$ for $i=1,\dots,k$. We may write $p=(p^0,p^1)=(p^0_1,\dots,p^0_m,p^1_1,\dots,p^1_k)$. We have $\operatorname{rk}\mathcal{H}_{F_Z}(p^0)\le 2$.

We distinguish two cases. If $\operatorname{rk}\mathcal{H}_{F_Z}(p^0)=2$, then $p^1=(0,\dots,0)$, and, in particular, E is numerically equivalent to f^*D for some pseudo-effective Cartier divisor D on Z. Since A is abelian, it follows that ρ^*D is a nef divisor. Thus, E is nef, a contradiction.

If $\operatorname{rk}\mathcal{H}_{F_Z}(p^0)\le 1$, then Lemma 3.1 implies that $p^0=0$. Thus,

$$E\equiv c_sE_s+c_tE_t$$

for some distinct $s,t\in\{1,\dots,k\}$ and rational numbers c_s,c_t. Since E is effective nontrivial, at least one of the c_i is positive. By symmetry we may assume that $c_s>0$. By the negativity lemma the divisor E_s is covered by rational curves C such that $E_s\cdot C<0$. Since E_s and E_t are disjoint, it follows that $E\cdot C<0$, which implies that C is contained in E. Thus, E_s is contained in E. Since E is prime, it follows that $E=E_s$ and $c_t=0$.

Finally, note that g contracts $E = E_s$ to a point since otherwise there exists a small contraction $\eta\colon Y \to Z$ and in particular Z is not $\mathbb{Q}$-factorial, a contradiction. Thus, $g\colon X \to Y$ is the contraction of E_s to the corresponding singular point on Z, which is again a contradiction. The claim follows. □

REMARK 3.2. As K. Oguiso kindly pointed out to us, the same proof shows that if $f\colon X \to Z$ is as in Theorem 1.2 and g is an automorphism on X, then the set of exceptional divisors of f is invariant with respect to g. Thus, there exists a positive integer m such that the power g^m descends to an automorphism on Z.

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