Michigan Math. J. 65 (2016), 489–509

The Parabolic Infinite-Laplace Equation in Carnot Groups

Thomas Bieske & Erin Martin

Abstract. By employing a Carnot parabolic maximum principle we show the existence and uniqueness of viscosity solutions to a class of equations modeled on the parabolic infinite Laplace equation in Carnot groups. We show the stability of solutions within the class and examine the limit as t goes to infinity.

1. Motivation

In Carnot groups, the following theorem has been established.

Theorem 1.1 [3; 14; 5]. *Let Ω be a bounded domain in a Carnot group, and let $v : \partial\Omega \to \mathbb{R}$ be a continuous function. Then the Dirichlet problem*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega \end{cases}$$

has a unique viscosity solution u_∞.

Our goal is to prove a parabolic version of Theorem 1.1 for a class of equations (defined in the next section), namely:

Conjecture 1.2. Let Ω be a bounded domain in a Carnot group, and let $T > 0$. Let $\psi \in C(\overline{\Omega})$ and $g \in C(\Omega \times [0, T))$ Then the Cauchy–Dirichlet problem

$$\begin{cases} u_t - \Delta_\infty^h u = 0 & \text{in } \Omega \times (0, T), \\ u(x, 0) = \psi(x) & \text{on } \overline{\Omega}, \\ u(x, t) = g(x, t) & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{1.1}$$

has a unique viscosity solution u.

In Sections 2 and 3, we review key properties of Carnot groups and parabolic viscosity solutions. In Section 4, we prove the uniqueness, and Section 5 covers the existence.

2. Calculus on Carnot Groups

We begin by denoting an arbitrary Carnot group in $\mathbb{R}^N$ by G and its corresponding Lie algebra by g. Recall that g is nilpotent and stratified, resulting in the

Received January 26, 2015. Revision received June 23, 2016.
The first author was partially supported by a University of South Florida Proposal Enhancement Grant.

decomposition

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_1, V_j] = V_{1+j}$. The Lie algebra g is associated with the group G via the exponential map $\exp : g \to G$. Since this map is a diffeomorphism, we can choose a basis for g so that it is the identity map. Denote this basis by

$$X_1, X_2, \dots, X_{n_1}, Y_1, Y_2, \dots, Y_{n_2}, Z_1, Z_2, \dots, Z_{n_3},$$

so that

$$\begin{aligned} V_1 &= \operatorname{span}\{X_1, X_2, \dots, X_{n_1}\}, \\ V_2 &= \operatorname{span}\{Y_1, Y_2, \dots, Y_{n_2}\}, \\ V_3 \oplus V_4 \oplus \cdots \oplus V_l &= \operatorname{span}\{Z_1, Z_2, \dots, Z_{n_3}\}. \end{aligned}$$

We endow g with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. Clearly, the Riemannian dimension of g (and so G) is $N = n_1 + n_2 + n_3$. However, we will also consider the homogeneous dimension of G, denoted $\mathcal{Q}$, which is given by

$$\mathcal{Q} = \sum_{i=1}^{l} i \cdot \dim V_i .$$

Before proceeding with the calculus, we recall the group and metric space properties. Since the exponential map is the identity, the group law is the Campbell–Hausdorff formula (see, e.g., [7]). For our purposes, this formula is given by

$$p \cdot q = p + q + \frac{1}{2}[p, q] + R(p, q), \tag{2.1}$$

where $R(p, q)$ are terms of order 3 or higher. The identity element of G will be denoted by 0 and called the origin. There is also a natural metric on G, which is the Carnot–Carathéodory distance, defined for the points p and q as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt,$$

where Γ is the set of all curves γ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma'(t) \in V_1$. By Chow's theorem (see, e.g., [2]) any two points can be connected by such a curve, which means that $d_C(p, q)$ is an honest metric. Define the Carnot–Carathéodory ball of radius r centered at a point p_0 by

$$B(p_0, r) = \{p \in G : d_C(p, p_0) < r\}.$$

In addition to the Carnot–Carathéodory metric, there is a smooth (off the origin) gauge. This gauge is defined for a point $p = (\zeta_1, \zeta_2, \dots, \zeta_l)$ with $\zeta_i \in V_i$ by

$$\mathcal{N}(p) = \left(\sum_{i=1}^{l} \|\zeta_i\|^{2l!/i} \right)^{1/(2l!)} \tag{2.2}$$

and induces the metric $d_{\mathcal{N}}$ that is bi-Lipschitz equivalent to the Carnot–Carathéodory metric and is given by

$$d_{\mathcal{N}}(p,q) = \mathcal{N}(p^{-1} \cdot q).$$

We define the gauge ball of radius r centered at a point p_0 by

$$B_{\mathcal{N}}(p_0, r) = \{p \in G : d_{\mathcal{N}}(p, p_0) < r\}.$$

In this environment, a smooth function $u : G \to \mathbb{R}$ has the horizontal derivative given by

$$\nabla_0 u = (X_1 u, X_2 u, \ldots, X_{n_1} u)$$

and the symmetrized horizontal second derivative matrix, denoted by $(D^2 u)^\star$, with entries

$$((D^2u)^\star)_{ij} = \frac{1}{2}(X_i X_j u + X_j X_i u)$$

for $i, j = 1, 2, \ldots, n_1$. We also consider the semihorizontal derivative given by

$$\nabla_1 u = (X_1 u, X_2 u, \ldots, X_{n_1} u, Y_1 u, Y_2 u, \ldots, Y_{n_2} u).$$

Using these derivatives, we define the h-homogeneous infinite Laplace operator for $h \geq 1$ by

$$\Delta_\infty^h f = \|\nabla_0 f\|^{h-3} \sum_{i,j=1}^{n_1} X_i f X_j f X_i X_j f = \|\nabla_0 f\|^{h-3} \langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle.$$

Given $T > 0$ and a function $u : G \times [0, T] \to \mathbb{R}$, we may define the analogous subparabolic infinite Laplace operator by

$$u_t - \Delta_\infty^h u,$$

and we consider the corresponding equation

$$u_t - \Delta_\infty^h u = 0. \tag{2.3}$$

We note that when $h \geq 3$, this operator is continuous. When $h = 3$, we have the subparabolic infinite Laplace equation analogous to the infinite Laplace operator in [5]. The Euclidean analog for $h = 1$ has been explored in [12], and the Euclidean analog for $1 < h < 3$ in [13].

We recall that for any open set $\mathcal{O} \subset G$, the function f is in the horizontal Sobolev space $W^{1,\mathrm{p}}(\mathcal{O})$ if f and $X_i f$ are in $L^{\mathrm{p}}(\mathcal{O})$ for $i = 1, 2, \ldots, n_1$. Replacing $L^{\mathrm{p}}(\mathcal{O})$ by $L^{\mathrm{p}}_{\mathrm{loc}}(\mathcal{O})$, the space $W^{1,\mathrm{p}}_{\mathrm{loc}}(\mathcal{O})$ is defined similarly. The space $W^{1,\mathrm{p}}_0(\mathcal{O})$ is the closure in $W^{1,\mathrm{p}}(\mathcal{O})$ of smooth functions with compact support. In addition, we recall that a function $u : G \to \mathbb{R}$ is $\mathcal{C}^2_{\mathrm{sub}}$ if $\nabla_1 u$ and $X_i X_j u$ are continuous for all $i, j = 1, 2, \ldots n_1$. Note that $\mathcal{C}^2_{\mathrm{sub}}$ is not equivalent to (Euclidean) C^2. For spaces involving time, the space $C(t_1, t_2; X)$ consists of all continuous functions $u : [t_1, t_2] \to X$ with $\max_{t_1 \leq t \leq t_2} \|u(\cdot, t)\|_X < \infty$. A similar definition holds for $L^{\mathrm{p}}(t_1, t_2; X)$.

Given an open box $\mathcal{O} = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)$, we define the parabolic space $\mathcal{O}_{t_1,t_2}$ to be $\mathcal{O} \times [t_1, t_2]$. Its parabolic boundary is given by $\partial_{\mathrm{par}} \mathcal{O}_{t_1,t_2} = (\overline{\mathcal{O}} \times \{t_1\}) \cup (\partial \mathcal{O} \times (t_1, t_2])$.

Finally, recall that if G is a Carnot group with homogeneous dimension $\mathcal{Q}$, then $G \times \mathbb{R}$ is again a Carnot group of homogeneous dimension $\mathcal{Q}+1$, where we have added an extra vector field $\frac{\partial}{\partial t}$ to the first layer of the grading. This allows us to give meaning to notations such as $W^{1,2}(\mathcal{O}_{t_1,t_2})$ and $\mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2})$, where we consider $\nabla_0 u$ to be $(X_1 u, X_2 u, \ldots, X_{n_1} u, \frac{\partial u}{\partial t})$.

3. Parabolic Jets and Viscosity Solutions

3.1. Parabolic Jets

In this subsection, we recall the definitions of the parabolic jets, as given in [6], but included here for completeness. We define the parabolic superjet of $u(p,t)$ at the point $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$, denoted $P^{2,+}u(p_0,t_0)$, by using triples $(a,\eta,X) \in \mathbb{R} \times V_1 \oplus V_2 \times S^{n_1}$ so that $(a,\eta,X) \in P^{2,+}u(p_0,t_0)$ if

$$\begin{aligned} u(p,t) \leq u(p_0,t_0) + a(t-t_0) + \langle \eta, \widehat{p_0^{-1}\cdot p}\rangle + \frac{1}{2}\langle X\overline{p_0^{-1}\cdot p}, \overline{p_0^{-1}\cdot p}\rangle \\ + o(|t-t_0| + |p_0^{-1}\cdot p|^2) \quad \text{as } (p,t) \to (p_0,t_0). \end{aligned}$$

We recall that S^k is the set of $k \times k$ symmetric matrices and $n_i = \dim V_i$. We define $\overline{p_0^{-1}\cdot p}$ as the first n_1 coordinates of $p_0^{-1}\cdot p$ and $\widehat{p_0^{-1}\cdot p}$ as the first n_1+n_2 coordinates of $p_0^{-1}\cdot p$. This definition is an extension of the superjet definition for subparabolic equations in the Heisenberg group [4]. We define the subjet $P^{2,-}u(p_0,t_0)$ by

$$P^{2,-}u(p_0,t_0) = -P^{2,+}(-u)(p_0,t_0).$$

We define the set-theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(p_0,t_0)$, by requiring $(a,\eta,X) \in \overline{P}^{2,+}u(p_0,t_0)$ exactly when there is a sequence $(a_n, p_n, t_n, u(p_n,t_n), \eta_n, X_n) \to (a, p_0, t_0, u(p_0,t_0), \eta, X)$ with the triple $(a_n,\eta_n,X_n) \in P^{2,+}u(p_n,t_n)$. A similar definition holds for the closure of the subjet.

We may also define jets using appropriate test functions. Given a function $u : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$, we consider the set $\mathcal{A}u(p_0,t_0)$ given by

$$\begin{aligned} \mathcal{A}u(p_0,t_0) = \{\phi \in \mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2}) : u(p,t) - \phi(p,t) \leq u(p_0,t_0) - \phi(p_0,t_0) = 0 \\ \forall (p,t) \in \mathcal{O}_{t_1,t_2}\}, \end{aligned}$$

consisting of all test functions that touch u from above at (p_0,t_0). We define the set of all test functions that touch from below, denoted $\mathcal{B}u(p_0,t_0)$, similarly.

The following lemma relates the test functions to jets. The proof is identical to that of Lemma 3.1 in [4] but uses the (smooth) gauge $\mathcal{N}(p)$ instead of Euclidean distance.

LEMMA 3.1.

$$P^{2,+}u(p_0,t_0) = \{(\phi_t(p_0,t_0), \nabla\phi(p_0,t_0), (D^2\phi(p_0,t_0))^\star) : \phi \in \mathcal{A}u(p_0,t_0)\}.$$

3.2. Jet Twisting

We recall that the set $V_1 = \operatorname{span}\{X_1, X_2, \dots, X_{n_1}\}$, and notationally, we will always denote n_1 by n. The vectors X_i at the point $p \in G$ can be written as

$$X_i(p) = \sum_{j=1}^{N} a_{ij}(p) \frac{\partial}{\partial x_j},$$

forming the $n \times N$ matrix $\mathbb{A}$ with smooth entries $\mathbb{A}_{ij} = a_{ij}(p)$. By linear independence of the X_i, $\mathbb{A}$ has rank n. Similarly,

$$Y_i(p) = \sum_{j=1}^{N} b_{ij}(p) \frac{\partial}{\partial x_j},$$

forming the $n_2 \times N$ matrix $\mathbb{B}$ with smooth entries $\mathbb{B}_{ij} = b_{ij}$. The matrix $\mathbb{B}$ has rank n_2. The following lemma differs from [5, Cor. 3.2] only in that there is now a parabolic term. This term, however, does not need to be twisted. The proof is then identical since only the space terms need twisting.

LEMMA 3.2. *Let $(a, \eta, X) \in \overline{P}^{2,+}_{\mathrm{eucl}} u(p,t)$. (Recall that $(\eta, X) \in \mathbb{R}^N \times S^N$.) Then*

$$(a, \mathbb{A} \cdot \eta \oplus \mathbb{B} \cdot \eta, \mathbb{A} X \mathbb{A}^T + \mathbb{M}) \in \overline{P}^{2,+} u(p,t).$$

Here the entries of the (symmetric) matrix $\mathbb{M}$ are given by

$$\mathbb{M}_{ij} = \begin{cases} \sum_{k=1}^{N} \sum_{l=1}^{N} (a_{il}(p) \frac{\partial}{\partial x_l} a_{jk}(p) + a_{jl}(p) \frac{\partial a_{ik}}{\partial x_l}(p)) \eta_k, & i \neq j, \\ \sum_{k=1}^{N} \sum_{l=1}^{N} a_{il}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \eta_k, & i = j. \end{cases}$$

3.3. Viscosity Solutions

We consider parabolic equations of the form

$$u_t + F(t, p, u, \nabla_1 u, (D^2 u)^\star) = 0 \tag{3.1}$$

for continuous and proper $F : [0, T] \times G \times \mathbb{R} \times g \times S^n \to \mathbb{R}$ [8]. We recall that S^n is the set of $n \times n$ symmetric matrices (where $\dim V_1 = n$) and the derivatives $\nabla_1 u$ and $(D^2 u)^\star$ are taken in the space variable p. We then use the jets to define subsolutions and supersolutions to Equation (3.1) in the usual way.

DEFINITION 1. Let $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ be as before. The upper semicontinuous function u is a *parabolic viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$, we have that $(a, \eta, X) \in \overline{P}^{2,+} u(p_0, t_0)$ produces

$$a + F(t_0, p_0, u(p_0, t_0), \eta, X) \leq 0.$$

A lower semicontinuous function u is a *parabolic viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$, we have that $(b, \nu, Y) \in \overline{P}^{2,-} u(p_0, t_0)$ produces

$$b + F(t_0, p_0, u(p_0, t_0), \nu, Y) \geq 0.$$

A continuous function u is a *parabolic viscosity solution* in $\mathcal{O}_{t_1,t_2}$ if it is both a parabolic viscosity subsolution and parabolic viscosity supersolution.

REMARK 3.3. In the special case where $F(t,p,u,\nabla_1 u,(D^2u)^\star) = F^h_\infty(\nabla_0 u, (D^2u)^\star) = -\Delta^h_\infty u$ for $h \geq 3$, we use the terms "parabolic viscosity h-infinite supersolution", and so on.

In the case where $1 \leq h < 3$, the definition above is insufficient due to the singularity occurring when the horizontal gradient vanishes. Therefore, following [12] and [13], we define viscosity solutions to Equation (2.3) when $1 \leq h < 3$ as follows.

DEFINITION 2. Let $\mathcal{O}_{t_1,t_2}$ be as before. A lower semicontinuous function $v : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$ is a *parabolic viscosity h-infinite supersolution* of $u_t - \Delta^h_\infty u = 0$ if whenever $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$ and $\phi \in \mathcal{B}u(p_0,t_0)$, we have

$$\begin{cases} \phi_t(p_0,t_0) - \Delta^h_\infty \phi(p_0,t_0) \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) \neq 0, \\ \phi_t(p_0,t_0) - \min_{\|\eta\|=1} \langle (D^2\phi)^\star(p_0,t_0)\eta,\eta\rangle \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and} \\ & h = 1, \\ \phi_t(p_0,t_0) \geq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and} \\ & 1 < h < 3. \end{cases}$$

An upper semicontinuous function $u : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$ is a *parabolic viscosity h-infinite subsolution* of $u_t - \Delta^h_\infty u = 0$ if whenever $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$ and $\phi \in \mathcal{A}u(p_0,t_0)$, we have

$$\begin{cases} \phi_t(p_0,t_0) - \Delta^h_\infty \phi(p_0,t_0) \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) \neq 0, \\ \phi_t(p_0,t_0) - \max_{\|\eta\|=1} \langle (D^2\phi)^\star(p_0,t_0)\eta,\eta\rangle \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and} \\ & h = 1, \\ \phi_t(p_0,t_0) \leq 0 & \text{when } \nabla_0\phi(p_0,t_0) = 0 \text{ and} \\ & 1 < h < 3. \end{cases}$$

A continuous function is a *parabolic viscosity h-infinite solution* if it is both a parabolic viscosity h-infinite subsolution and parabolic viscosity h-infinite subsolution.

REMARK 3.4. When $1 < h < 3$, we can actually consider the continuous operator

$$F^h_\infty(\nabla_0 u,(D^2u)^\star) = \begin{cases} -\|\nabla_0 u\|^{h-3}\langle (D^2u)^\star \nabla_0 u, \nabla_0 u\rangle = -\Delta^h_\infty u, & \nabla_0 u \neq 0, \\ 0, & \nabla_0 u = 0. \end{cases} \tag{3.2}$$

Definitions 1 and 2 would then agree (cf. [13]).

We also wish to define what [11] refers to as parabolic viscosity solutions. We first need to consider the set

$$\mathcal{A}^- u(p_0,t_0) = \{\phi \in C^2(\mathcal{O}_{t_1,t_2}) : u(p,t) - \phi(p,t) \leq u(p_0,t_0) - \phi(p_0,t_0) = 0 \\ \text{for } p \neq p_0, t < t_0\},$$

consisting of all functions that touch from above only when $t < t_0$. Note that this set is larger than $\mathcal{A}u$ and corresponds physically to the past alone playing a role in determining the present. We define $\mathcal{B}^- u(p_0, t_0)$ similarly. We then have the following definition.

DEFINITION 3. An upper semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *past parabolic viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{A}^- u(p_0, t_0)$ produces

$$\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0), \nabla_1 \phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \leq 0.$$

An lower semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *past parabolic viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{B}^- u(p_0, t_0)$ produces

$$\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0), \nabla_1 \phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \geq 0.$$

A continuous function is a *past parabolic viscosity solution* if it is both a past parabolic viscosity supersolution and subsolution.

We have the following proposition whose proof is obvious. The analogous theorem and its converse for the Euclidean case can be found in [11]. We will address the converse in the Carnot group case in the next section.

PROPOSITION 3.5. *Past parabolic viscosity sub(super)solutions are parabolic viscosity sub(super)solutions. In particular, past parabolic viscosity h-infinite sub(super)solutions are parabolic viscosity h-infinite sub(super)solutions for $h \geq 1$.*

3.4. The Carnot Parabolic Maximum Principle

In this subsection, we recall the Carnot parabolic maximum principle and key corollaries, as proved in [6].

LEMMA 3.6 (Carnot parabolic maximum principle). *Let u be a viscosity subsolution to Equation (3.1), and v be a viscosity supersolution to Equation (3.1) in the bounded parabolic set $\Omega \times (0, T)$ where Ω is a (bounded) domain, and let τ be a positive real parameter. Let $\phi(p, q, t) = \varphi(p \cdot q^{-1}, t)$ be a C^2 function in the space variables p and q, and a C^1 function in t. Suppose that the local maximum*

$$M_\tau \equiv \max_{\overline{\Omega}\times\overline{\Omega}\times[0,T]} \{u(p, t) - v(q, t) - \tau\phi(p, q, t)\} \tag{3.3}$$

occurs at the interior point (p_τ, q_τ, t_τ) of the parabolic set $\Omega \times \Omega \times (0, T)$. Define the $n \times n$ matrix W by

$$W_{ij} = X_i(p) X_j(q) \phi(p_\tau, q_\tau, t_\tau).$$

Let the $2n \times 2n$ matrix $\mathfrak{W}$ be given by

$$\mathfrak{W} = \begin{pmatrix} 0 & \frac{1}{2}(W - W^T) \\ \frac{1}{2}(W^T - W) & 0 \end{pmatrix}, \tag{3.4}$$

and let the matrix $\mathcal{W} \in S^{2N}$ be given by

$$\mathcal{W} = \begin{pmatrix} D^2_{pp}\phi(p_\tau, q_\tau, t_\tau) & D^2_{pq}\phi(p_\tau, q_\tau, t_\tau) \\ \\ D^2_{qp}\phi(p_\tau, q_\tau, t_\tau) & D^2_{qq}\phi(p_\tau, q_\tau, t_\tau) \end{pmatrix}. \tag{3.5}$$

Suppose that

$$\lim_{\tau\to\infty} \tau\phi(p_\tau, q_\tau, t_\tau) = 0.$$

Then for each $\tau > 0$, there exists real numbers a_1 and a_2, symmetric matrices $\mathcal{X}_\tau$ and $\mathcal{Y}_\tau$, and a vector $\Upsilon_\tau \in V_1 \oplus V_2$, namely $\Upsilon_\tau = \nabla_1(p)\phi(p_\tau, q_\tau, t_\tau)$, such that the following hold:

A) $(a_1, \tau\Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ *and* $(a_2, \tau\Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$.
B) $a_1 - a_2 = \phi_t(p_\tau, q_\tau, t_\tau)$.
C) *For any vectors $\xi, \epsilon \in V_1$, we have*

$$\begin{aligned} \langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\epsilon, \epsilon\rangle \le{}& \tau\langle (D_p^2\phi)^\star(p_\tau, q_\tau, t_\tau)(\xi - \epsilon), (\xi - \epsilon)\rangle \\ &+ \tau\langle \mathfrak{W}(\xi \oplus \epsilon), (\xi \oplus \epsilon)\rangle \\ &+ \tau\|\mathcal{W}\|^2\|\mathbb{A}(\hat{p})^T\xi \oplus \mathbb{A}(\hat{q})^T\epsilon\|^2. \end{aligned} \tag{3.6}$$

In particular,

$$\langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\xi, \xi\rangle \lesssim \tau\|\mathcal{W}\|^2\|\xi\|^2. \tag{3.7}$$

COROLLARY 3.7. *Let $\phi(p, q, t) = \phi(p, q) = \varphi(p \cdot q^{-1})$ be a nonnegative function independent of t. Suppose that $\phi(p, q) = 0$ exactly when $p = q$. Then*

$$\lim_{\tau\to\infty} \tau\phi(p_\tau, q_\tau) = 0.$$

In particular, if

$$\phi(p, q, t) = \frac{1}{m}\sum_{i=1}^{N}((p \cdot q^{-1})_i)^m \tag{3.8}$$

*for some **even** integer $m \ge 4$ where $(p \cdot q^{-1})_i$ is the ith component of the Carnot group multiplication group law, then for the vector Υ_τ and matrices $\mathcal{X}_\tau$, $\mathcal{Y}_\tau$ from the lemma, we have*:

A) $(a_1, \tau\Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ *and* $(a_1, \tau\Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$.
B) *The vector Υ_τ satisfies*

$$\|\Upsilon_\tau\| \sim \phi(p_\tau, q_\tau)^{(m-1)/m}.$$

C) *For any fixed vector $\xi \in V_1$, we have*

$$\langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\xi, \xi\rangle \lesssim \tau\|\mathcal{W}\|^2\|\xi\|^2 \lesssim \tau(\phi(p_\tau, q_\tau))^{(2m-4)/m}\|\xi\|^2. \tag{3.9}$$

4. Uniqueness of Viscosity Solutions

We wish to formulate a comparison principle for the following problem.

PROBLEM 4.1. Let $h \geq 1$. Let Ω be a bounded domain, and let $\Omega_T = \Omega \times [0, T)$. Let $\psi \in C(\overline{\Omega})$ and $g \in C(\overline{\Omega_T})$. We consider the following boundary and initial value problem:

$$\begin{cases} u_t + F_\infty^h(\nabla_0 u, (D^2u)^\star) = 0 & \text{in } \Omega \times (0, T), \quad \text{(E)} \\ u(p, t) = g(p, t), & p \in \partial\Omega, t \in [0, T), \quad \text{(BC)} \\ u(p, 0) = \psi(p), & p \in \overline{\Omega}. \quad \text{(IC)} \end{cases} \tag{4.1}$$

We also adopt the definition that a subsolution $u(p, t)$ to Problem 4.1 is a viscosity subsolution to (E), $u(p, t) \leq g(p, t)$ on $\partial\Omega$ with $0 \leq t < T$, and $u(p, 0) \leq \psi(p)$ on $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous matter.

Because our solution u will be continuous, we offer the following remark.

REMARK 4.2. The functions ψ and g may be replaced by one function $g \in C(\overline{\Omega_T})$. This combines conditions (E) and (BC) into one condition

$$u(p, t) = g(p, t), \quad (p, t) \in \partial_{\text{par}}\Omega_T. \quad \text{(IBC)} \tag{4.2}$$

THEOREM 4.3. *Let Ω be a bounded domain in G, and let $h \geq 1$. If u is a parabolic viscosity subsolution and v a parabolic viscosity supersolution to Problem 4.1, then $u \leq v$ on $\Omega_T \equiv \Omega \times [0, T)$.*

Proof. Our proof follows that of [8, Thm. 8.2], and so we discuss only the main parts.

For $\varepsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$u_t + F_\infty^h(\nabla_0 u, (D^2u)^\star) \leq -\frac{\varepsilon}{T^2} < 0, \tag{4.3}$$

$$\lim_{t \uparrow T} u(p, t) = -\infty \quad \text{uniformly on } \overline{\Omega}, \tag{4.4}$$

and take limits to obtain the desired result. Assume that the maximum occurs at $(p_0, t_0) \in \Omega \times (0, T)$ with

$$u(p_0, t_0) - v(p_0, t_0) = \delta > 0.$$

Case 1: $h > 1$.

Let $H \geq h + 3$ be an even number. As in Equation (3.8), we let

$$\phi(p, q) = \frac{1}{H} \sum_{i=1}^{N} ((p \cdot q^{-1})_i)^H$$

where $(p \cdot q^{-1})_i$ is the ith component of the Carnot group multiplication group law. Let

$$M_\tau = u(p_\tau, t_\tau) - v(q_\tau, t_\tau) - \tau\phi(p_\tau, q_\tau)$$

with (p_τ, q_τ, t_τ) the maximum point in $\overline{\Omega} \times \overline{\Omega} \times [0,T)$ of $u(p,t) - v(q,t) - \tau\phi(p,q)$.

If $t_\tau = 0$, then we have

$$0 < \delta \le M_\tau \le \sup_{\overline{\Omega}\times\overline{\Omega}} (\psi(p) - \psi(q) - \tau\phi(p,q)),$$

leading to a contradiction for large τ. We therefore conclude that $t_\tau > 0$ for large τ. Since $u \le v$ on $\partial\Omega \times [0,T)$ by Equation (BC) of Problem 4.1, we conclude that for large τ, we have that (p_τ, q_τ, t_τ) is an interior point, that is, $(p_\tau, q_\tau, t_\tau) \in \Omega \times \Omega \times (0,T)$. Using Corollary 3.7, Property A, we obtain

$$\begin{aligned} &(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}_\tau) \in \overline{P}^{2,+} u(p_\tau, t_\tau) \\ \text{and} \quad &(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \in \overline{P}^{2,-} v(q_\tau, t_\tau), \end{aligned}$$

satisfying the equations

$$\begin{aligned} &a + F_\infty^h(\tau\Upsilon(p_\tau, q_\tau), \mathcal{X}_\tau) \le -\frac{\varepsilon}{T^2} \\ \text{and} \quad &a + F_\infty^h(\tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \ge 0. \end{aligned}$$

If there is a subsequence $\{p_\tau, q_\tau\}_{\tau>0}$ such that $p_\tau \ne q_\tau$, we subtract, and using Corollary 3.7, we have

$$\begin{aligned} 0 &< \frac{\varepsilon}{T^2} \\ &\le (\tau\Upsilon(p_\tau, q_\tau))^{h-3}\tau^2(\langle \mathcal{X}_\tau \Upsilon(p_\tau, q_\tau), \Upsilon(p_\tau, q_\tau)\rangle \\ &\quad - \langle \mathcal{Y}_\tau \Upsilon(p_\tau, q_\tau), \Upsilon(p_\tau, q_\tau)\rangle) \\ &\lesssim \tau^h (\varphi(p_\tau, q_\tau)^{(H-1)/H})^{h-3} (\varphi(p_\tau, q_\tau))^{(2H-4)/H} (\varphi(p_\tau, q_\tau))^{(2H-2)/H} \quad (4.5) \\ &= \tau^h (\varphi(p_\tau, q_\tau))^{(Hh+H-h-3)/H} = (\tau\varphi(p_\tau, q_\tau))^h \varphi(p_\tau, q_\tau)^{(H-h-3)/H}. \quad (4.6) \end{aligned}$$

Because $H > h + 3$, we arrive at a contradiction as $\tau \to \infty$.

If we have $p_\tau = q_\tau$, then we arrive at a contradiction since

$$F_\infty^h(\tau\Upsilon(p_\tau, q_\tau), \mathcal{X}_\tau) = F_\infty^h(\tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) = 0.$$

Case 2: $h = 1$.

We follow the proof of Theorem 3.1 in [12]. Let

$$\varphi(p,q,t,s) = \frac{1}{4}\sum_{i=1}^{N}((p \cdot q^{-1})_i)^4 + \frac{1}{2}(t-s)^2,$$

and let $(p_\tau, q_\tau, t_\tau, s_\tau)$ be the maximum of

$$u(p,t) - v(q,s) - \tau\phi(p,q,t,s).$$

Again, for large τ, this point is an interior point. If we have a sequence where $p_\tau \ne q_\tau$, then Lemma 3.2 yields

$$\begin{aligned} &(\tau(t_\tau - s_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}_\tau) \in \overline{P}^{2,+} u(p_\tau, t_\tau) \\ \text{and} \quad &(\tau(t_\tau - s_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \in \overline{P}^{2,-} v(q_\tau, s_\tau), \end{aligned}$$

satisfying the equations

$$\begin{aligned} &\tau(t_\tau - s_\tau) + F^h_\infty(\tau\Upsilon(p_\tau, q_\tau), \mathcal{X}_\tau) \leq -\frac{\varepsilon}{T^2} \\ \text{and} \quad &\tau(t_\tau - s_\tau) + F^h_\infty(\tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \geq 0. \end{aligned}$$

As in the first case, we subtract to obtain

$$\begin{aligned} 0 &< \frac{\varepsilon}{T^2} \\ &\leq (\tau\Upsilon(p_\tau, q_\tau))^{-2}\tau^2(\langle \mathcal{X}_\tau\Upsilon(p_\tau, q_\tau), \Upsilon(p_\tau, q_\tau)\rangle - \langle \mathcal{Y}_\tau\Upsilon(p_\tau, q_\tau), \Upsilon(p_\tau, q_\tau)\rangle) \\ &\lesssim \varphi(p_\tau, q_\tau)^{-3/2}(\tau\varphi(p_\tau, q_\tau)\varphi(p_\tau, q_\tau)^{3/2}) = \tau\varphi(p_\tau, q_\tau). \end{aligned}$$

We arrive at a contradiction as $\tau \to \infty$.

If $p_\tau = q_\tau$, then $v(q,s) - \beta^v(q,s)$ has a local minimum at (q_τ, s_τ) where

$$\beta^v(q,s) = -\frac{\tau}{4}\sum_{i=1}^N((p_\tau \cdot q^{-1})_i)^4 - \frac{\tau}{2}(t_\tau - s)^2.$$

We then have

$$0 < \varepsilon(T - s_\tau)^{-2} \leq \beta^v_s(q_\tau, s_\tau) - \min_{\|\eta\|=1}\langle (D^2\beta^v)^\star(q_\tau, s_\tau)\eta, \eta\rangle.$$

Similarly, $u(p,t) - \beta^u(p,t)$ has a local maximum at (p_τ, t_τ) where

$$\beta^u(p,t) = \frac{\tau}{4}\sum_{i=1}^N((p \cdot q_\tau^{-1})_i)^4 + \frac{\tau}{2}(t - s_\tau)^2.$$

We then have

$$0 \geq \beta^u_t(p_\tau, t_\tau) - \max_{\|\eta\|=1}\langle (D^2\beta^u)^\star(p_\tau, t_\tau)\eta, \eta\rangle,$$

and subtraction gives us

$$\begin{aligned} 0 &< \varepsilon(T - s_\tau)^{-2} \\ &\leq \max_{\|\eta\|=1}\langle (D^2\beta^u)^\star(p_\tau, t_\tau)\eta, \eta\rangle - \min_{\|\eta\|=1}\langle (D^2\beta^v)^\star(q_\tau, s_\tau)\eta, \eta\rangle \\ &\quad + \beta^v_s(q_\tau, s_\tau) - \beta^u_t(p_\tau, t_\tau) \\ &= \tau \max_{\|\eta\|=1}\langle (D^2_{pp}\varphi(p \cdot q_\tau^{-1}))^\star(p_\tau, t_\tau)\eta, \eta\rangle \\ &\quad - \tau \min_{\|\eta\|=1}\langle (D^2_{qq}\varphi(p_\tau \cdot q^{-1}))^\star(q_\tau, s_\tau)\eta, \eta\rangle \\ &\quad + \tau(t_\tau - s_\tau) - \tau(t_\tau - s_\tau) \\ &= 0. \end{aligned}$$

Here, the last equality comes from the fact that $p_\tau = q_\tau$ and from the definition of $\varphi(p \cdot q^{-1})$. $\square$

The comparison principle has the following consequences concerning properties of solutions.

COROLLARY 4.4. *Let $h \geq 1$. The past parabolic viscosity h-infinite solutions are exactly the parabolic viscosity h-infinite solutions.*

Proof. By Proposition 3.5 past parabolic viscosity h-infinite sub(super)solutions are parabolic viscosity h-infinite sub(super)solutions. To prove the converse, we will follow the proof of the subsolution case found in [11], highlighting the main details. Assume that u is not a past parabolic viscosity h-infinite subsolution. Let $\phi \in \mathcal{A}^- u(p_0, t_0)$ have the property that

$$\phi_t(p_0, t_0) - \Delta_\infty^h \phi(p_0, t_0) \geq \epsilon > 0$$

for a small parameter ϵ. We may assume that p_0 is the origin. Let $r > 0$ and define $S_r = B_{\mathcal{N}}(r) \times (t_0 - r, t_0)$, and let ∂S_r be its parabolic boundary. Then the function

$$\tilde{\phi}_r(p,t) = \phi(p,t) + (t_0 - t)^{8l!} - r^{8l!} + (\mathcal{N}(p))^{8l!}$$

is a classical supersolution for sufficiently small r. We then observe that $u \leq \tilde{\phi}_r$ on ∂S_r but $u(0, t_0) > \tilde{\phi}(0, t_0)$. Thus, the comparison principle, Theorem 4.3, does not hold. Thus, u is not a parabolic viscosity h-infinite subsolution. The supersolution case is identical and omitted. □

The following corollary has a proof similar to that of [12, Lemma 3.2].

COROLLARY 4.5. *Let $u : \Omega_T \to \mathbb{R}$ be upper semicontinuous. Let $\phi \in \mathcal{A}u(p_0, t_0)$. If*

$$\begin{cases} \phi_t(p_0, t_0) - \Delta_\infty^1 \phi(p_0, t_0) \leq 0 & \textit{when } \nabla_0 \phi(p_0, t_0) \neq 0, \\ \phi_t(p_0, t_0) \leq 0 & \textit{when } \nabla_0 \phi(p_0, t_0) = 0, \\ & (D^2\phi)^\star(p_0, t_0) = 0, \end{cases} \tag{4.7}$$

then u is a viscosity subsolution to (E) of Problem 4.1.

We also have the following function estimates with respect to boundary data.

COROLLARY 4.6. *Let $h \geq 1$. Let $g_1, g_2 \in C(\overline{\Omega_T})$ and u_1, u_2 be parabolic viscosity solutions to Equation (4.1) with boundary data g_1 and g_2, respectively. Then*

$$\sup_{(p,t) \in \Omega_T} |u_1(p,t) - u_2(p,t)| \leq \sup_{(p,t) \in \partial_{\text{par}} \Omega_T} |g_1(p,t) - g_2(p,t)|.$$

Proof. The function $u^+(p,t) = u_2(p,t) + \sup_{(p,t) \in \partial_{\text{par}} \Omega_T} |g_1(p,t) - g_2(p,t)|$ is a parabolic viscosity supersolution with boundary data g_1, and the function $u^-(p,t) = u_2(p,t) - \sup_{(p,t) \in \partial_{\text{par}} \Omega_T} |g_1(p,t) - g_2(p,t)|$ is a parabolic viscosity subsolution with boundary data g_1. Moreover, $u^- \leq u_1 \leq u^+$ on $\partial_{\text{par}} \Omega_T$, and by Theorem 4.3 $u^- \leq u_1 \leq u^+$ in Ω_T. □

COROLLARY 4.7. *Let $h \geq 1$. Let $g \in C(\overline{\Omega_T})$. Then every parabolic viscosity solution to Problem 4.1 satisfies*

$$\sup_{(p,t) \in \Omega_T} |u(p,t)| \leq \sup_{(p,t) \in \partial_{\text{par}} \Omega_T} |g(p,t)|.$$

Proof. The proof is similar to the previous corollary but using the functions $u^{\pm}(p,t) = \pm \sup_{(p,t)\in\partial_{\text{par}}\Omega_T} |g(p,t)|$ instead. □

5. Existence of Viscosity Solutions

5.1. Parabolic Viscosity Infinite Solutions: The Continuity Case

As before, we will focus on the equations of the form (3.1) for continuous and proper $F : [0,T] \times G \times \mathbb{R} \times g \times S^{n_1} \to \mathbb{R}$ that possess a comparison principle such as Theorem 4.3 or [6, Thm. 3.6]. We will use Perron's method combined with the Carnot parabolic maximum principle to yield the desired existence theorem. In particular, the following proofs are similar to those found in [10, Chap. 2] except that the Euclidean derivatives have been replaced with horizontal derivatives and the Euclidean norms have been replaced with the gauge norm.

LEMMA 5.1. *Let $\mathcal{L}$ be a collection of parabolic viscosity supersolutions to* (3.1), *and let* $u(p,t) = \inf\{v(p,t) : v \in \mathcal{L}\}$. *If u is finite in a dense subset of* $\Omega_T = \Omega \times [0,T)$, *then u is a parabolic viscosity supersolution to* (3.1).

Proof. First, note that u is lower semicontinuous since every $v \in \mathcal{L}$ is. Let $(p_0,t_0) \in \Omega_T$ and $\phi \in \mathcal{A}u(p_0,t_0)$. Now let

$$\psi(p,t) = \phi(p,t) - (d_{\mathcal{N}}(p_0,p))^{2l!} - |t-t_0|^2$$

and notice that $\psi \in \mathcal{A}u(p_0,t_0)$. Then

$$\begin{aligned}(u-\psi)(p,t) - (d_{\mathcal{N}}(p_0,p))^{2l!} - |t-t_0|^2 &= (u-\phi)(p,t)\\ &\geq (u-\phi)(p_0,t_0)\\ &= (u-\psi)(p_0,t_0)\\ &= 0\end{aligned}$$

yields

$$(u-\psi)(p,t) \geq (d_{\mathcal{N}}(p_0,p))^{2l!} + |t-t_0|^2. \tag{5.1}$$

Since u is lower semicontinuous, there exists a sequence $\{(p_k,t_k)\}$ with $t_k < t_0$ converging to (p_0,t_0) as $k \to \infty$ such that

$$(u-\psi)(p_k,t_k) \to (u-\psi)(p_0,t_0) = 0.$$

Since $u(p,t) = \inf\{v(p,t) : v \in \mathcal{L}\}$, there exists a sequence $\{v_k\} \subset \mathcal{L}$ such that $v_k(p_k,t_k) < u(p_k,t_k) + 1/k$ for $k = 1,2,\ldots$. Since $v_k \geq u$, Equation (5.1) gives us

$$(v_k-\psi)(p,t) \geq (u-\psi)(p,t) \geq (d_{\mathcal{N}}(p_0,p))^{2l!} + |t-t_0|^2. \tag{5.2}$$

Let $B \subset \Omega$ denote a compact neighborhood of (p_0,t_0). Since $v_k - \psi$ is lower semicontinuous, it attains a minimum in B at a point $(q_k,s_k) \in B$. Then by (5.1) and (5.2) we have

$$\begin{aligned}(u-\psi)(p_k,t_k) + 1/k > (v_k-\psi)(p_k,t_k) &\geq (v_k-\psi)(q_k,s_k)\\ &\geq (d_{\mathcal{N}}(p_0,q_k))^{2l!} + |s_k-t_0|^2 \geq 0\end{aligned}$$

for sufficiently large k such that $(p_k, t_k) \in B$. By the squeeze theorem, $(q_k, s_k) \to (p_0, t_0)$ as $k \to \infty$. Let $\eta = \psi - (d_{\mathcal{N}}(q_k, p))^{2l!} - |s_k - t|^2$. Then $\eta \in \mathcal{A}v_k(q_k, s_k)$, and we have that

$$\eta_t(q_k, s_k) + F(s_k, q_k, v_k(q_k, s_k), \nabla_1 \eta(q_k, s_k), (D^2\eta(q_k, s_k))^\star) \geq 0.$$

This implies

$$\psi_t(q_k, s_k) + F(s_k, q_k, v_k(q_k, s_k), \nabla_1 \psi(q_k, s_k), (D^2\psi(s_k, s_k))^\star) \geq 0.$$

Letting $k \to \infty$ yields

$$\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0)\nabla_1 \phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \geq 0$$

and that u is a parabolic viscosity supersolution, as desired. $\square$

A similar argument yields the following:

LEMMA 5.2. *Let $\mathcal{L}$ be a collection of parabolic viscosity subsolutions to* (3.1), *and let $u(p, t) = \sup\{v(p, t) : v \in \mathcal{L}\}$. If u is finite in a dense subset of Ω_T, then u is a parabolic viscosity subsolution to* (3.1).

For the following lemmas, we need to recall the following definition.

DEFINITION 4. The *upper and lower semicontinuous envelopes* of a function u are given by

$$u^*(p, t) := \lim_{r \downarrow 0} \sup\{u(q, s) : |q^{-1}p|_g + |s - t| \leq r\}$$

and

$$u_*(p, t) := \lim_{r \downarrow 0} \inf\{u(q, s) : |q^{-1}p|_g + |s - t| \leq r\},$$

respectively.

LEMMA 5.3. *Let h be a parabolic viscosity supersolution to* (3.1) *in Ω_T. Let $\mathcal{S}$ be the collection of all parabolic viscosity subsolutions v of* (3.1) *satisfying $v \leq h$. If for $\hat{v} \in \mathcal{S}$, $\hat{v}_*$ is not a parabolic viscosity supersolution of* (3.1), *then there are a function $w \in \mathcal{S}$ and a point (p_0, t_0) such that $\hat{v}(p_0, t_0) < w(p_0, t_0)$.*

Proof. Let $\hat{v} \in \mathcal{S}$ such that $\hat{v}_*$ is not a parabolic viscosity supersolution of (3.1). Then there exist $(\hat{p}, \hat{t}) \in \Omega_T$ and $\phi \in \mathcal{A}\hat{v}_*(\hat{p}, \hat{t})$ such that

$$\phi_t(p, t) + F(t, p, \hat{v}_*(p, t), \nabla_1 \phi(p, t), (D^2\phi(p, t))^\star) > 0. \tag{5.3}$$

Let

$$\psi(p, t) = \phi(p, t) - (d_{\mathcal{N}}(\hat{p}, p))^{2l!} - |t - \hat{t}|^2$$

and notice that $\psi \in \mathcal{A}\hat{v}_*(\hat{p}, \hat{t})$. As in Lemma 5.1,

$$(\hat{v}_* - \psi)(p, t) \geq (d_{\mathcal{N}}(\hat{p}, p))^{2l!} + |t - \hat{t}|^2. \tag{5.4}$$

Let B denote a compact neighborhood of $(\hat{p}, \hat{t})$, and let

$$B_{k\epsilon} = B \cap \{(p, t) : (d_{\mathcal{N}}(\hat{p}, p))^{2l!} \leq k\epsilon \text{ and } |t - \hat{t}|^2 \leq k\epsilon\}.$$

Since $\hat{v} \in \mathcal{S}$, we have that $\hat{v} \le h$, and thus $\psi(\hat{p},\hat{t}) = \hat{v}_*(\hat{p},\hat{t}) \le \hat{v}(\hat{p},\hat{t}) \le h(\hat{p},\hat{t})$. However, if $\psi(\hat{p},\hat{t}) = h(\hat{p},\hat{t})$, then $\psi \in \mathcal{A}h(\hat{p},\hat{t})$, and inequality (5.3) would be contradictory. Thus,

$$\psi(\hat{p},\hat{t}) < h(\hat{p},\hat{t}).$$

Since ψ is continuous and h is lower semicontinuous, there exists $\epsilon > 0$ such that

$$\psi(p,t) + 4\epsilon \le h(p,t)$$

for $(p,t) \in B_{2\epsilon}$. Notice that $\psi + 4\epsilon$ is a subsolution of (3.1) on the interior of $B_{2\epsilon}$. Further, by (5.4)

$$\hat{v}(p,t) \ge \hat{v}_*(p,t) \ge \psi(p,t) + 4\epsilon \quad \text{for } (p,t) \in B_{2\epsilon} \backslash B_\epsilon. \tag{5.5}$$

We now define ω by

$$\omega = \begin{cases} \max\{\psi(p,t) + 4\epsilon, \hat{v}(p,t)\}, & (p,t) \in B_\epsilon, \\ \hat{v}(p,t), & (p,t) \in \Omega_T \backslash B_\epsilon. \end{cases}$$

But by (5.5)

$$\omega(p,t) = \max\{\psi(p,t) + 4\epsilon, \hat{v}(p,t)\} \quad \text{for } (p,t) \in B_{2\epsilon},$$

not just for $(p,t) \in B_\epsilon$. Then by Lemma 5.2, ω is a subsolution in the interior of $B_{2\epsilon}$ and thus a subsolution in Ω_T. Therefore, $\omega \in \mathcal{S}$. Since

$$0 = (\hat{v}_* - \psi)(\hat{p},\hat{t}) = \liminf_{r \downarrow 0}\{(\hat{v} - \psi)(p,t) : (p,t) \in B_r\},$$

there is a point $(p_0,t_0) \in B_\epsilon$ that satisfies

$$\hat{v}(p_0,t_0) - \psi(p_0,t_0) < 4\epsilon,$$

which yields

$$\hat{v}(p_0,t_0) < \psi(p_0,t_0) + 4\epsilon = \omega(p_0,t_0).$$

Thus, we have constructed $\omega \in \mathcal{S}$ that satisfies $\hat{v}(p_0,t_0) < \omega(p_0,t_0)$. □

We then have the following existence theorem concerning parabolic viscosity solutions.

THEOREM 5.4. *Let f be a parabolic viscosity subsolution to (3.1), and g be a parabolic viscosity supersolution to (3.1) satisfying $f \le g$ on Ω_T and $f_* = g^*$ on $\partial_{\mathrm{par}}\mathcal{O}_{0,T}$. Then there is a parabolic viscosity solution u to (3.1) satisfying $u \in C(\overline{O_T})$. Explicitly, there exists a unique parabolic viscosity infinite solution to Problem 4.1 when $h > 1$.*

Proof. Let

$$S = \{\nu : \nu \text{ is a parabolic viscosity subsolution to (3.1) in } \Omega_T \text{ with } \nu \le g \text{ in } \Omega_T\}$$

and

$$u(p,t) = \sup\{\nu(p,t) : \nu \in S\}.$$

Since $f \le g$, the set S is nonempty. Notice that $f \le u \le g$ by construction. By Lemma 5.2, u is a parabolic viscosity subsolution. Suppose u_* is not a parabolic viscosity supersolution. Then by Lemma 5.3 there exist a function $w \in S$ and a point $(p_0, t_0) \in \Omega_T$ such that $u(p_0, t_0) < w(p_0, t_0)$. But this contradicts the definition of u at (p_0, t_0). Thus, u_* is a parabolic viscosity supersolution. By our assumptions on f and g on $\partial_{\mathrm{par}}\mathcal{O}_{0,T}$,

$$u = u^* \le g^* = f_* \le u_*$$

on $\partial_{\mathrm{par}}\mathcal{O}_{0,T}$. Then by the (assumed) comparison principle, $u \le u_*$ on Ω_T. Thus, we have that u is a parabolic viscosity solution such that $u \in C(\overline{O_T})$. □

5.2. *The $h = 1$ Case*

We begin by recalling the definition of upper and lower relaxed limits of a function [8; 10].

DEFINITION 5. For $\varepsilon > 0$, consider the function $\mathfrak{h}_\varepsilon : O_T \subset G \to \mathbb{R}$. The *upper relaxed limit* $\overline{\mathfrak{h}}(p,t)$ and the lower relaxed limit $\underline{\mathfrak{h}}(p,t)$ are given by

$$\begin{aligned}
\overline{\mathfrak{h}}(p,t) &= \limsup_{\hat{p}\to p,\hat{t}\to t,\varepsilon\to 0} \mathfrak{h}_\varepsilon(\hat{p},\hat{t}) \\
&= \lim_{\varepsilon\to 0} \sup_{0<\delta<\varepsilon} \{\mathfrak{h}_\delta(\hat{p},\hat{t}) : O_T \cap B_\varepsilon(\hat{p},\hat{t})\} \\
\text{and} \quad \underline{\mathfrak{h}}(p,t) &= \liminf_{\hat{p}\to p,\hat{t}\to t,\varepsilon\to 0} \mathfrak{h}_\varepsilon(\hat{p},\hat{t}) \\
&= \lim_{\varepsilon\to 0} \inf_{0<\delta<\varepsilon} \{\mathfrak{h}_\delta(\hat{p},\hat{t}) : O_T \cap B_\varepsilon(\hat{p},\hat{t})\}.
\end{aligned}$$

Taking the relaxed limits as $h \to 1^+$ of the operator $F_\infty^h(\nabla_0 u, (D^2u)^\star)$ in Equation (3.2), we have via the continuity of the operator

$$\begin{aligned}
\overline{F}_\infty^1(\nabla_0 u, (D^2u)^\star) &= \underline{F}_\infty^1(\nabla_0 u, (D^2u)^\star) \\
&= \begin{cases} -\|\nabla_0 u\|^{-2}\langle (D^2u)^\star \nabla_0 u, \nabla_0 u\rangle, & \nabla_0 u \neq 0, \\ 0, & \nabla_0 u = 0. \end{cases}
\end{aligned}$$

We give this operator the label $\mathcal{F}(\nabla_0 u, (D^2u)^\star)$. Consider the relaxed limits $\overline{u}(p,t)$ and $\underline{u}(p,t)$ of the sequence of unique (continuous) viscosity solutions to Problem 4.1 $\{u_h(p,t)\}$ as $h \to 1^+$. By [10, Thm. 2.2.1] we have that $\overline{u}(p,t)$ is a viscosity subsolution and $\underline{u}(p,t)$ is a viscosity supersolution to

$$u_t + \mathcal{F}(\nabla_0 u, (D^2u)^\star) = 0.$$

We have the following comparison principle, whose proof is similar to that of Theorem 4.3 in the case $h = 1$ and is omitted.

LEMMA 5.5. *Let Ω be a bounded domain in G. If $\mathfrak{u}$ is a parabolic viscosity subsolution and $\mathfrak{v}$ a parabolic viscosity supersolution to*

$$u_t + \mathcal{F}(\nabla_0 u, (D^2u)^\star) = 0,$$

then $\mathfrak{u} \le \mathfrak{v}$ on $\Omega_T \equiv \Omega \times [0,T)$.

COROLLARY 5.6. $\overline{u}(p,t) = \underline{u}(p,t)$.

Proof. By construction, $\underline{u}(p,t) \leq \overline{u}(p,t)$. By the lemma, $\underline{u}(p,t) \geq \overline{u}(p,t)$. □

REMARK 5.7. Using the corollary, we will call this common relaxed limit $u^1(p,t)$. By [10, Chap. 2] and [8, Sect. 6], it is continuous, and the sequence $\{u_h(p,t)\}$ converges locally uniformly to $u^1(p,t)$ as $h \to 1^+$.

We then have the following theorem.

THEOREM 5.8. *There exists a unique parabolic viscosity infinite solution to Problem 4.1 when $h = 1$.*

Proof. Let $\{u_h(p,t)\}$ and $u^1(p,t)$ be as before. Let $\{h_j\}$ be a subsequence with $h_j \to 1^+$ where $u_h(p,t) \to u^1(p,t)$ uniformly. We may assume that $h_j < 3$.

Let $\phi \in \mathcal{A}u_1(p_0,t_0)$. Using the uniform convergence, there is a sequence $\{p_j,t_j\} \to (p_0,t_0)$ such that $\phi \in \mathcal{A}u_{h_j}(p_j,t_j)$. If $\nabla_0\phi(p_0,t_0) \neq 0$, then we have $\nabla_0\phi(p_j,t_j) \neq 0$ for sufficiently large j. We then have

$$\phi_t(p_j,t_j) - \Delta_\infty^{h_j}\phi(p_j,t_j) \leq 0,$$

and letting $j \to \infty$ yields

$$\phi_t(p_0,t_0) - \Delta_\infty^1\phi(p_0,t_0) \leq 0.$$

Suppose $\nabla_0\phi(p_0,t_0) = 0$. By Corollary 4.5 we may assume that $(D^2\phi)^\star(p_0, t_0) = 0$. Passing to a subsequence if needed, we have $\nabla_0\phi(p_j,t_j) \neq 0$. Then

$$\phi_t(p_j,t_j) - \max_{\|\eta\|=1} \langle (D^2\phi)^\star(p_j,t_j)\eta,\eta\rangle \leq \phi_t(p_j,t_j) - \Delta_\infty^{h_j}\phi(p_j,t_j) \leq 0.$$

Letting $j \to \infty$ yields

$$\phi_t(p_0,t_0) = \phi_t(p_j,t_j) - \max_{\|\eta\|=1} \langle (D^2\phi)^\star(p_0,t_0)\eta,\eta\rangle \leq 0.$$

In the case $\nabla_0\phi(p_j,t_j) = 0$, since $h_j < 3$, we have $\phi_t(p_j,t_j) \leq 0$, and letting $j \to \infty$ yields $\phi_t(p_0,t_0) \leq 0$. We conclude that u_1 is a parabolic viscosity h-infinite subsolution. Similarly, u_1 is a parabolic viscosity h-infinite supersolution. □

6. The Limit as $t \to \infty$

We now focus our attention on the asymptotic limits of the parabolic viscosity h-infinite solutions. We wish to show that for $1 \leq h$, we have that the (unique) viscosity solution to $u_t - \Delta_\infty^h u = 0$ approaches the viscosity solution of $-\Delta_\infty^h u = 0$ as $t \to \infty$. Our goal is the following theorem.

THEOREM 6.1. *Let $h > 1$, and let $u \in C(\overline{\Omega} \times [0,\infty))$ be a viscosity solution of*

$$\begin{cases} u_t - \Delta_\infty^h u = 0 & \text{in } \Omega \times (0,\infty), \\ u(p,t) = g(p) & \text{on } \partial_{\text{par}}(\Omega \times (0,\infty)) \end{cases} \tag{6.1}$$

with $g : \overline{\Omega} \to \mathbb{R}$ continuous and assuming that $\partial\Omega$ satisfies the property of positive geometric density (see [11, p. 2,909]). Then $u(p,t) \to U(p)$ uniformly in Ω

as $t \to \infty$ where $U(p)$ is the unique viscosity solution of $-\Delta_\infty^h U = 0$ with the Dirichlet boundary condition $\lim_{q\to p} U(q) = g(p)$ for all $p \in \partial\Omega$.

We first must establish the uniqueness of viscosity solutions to the limit equation. Note that for future reference, we include the case $h = 1$.

THEOREM 6.2. *Let $1 \le h < \infty$, and let Ω be a bounded domain. Let u be a viscosity subsolution to $\Delta_\infty^h u = 0$, and let v be a viscosity supersolution to $-\Delta_\infty^h u = 0$. Then,*

$$\sup_{p\in\overline{\Omega}}(u(p) - v(p)) = \sup_{p\in\partial\Omega}(u(p) - v(p)).$$

Proof. Let u be a viscosity subsolution to $-\Delta_\infty^h u = 0$. Then choose $\phi \in \mathcal{C}^2_{\text{sub}}(\Omega)$ such that $0 = \phi(p_0) - u(p_0) < \phi(p) - u(p)$ for $p \in \Omega$, $p \neq p_0$. If $\|\nabla_0\phi(p_0)\| = 0$, then $-\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle = 0 \le 0$. If $\|\nabla_0\phi(p_0)\| \neq 0$, we then have

$$-\Delta_\infty^h \phi(p_0) = -\|\nabla_0\phi(p_0)\|^{h-3}\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle \le 0.$$

Dividing, we have $-\langle (D^2\phi)^\star(p_0)\nabla_0\phi(p_0), \nabla_0\phi(p_0)\rangle \le 0$. In either case, u is a viscosity subsolution to $-\Delta_\infty^3 u = 0$. Similarly, v is a viscosity supersolution to $-\Delta_\infty^3 u = 0$. The theorem follows from the corresponding result for $-\Delta_\infty^3 u = 0$ in [5; 3; 14]. □

We state some obvious corollaries.

COROLLARY 6.3. *Let $1 \le h < \infty$, and let $g : \partial\Omega \to \mathbb{R}$ be continuous. Then there is exactly one solution to*

$$\begin{cases} -\Delta_\infty^h u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

COROLLARY 6.4. *Let $1 \le h < \infty$, and let $g : \partial\Omega \to \mathbb{R}$ be continuous. The unique viscosity solution to*

$$\begin{cases} -\Delta_\infty^h u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

is the unique viscosity solution to

$$\begin{cases} -\Delta_\infty^3 u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Our method of proof for Theorem 6.1 follows that of [11, Thm. 2], the core of which hinges on the construction of a parabolic test function from an elliptic one. In order to construct such a parabolic test function, we need to examine the homogeneity of Equation (6.1). A quick calculation shows that for a fixed $h > 1$, $k^{1/(h-1)}u(x, kt)$ is a $\mathcal{C}^2_{\text{sub}}$ solution to Equation (6.1) if $u(x,t)$ is a $\mathcal{C}^2_{\text{sub}}$ solution. A routine calculation then shows that parabolic viscosity h-infinite solutions share this homogeneity. We use this property in the following lemma, the proof of which can be found in [9, p. 170]. (Also, cf. [6, Lemma 6.2] and [11].)

LEMMA 6.5. *Let u be as in Theorem 6.1, and $h > 1$. Then for every $(x,t) \in \Omega \times (0,\infty)$ and for $0 < \mathcal{T} < t$, we have*

$$|u(x,t-\mathcal{T}) - u(x,t)| \leq \frac{2\|g\|_{\infty,\Omega}}{h-1}\left(1-\frac{\mathcal{T}}{t}\right)^{h/(1-h)}\frac{\mathcal{T}}{t}.$$

Proof of Theorem 6.1. Fix $h > 1$. Let u be a viscosity solution of (6.1). The results of [9, Chap. III] imply that the family $\{u(\cdot,t) : t \in (0,\infty)\}$ is equicontinuous. Since it is uniformly bounded due to the boundedness of g, Arzelà–Ascoli's theorem yields that there exists a sequence $t_j \to \infty$ such that $u(\cdot,t_j)$ converge uniformly in $\overline{\Omega}$ to a function $U \in C(\overline{\Omega})$ for which $U(p) = g(p)$ for all $p \in \partial\Omega$. By Corollary 6.3 it suffices to show that U is a viscosity subsolution to $-\Delta_\infty^h U = 0$ on Ω. With that in mind, let $p_0 \in \Omega$ and choose $\phi \in \mathcal{C}^2_{\mathrm{sub}}(\Omega)$ such that $0 = \phi(p_0) - U(p_0) < \phi(p) - U(p)$ for $p \in \Omega$, $p \neq p_0$. Using the uniform convergence, we can find a sequence $p_j \to p_0$ such that $u(\cdot,t_j) - \phi$ has a local maximum at p_j. Now define

$$\phi_j(p,t) = \phi(p) + C\left(\frac{t}{t_j}\right)^{h/(1-h)}\frac{t_j - t}{t_j},$$

where $C = 2\|g\|_{\infty,\Omega}/(h-1)$. Note that $\phi_j(p,t) \in \mathcal{C}^2_{\mathrm{sub}}(\Omega \times (0,\infty))$. Then using Lemma 6.5, we have

$$\begin{aligned} u(p_j,t_j) - \phi_j(p_j,t_j) &= u(p_j,t_j) - \phi(p_j) \geq u(p,t_j) - \phi(p) \\ &\geq u(p,t) - \phi(p) - C\left(\frac{t}{t_j}\right)^{h/(1-h)}\frac{t_j - t}{t_j} \\ &= u(p,t) - \phi_j(p,t) \end{aligned}$$

for any $p \in \Omega$ and $0 < t < t_j$. Thus, we have that ϕ_j is an admissible test function at (p_j,t_j) on $\Omega \times [0,T]$. Therefore,

$$(\phi_j)_t(p_j,t_j) - \Delta_\infty^h \phi_j(p_j,t_j) \leq 0.$$

This yields

$$-\Delta_\infty^h \phi(p_j) \leq \frac{C}{t_j}.$$

The theorem follows by letting $j \to \infty$. □

Combining the results of the previous sections, we have the following theorem.

THEOREM 6.6. *Let Ω be a bounded domain where $\partial\Omega$ satisfies the property of positive geometric density. Let $h \geq 1$, and let $g : \overline{\Omega} \to \mathbb{R}$ be continuous. Let $u^{h,t}$ be the unique viscosity solution to*

$$\begin{cases} u_t^{h,t} - \Delta_\infty^h u^{h,t} = 0 & \text{in } \Omega \times (0,\infty), \\ u^{h,t}(p,t) = g(p) & \text{on } \partial_{\mathrm{par}}(\Omega \times (0,\infty)), \end{cases}$$

and let $u^{h,\infty}$ *be the unique viscosity solution to*

$$\begin{cases} -\Delta_\infty^h u^{h,\infty} = 0 & \text{in } \Omega, \\ u^{h,\infty} = g & \text{on } \partial\Omega \end{cases}$$

with the Dirichlet boundary condition $\lim_{q\to p} u^{h,\infty}(q) = g(p)$ *for all* $p \in \partial\Omega$. *Then*

$$\lim_{\substack{h\to 1^+ \\ t\to\infty}} u^{h,t} = \lim_{h\to 1^+}\lim_{t\to\infty} u^{h,t} = \lim_{t\to\infty}\lim_{h\to 1^+} u^{h,t} = u^{1,\infty}.$$

That is, the following diagram commutes:

$$\begin{tikzcd} u_t^{h,t} - \Delta_\infty^h u^{h,t} = 0 \arrow[r, "h\to 1^+"'] \arrow[d, "t\to\infty"] & u_t^{1,t} - \Delta_\infty^1 u^{1,t} = 0 \arrow[d, "t\to\infty"] \\ -\Delta_\infty^h u^{h,\infty} = 0 \arrow[r, "h\to 1^+"'] & -\Delta_\infty^1 u^{1,\infty} = 0 \end{tikzcd}$$

Proof. By Theorem 6.1,

$$\lim_{t\to\infty} u^{h,t} = u^{h,\infty}, \tag{6.2}$$

and the convergence is uniform. By Corollary 6.4,

$$\lim_{h\to 1^+} u^{h,\infty} = u^{1,\infty},$$

and this convergence is clearly uniform. We thus have the iterated limit

$$\lim_{h\to 1^+}\lim_{t\to\infty} u^{h,t} = u^{1,\infty}$$

with both limits converging uniformly. By Remark 5.7 we have

$$\lim_{h\to 1^+} u^{h,t} = u^{1,t},$$

and this convergence is locally uniform. By the proof of Theorem 6.1 we have

$$\lim_{t\to\infty} u^{1,t} = f \tag{6.3}$$

for some function f, and the convergence is uniform. We then have

$$\lim_{t\to\infty}\lim_{h\to 1^+} u^{h,t} = f$$

with the inner limit locally uniform and the outer limit uniform. By the results of iterated limits in [1, Sect. 19] we then have that the full (double) limit exists. In addition, the full limit and both iterated limits equal. That is, $f = u^{1,\infty}$ and

$$\lim_{\substack{h\to 1^+ \\ t\to\infty}} u^{h,t} = \lim_{h\to 1^+}\lim_{t\to\infty} u^{h,t} = \lim_{t\to\infty}\lim_{h\to 1^+} u^{h,t} = u^{1,\infty}.$$

□

References

[1] R. Bartle, *The elements of real analysis,* 2nd edition, John Wiley and Sons, New York, 1976.

[2] A. Bellaïche, *The tangent space in sub-Riemannian geometry,* Sub-Riemannian geometry (A. Bellaïche, J.-J. Risler, eds.), Progr. Math., 144, pp. 1–78, Birkhäuser, Basel, Switzerland, 1996.

[3] T. Bieske, *On infinite harmonic functions on the Heisenberg group,* Comm. Partial Differential Equations 27 (2002), no. 3&4, 727–762.

[4] ______, *Comparison principle for parabolic equations in the Heisenberg group,* Electron. J. Differ. Equ. 2005 (2005), no. 95, 1–11.

[5] ______, *A sub-Riemannian maximum principle and its application to the p-Laplacian in Carnot groups,* Ann. Acad. Sci. Fenn. Math. 37 (2012), 119–134.

[6] T. Bieske and E. Martin, *The parabolic p-Laplace equation in Carnot groups,* Ann. Acad. Sci. Fenn. Math. 39 (2014), 605–623.

[7] N. Bourbaki, *Lie groups and Lie algebras, Chapters 1–3,* Elements of Mathematics, Springer-Verlag, Berlin, 1989.

[8] M. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc. 27 (1992), no. 1, 1–67.

[9] E. DiBenedetto, *Degenerate parabolic equations,* Springer-Verlag, New York, 1993.

[10] Y. Giga, *Surface evolution equations: a level set approach,* Monogr. Math., 99, Birkhäuser Verlag, Basel, Switzerland, 2006.

[11] P. Juutinen, *On the definition of viscosity solutions for parabolic equations,* Proc. Amer. Math. Soc. 129 (2001), no. 10, 2907–2911.

[12] P. Juutinen and B. Kawohl, *On the evolution governed by the infinite Laplacian,* Math. Ann. 335 (2006), no. 4, 819–851.

[13] M. Portilheiro, Vázquez, and J. Luis, *Degenerate homogeneous parabolic equations associated with the infinity-Laplacian,* Calc. Var. Partial Differential Equations 46 (2013), no. 3–4, 705–724.

[14] C. Y. Wang, *The Aronsson equation for gradient minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition,* Trans. Amer. Math. Soc. 359 (2007), 91–113.

T. Bieske
Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Ave. CMC342
Tampa, FL 33620
USA

tbieske@usf.edu

E. Martin
Department of Mathematics and Physics
Westminster College
501 Westminster Ave
Fulton, MO 65251
USA

Erin.Martin@westminster-mo.edu