Michigan Math. J. 65 (2016), 413–440

Newton–Okounkov Bodies of Bott–Samelson Varieties and Grossberg–Karshon Twisted Cubes

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Abstract. We describe, under certain conditions, the Newton–Okounkov body of a Bott–Samelson variety as a lattice polytope defined by an explicit list of inequalities. The valuation that we use to define the Newton–Okounkov body is different from that used previously in the literature. The polytope that arises is a special case of the Grossberg–Karshon twisted cubes studied by Grossberg and Karshon in connection to character formulae for irreducible G-representations and also studied previously by the authors in relation to certain toric varieties associated to Bott–Samelson varieties. In particular, the Grossberg–Karshon twisted cubes that appear in the present manuscript are in fact untwisted (though possibly degenerate).

Introduction

The main result of this paper is an explicit computation of a Newton–Okounkov body associated to a Bott–Samelson variety under certain hypotheses. To place our result in context, recall that the recent theory of Newton–Okounkov bodies, introduced independently by Kaveh and Khovanskii [10] and Lazarsfeld and Mustata [15], associates to a complex algebraic variety X (equipped with some auxiliary data) a convex body of dimension $n = \dim_{\mathbb{C}}(X)$. In some cases, this convex body (the *Newton–Okounkov body,* also called *Okounkov body*) is a rational polytope; indeed, if X is a projective toric variety, then we can recover the usual moment polytope of X as a Newton–Okounkov body. These Newton–Okounkov bodies have been shown to be related to many other research areas, including (but certainly not limited to) toric degenerations [1], representation theory [8], symplectic geometry [6], and Schubert calculus [11; 12]. However, relatively few explicit examples of Newton–Okounkov bodies have been computed so far, and thus it is an interesting problem to give new and concrete examples.

Motivated by all this, in this paper we study the Newton–Okounkov bodies of Bott–Samelson varieties; these varieties are well known and studied in representation theory due to their relation to Schubert varieties and flag varieties (see e.g. [3]) and have been studied in the context of Newton–Okounkov bodies. For instance, Anderson computed a Newton–Okounkov body for an $\mathrm{SL}(3, \mathbb{C})$ example in [1], they appear in the proof of Kaveh's identification of Newton–Okounkov bodies as string polytopes in [8], and Kiritchenko conjectures a description of

Received April 24, 2015. Revision received October 7, 2015.

some Newton–Okounkov bodies of Bott–Samelson varieties using her divided-difference operators in [11]. Moreover, the global Newton–Okounkov body of Bott–Samelson varieties is studied by Schmitz and Seppänen [20], who show that it is rational polyhedral and also give an inductive description of it. Additionally, during the preparation of this manuscript, we learned that Fujita has also (independently) computed the Newton–Okounkov bodies of Bott–Samelson varieties [4]. However, the valuation we use in this paper (part of the auxiliary data necessary for the definition of a Newton–Okounkov body) is *different* from that associated to the "vertical flag" considered by Schmitz and Seppänen [20], the highest-term valuation used by Fujita and Kaveh [4; 8] and the geometric valuation used by Anderson and Kiritchenko in [1; 11] (cf. also Remark 3.3).

We now briefly recall the geometric objects of interest; for details, see Section 1. Let G be a complex semisimple connected and simply connected linear algebraic group, and let $\{\alpha_1, \dots, \alpha_r\}$ denote the set of simple roots of G. Let $\mathbf{i} = (i_1, \dots, i_n) \in \{1, 2, \dots, r\}^n$ be a *word* that specifies a sequence of simple roots $\{\alpha_{i_1}, \dots, \alpha_{i_n}\}$. We say that a word is reduced if the corresponding sequence of simple roots gives a reduced word decomposition $s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}$ of an element in the Weyl group. Also let $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ be a *multiplicity list;* this specifies a sequence of weights $\{\lambda_1 := m_1\varpi_{\alpha_{i_1}}, \dots, \lambda_n := m_n\varpi_{\alpha_{i_n}}\}$ in the weight lattice of G. Associated to $\mathbf{i}$ and $\mathbf{m}$, we can define a Bott–Samelson variety $Z_{\mathbf{i}}$ (cf. Definition 1.1) and a line bundle $L_{\mathbf{i},\mathbf{m}}$ over it (cf. Definition 1.2). The spaces of global sections $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ appear in representation theory as so-called *generalized Demazure modules.* We also consider a certain natural flag of subvarieties $Y_\bullet : Z_{\mathbf{i}} = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{\mathrm{pt}\}$ in $Z_{\mathbf{i}}$ and consider a valuation $\nu_{Y_\bullet}$ on the spaces of sections $H^0(Z_{\mathbf{i}}, L^{\otimes k}_{\mathbf{i},\mathbf{m}})$ associated to $Y_\bullet$ (for details, see Section 3). Our main result is the following; a more precise statement is given in Theorem 3.4. The polytope $P(\mathbf{i},\mathbf{m})$ and the "condition (P)" mentioned in the statement of the theorem are discussed further.

THEOREM. *Let $\mathbf{i} = (i_1, \dots, i_n) \in \{1, 2, \dots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ be a multiplicity list. Let $Z_{\mathbf{i}}$ and $L_{\mathbf{i},\mathbf{m}}$ be the associated Bott–Samelson variety and line bundle. Suppose that $\mathbf{i}$ is reduced and the pair $(\mathbf{i},\mathbf{m})$ satisfies condition* (P). *Then the Newton–Okounkov body $\Delta(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}, \nu_{Y_\bullet})$ of $Z_{\mathbf{i}}$, with respect to the line bundle $L_{\mathbf{i},\mathbf{m}}$ and the geometric valuation $\nu_{Y_\bullet}$, is equal to $P(\mathbf{i},\mathbf{m})$* (*up to a reordering of coordinates*).

Both the polytope $P(\mathbf{i},\mathbf{m})$ and the "condition (P)" (defined precisely in Section 2) mentioned in the theorem have appeared previously in the literature. Indeed, the polytope $P(\mathbf{i},\mathbf{m})$ is a special case of the *Grossberg–Karshon twisted cubes,* which yield character formulae (possibly with sign) for irreducible G-representations [5]. Specifically, we showed in [7, Prop. 2.1] that if the pair $(\mathbf{i},\mathbf{m})$ satisfies condition (P), then the Grossberg–Karshon twisted cube is equal to the polytope $P(\mathbf{i},\mathbf{m})$ and that the Grossberg–Karshon character formula from [5] corresponding to $\mathbf{i}$ and $\mathbf{m}$ is a *positive* formula (that is, with no negative signs). Condition (P) can also be stated geometrically. Namely, we showed in [7] that $(\mathbf{i},\mathbf{m})$

satisfies condition (P) if and only if a certain torus-invariant divisor $D(c, \ell)$ in a toric variety $X(c)$ is basepoint-free, where we follow the notation of [7]. Here, $X(c)$ and $D(c, \ell)$ are obtained as the special fiber and accompanying line bundle of a toric degeneration of $Z_{\mathbf{i}}$ constructed from the data of the pair $(\mathbf{i}, \mathbf{m})$. For the purposes of the present manuscript, it is also significant that the polytope $P(\mathbf{i}, \mathbf{m})$ is a *lattice* polytope (not just a rational polytope) whose vertices can be easily described as the Cartier data of the torus-invariant divisor D mentioned before [7, Thm. 2.4]. Thus, our theorem gives a computationally efficient description of the Newton–Okounkov body $\Delta(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}, \nu_{Y_\bullet})$.

We note that condition (P) is rather restrictive. For instance, suppose $L_{\mathbf{i},\mathbf{m}}$ is the pullback of a line bundle L_λ over G/B via the usual morphism $Z_{\mathbf{i}} \to G/B$, where $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_r \varpi_r$ is a dominant weight (here $\{\varpi_j\}$ are the fundamental weights corresponding to the simple roots $\{\alpha_j\}$, so $\lambda_j \geq 0$ for all j). In this situation we show in [7, Thm. 4.2] that if $(\mathbf{i}, \mathbf{m})$ satisfies condition (P), then for any simple root α_j that appears more than once in the word $\mathbf{i}$, we must have $\lambda_j = 0$. On the other hand, for a given word $\mathbf{i}$, it is not difficult to explicitly construct (either directly from the definition or by using the other equivalent characterizations of condition (P) in [7, Prop. 2.1]) infinitely many choices of $\mathbf{m}$ such that $(\mathbf{i}, \mathbf{m})$ satisfies condition (P).

We now sketch the main ideas in the proof of our main result (Theorem 3.4). To place the discussion in context, it may be useful to recall that an essential step in the computation of a Newton–Okounkov body of a variety X is to compute a certain semigroup $S = S(R, \nu)$ associated to the (graded) ring of sections $R = \bigoplus_k H^0(X, L^{\otimes k})$ for L a line bundle over X and a choice of valuation ν. In general, this computation can be quite subtle; one of the main difficulties is that the semigroup may not even be finitely generated. (The issue of finite generation, in the context of Newton–Okounkov bodies, is studied in [1].) Even when S is finitely generated, finding explicit generators is related to the problem of finding a "SAGBI basis" for R with respect to the valuation,[1] which appears to be nontrivial in practice. In this manuscript, we are able to sidestep this subtle issue and compute S directly by a simple observation, which we now explain. It is a general fact that the valuations arising from flags of subvarieties $Y_\bullet$ such as those before have *one-dimensional leaves* (cf. Definition 3.1). It is also an elementary fact that a valuation ν with one-dimensional leaves, defined on a finite-dimensional vector space V, satisfies $|\nu(V \setminus \{0\})| = \dim_{\mathbb{C}}(V)$ [10, Prop. 2.6]. As it happens, in our setting the vector spaces in question are precisely the generalized Demazure modules $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ mentioned before, and Lakshmibai, Littelmann, and Magyar [13] prove that $\dim_{\mathbb{C}}(H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})) = |\mathcal{T}(\mathbf{i}, \mathbf{m})|$ where $\mathcal{T}(\mathbf{i}, \mathbf{m})$ is the set of *standard tableaux* associated with $\mathbf{i}$ and $\mathbf{m}$. Armed with this key theorem of Lakshmibai, Littelmann, and Magyar, we are able to compute our semigroup S and hence the Newton–Okounkov body explicitly in two steps. On the one hand, we show in Proposition 3.7 that, assuming that $\mathbf{i}$ is reduced, our geometric valuation $\nu_{Y_\bullet}$ defined on $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \setminus \{0\}$ takes values in the polytope $P(\mathbf{i}, \mathbf{m})$ (up

[1] Such a basis is also called a *Khovanskii basis* in [6, Section 8], cf. also [9, Section 5.6].

to reordering coordinates). On the other hand, we show in Proposition 2.4 that, assuming that $(\mathbf{i}, \mathbf{m})$ satisfies condition (P), there is a bijection between the lattice points in $P(\mathbf{i}, \mathbf{m})$ and the set of standard tableaux $\mathcal{T}(\mathbf{i}, \mathbf{m})$, so in particular $|P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n| = |\mathcal{T}(\mathbf{i}, \mathbf{m})|$. Now a simple counting argument and the fact that $P(\mathbf{i}, \mathbf{m})$ is a lattice polytope finishes the proof of the main theorem.

We now outline the contents of the manuscript. In Section 1 we establish the basic terminology and notation and also state the key result of Lakshmibai, Littelmann, and Magyar (Theorem 1.8). The statement and proof of the bijection between $\mathcal{T}(\mathbf{i}, \mathbf{m})$ and the lattice points in $P(\mathbf{i}, \mathbf{m})$ occupies Section 2. In the process we introduce a separate "condition (P′)", stated directly in the language of paths and root operators as in [16; 17; 13], and prove in Proposition 2.10 that our polytope-theoretic condition (P) implies condition (P′). It is then straightforward to see that condition (P′) implies that $|P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n| = |\mathcal{T}(\mathbf{i}, \mathbf{m})|$. In Section 3 we briefly recall the definition of a Newton–Okounkov body and define our geometric valuation $\nu_{Y_\bullet}$ with respect to a certain flag of subvarieties. We then prove in Proposition 3.7 that $\nu_{Y_\bullet}$ takes values in our polytope; as already explained, by using the bijection from Section 2 our main theorem then readily follows. Concrete examples and pictures for $G = \mathrm{SL}(3, \mathbb{C})$ and $G = \mathrm{Sp}(4, \mathbb{C})$ are contained in Section 4.

We take a moment to comment on the combinatorics in Section 2. It may be that our polytope $P(\mathbf{i}, \mathbf{m})$, our conditions (P) and (P′), and our Proposition 2.4 are well known or are minor variations on standard arguments in combinatorial representation theory. We welcome comments from the experts. At any rate, as the previous discussion indicates, Proposition 2.4 is only a stepping stone to our main result (Theorem 3.4). One final comment: in Section 2, we chose to explain conditions (P) and (P′) separately and to explicitly state and prove the relation between them in Proposition 2.10 because we suspect that condition (P′) may be more familiar to experts in representation theory, whereas our condition (P) arises from the toric-geometric considerations in [7]. Put another way, our condition (P) is a *geometrically* motivated condition on $\mathbf{i}$ and $\mathbf{m}$, which suffices to guarantee condition (P′).

Finally, we mention some directions for future work. Firstly, we hope to better understand the relation between our computations and those in [4]. Secondly, our condition (P) on the pairs $(\mathbf{i}, \mathbf{m})$ is rather restrictive, and the corresponding Newton–Okounkov bodies are combinatorially extremely simple (they are essentially cubes, though they can sometimes degenerate). Hence, it is a natural problem to ask for the relation, if any, between the Newton–Okounkov bodies computed in this paper and those for the line bundles that do *not* satisfy condition (P). It may be possible to analyze such a relationship using some results of Anderson [1], and we hope to take this up in a future paper. Thirdly, it would be of interest to examine the relation between our polytopes $P(\mathbf{i}, \mathbf{m})$ and the polytopes arising from Kiritchenko's divided-difference operators, particularly in relation to her "degeneration of string spaces" in [11, Sect. 4].

1. Preliminaries

In this section we record the basic notation in Section 1.1, recall the definitions of the central geometric objects in Section 1.2, and state a key result (Theorem 1.8) of Lakshmibai, Littelmann, and Magyar in Section 1.3.

1.1. Notation

We list here some notation and conventions to be used in the manuscript.

- We let G denote a complex semisimple connected and simply connected algebraic group over $\mathbb{C}$, and $\mathfrak{g}$ denotes its Lie algebra.
- We let H denote a Cartan subgroup of G.
- We let B denote a Borel subgroup of G with $H \subset B \subset G$.
- We let r denote the rank of G.
- We let X denote the weight lattice of G, and $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$ is its real form. The Killing form[2] on $X_{\mathbb{R}}$ is denoted by $\langle \alpha, \beta \rangle$.
- For a weight $\alpha \in X$, we let e^{α} denote the corresponding multiplicative character $e^{\alpha} : H \to \mathbb{C}^*$.
- We let $\{\alpha_1, \ldots, \alpha_r\}$ denote the set of positive simple roots (with an ordering) with respect to the choices $H \subset B \subset G$, and $\{\alpha_1^{\vee}, \ldots, \alpha_r^{\vee}\}$ are the corresponding coroots. Recall that the coroots satisfy

$$\alpha^{\vee} := \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

  In particular, $\langle \alpha, \alpha^{\vee} \rangle = 2$ for any simple root α.
- For a simple root α, let $s_{\alpha} : X \to X$, $\lambda \mapsto \lambda - \langle \lambda, \alpha^{\vee} \rangle \alpha$, be the associated simple reflection; these generate the Weyl group W.
- We let $\{\varpi_1, \ldots, \varpi_r\}$ denote the set of fundamental weights satisfying $\langle \varpi_i, \alpha_j^{\vee} \rangle = \delta_{i,j}$.
- For a simple root α, $P_{\alpha} := B \cup B s_{\alpha} B$ is the minimal parabolic subgroup containing B associated to α.

1.2. Bott–Samelson Varieties

In this section, we briefly recall the definition of Bott–Samelson varieties and some facts about line bundles on Bott–Samelson varieties. Further details may be found, for instance, in [5]. Note that the literature uses many different notational conventions.

With the notation in Section 1.1 in place, suppose given an arbitrary *word* in $\{1, 2, \ldots, r\}$, that is, a sequence $\mathbf{i} = (i_1, \ldots, i_n)$ with $1 \leq i_j \leq r$. This specifies an associated sequence of simple roots $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}\}$. To simplify notation, we define $\beta_j := \alpha_{i_j}$, so the sequence can be denoted $\{\beta_1, \ldots, \beta_n\}$. Note that we do not assume here that the corresponding expression $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_n}$ is reduced; in

[2] The Killing form is naturally defined on the Lie algebra of G, but its restriction to the Lie algebra $\mathfrak{h}$ of H is positive-definite, so we may identify $\mathfrak{h} \cong \mathfrak{h}^*$.

particular, there may be repetitions. (However, we will add the reducedness as a hypothesis in Section 3.)

Definition 1.1. The *Bott–Samelson variety* corresponding to a word $\mathbf{i} = (i_1, \dots, i_n) \in \{1, 2, \dots, r\}^n$ is the quotient

$$Z_{\mathbf{i}} := (P_{\beta_1} \times \cdots \times P_{\beta_n})/B^n,$$

where $\beta_j = \alpha_{i_j}$, and B^n acts on the right on $P_{\beta_1} \times \cdots \times P_{\beta_n}$ by

$$(p_1, \dots, p_n) \cdot (b_1, \dots, b_n) := (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n).$$

It is known that $Z_{\mathbf{i}}$ is a smooth projective algebraic variety of dimension n. By convention, if $n = 0$ and $\mathbf{i}$ is the empty word, we set $Z_{\mathbf{i}}$ equal to a point.

We next describe certain line bundles over a Bott–Samelson variety. Suppose given a sequence $\{\lambda_1, \dots, \lambda_n\}$ of weights $\lambda_j \in X$. We let $\mathbb{C}_{(-\lambda_1,\dots,-\lambda_n)}$ denote the one-dimensional representation of B^n defined by

$$(b_1, \dots, b_n) \cdot k := e^{-\lambda_1}(b_1) \cdots e^{-\lambda_n}(b_n) k. \tag{1.1}$$

Definition 1.2. Let $\lambda_1, \dots, \lambda_n$ be a sequence of weights. We define the line bundle $L_{\mathbf{i}}(\lambda_1, \dots, \lambda_n)$ over $Z_{\mathbf{i}}$ to be

$$L_{\mathbf{i}}(\lambda_1, \dots, \lambda_n) := (P_{\beta_1} \times \cdots \times P_{\beta_n}) \times_{B^n} \mathbb{C}_{(-\lambda_1,\dots,-\lambda_n)}, \tag{1.2}$$

where the equivalence relation is given by

$$((p_1, \dots, p_n) \cdot (b_1, \dots, b_n), k) \sim ((p_1, \dots, p_n), (b_1, \dots, b_n) \cdot k)$$

for $(p_1, \dots, p_n) \in P_{\beta_1} \times \cdots \times P_{\beta_n}$, $(b_1, \dots, b_n) \in B^n$, and $k \in \mathbb{C}$. The projection $L_{\mathbf{i}}(\lambda_1, \dots, \lambda_n) \to Z_{\mathbf{i}}$ to the base space is given by taking the first factor $[(p_1, \dots, p_n, k)] \mapsto [(p_1, \dots, p_n)] \in Z_{\mathbf{i}}$.

In what follows, we will frequently choose the weights λ_j to be of a special form. Specifically, suppose given a *multiplicity list* $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$. Then we may define a sequence of weights $\{\lambda_1, \dots, \lambda_n\}$ associated to the word $\mathbf{i}$ and the multiplicity list $\mathbf{m}$ by setting

$$\lambda_1 := m_1 \varpi_{i_1}, \qquad \dots, \qquad \lambda_n = m_n \varpi_{i_n}. \tag{1.3}$$

For such λ_i, we will use the notation

$$L_{\mathbf{i},\mathbf{m}} := L_{\mathbf{i}}(m_1 \varpi_{\beta_{i_1}}, \dots, m_n \varpi_{\beta_{i_n}}). \tag{1.4}$$

In this manuscript we will study the space of global sections of these line bundles. Note that the Borel subgroup acts on both $Z_{\mathbf{i}}$ and $L_{\mathbf{i},\mathbf{m}}$ by left multiplication on the first coordinate: indeed, for $b \in B$, the equation $b \cdot [(p_1, \dots, p_n)] := [(bp_1, p_2, \dots, p_n)]$ defines the action on $Z_{\mathbf{i}}$, and $b \cdot [(p_1, \dots, p_n, k)] := [(bp_1, p_2, \dots, p_n, k)]$ defines the action on $L_{\mathbf{i},\mathbf{m}}$. It is straightforward to check that both are well defined. The space of global sections $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ is then naturally a B-module; these are called *generalized Demazure modules* (cf., for instance, [13]).

1.3. Paths and Root Operators

We use the machinery of paths and root operators as in [13] (cf. also [16; 17]), so in this section we briefly recall some necessary definitions and basic properties.

Let $X_{\mathbb{R}} := X \otimes_{\mathbb{Z}} \mathbb{R}$ denote the real form of the weight lattice. By a *path* we will mean a piecewise-linear map $\pi : [0, 1] \to X_{\mathbb{R}}$ (up to reparameterization) with $\pi(0) = 0$. We consider the set $\Pi \cup \{\mathbf{O}\}$ where Π denotes the set of all paths and $\mathbf{O}$ is a formal symbol. For a weight $\lambda \in X$, we let π^{λ} denote the straight-line path: $\pi^{\lambda}(t) := t\lambda$. By the symbol $\pi_1 \star \pi_2$ we mean the concatenation of two paths; more precisely, $\pi(t) = (\pi_1 \star \pi_2)(t)$ is defined by

$$\pi(t) := \begin{cases} \pi_1(2t) & \text{if } 0 \le t \le 1/2, \\ \pi_1(1) + \pi_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases} \tag{1.5}$$

By convention we take $\pi \star \mathbf{O} := \pi$ for any element $\pi \in \Pi \cup \{\mathbf{O}\}$. For a simple root α and a path π, we define $s_\alpha(\pi)$ to be the path given by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$, that is, the path π is reflected by s_α. We pay particular attention to endpoints, so we give it a name: given π, we say the *weight* of π is its endpoint, $\mathrm{wt}(\pi) := \pi(1)$ (also denoted $v(\pi)$ in the literature; see [16]). The following is immediate from the definitions.

Lemma 1.3. *Let* π, π_1, π_2 *be paths in* Π, *and* α *a simple root. Then* $\mathrm{wt}(\pi_1 \star \pi_2) = \mathrm{wt}(\pi_1) + \mathrm{wt}(\pi_2)$ *and* $\mathrm{wt}(s_\alpha(\pi)) = s_\alpha(\mathrm{wt}(\pi))$.

Fix a simple root α. We now briefly recall the definitions of the *raising operator* e_α and *lowering operator* f_α on the set $\Pi \cup \{\mathbf{O}\}$, for which we need some preparation of notation. Fix a path $\pi \in \Pi$. We cut π into three pieces according to the behavior of the path π under the projection with respect to α. More precisely, define the function

$$h_\alpha : [0, 1] \to \mathbb{R}, \quad t \mapsto \langle \pi(t), \alpha^\vee \rangle$$

and let Q denote the smallest integer attained by h_α, that is,

$$Q := \min\{\mathrm{image}(h_\alpha) \cap \mathbb{Z}\}.$$

Note that since $\pi(0) = 0$ by definition, we always have $Q \le 0$. Now let $q := \min\{t \in [0, 1] : h_\alpha(t) = \langle \pi(t), \alpha^\vee \rangle = Q\}$ be the "first" time t at which the minimum integer value of h_α is attained. Next, in the case that $Q \le -1$ (note that if $Q = 0$, then, since $\pi(0) = 0$, the value q must be 0, and the following discussion is not applicable), we define y to be the "last time before q" when the value $Q + 1$ is attained. More precisely, y is defined by the conditions

$$h_\alpha(y) = Q + 1 \quad \text{and} \quad Q < h_\alpha(t) < Q + 1 \quad \text{for } y < t < q.$$

We now define three paths π_1, π_2, π_3 in such a way that π is by definition the concatenation $\pi = \pi_1 \star \pi_2 \star \pi_3$, where π_1 is the path π "up to time y", π_2 is the path π "between y and q", and π_3 is the path π "after time q". More precisely, we define

$$\pi_1(t) := \pi(ty), \qquad \pi_2(t) := \pi(y + t(q-y)) - \pi(y), \quad \text{and}$$
$$\pi_3(t) := \pi(q + t(1-q)) - \pi(q).$$

See [16, Sect. 1.2, Example] for a figure illustrating an example in rank 2. Given this decomposition of π into "pieces", we may now define the *raising (root) operator* e_α as follows.

Definition 1.4. Fix a path π. If $Q = 0$, that is, if the path π lies entirely in the closed half-space defined by $\{h_\alpha > -1\}$, then $e_\alpha(\pi) = \mathbf{O}$, where $\mathbf{O}$ is the formal symbol in $\Pi \cup \{\mathbf{O}\}$. If $Q < 0$, then we define $e_\alpha(\pi) := \pi_1 \star s_\alpha(\pi_2) \star \pi_3$, that is, we "reflect across α" the portion of the path π between time y and time q. We also define $e_\alpha(\mathbf{O}) = \mathbf{O}$.

The lowering (root) operator f_α may be defined similarly. This time, let p denote the *maximal* real number in [0, 1] such that $h_\alpha(p) = Q$, that is, it is the "last" time t at which the minimal value Q is attained. Then let P denote the integral part of $h_\alpha(1) - Q$. If $P \geq 1$, then let x denote the first time after p that h_α achieves the value $Q + 1$; more precisely, let x be the unique element in $(p, 1]$ satisfying

$$h_\alpha(x) = Q + 1 \quad \text{and} \quad Q < h_\alpha(t) < Q + 1 \quad \text{for } p < t < x.$$

Once again, we may decompose the path π into three components, $\pi = \pi_1 \star \pi_2 \star \pi_3$ by the equations

$$\begin{aligned} &\pi_1(t) := \pi(tp), \qquad \pi_2(t) := \pi(p + t(x - p)) - \pi(p), \quad \text{and} \\ &\pi_3(t) := \pi(x + t(1 - x)) - \pi(x). \end{aligned} \tag{1.6}$$

Given this decomposition, we define the *lowering (root) operator* f_α as follows.

Definition 1.5. Fix a path π as before. If $P \geq 1$, then we define $f_\alpha(\pi) := \pi_1 \star s_\alpha(\pi_2) \star \pi_3$, so we "reflect across α" the portion of the path π between time p and x. If $P = 0$, then $f_\alpha(\pi) = \mathbf{O}$. Finally, we define $f_\alpha(\mathbf{O}) = \mathbf{O}$.

The following basic properties of the root operators are recorded in [16, Sect. 1.4].

LEMMA 1.6. *Let $\pi \in \Pi$ be a path.*

(1) *If $e_\alpha(\pi) \neq \mathbf{O}$, then $\mathrm{wt}(e_\alpha(\pi)) = \mathrm{wt}(\pi) + \alpha$, and if $f_\alpha(\pi) \neq \mathbf{O}$, then $\mathrm{wt}(f_\alpha(\pi)) = \mathrm{wt}(\pi) - \alpha$.*
(2) *If $e_\alpha(\pi) \neq \mathbf{O}$, then $f_\alpha(e_\alpha(\pi)) = \pi$. If $f_\alpha(\pi) \neq \mathbf{O}$, then $e_\alpha(f_\alpha(\pi)) = \pi$.*
(3) *We have $e_\alpha^n(\pi) = \mathbf{O}$ if and only if $n > -Q$, and $f_\alpha^n \pi = \mathbf{O}$ if and only if $n > P$.*

We now recall a result (Theorem 1.8) of Lakshmibai, Littelmann, and Magyar [13], which is crucial to our arguments in the remainder of this paper. Specifically, Theorem 1.8 gives a bijective correspondence between a certain set $\mathcal{T}(\mathbf{i}, \mathbf{m})$ of standard tableaux, defined further using paths and the root operators, and a basis of the vector space $H^0(Z_\mathbf{i}, L_{\mathbf{i},\mathbf{m}})$ of global sections of $L_{\mathbf{i},\mathbf{m}}$ over $Z_\mathbf{i}$. Our main result in Section 2 is that—under certain conditions on the word $\mathbf{i}$ and the multiplicity list $\mathbf{m}$—there exists, in turn, a bijection between $\mathcal{T}(\mathbf{i}, \mathbf{m})$ and the set of integer lattice points in a certain polytope. This then allows us to compute Newton–Okounkov bodies associated to $Z_\mathbf{i}$ and $L_{\mathbf{i},\mathbf{m}}$ in Section 3.

We now recall the definition of standard tableaux. Suppose given a word $\mathbf{i}$ and multiplicity list $\mathbf{m}$ as before. Let $\{\beta_1 = \alpha_{i_1}, \ldots, \beta_n = \alpha_{i_n}\}$ be the sequence of simple roots associated to $\mathbf{i}$ and set $\lambda_j := m_j\beta_j$ for $1 \le j \le n$. The following is from [13, Sect. 1.2].

Definition 1.7. A path $\pi \in \Pi$ is called a *(constructable) standard tableau of shape* $\lambda = (\lambda_1, \dots, \lambda_n)$ if there exist integers $\ell_1, \dots, \ell_n \in \mathbb{Z}_{\ge 0}$ such that

$$\pi = f_{\beta_1}^{\ell_1}(\pi^{\lambda_1} \star f_{\beta_2}^{\ell_2}(\pi^{\lambda_2} \star \cdots \star f_{\beta_n}^{\ell_n}(\pi^{\lambda_n})\cdots)),$$

where the f_{β_j} are the lowering operators defined previously. Given a word $\mathbf{i} = (i_1, \dots, i_n)$ and multiplicity list $\mathbf{m} = (m_1, \dots, m_n)$, we denote by $\mathcal{T}(\mathbf{i}, \mathbf{m})$ the set of standard tableaux of shape $(\lambda_1 = m_1\varpi_{\beta_1}, \dots, \lambda_n = m_n\varpi_{\beta_n})$.

It turns out that there are only finitely many standard tableaux of a given shape associated to a given pair $(\mathbf{i}, \mathbf{m})$. In fact, Lakshmibai, Littelmann, and Magyar prove [13, Thms. 4 and 6] the following.

THEOREM 1.8. *Let* $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, r\}^n$ *be a word, and* $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\ge 0}^n$ *be a multiplicity list. Let* $\{\beta_1 = \alpha_{i_1}, \dots, \beta_n = \alpha_{i_n}\}$ *be the sequence of simple roots associated to* $\mathbf{i}$ *and set* $\lambda_j := m_j\beta_j$ *for* $1 \le j \le n$. *Then*

$$|\mathcal{T}(\mathbf{i}, \mathbf{m})| = \dim_{\mathbb{C}} H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}).$$

2. A Bijection between Standard Tableaux and Lattice Points in a Polytope

The main result of this section (Proposition 2.4) is that, under a certain assumption on the word $\mathbf{i}$ and the multiplicity list $\mathbf{m}$, there is a bijection between the set of integer lattice points within a certain lattice polytope $P(\mathbf{i}, \mathbf{m})$ and the set of standard tableaux $\mathcal{T}(\mathbf{i}, \mathbf{m})$. Together with Theorem 1.8, this then implies that the cardinality of $P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n$ is equal to the dimension of the space $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ of sections of the line bundle $L_{\mathbf{i},\mathbf{m}}$ over the Bott–Samelson variety $Z_{\mathbf{i}}$. This then allows us to compute Newton–Okounkov bodies in the next section. The necessary hypothesis on $\mathbf{i}$ and $\mathbf{m}$, which we call “condition (P)”, also appeared in our previous work [7] connecting the polytopes $P(\mathbf{i}, \mathbf{m})$ with representation theory and toric geometry (cf. Remark 2.2).

We begin with the definition of the polytope $P(\mathbf{i}, \mathbf{m})$ by an explicit set of inequalities.

Definition 2.1. Let $\mathbf{i} = (i_1, \dots, i_n) \in \{1, \dots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\ge 0}^n$ be a multiplicity list. Then the polytope $P(\mathbf{i}, \mathbf{m})$ is defined to be the set of all real points $(x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying the following inequalities:

$$\begin{aligned}
&0 \le x_n \le A_n := m_n,\\
&0 \le x_{n-1} \le A_{n-1}(x_n) := \langle m_{n-1}\varpi_{\beta_{n-1}} + m_n\varpi_{\beta_n} - x_n\beta_n, \beta_{n-1}^\vee\rangle,\\
&0 \le x_{n-2} \le A_{n-2}(x_{n-1}, x_n)\\
&\quad := \langle m_{n-2}\varpi_{\beta_{n-2}} + m_{n-1}\varpi_{\beta_{n-1}} + m_n\varpi_{\beta_n} - x_{n-1}\beta_{n-1} - x_n\beta_n, \beta_{n-2}^\vee\rangle,\\
&\qquad \vdots
\end{aligned}$$

$$\begin{aligned} 0 &\le x_1 \le A_1(x_2, \ldots, x_n) \\ &:= \langle m_1 \varpi_{\beta_1} + m_2 \varpi_{\beta_2} + \cdots + m_n \varpi_{\beta_n} - x_2\beta_2 - \cdots - x_n\beta_n, \beta_1^\vee \rangle. \end{aligned}$$

Remark 2.2. • The polytopes $P(\mathbf{i}, \mathbf{m})$ have appeared previously in the literature and have connections to toric geometry and representation theory. Specifically, under a hypothesis on $\mathbf{i}$ and $\mathbf{m}$, which we call "condition (P)" (see Definition 2.3), we show in [7] that $P(\mathbf{i}, \mathbf{m})$ is exactly a so-called *Grossberg–Karshon twisted cube*. These twisted cubes were introduced in [5] in connection with Bott towers and character formulae for irreducible G-representations. Our proof of this fact in [7] used a certain torus-invariant divisor in a toric variety associated to Bott–Samelson varieties studied by Pasquier [18].

- The functions $A_k(x_{k+1}, \ldots, x_n)$ appearing in Definition 2.1 also have a natural interpretation in terms of paths, as we shall see in Lemma 2.7; this is useful in our proof of Proposition 2.4.

In the statement of our main proposition of this section, we need the following technical hypothesis on the word and the multiplicity list. As already noted, the same condition appeared in our previous work [7], which related the polytope $P(\mathbf{i}, \mathbf{m})$ to toric geometry and representation theory.

Definition 2.3. Let $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\ge 0}$ be a multiplicity list. We say that the pair $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P) if for every integer k with $1 \le k \le n-1$, the following statement, which we refer to as condition (P-k), holds:

(P-k) if $(x_{k+1}, \ldots, x_n)$ satisfies

$$\begin{aligned} &0 \le x_n \le A_n, \\ &0 \le x_{n-1} \le A_{n-1}(x_n), \\ &\qquad \vdots \\ &0 \le x_{k+1} \le A_{k+1}(x_{k+2}, \ldots, x_n), \end{aligned}$$

then

$$A_k(x_{k+1}, \ldots, x_n) \ge 0.$$

We may now state the main result of this section.

PROPOSITION 2.4. *If* $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P), *then there exists a bijection between the set of integer lattice points in the polytope* $P(\mathbf{i}, \mathbf{m})$ *and the set of standard tableaux* $\mathcal{T}(\mathbf{i}, \mathbf{m})$. *Therefore,*

$$|P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n| = |\mathcal{T}(\mathbf{i}, \mathbf{m})|.$$

To prove Proposition 2.4, we need some preliminaries. Let $\mathbf{i}, \mathbf{m}$ be as before. For any k with $1 \le k \le n$, we define the notation

$$\mathbf{i}[k] := (i_k, i_{k+1}, \ldots, i_n), \qquad \mathbf{m}[k] := (m_k, m_{k+1}, \ldots, m_n),$$

so $\mathbf{i}[k]$ and $\mathbf{m}[k]$ are obtained from $\mathbf{i}$ and $\mathbf{m}$ by deleting the left-most $k-1$ coordinates. The following lemma is immediate from the inductive nature of the definitions of the polytopes $P(\mathbf{i}, \mathbf{m})$ and of condition (P).

LEMMA 2.5. *Let* $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ *be a word, and* $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ *be a multiplicity list.*

(1) *Suppose* $(x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m})$. *For any* k *with* $1 \leq k \leq n-1$, *we have*

$$(x_{k+1}, \ldots, x_n) \in P(\mathbf{i}[k+1], \mathbf{m}[k+1]).$$

(2) *The pair* $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P) *if and only if for any* k *with* $1 \leq k \leq n-1$ *and any* $(x_{k+1}, \ldots, x_n) \in P(\mathbf{i}[k+1], \mathbf{m}[k+1])$, *the vector* $(0, \ldots, 0, x_{k+1}, \ldots, x_n)$ *lies in* $P(\mathbf{i}, \mathbf{m})$, *where* $(0, \ldots, 0, x_{k+1}, \ldots, x_n)$ *is the vector obtained by adding* k *zeroes to the left.*

(3) *If* $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P), *then for any* k *with* $1 \leq k \leq n$, *the pair* $(\mathbf{i}[k], \mathbf{m}[k])$ *also satisfies condition* (P).

To prove Proposition 2.4, the plan is to first explicitly construct a map from $P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n$ to $\mathcal{T}(\mathbf{i}, \mathbf{m})$ and then prove that it is a bijection. In fact, it will be convenient to define a sequence of maps $\varphi_k : \mathbb{Z}^{n-k+1}_{\geq 0} \to \Pi \cap \{\mathbf{O}\}$; the map $\varphi := \varphi_1$ will be the desired bijection between $P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n$ and $\mathcal{T}(\mathbf{i}, \mathbf{m})$.

Definition 2.6. Let $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ be a multiplicity list. Let k be an integer with $1 \leq k \leq n$. We define a map $\varphi_k : \mathbb{Z}^{n-k+1}_{\geq 0} \to \Pi \cup \{\mathbf{O}\}$ associated with $\mathbf{i}$ and $\mathbf{m}$ by

$$\varphi_k(x_k, \ldots, x_n) := f_{\beta_k}^{x_k}(\pi^{\lambda_k} \star f_{\beta_{k+1}}^{x_{k+1}}(\pi^{\lambda_{k+1}} \star \cdots \star f_{\beta_n}^{x_n}(\pi^{\lambda_n}) \cdots)), \tag{2.1}$$

where $\lambda_k := m_k \varpi_{\beta_k}$ for $1 \leq k \leq n$. (Although the map φ_k depends on $\mathbf{i}$ and $\mathbf{m}$, for simplicity, we omit it from the notation.)

From the definition it is immediate that the φ_k are related to one another by the equation

$$\varphi_k(x_k, \ldots, x_n) = f_{\beta_k}^{x_k}(\pi^{\lambda_k} \star \varphi_{k+1}(x_{k+1}, \ldots, x_n))$$

for $1 \leq k < n$. It will be also useful to introduce the notation

$$\tau_k(x_{k+1}, \ldots, x_n) := \pi^{\lambda_k} \star \varphi_{k+1}(x_{k+1}, \ldots, x_n) \tag{2.2}$$

for $1 \leq k < n$, and we set $\tau_n := \pi^{\lambda_n}$, from which it immediately follows that

$$\varphi_k(x_k, \ldots, x_n) = f_{\beta_k}^{x_k}(\tau_k(x_{k+1}, \ldots, x_n)). \tag{2.3}$$

With this notation in place, we can interpret the functions A_k appearing in the definition of $P(\mathbf{i}, \mathbf{m})$ naturally in terms of paths. Recall that the endpoint $\pi(1)$ of a path $\pi \in \Pi$ is called its *weight,* and we denote it by $\mathrm{wt}(\pi) := \pi(1)$.

LEMMA 2.7. *Let* $(x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq 0}$, *and let* k *be an integer,* $0 \leq k \leq n-1$. *If* $\varphi_{k+1}(x_{k+1}, \ldots, x_n) \neq \mathbf{O}$, *then*

$$\mathrm{wt}(\varphi_{k+1}(x_{k+1}, \ldots, x_n)) = m_{k+1}\varpi_{\beta_{k+1}} + \cdots + m_n \varpi_{\beta_n} - x_{k+1}\beta_{k+1} - \cdots - x_n \beta_n.$$

Moreover, if in addition $k \geq 1$, *then* $\tau_k(x_{k+1}, \ldots, x_n) \neq \mathbf{O}$ *and*

$$\begin{aligned}&\mathrm{wt}(\tau_k(x_{k+1}, \ldots, x_n)) \\ &\quad = m_k \varpi_{\beta_k} + m_{k+1}\varpi_{\beta_{k+1}} + \cdots + m_n \varpi_{\beta_n} - x_{k+1}\beta_{k+1} - \cdots - x_n \beta_n,\end{aligned}$$

so, in particular,

$$A_k(x_{k+1}, \dots, x_n) = \langle \mathrm{wt}(\tau_k(x_{k+1}, \dots, x_n)), \beta_k^\vee \rangle. \tag{2.4}$$

Proof. Under the hypothesis that $\varphi_{k+1}(x_{k+1}, \dots, x_n)$ is an honest path (that is, it is not $\mathbf{O}$), the first statement of the lemma is immediate from the definition of φ_k, Lemma 1.3, and Lemma 1.6(1). The other statements of the lemma are then straightforward from the definitions. □

In words, equation (2.4) says that the functions A_k measure the pairing of the end-point of $\tau_k(x_{k+1}, \dots, x_n)$ against the coroot $\beta_k^\vee$ (assuming that $\tau_k(x_{k+1}, \dots, x_n)$ is an honest path).

Now we show that when φ_k is restricted to the subset $P(\mathbf{i}[k], \mathbf{m}[k]) \cap \mathbb{Z}^{n-k+1}$, the output is an honest path in Π (that is, it is not the formal symbol $\mathbf{O}$). From the definition of standard tableaux it immediately follows that the output is also in fact an element in $\mathcal{T}(\mathbf{i}[k], \mathbf{m}[k])$.

LEMMA 2.8. *Let k be an integer with $1 \le k \le n$. The map φ_k restricts to a map*

$$\varphi_k : P(\mathbf{i}[k], \mathbf{m}[k]) \cap \mathbb{Z}^{n-k+1} \to \mathcal{T}(\mathbf{i}, \mathbf{m}).$$

Proof. We first show that the outputs of the maps φ_k are honest paths (that is, $\neq \mathbf{O}$). We argue by induction, and since the definition of the φ_k is a composition of operators starting with f_{β_n} (not f_{β_1}), the base case is $k = n$. From the definition of $P(\mathbf{i}, \mathbf{m})$ we know that $x_n \le m_n = \langle \pi^{\lambda_n}(1), \beta_n^\vee \rangle$, so it suffices to prove that, for such x_n, we have $f_{\beta_n}^{x_n}(\pi^{\lambda_n} = \pi^{m_n \varpi_{\beta_n}}) \neq \mathbf{O}$. Since π^{λ_n} is a straight-line path from 0 to $\lambda_n = m_n \varpi_{\beta_n}$, the constants Q and P in the definition of f_{β_n} (applied to π^{λ_n}) are 0 and $h_{\beta_n}(1) - Q = \langle m_n \varpi_{\beta_n}, \beta_n^\vee \rangle = m_n$, respectively. Thus, by Lemma 1.6(3) we may conclude $\varphi_n(x_n) := f_{\beta_n}^{x_n}(\pi^{\lambda_n}) \neq \mathbf{O}$, which completes the base case. Now suppose that $1 \le k < n$ and $\varphi_{k+1}(x_{k+1}, \dots, x_n) \neq \mathbf{O}$, which in turn implies $\tau_k(x_{k+1}, \dots, x_n) \neq \mathbf{O}$ since concatenation of paths always results in a path. We must show that $\varphi_k(x_k, \dots, x_n) = f_{\beta_k}^{x_k}(\tau_k) \neq \mathbf{O}$. Since τ_k is a path starting at the origin 0, the constants Q and P in the definition of f_{β_1} (applied to $\tau_k(x_{k+1}, \dots, x_n)$) are ≤ 0 and $\ge \langle \mathrm{wt}(\tau_k(x_{k+1}, \dots, x_n)), \beta_k^\vee \rangle$, respectively. In particular, again by Lemma 1.6(3) it suffices to show that $x_k \le \langle \mathrm{wt}(\tau_k(x_{k+1}, \dots, x_n)), \beta_k^\vee \rangle$. Since $\tau_k(x_{k+1}, \dots, x_n) \neq \mathbf{O}$ and $(x_k, \dots, x_n) \in P(\mathbf{i}[k], \mathbf{m}[k])$, the result then holds by definition of $P(\mathbf{i}[k], \mathbf{m}[k])$ and the interpretation of the A_k given in Lemma 2.7. It remains to check that the paths $\varphi_k(x_{k+1}, \dots, x_n) \in \Pi$ are standard tableaux, but this follows directly from Definition 1.7. □

From the preceding discussion we have a well-defined map

$$\varphi := \varphi_1 : P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n \to \mathcal{T}(\mathbf{i}, \mathbf{m}). \tag{2.5}$$

We need to prove that φ is a bijection. For this, it is useful to introduce another condition on $(\mathbf{i}, \mathbf{m})$, which we call condition (P′); it is formulated in terms of the paths τ_k and the raising operators e_{β_k}.

Definition 2.9. Let $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ be a multiplicity list. We say that the pair $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P′) if for all $(x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m}) \cap \mathbb{Z}^n_{\geq 0}$ and all k with $1 \leq k \leq n$, we have $e_{\beta_k}(\tau_k(x_{k+1}, \ldots, x_n)) = \mathbf{O}$.

It may be conceptually helpful to note that, from our interpretation of the functions A_k in Lemma 2.7 and the definitions of $P(\mathbf{i}, \mathbf{m})$ and τ_k, we may think of condition (P) as saying that the *endpoints* of certain paths τ_k are always contained in the affine half-space defined by $\{\langle \cdot, \beta_k \rangle \geq 0\}$ (that is, the half-space pairing nonnegatively against the coroot $\beta_k^\vee$). Moreover, from Lemma 1.6(3) we see that in order to show $e_{\beta_k}(\tau_k) = \mathbf{O}$ for a given path τ_k, it suffices to show that the *entire path* τ_k lies in the same affine half-space. Thus, roughly speaking, condition (P) is about endpoints, whereas condition (P′) is about the entire path.

Proposition 2.10. *Let* $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, \ldots, r\}^n$ *be a word, and* $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ *be a multiplicity list. If the pair* $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P), *then* $(\mathbf{i}, \mathbf{m})$ *satisfies condition* (P′).

Since condition (P′) is phrased in terms of the e_{β_k} and because the raising and lowering operators act as inverses (provided that the composition makes sense) as in Lemma 1.6(2), once we know Proposition 2.10, it is straightforward to show that φ is a bijection. Indeed, we suspect that the argument given further is standard for the experts, but we include it for completeness.

Proof of Proposition 2.4 (assuming Proposition 2.10). By Proposition 2.10 we may assume that condition (P′) holds. First, we prove by induction that φ_k is injective for each k, starting with the base case $k = n$. Suppose that

$$\varphi_n(x_n) = f_{\beta_n}^{x_n}(\pi^{\lambda_n}) = f_{\beta_n}^{y_n}(\pi^{\lambda_n}) = \varphi_n(y_n) \tag{2.6}$$

and also suppose for a contradiction that $x_n < y_n$. Applying $e_{\beta_n}^{x_n+1}$ to the LHS of (2.6) yields $e_{\beta_n}(\pi^{\lambda_n})$ since by Lemma 1.6(2) we know that e_{β_n} is inverse to f_{β_n} whenever the image of f_{β_n} is $\neq \mathbf{O}$. By condition (P′), $e_{\beta_n}(\pi^{\lambda_n}) = e_{\beta_n}(\tau_n) = \mathbf{O}$. On the other hand, applying $e_{\beta_n}^{x_n+1}$ to the RHS of (2.6) yields $f_{\beta_n}^{y_n - x_n - 1}(\pi^{\lambda_n})$, which is $\neq\mathbf{O}$ since $y_n - x_n - 1 \geq 0$ by assumption. This contradicts (2.6), and so $x_n = y_n$, and we conclude that φ_n is injective. This completes the base case. Now suppose by induction that φ_{k+1} is injective; we need to show that φ_k is injective. Assume that

$$\begin{aligned}\varphi_k(x_k, \ldots, x_n) &= f_{\beta_k}^{x_k}(\tau_k(x_{k+1}, \ldots, x_n)) = f_{\beta_k}^{y_k}(\tau_k(y_{k+1}, \ldots, y_n)) \\ &= \varphi_k(y_k, \ldots, y_n)\end{aligned}$$

and suppose also that $x_k < y_k$. The same argument as before, namely applying $e_{\beta_k}^{x_k+1}$ to both sides, yields a contradiction due to condition (P′). Thus, $x_k = y_k$. Applying $e_{\beta_k}^{x_k}$ to both sides of the equation, we obtain $\tau_k(x_{k+1}, \ldots, x_n) = \tau_k(y_{k+1}, \ldots, y_n)$. Concatenation by π^{λ_k} is evidently injective, so $\varphi_{k+1}(x_{k+1}, \ldots, x_n) = \varphi_{k+1}(y_{k+1}, \ldots, y_n)$, but then by the inductive assumption we have

$(x_{k+1}, \dots, x_n) = (y_{k+1}, \dots, y_n)$. This proves $(x_k, \dots, x_n) = (y_k, \dots, y_n)$ and hence that φ_k is injective, as desired.

Now we claim that φ_k is surjective for each k. We argue by induction on the size of n. First, consider the base case $n = 1$, so $w = (\beta_1 = \beta)$, $m = (m_1 = m)$, and $P(w, m) = [0, m]$. By definition a standard tableau of shape $\lambda = m\varpi_\beta$ is of the form $f_\beta^x(\pi^\lambda)$ for some $x \in \mathbb{Z}_{\geq 0}$. Since π^λ is a straight-line path from 0 to $m\beta$, the constants Q and P in the definition of f_β applied to π^λ are 0 and m, respectively. Then for x a nonnegative integer, we know by Lemma 1.6(3) that $f_\beta^x(\pi^\lambda) \neq \mathbf{O}$ if and only if $x \leq m$. Since $P(\mathbf{i}, \mathbf{m}) = [0, m]$ in this case, we conclude that φ_1 is surjective if $n = 1$, as desired.

Now assume by induction that each φ_k is surjective (hence bijective) for words of length $< n$. From Lemma 2.5(3) we know that $(\mathbf{i}[k], \mathbf{m}[k])$ satisfies condition (P) (and hence condition (P$'$)). By the inductive assumption we may therefore assume that $\varphi_k : P(\mathbf{i}[k], \mathbf{m}[k]) \cap \mathbb{Z}^{n-k+1} \to \mathcal{T}(\mathbf{i}[k], \mathbf{m}[k])$ is a bijection for $k > 1$, and we wish to show that $\varphi = \varphi_1$ is surjective. By the definition of the standard tableaux any element in $\mathcal{T}(\mathbf{i}, \mathbf{m})$ is of the form $f_{\beta_1}^{\ell_1}(\pi^{\lambda_1} \star \tau')$ for some $\tau' \in \mathcal{T}(w[2], m[2])$ and some $\ell_1 \in \mathbb{Z}_{\geq 0}$. By the inductive assumption we know that there exists some $(x_2, \dots, x_n) \in P(\mathbf{i}[2], \mathbf{m}[2])$ such that $\tau' = \varphi_2(x_2, \dots, x_n)$. From the definition of $P(\mathbf{i}, \mathbf{m})$, in order to prove the surjectivity, it would suffice to show that

$$\begin{aligned} &f_{\beta_1}^{\ell_1}(\pi^{m_1\varpi_{\beta_1}} \star \varphi_2(x_2, \dots, x_n)) = f_{\beta_1}(\tau_1(x_2, \dots, x_n)) \neq \mathbf{O} \\ &\quad \Rightarrow \quad \ell_1 \leq A_1(x_2, \dots, x_n). \end{aligned}$$

From Lemma 1.6(3) we know $f_{\beta_1}^{\ell_1}(\tau_1) \neq \mathbf{O} \Leftrightarrow \ell_1 \leq P$, where P is defined to be the integral part of $\langle \mathrm{wt}(\tau_1(x_2, \dots, x_n)), \beta_1^\vee \rangle - Q$, and $Q = \min_{t \in [0,1]} \langle \tau_1(x_2, \dots, x_n)(t), \beta_1^\vee \rangle$. Since $\tau_1(x_2, \dots, x_n) \neq \mathbf{O}$ by assumption, we know from (2.4) that $A_1(x_2, \dots, x_n) = \langle \mathrm{wt}(\tau_1(x_2, \dots, x_n)), \beta_1^\vee \rangle$, and it is evident from the definition of A_1 that for $(x_2, \dots, x_n) \in \mathbb{Z}^{n-1}$, the value $A_1(x_2, \dots, x_n)$ is integral. Hence, it suffices to show that $Q = 0$, and again by Lemma 1.6(3) this is equivalent to showing that $e_{\beta_1}(\tau_k(x_2, \dots, x_n)) = \mathbf{O}$. Note that the vector $(0, x_2, \dots, x_n)$ lies in $P(\mathbf{i}, \mathbf{m})$ by Lemma 2.5(2). By applying the statement of condition (P$'$) to $(0, x_2, \dots, x_n)$ and $k = 1$ we obtain that $e_{\beta_1}(\tau_k(x_2, \dots, x_n)) = \mathbf{O}$, as desired. This completes the proof. $\square$

It remains to justify Proposition 2.10. The following simple lemma will be helpful.

LEMMA 2.11. *Let $\pi \in \Pi$ be a piecewise linear path in $X_\mathbb{R}$.*

(1) *Let π^λ be a linear path for some $\lambda \in X_\mathbb{R}$. Then for any $t \in [0, 1]$, there exist nonnegative real constants $a, c \geq 0$ and $s \in [0, 1]$ such that $(\pi^\lambda \star \pi)(t) = a\lambda + c\pi(s)$.*

(2) *Let β be a simple root. Let x be a positive integer and assume that $f_\beta^x(\pi) \neq \mathbf{O}$. Then for any $t \in [0, 1]$, there exists $b \in \mathbb{R}$ with $0 \leq b \leq x$ such that $f_\beta^x(\pi)(t) = \pi(t) + b(-\beta)$.*

(3) *Let $\pi \in \Pi$ be a path in $X_{\mathbb{R}}$. Let $\{\beta_1, \ldots, \beta_j\}$ be any sequence of simple roots, and $n_1, \ldots, n_j \in \mathbb{Z}_{\geq 0}$ any sequence of nonnegative integers. Then any point along the path $f_{\beta_1}^{n_1}(\pi^{n_1 \varpi_{\beta_1}} \star f_{\beta_2}^{n_2}(\pi^{n_2 \varpi_{\beta_2}} \star \cdots \star f_{\beta_j}^{n_j}(\pi^{n_j \varpi_{\beta_j}} \star \pi) \cdots))$ can be expressed as a linear combination*

$$\sum_{\ell=1}^{j} a_\ell n_\ell \varpi_{\alpha_\ell} + \sum_{\ell=1}^{j} b_\ell(-\beta_\ell) + c\pi(s)$$

for some constants $a_\ell, b_\ell, c \geq 0$ and some $s \in [0, 1]$.

(4) *Let $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n$ and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ be a word and multiplicity list, and let k be an integer with $1 \leq k \leq n$. Let φ_k denote the map associated with $\mathbf{i}, \mathbf{m}$ as in Definition 2.6. Then any point along the path $\varphi_k(x_k, \ldots, x_n)$ can be expressed as a linear combination*

$$\sum_{\ell=k}^{n} a_\ell m_\ell \varpi_{\beta_\ell} + \sum_{\ell=k}^{n} b_\ell(-\beta_\ell), \tag{2.7}$$

where $a_\ell, b_\ell \geq 0$.

Proof. First, we prove (1). From definition (1.5) of paths and the definition of a straight-line path π^λ it follows that for $t \in [0, \frac{1}{2}]$, we may take $a = 2t$ and $c = 0$ since $(\pi^\lambda \star \pi)(t) = \pi^\lambda(2t) = 2t\lambda$ in this case. On the other hand, if $t \in [\frac{1}{2}, 1]$, then we may take $a = 1$, $c = 1$ and $s = 2t - 1$ since by (1.5) we have $(\pi^\lambda \star \pi)(t) := \pi^\lambda(1) + \pi(2t - 1) = \lambda + \pi(2t - 1)$. This proves the claim.

Next, we prove (2). Recall that the reflection operator s_β acts by $s_\beta(\alpha) := \alpha - \langle \alpha, \beta^\vee \rangle \beta$, so for any path π and for any time t, we have $s_\beta(\pi)(t) := s_\beta(\pi(t)) = \pi(t) - \langle \pi(t), \beta^\vee \rangle \beta = \pi(t) + \langle \pi(t), \beta^\vee \rangle(-\beta)$, and, in particular, $s_\beta(\pi)(t)$ is a linear combination of $\pi(t)$ and $-\beta$. Additionally, from Definition 1.5 we know that $f_\beta(\pi) := \pi_1 \star s_\beta(\pi_2) \star \pi_3$ where π_1, π_2, and π_3 are defined in (1.6), and from the discussion preceding Definition 1.5 that defines p and x it follows that $\langle \pi_2(t), \beta^\vee \rangle \in [0, 1]$ for all t. To prove the claim, we begin with the base case $x = 1$. Consider each of the three components of $f_\beta(\pi)$ in turn. For the first portion of the path (corresponding to π_1), the operator f_β does not alter the path at all, so for such t, we have $f_\beta(\pi)(t) = \pi(t)$, and the claim of the lemma holds with $b = 0$. For t in the second portion of the path, we have $\pi(t) = \pi_1(p) + \pi_2(t')$ (here t' is determined by t by some reparameterization coming from the concatenation operation) and $f_\beta(\pi)(t) = \pi_1(p) + s_\beta(\pi_2(t')) = \pi_1(p) + \pi_2(t') + \langle \pi_2(t'), \beta^\vee \rangle(-\beta) = \pi(t) + \langle \pi_2(t'), \beta^\vee \rangle(-\beta)$. As we have already seen, $\langle \pi_2(t'), \beta^\vee \rangle \in [0, 1]$, so choosing $b = \langle \pi_2(t'), \beta^\vee \rangle$ does the job. Finally, again from the discussion preceding the definitions of π_1, π_2, and π_3 it follows that $\langle \pi_2(1), \beta^\vee \rangle = 1$, so for the last (third) portion of the path, we have that $f_\beta(\pi)(t) = (\pi(x) - \beta) + \pi_3(t'') = \pi(x) + \pi_3(t'') - \beta = \pi(t) - \beta$, where again t'' is determined by t by a reparameterization. By choosing $b = 1$ we see that the claim holds in this case also. Applying the same argument x times yields the result.

Statements (3) and (4) follow straightforwardly by applying (1) and (2) repeatedly. □

The following elementary observation is also conceptually useful. For two simple positive roots α, β, we say that α and β are *adjacent* if they are distinct and they correspond to two adjacent nodes in the corresponding Dynkin diagram. (From properties of the Cartan matrix, α and β are adjacent precisely when the value of the pairing $\langle \alpha, \beta^\vee \rangle$ is strictly negative.) Then it is immediate that $A_k(x_{k+1}, \ldots, x_n)$ can be interpreted as

$$\begin{aligned} &A_k(x_{k+1}, \ldots, x_n) \\ &\quad = m_k + \Bigg(\sum_{\substack{j>k \\ \beta_j = \beta_k}} (m_j - 2x_j) \Bigg) - \Bigg(\sum_{\substack{j>k \\ \beta_j \text{ adjacent to } \beta_k}} x_j \langle \beta_j, \beta_k^\vee \rangle \Bigg). \end{aligned} \tag{2.8}$$

Proof of Proposition 2.10. We begin by noting that the path τ_n is by definition π^{λ_n} where $\lambda_n := m_n \beta_n$. Thus, $Q = 0$ in this case, and by Lemma 1.6(3) we conclude $e_{\beta_n}(\tau_n) = \mathbf{O}$. So it remains to check the cases $k < n$. As in the discussion before, by Lemma 1.6(3) and by the definition of the raising operators, in order to prove the claim, it suffices to prove that for any $(x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m})$ and any k with $1 \le k \le n-1$, we have

$$\min_{t \in [0,1]} \{ \langle \tau_k(x_{k+1}, \ldots, x_n)(t), \beta_k^\vee \rangle \} \ge 0, \tag{2.9}$$

which is equivalent to

$$\min_{t \in [0,1]} \{ \langle \varphi_{k+1}(x_{k+1}, \ldots, x_n)(t), \beta_k^\vee \rangle \} \ge -m_k \tag{2.10}$$

by the definition of the τ_k and φ_k.

We use induction on the size of n. We already proved the case $n = 1$, so the base case is $n = 2$ and $k = 1$. Let $\mathbf{i} = (i_1, i_2)$ with associated sequence of simple roots (β_1, β_2) and $\mathbf{m} = (m_1, m_2)$. Let $(x_1, x_2) \in P(\mathbf{i}, \mathbf{m})$. Then we have $0 \le x_2 \le m_2$, so an explicit computation shows $\varphi_2(x_2) = f_{\beta_2}^{x_2}(\pi^{m_2 \varpi_{\beta_2}}) = \pi^{x_2(\varpi_{\beta_2} - \beta_2)} \star \pi^{(m_2 - x_2)\varpi_{\beta_2}}$. Hence, we wish to show that

$$\min_{t \in [0,1]} \{ \langle \varphi_2(x_2) = \pi^{x_2(\varpi_{\beta_2} - \beta_2)} \star \pi^{(m_2 - x_2)\varpi_{\beta_2}}(t), \beta_1^\vee \rangle \} \ge -m_1.$$

First, consider the case $\beta_1 \ne \beta_2$. Since $\langle \varpi_{\beta_2}, \beta_1^\vee \rangle = 0$ and $\langle \beta_2, \beta_1^\vee \rangle \le 0$ for any two distinct simple roots, and $x_2 \ge 0$ by assumption, we can see that $\langle \pi^{x_2(\varpi_{\beta_2} - \beta_2)} \star \pi^{(m_2 - x_2)\varpi_{\beta_2}}(t), \beta_1^\vee \rangle \ge 0$ for all t. In particular, the minimum value taken over all t is 0, which is greater than or equal to $-m_1$, as desired (since $m_1 \ge 0$ by assumption). Next consider the case $\beta_1 = \beta_2$. In this case, the inequalities defining $P(\mathbf{i}, \mathbf{m})$ are

$$0 \le x_2 \le m_2 \quad \text{and} \quad 0 \le x_1 \le \langle m_1 \varpi_{\beta_1} + m_2 \varpi_{\beta_2} - x_2 \beta_2, \beta_1^\vee \rangle = m_1 + m_2 - 2x_2.$$

By condition (P), for any choice of x_2 with $0 \le x_2 \le m_2$, we must have $A_1(x_2) = m_1 + m_2 - 2x_2 \ge 0$. In particular, for $x_2 = m_2$, we must have $m_1 - m_2 \ge 0$, from which it follows $m_1 \geq m_2$. Next notice that since the vector $(m_2 - x_2)\varpi_{\beta_2}$ pairs nonnegatively with $\beta_1^\vee = \beta_2^\vee$, the minimum value of the function

$$t \mapsto \langle \pi^{x_2(\varpi_{\beta_2} - \beta_2)} \star \pi^{(m_2 - x_2)\varpi_{\beta_2}}(t), \beta_1^\vee \rangle$$

occurs at the endpoint of $\pi^{x_2(\varpi_{\beta_2}-\beta_2)}$, where the value is $-x_2$. From the assumptions we know $x_2 \leq m_2$, so $-x_2 \geq -m_2$. Also, we have seen that $m_1 \geq m_2$, so $-m_2 \geq -m_1$, and finally we obtain $-x_2 \geq -m_1$. This completes the base case.

We now assume by induction that the statement of the proposition holds for words and multiplicity lists of length $\leq n-1$, and we must prove the statement for n. As before, we already know that the statement holds for $k = n$. Next, suppose $1 < k < n$. By Lemma 2.5 we know that $(\mathbf{i}[k], \mathbf{m}[k])$ satisfies condition (P) and $(x_k, \ldots, x_n)$ lies in $P(\mathbf{i}[k], \mathbf{m}[k])$. Since $\mathbf{i}[k], \mathbf{m}[k]$ have length strictly less than n, by the inductive assumption we know that the statement holds for such k. Thus, it remains to check the case $k = 1$, that is, that $e_{\beta_1}(\tau_1(x_2, \ldots, x_n)) = \mathbf{O}$ for $(x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m})$. First, consider the case in which the simple root β_1 does not appear in the word $(\beta_2, \ldots, \beta_n)$. By Lemma 2.11(4) any point along the path $\varphi_2(x_2, \ldots, x_n)$ can be written in the form $\sum_{\ell=2}^n a_\ell \varpi_{\beta_\ell} + \sum_{\ell=2}^n b_\ell(-\beta_\ell)$, where $a_\ell, b_\ell \geq 0$ are real constants, and all the simple roots β_ℓ are distinct from β_1. Then for any time t, we have $\langle \varphi_2(x_2, \ldots, x_n)(t), \beta_1^\vee \rangle = \langle \sum_{\ell=2}^n a_\ell \varpi_{\beta_\ell} + \sum_{\ell_2}^n b_\ell(-\beta_\ell), \beta_1^\vee \rangle = \langle -\sum_{\ell=2}^n b_\ell \beta_\ell, \beta_1^\vee \rangle = -\sum_{\ell=2}^n b_\ell \langle \beta_\ell, \beta_1^\vee \rangle \geq 0$, where the second equality is because $\langle \varpi_{\beta_\ell}, \beta_1^\vee \rangle = 0$ for $\beta_\ell \neq \beta_1$, and the last inequality is because $\langle \beta_\ell, \beta_1^\vee \rangle \leq 0$ for $\beta_\ell \neq \beta_1$. Since $m_1 \geq 0$ by assumption, we conclude that $\langle \varphi(x_2, \ldots, x_n)(t), \beta_1^\vee \rangle \geq 0 \geq -m_1$ for all t, which yields the desired result.

Next we consider the case where β_1 occurs in the sequence $(\beta_2, \ldots, \beta_n)$. Let s be the smallest index with $s \geq 2$ such that $\beta_s = \beta_1$, that is, it is the first place after β_1 where the repetition occurs. Since the length of $\mathbf{i}[s]$ is $n-1$, from the inductive assumption we know that $\min_{t\in[0,1]}\{\langle \tau_s(x_{s+1}, \ldots, x_n)(t), \beta_s^\vee = \beta_1^\vee \rangle\} \geq 0$. Note also that the path τ_s has the property that the minimum value $\min_{t\in[0,1]}\{\langle \tau_s(x_{s+1}, \ldots, x_n)(t), \beta_s^\vee = \beta_1^\vee \rangle\}$ and the endpoint pairing $\langle \mathrm{wt}(\tau_s), \beta_s^\vee \rangle$ are both integers; this follows from its construction. Also by definition, the operator f_{β_s} preserves these properties; moreover, for such a path τ', it follows from the definition of f_{β_s} that $\min_{t\in[0,1]}\{\langle f_{\beta_s}(\tau')(t), \beta_s^\vee \rangle\} = \min_{t\in[0,1]}\{\langle \tau'(t), \beta_s^\vee \rangle\} - 1$, that is, the minimum decreases by precisely 1. From this we conclude that $\varphi_s(x_s, \ldots, x_n) = f_{\beta_s=\beta_1}^{x_s}(\tau_s)$ satisfies

$$\langle \varphi_s(x_s, \ldots, x_n)(t), \beta_1^\vee = \beta_s^\vee \rangle \geq -x_s \quad \text{for all } t \in [0, 1]. \tag{2.11}$$

By definition $\varphi_2(x_2, \ldots, x_n)$ is obtained from $\varphi_s(x_s, \ldots, x_n)$ by

$$\varphi_2(x_2, \ldots, x_n) := f_{\beta_2}^{x_2}(\pi^{m_2\varpi_{\beta_2}} \star (\cdots f_{\beta_{s-1}}^{x_{s-1}}(\pi^{m_{s-1}\varpi_{\beta_{s-1}}} \star \varphi_s(x_s, \ldots, x_n)) \cdots)).$$

By assumption, β_1 is distinct from all the roots β_ℓ for $2 \leq \ell \leq s-1$. Thus, $\langle \varpi_{\beta_\ell}, \beta_1^\vee \rangle = 0$ and $\langle -\beta_\ell, \beta_1^\vee \rangle \geq 0$ for $2 \leq \ell \leq s-1$, and from Lemma 2.11(3) it follows that

$$\min_{t\in[0,1]}\{\langle \varphi_2(x_2, \ldots, x_n)(t), \beta_1^\vee \rangle\} \geq \min_{t\in[0,1]}\{\langle \varphi_s(x_s, \ldots, x_n)(t), \beta_1^\vee \rangle\}.$$

Since we know from (2.11) that the RHS is $\geq -x_s$, it now suffices to prove that $x_s \leq m_1$. Since $(x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m})$, we know that $(y_s, x_{s+1}, \ldots, x_n) \in P(\mathbf{i}[s], \mathbf{m}[s])$ if $0 \leq y_s \leq A_s(x_{s+1}, \ldots, x_n)$. Also, since $(\mathbf{i}, \mathbf{m})$ satisfies condition (P), from Lemma 2.5(2) we know that $(y_2, \ldots, y_n) \in P(\mathbf{i}[2], \mathbf{m}[2])$, where $y_2 = \cdots = y_{s-1} = 0$, $y_s = A_s(x_{s+1}, \ldots, x_n)$, and $y_k = x_k$ for $k \geq s+1$. Then from condition (P) we conclude that

$$\begin{aligned} A_1(y_2, \ldots, y_n) &= m_1 + (m_s - 2y_s) + \Bigg(\sum_{\substack{k>s \\ \beta_k=\beta_1=\beta_s}} (m_k - 2x_k) \Bigg) \\ &\quad - \Bigg(\sum_{\substack{k>s \\ \beta_k \text{ adjacent to } \beta_1=\beta_s}} x_k \langle \beta_k, \beta_1^\vee = \beta_s^\vee \rangle \Bigg) \\ &= m_1 + A_s(x_{s+1}, \ldots, x_n) - 2y_s = m_1 - A_s(x_{s+1}, \ldots, x_n) \geq 0 \end{aligned}$$

or, in other words, $m_1 \geq A_s(x_{s+1}, \ldots, x_n)$. But the original x_s was required to satisfy the inequality $x_s \leq A_1(x_{s+1}, \ldots, x_n)$, from which it follows that $x_s \leq m_1$, as was to be shown. This completes the inductive argument and hence the proof. □

3. Newton–Okounkov Bodies of Bott–Samelson Varieties

The main result of this manuscript is Theorem 3.4, which gives an explicit description of the Newton–Okounkov body of $(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ with respect to a certain geometric valuation (to be further described in detail), provided that the word $\mathbf{i}$ corresponds to a reduced word decomposition and the pair $(\mathbf{i}, \mathbf{m})$ satisfies condition (P).

We first very briefly recall the ingredients in the definition of a Newton–Okounkov body. For details, we refer the reader to [10; 15]. We begin with the definition of a valuation (in our setting). We equip $\mathbb{Z}^n$ with the lexicographic order.

Definition 3.1. (1) Let V be a $\mathbb{C}$- vector space. A *prevaluation* on V is a function

$$\nu : V \setminus \{0\} \to \mathbb{Z}^n$$

satisfying the following:

(a) $\nu(cf) = \nu(f)$ for all $f \in V \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$,

(b) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ for all $f, g \in V \setminus \{0\}$ with $f + g \neq 0$.

(2) Let A be a $\mathbb{C}$-algebra. A *valuation* on A is a prevaluation on A, $\nu : A \setminus \{0\} \to \mathbb{Z}^n$, which also satisfies the following: $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in A \setminus \{0\}$.

(3) The image $\nu(A \setminus \{0\}) \subset \mathbb{Z}^n$ of a valuation ν on a $\mathbb{C}$-algebra A is clearly a semigroup and is called the *value semigroup* of the pair (A, ν).

(4) Moreover, if in addition the valuation has the property that for any pair $f, g \in A \setminus \{0\}$ with same value $\nu(f) = \nu(g)$, there exists a nonzero constant $c \neq 0 \in \mathbb{C}$ such that either $\nu(g - cf) > \nu(g)$ or else $g - cf = 0$, then we say that the valuation has *one-dimensional leaves.*

In the construction of Newton–Okounkov bodies, we consider valuations on rings of sections of line bundles. More specifically, let X be a complex-n-dimensional algebraic variety over $\mathbb{C}$, equipped with a line bundle $L = \mathcal{O}_X(D)$ for some (Cartier) divisor D. Consider the corresponding (graded) $\mathbb{C}$-algebra of sections $R = R(L) := \bigoplus_{k \geq 0} R_k$ where $R_k := H^0(X, L^{\otimes k})$. We now describe a way to geometrically construct a special kind of valuation. Suppose given a flag

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{\text{pt}\}$$

of irreducible subvarieties of X where $\text{codim}_{\mathbb{C}}(Y_\ell) = \ell$ and each Y_ℓ is nonsingular at the point $Y_n = \{\text{pt}\}$ ($\ell = 0, 1, \ldots, n$). Such a flag defines a valuation $\nu_{Y_\bullet} : H^0(X, L) \setminus \{0\} \to \mathbb{Z}^n$ by an inductive procedure involving restricting sections to each subvariety and considering its order of vanishing along the next (smaller) subvariety as follows. We will assume that all Y_i are smooth (though this is not necessary, cf. [15]). Given a nonzero section $s \in H^0(X, L = \mathcal{O}_X(D))$, we define

$$\nu_1 := \text{ord}_{Y_1}(s),$$

that is, the order of vanishing of s along Y_1. By choosing a local equation for Y_1 in X we can construct a section $\tilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \nu_1 Y_1))$ that does not vanish identically on Y_1. By restricting we obtain a nonzero section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1))$ and define $\nu_2 := \text{ord}_{Y_2}(s_1)$. We define each ν_i by proceeding inductively in the same fashion. It is not difficult to see that $\nu_{Y_\bullet}$ thus defined gives a valuation with one-dimensional leaves on each R_k.

Given such a valuation ν, we may then define

$$S(R) = S(R, \nu) := \bigcup_{k>0} \{(k, \nu(\sigma)) \mid \sigma \in R_k \setminus \{0\}\} \subset \mathbb{N} \times \mathbb{Z}^n$$

(cf. also [15, Def. 1.6], where the notation slightly differs), which can be seen to be an additive semigroup. Now define $C(R) \subseteq \mathbb{R} \times \mathbb{R}^n$ to be the cone generated by the semigroup $S(R)$, that is, it is the smallest closed convex cone centered at the origin containing $S(R)$. We can now define the central object of interest.

Definition 3.2. Let $\Delta = \Delta(R) = \Delta(R, \nu)$ be the slice of the cone $C(R)$ at level 1, that is, $C(R) \cap (\{1\} \times \mathbb{R}^n)$, projected to $\mathbb{R}^n$ via the projection to the second factor $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. We have

$$\Delta = \overline{\text{conv}\left(\bigcup_{k>0} \left\{ \frac{x}{k} : (k, x) \in S(R) \right\} \right)}.$$

The convex body Δ is called the *Newton–Okounkov body of R* with respect to the valuation ν.

In the current manuscript, the geometric objects under study are the Bott–Samelson variety $Z_{\mathbf{i}}$ and the line bundle $L_{\mathbf{i},\mathbf{m}}$ over it. Following the notation, we wish to study the Newton–Okounkov body of $R(L_{\mathbf{i},\mathbf{m}}) = \bigoplus_{k>0} H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}^{\otimes k})$. We begin with a description of the flag $Y_\bullet$ of subvarieties with respect to which we will define a valuation. Given ℓ with $1 \leq \ell \leq n$, we define a subvariety Y_ℓ of $Z_{\mathbf{i}}$ of codimension ℓ by

$$Y_\ell := \{[(p_1, \ldots, p_n)] : \\ p_s = e \text{ for the last } \ell \text{ coordinates, that is, for } n - \ell + 1 \leq s \leq n\}.$$

The subvariety Y_ℓ is smooth since it is evidently isomorphic to the Bott–Samelson variety $Z_{(i_1,\ldots,i_{n-\ell})}$. In Kaveh's work on Newton–Okounkov bodies and crystal bases [8], he introduces a set of coordinates, which he denotes $(t_1, \ldots, t_n)$ near the point $Y_0 = \{[(e, e, \ldots, e)]\}$. Near Y_0, our flag $Y_\bullet$ can be described using Kaveh's coordinates as

$$\{t_n = 0\} \supset \{t_n = t_{n-1} = 0\} \supset \cdots \supset \{t_n = \cdots = t_2 = 0\} \supset \{(0, 0, \ldots, 0)\}.$$

Remark 3.3. In particular, with respect to Kaveh's coordinates, our geometric valuation $\nu_{Y_\bullet}$ is the lowest-term valuation on polynomials in $t_1, \ldots, t_n$ with respect to the lexicographic order with $t_1 < t_2 < \cdots < t_n$. Thus, our valuation is different from the valuation used by Kaveh [8] and Fujita [4] since they take the highest-term valuation with respect to the lexicographic order with the variables in the reverse order, $t_1 > t_2 > \cdots > t_n$. In general, it seems to be a rather subtle problem to understand the dependence of the Newton–Okounkov body on the choice of valuation; cf., for instance, the discussion in [8, Rem. 2.3].

We now state the main theorem of this section, which is also the main result of this manuscript. Let $P(\mathbf{i}, \mathbf{m})$ denote the polytope of Definition 2.1. In Theorem 3.4, $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}}$ denotes the points in $P(\mathbf{i}, \mathbf{m})$ with coordinates reversed, that is, $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}} := \{(x_n, \ldots, x_1) : (x_1, \ldots, x_n) \in P(\mathbf{i}, \mathbf{m})\}$. (The reversal of the ordering on coordinates arises because, locally near $Y_n = \{[e, e, \ldots, e]\}$ and in Kaveh's coordinates, Y_i is given by the equations $\{t_{n-i+1} = \cdots = t_n = 0\}$, that is, the *last* coordinates are 0. So, for example, $\nu_1(s)$ is the order of vanishing of s along $\{t_n = 0\}$, not $\{t_1 = 0\}$.)

THEOREM 3.4. *Let $\mathbf{i} = (i_1, \ldots, i_n) \in \{1, 2, \ldots, r\}^n$ be a word, and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ be a multiplicity list. Let $Z_{\mathbf{i}}$ and $L_{\mathbf{i},\mathbf{m}}$ denote the associated Bott–Samelson variety and line bundle, respectively. Suppose that $\mathbf{i}$ corresponds to a reduced word decomposition and that $(\mathbf{i}, \mathbf{m})$ satisfies condition* (P). *Consider the valuation $\nu_{Y_\bullet}$ previously defined and let $S(R(L_{\mathbf{i},\mathbf{m}}))$ denote the corresponding value semigroup. Then*

(1) *the degree-*1 *piece $S_1 := S(R(L_{\mathbf{i},\mathbf{m}})) \cap \{1\} \times \mathbb{Z}^n$ of $S(R(L_{\mathbf{i},\mathbf{m}}))$ is equal to $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$ (where we identify $\{1\} \times \mathbb{Z}^n$ with $\mathbb{Z}^n$ by projection to the second factor*),
(2) *$S(R(L_{\mathbf{i},\mathbf{m}}))$ is generated by S_1, so, in particular, it is finitely generated, and*
(3) *the Newton–Okounkov body $\Delta = \Delta(R(L_{\mathbf{i},\mathbf{m}}))$ of $Z_{\mathbf{i}}$ and $L_{\mathbf{i},\mathbf{m}}$ with respect to $\nu_{Y_\bullet}$ is equal to the polytope $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}}$.*

Before diving into the proof of Theorem 3.4, we explain the basic structure of our argument. Our first step is Proposition 3.7, where we show that the image of $\nu_{Y_\bullet}$ is always a subset of the polytope $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}}$. This is the most important step in our argument; here we need that $\mathbf{i}$ is reduced. Then, under the additional assumption that $(\mathbf{i}, \mathbf{m})$ satisfies condition (P), the results of Section 2 allow us to quickly conclude that $\nu_{Y_\bullet}$ gives a surjection from S_1 to $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$, from which the theorem follows.

We need some preliminaries. For each j with $1 \le j \le n$, let C_j denote the curve in $Z_{\mathbf{i}}$ given by setting all but the jth coordinate in $[(p_1, \ldots, p_n)] \in Z_{\mathbf{i}}$ equal to e. Note that the curves are isomorphic to $\mathbb{P}^1$. The lemma below is from [5, Sect. 3.7].

Lemma 3.5. *Let $\lambda_1, \ldots, \lambda_n$ be a sequence of weights. The degree of the restriction of the line bundle $L_{\mathbf{i}}(\lambda_1, \ldots, \lambda_n)$ on $Z_{\mathbf{i}}$ to the curve C_n is equal to $\langle \lambda_n, \beta_n^\vee \rangle$.*

In what follows, we also need the following codimension-1 subvarieties (divisors) on $Z_{\mathbf{i}}$. For $1 \le j \le n$, let $Z_{\mathbf{i}(j)}$ denote the subvariety of $Z_{\mathbf{i}}$ obtained by requiring the jth coordinate of $[(p_1, \ldots, p_n)] \in Z_{\mathbf{i}}$ to be equal to e. Notice that $Z_{\mathbf{i}(n)}$ is the same as our Y_1 before and is also naturally isomorphic to the smaller Bott–Samelson variety $Z_{(i_1,\ldots,i_{n-1})}$ associated with the word obtained by deleting the last entry in $\mathbf{i}$. Also note that since $Z_{\mathbf{i}(n)}$ is an irreducible subvariety of codimension 1, it determines a line bundle $\mathcal{O}(Z_{\mathbf{i}(n)})$. We will need the following lemma, which computes the restriction of certain line bundles on $Z_{\mathbf{i}}$ to $Z_{\mathbf{i}(n)}$.

Lemma 3.6. *Let $\lambda_1, \ldots, \lambda_n$ be a sequence of weights. Then the restriction to $Z_{\mathbf{i}(n)}$ of the line bundle $L_{\mathbf{i}}(\lambda_1, \ldots, \lambda_n)$ is isomorphic to $L_{\mathbf{i}(n)}(\lambda_1, \ldots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n)$ on $Z_{(i_1,\ldots,i_{n-1})}$. Moreover, the restriction of $\mathcal{O}(Z_{\mathbf{i}(n)})$ to $Z_{\mathbf{i}(n)}$ is isomorphic to $L_{(i_1,\ldots,i_{n-1})}(0, \ldots, 0, \beta_n)$ on $Z_{(i_1,\ldots,i_{n-1})}$.*

Proof. Consider the map $\varphi : L_{\mathbf{i}(n)}(\lambda_1, \ldots, \lambda_{n-1} + \lambda_n) \to L_{\mathbf{i}}(\lambda_1, \ldots, \lambda_n)|_{Z_{\mathbf{i}(n)}}$ given by $[(p_1, \ldots, p_{n-1}, k)] \mapsto [(p_1, \ldots, p_{n-1}, e, k)]$. Then φ gives the required isomorphism. Indeed, φ is well defined as can be seen by the computation

$$
\begin{aligned}
&[(p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-2}^{-1} p_{n-1} b_{n-1}, e = b_{n-1}^{-1} b_{n-1}), k] \\
&\quad = [(p_1, p_2, \ldots, p_{n-1}, e, e^{-\lambda_1}(b_1) \cdots e^{-\lambda_{n-1}}(b_{n-1}) e^{-\lambda_n}(b_{n-1}) k)] \\
&\quad = [(p_1, p_2, \ldots, p_{n-1}, e, e^{-\lambda_1}(b_1) \cdots e^{-(\lambda_{n-1}+\lambda_n)}(b_{n-1}) k)]
\end{aligned}
$$

in $L_{\mathbf{i}}(\lambda_1, \ldots, \lambda_n)$. It can be checked similarly that φ is injective, and the surjectivity is immediate from its definition.

For the second claim, recall that the restriction $\mathcal{O}(D)|_D$ is the normal bundle to D (see e.g. [21, Exer. 21.2H]). Applying this to $Z_{\mathbf{i}(n)}$, it suffices to show that the normal bundle to $Z_{\mathbf{i}(n)}$ in $Z_{\mathbf{i}}$ is isomorphic to $L_{(i_1,\ldots,i_{n-1})}(0, \ldots, 0, \beta_n)$. Now note $Z_{\mathbf{i}}$ is a P_{β_n}/B-bundle over $Z_{\mathbf{i}(n)} \cong Z_{(i_1,\ldots,i_{n-1})}$, and since $Z_{\mathbf{i}(n)}$ is defined by setting the last coordinate equal to e, the normal bundle in question can be identified with $Z_{(i_1,\ldots,i_{n-1})} \times_B T_{eB}(P_{\beta_n}/B)$. The weight of the action of B on the tangent space $T_{eB}(P_{\beta_n}/B)$ at the identity coset eB of P_{β_n}/B is $-\beta_n$. Thus, the normal bundle is precisely $L_{(i_1,\ldots,i_{n-1})}(0, \ldots, 0, \beta_n)$, as desired. □

The important step toward the proof of the main result is the following, which states that the image of the valuation is contained inside the polytope $P(\mathbf{i}, \mathbf{m})^{\mathrm{op}}$.

PROPOSITION 3.7. *Let* $\mathbf{i} = (i_1, \dots, i_n) \in \{1, 2, \dots, r\}^n$ *be a word, and* $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$ *a multiplicity list. Let* $Z_{\mathbf{i}}$ *and* $L_{\mathbf{i},\mathbf{m}}$ *be the Bott–Samelson variety and line bundle specified by* $\mathbf{i}, \mathbf{m}$, *and let* $\nu_{Y_\bullet}$ *denote the geometric valuation specified by the flag* $Y_\bullet$ *given before. Assume that* $\mathbf{i}$ *corresponds to a reduced word decomposition. Then*

$$\nu_{Y_\bullet}(H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \setminus \{0\}) \subseteq P(\mathbf{i}, \mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n.$$

Proof. Let $0 \neq s \in H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ with $\nu_{Y_\bullet}(s) = (x_n, x_{n-1}, \dots, x_1)$. We wish to show that $(x_1, \dots, x_n) \in P(\mathbf{i}, \mathbf{m})$, for which it is enough to show that $x_n \leq m_n$ and $x_k \leq A_k(x_{k+1}, \dots, x_n)$ for $1 \leq k \leq n-1$.

We first prove that $x_n \leq m_n$. Since $m_i \geq 0$ for all i, by [14, Cor. 3.3] the bundle $L_{\mathbf{i},\mathbf{m}}$ is globally generated and hence effective. Moreover, $\mathbf{i}$ is reduced by assumption, so we can conclude from [14, Prop. 3.5] that

$$L_{\mathbf{i},\mathbf{m}} \cong \mathcal{O}\left(\sum_{k=1}^{n} a_k Z_{\mathbf{i}(k)}\right)$$

for some integers $a_k \geq 0$, $1 \leq k \leq n$. Also, since $x_n = \nu_1(s) = \mathrm{ord}_{Z_{\mathbf{i}(n)}}(s)$ is the order of vanishing of s along $Y_1 = Z_{\mathbf{i}(n)}$, we know that $\mathrm{div}(s) = x_n Z_{\mathbf{i}(n)} + E$ for some effective divisor E. Since $\mathrm{div}(s)$ is linearly equivalent to $\sum_{k=1}^n a_k Z_{\mathbf{i}(k)}$, we may conclude

$$E \sim -x_n Z_{\mathbf{i}(n)} + \sum_{k=1}^{n} a_k Z_{\mathbf{i}(k)}, \tag{3.1}$$

where $\sim$ denotes linear equivalence. Considering now the corresponding Chow classes, we may compare the (intersection) product of both sides of (3.1) with the class $[C_n] \in A^*(Z_{\mathbf{i}})$. The Chow ring $A^*(Z_{\mathbf{i}})$ and the classes $[Z_{\mathbf{i}(k)}]$ have been extensively studied, and it is known (cf. [3; 14], see also [19, Prop. 2.11]) that $[C_n] \cdot [Z_{\mathbf{i}(j)}] = \delta_{jn}$. Thus, we obtain that the product (RHS of (3.1)) $\cdot [C_n] = -x_n + a_n$, whereas the product (LHS of (3.1)) $\cdot [C_n] = b_n \geq 0$ since E is effective. Hence, $x_n \leq a_n$. Furthermore, from [19, Prop. 2.11] and from basic properties of intersection products we may also conclude that a_n is the degree of the restriction $L_{\mathbf{i},\mathbf{m}}|_{C_n}$ of the line bundle $L_{\mathbf{i},\mathbf{m}}$ to the curve C_n (which is isomorphic to $\mathbb{P}^1$, so $A_0(C_n) \cong \mathbb{Z}$). By Lemma 3.5, this degree is precisely equal to $\langle m_n \varpi_n, \beta_n^\vee \rangle = m_n$. Thus, $x_n \leq m_n$ as was to be shown.

Next, we consider $x_{n-1} = \nu_2(s) = \mathrm{ord}_{Y_2}(s_1)$, where $0 \neq s_1 \in H^0(Y_1 = Z_{\mathbf{i}(n)}, L_{\mathbf{i},\mathbf{m}} \otimes \mathcal{O}(-x_n Z_{\mathbf{i}(n)})|_{Y_1 = Z_{\mathbf{i}(n)}})$, and s_1 is constructed from s in the fashion described previously. Note that $Z_{\mathbf{i}(n)} \cong Z_{(i_1,\dots,i_{n-1})}$. Thus, repeating the same argument as given before, we may deduce that x_{n-1} is at most the degree of the restriction of the line bundle $L_{\mathbf{i},\mathbf{m}} \otimes \mathcal{O}(-x_n Z_{\mathbf{i}(n)})|_{Y_1 = Z_{\mathbf{i}(n)}}$ to the curve C_{n-1}.

From Lemma 3.6 we know that the restriction of $L_{\mathbf{i},\mathbf{m}}$ to $Z_{\mathbf{i}(n)} \cong Z_{(i_1,\dots,i_{n-1})}$ is isomorphic to the line bundle $L_{(i_1,\dots,i_{n-1})}(m_1 \varpi_{\beta_1}, \dots, m_{n-2} \varpi_{\beta_{n-2}}, m_{n-1} \varpi_{\beta_{n-1}} + m_n \varpi_{\beta_n})$ in the notation of (1.2), and also from Lemma 3.6 we know $\mathcal{O}(Z_n)|_{Z_n} \cong$

$L_{(i_1,\dots,i_{n-1})}(0,\dots,0,\beta_n)$. Thus, we have

$$L_{\mathbf{i},\mathbf{m}} \otimes \mathcal{O}(-x_n Z_{\mathbf{i}(n)})|_{Y_1=Z_{\mathbf{i}(n)}} \cong L_{(i_1,\dots,i_{n-1})}(m_1\varpi_1,\dots,m_{n-2}\varpi_{\beta_{n-2}}, m_{n-1}\varpi_{\beta_{n-1}} + m_n\varpi_{\beta_n} - x_n\beta_n). \tag{3.2}$$

Since s_1 is a nonzero global section, the line bundle in (3.2) is effective. Thus, by again applying [14, Prop. 3.5] we can write it as $\mathcal{O}(\sum_k a'_k Z_k)$ where $a'_k \geq 0$. By proceeding with the same argument as before, since the degree of (3.2) along C_{n-1} is precisely

$$\langle m_{n-1}\varpi_{n-1} + m_n\varpi_n - x_n\beta_n, \beta_{n-1}^\vee\rangle = A_{n-1}(x_n),$$

we may conclude that $x_{n-1} \leq A_{n-1}(x_n)$. Continuing similarly, we obtain $(x_1,\dots,x_n) \in P(\mathbf{i},\mathbf{m})$, as desired. □

Remark 3.8. Note that since a scalar multiple $r\mathbf{m}$ is also a multiplicity list for any positive integer r, it immediately follows from Proposition 3.7 that

$$\nu_{Y_\bullet}(H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}^{\otimes r}) \setminus \{0\}) \subseteq P(\mathbf{i}, r\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$$

for any $r \in \mathbb{N}$.

To complete the argument, we need to recall the following fact from [7].

Proposition 3.9. *If* $(\mathbf{i},\mathbf{m})$ *satisfies condition* (P), *then* $P(\mathbf{i},\mathbf{m})$ *is a* lattice *polytope*.

We are finally ready to prove the main result.

Proof of Theorem 3.4. We begin with the first claim of the theorem. It is elementary that if a valuation $\nu : V \setminus \{0\} \to \mathbb{Z}^n$ (for V a finite-dimensional complex vector space) has one-dimensional leaves, then the cardinality $|\nu(V \setminus \{0\})|$ of the image of ν is equal to $\dim_{\mathbb{C}}(V)$ [10, Prop. 2.6]. Since our valuation $\nu_{Y_\bullet}$ has one-dimensional leaves on R_1, we conclude that $|\nu_{Y_\bullet}(R_1 \setminus \{0\})| = \dim_{\mathbb{C}}(R_1) = \dim_{\mathbb{C}}(H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}))$. On the other hand, we know from Proposition 3.7 that the image of $\nu_{Y_\bullet}$ on $R_1 = H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ must lie in $P(\mathbf{i},\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$. Proposition 2.4 implies $|P(\mathbf{i},\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n| = |P(\mathbf{i},\mathbf{m}) \cap \mathbb{Z}^n| = \dim_{\mathbb{C}}(H^0(Z_{\mathbf{i}}, L(\mathbf{i},\mathbf{m})))$, so we conclude that $S_1 := S(R) \cap \{1\} \times \mathbb{Z}^n$ (which by definition is the image of $\nu_{Y_\bullet} : R_1 \setminus \{0\} \to P(\mathbf{i},\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$) is precisely $P(\mathbf{i},\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$. Here we identify $\{1\} \times \mathbb{Z}^n$ with $\mathbb{Z}^n$ by projection to the second factor. This proves the first statement of the theorem.

By Remark 3.8 we also conclude that S_r is equal to $P(\mathbf{i}, r\mathbf{m})^{\mathrm{op}} \cap \mathbb{Z}^n$. From the definition of the polytopes $P(\mathbf{i},\mathbf{m})$ it follows that $P(\mathbf{i}, r\mathbf{m}) = r \cdot P(\mathbf{i},\mathbf{m})$. This justifies the second statement of the theorem. Finally, the last statement of the theorem now follows directly from Definition 3.2 and Proposition 3.9. □

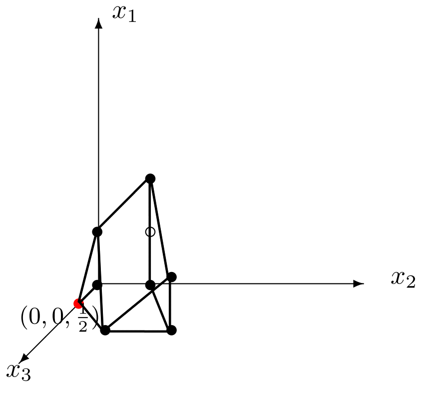
4. Examples

In this section, we give several concrete examples in order to illustrate our results. The first three examples are in Lie type A, and the last is in Lie type C.

First, we let $G = \mathrm{SL}(3,\mathbb{C})$ with Borel subgroup B the upper-triangular matrices and T the diagonal subgroup. The rank r is 2 in this case, and we let $\{\alpha_1, \alpha_2\}$ be the usual positive simple roots corresponding to the simple transpositions $s_1 = (12)$ and $s_2 = (23)$ in the Weyl group $W = S_3$. For the first three examples, we consider the Bott–Samelson variety $Z_{\mathbf{i}}$ where $\mathbf{i} = (1,2,1)$ corresponds to the reduced word decomposition $s_1 s_2 s_1$ of the longest element w_0 in $W = S_3$.

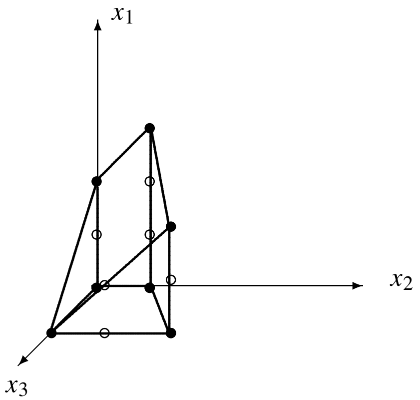
In Example 4.1, we give a pair $(\mathbf{i},\mathbf{m})$ for which the corresponding $L_{\mathbf{i},\mathbf{m}}$ is a pullback from $\mathcal{F}\ell(\mathbb{C}^3) \cong \mathrm{SL}(3,\mathbb{C})/B$, and $(\mathbf{i},\mathbf{m})$ does *not* satisfy condition (P). In Example 4.2, the pair $(\mathbf{i},\mathbf{m})$ is not a pullback from $\mathcal{F}\ell(\mathbb{C}^3)$ and satisfies condition (P), but the polytope $P(\mathbf{i},\mathbf{m})$ is not simple. In Example 4.3, we give an infinite family of pairs $(\mathbf{i},\mathbf{m})$ that are not pullbacks from $\mathcal{F}\ell(\mathbb{C}^3)$, satisfy condition (P), and the corresponding polytopes are simple and in fact smooth (in the sense of [2, Def. 2.4.2]).

Example 4.1. Let $\mathbf{m} = (0,1,1)$. Then $L_{\mathbf{i},\mathbf{m}}$ is in fact the pullback of $L_{\alpha_1+\alpha_2}$ on $\mathcal{F}\ell(\mathbb{C}^3) \cong \mathrm{SL}(3,\mathbb{C})/B$. Then we can check easily that $(\mathbf{i},\mathbf{m})$ does not satisfy condition (P). The polytope $P(\mathbf{i},\mathbf{m})$ is illustrated further. Note that $P(\mathbf{i},\mathbf{m})$ is not a lattice polytope and its volume is $\frac{13}{12}$. In this case, the expected volume of any Newton–Okounkov body of $Z_{\mathbf{i}}$ associated with this line bundle $L_{\mathbf{i},\mathbf{m}}$ is 1, so we see that $P(\mathbf{i},\mathbf{m})$ cannot be a Newton–Okounkov body. However, the convex hull of the eight lattice points in $P(\mathbf{i},\mathbf{m})$ has volume 1. Hence, from Proposition 3.7 it follows that the Newton–Okounkov body of $Z_{(1,2,1)}$ for $L_{(1,2,1),(0,1,1)}$ with respect to our valuation $\nu_{Y_\bullet}$ from Section 3 is precisely the convex hull of these eight lattice points.

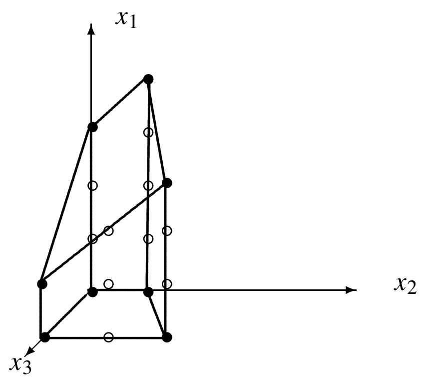


Example 4.2. Let $\mathbf{m} = (1,1,1)$. Then it can be easily checked that $(\mathbf{i},\mathbf{m})$ satisfies condition (P). The figure below illustrates the polytope $P(\mathbf{i},\mathbf{m})$ that is (up to a

reordering of coordinates) the Newton–Okounkov body of $Z_{(1,2,1)}$ with line bundle $L_{(1,2,1),(1,1,1)}$ with respect to our valuation $\nu_{Y_\bullet}$. For visualization purposes, the vertices of the polytope are indicated by black dots, whereas the other lattice points are indicated by white dots. The polytope $P(\mathbf{i}, \mathbf{m})$ is not simple since there are four edges emanating from the vertex $(0, 0, 1)$.



Example 4.3. Let $\mathbf{m} = (a, 1, 1)$ for any integer a, $a \geq 2$. Again, it can be checked easily that $(\mathbf{i}, \mathbf{m})$ for such a choice of $\mathbf{m}$ satisfies condition (P). The polytope $P(\mathbf{i}, \mathbf{m})$, that is, the Newton–Okounkov body of $Z_{\mathbf{i}}$ and $L_{\mathbf{i},\mathbf{m}}$ with respect to $\nu_{Y_\bullet}$ (again up to reordering), is illustrated below for the case $a = 2$. Now the polytope $P(\mathbf{i}, \mathbf{m})$ is simple and in fact smooth, and it is combinatorially a cube.



In our last example, we consider the case $G = \mathrm{Sp}(4, \mathbb{C})$. Let α_1, α_2 be the simple roots, where α_1 is the short root, and α_2 is the long root.

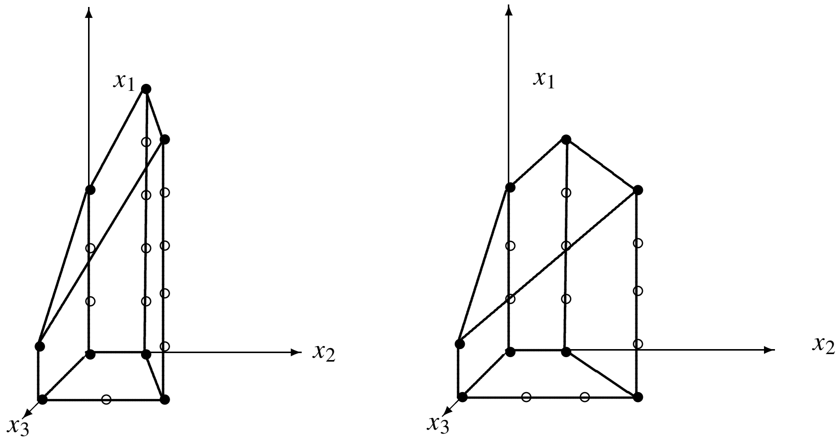
Example 4.4. We compute the polytopes $P(\mathbf{i}, \mathbf{m})$ for two choices of $\mathbf{i}$: namely, $\mathbf{i}_1 = (1, 2, 1)$ (the left figure) and $\mathbf{i}_2 = (2, 1, 2)$ (the right figure). For both cases, we choose $\mathbf{m} = (2, 1, 1)$; it is easily checked that, with these choices, both pairs $(\mathbf{i}, \mathbf{m})$ satisfy condition (P). The corresponding polytopes are illustrated further. Explicitly (and for comparison with the type A case), the inequalities for $P(\mathbf{i}_1, \mathbf{m})$

are

$$0 \le x_3 \le 1, \qquad 0 \le x_2 \le 1 + x_3, \qquad 0 \le x_1 \le 3 + 2x_2 - 2x_3.$$

The inequalities for $P(\mathbf{i}_2, \mathbf{m})$ are

$$0 \le x_3 \le 1, \qquad 0 \le x_2 \le 1 + 2x_3, \qquad 0 \le x_1 \le 3 + x_2 - 2x_3.$$



ACKNOWLEDGMENTS. We are grateful to Lauren DeDieu, Naoki Fujita, Dmitry Kerner, Eunjeong Lee, and Satoshi Naito for useful conversations and to Henrik Seppänen for explaining his work to us. We especially thank Dave Anderson for help with our arguments in Section 3 and to Kiumars Kaveh for pointing out a critical error in a previous version of this manuscript. The first author was partially supported by an NSERC Discovery Grant, a Canada Research Chair (Tier 2) Award, an Ontario Ministry of Research and Innovation Early Researcher Award, an Association for Women in Mathematics Ruth Michler Award, and a Japan Society for the Promotion of Science Invitation Fellowship for Research in Japan (Fellowship ID L-13517). Both authors thank the Osaka City University Advanced Mathematics Institute for its hospitality while part of this research was conducted. The first author also thanks the Department of Mathematics at Cornell University for its hospitality during her tenure as the Ruth Michler Visiting Fellow when portions of this manuscript were written. The second author also thanks Dong Youp Suh for his hospitality during her visit to the Department of Mathematical Sciences, KAIST. Finally, we are grateful to the two anonymous referees for their careful reading and helpful suggestions, which significantly improved the manuscript.

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