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Volume and Hilbert Function of $\mathbb{R}$-Divisors

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1. Introduction

Let X be a proper, normal algebraic variety of dimension n over a field K, and D an $\mathbb{R}$-divisor on X. The *Hilbert function* of D is the function

$$\mathcal{H}(X, D) : m \mapsto h^0(mD) := \dim_K H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$$

defined for all $m \in \mathbb{R}$. If D is an ample Cartier divisor, then $\mathcal{H}(X, D)$ agrees with the usual Hilbert polynomial whenever $m \gg 1$ is an integer, but in general $\mathcal{H}(X, D)$ is not a polynomial, not even if D is a $\mathbb{Z}$-divisor and $m \in \mathbb{Z}$. The simplest numerical invariant associated to the Hilbert function is the *volume* of D, defined as

$$\operatorname{vol}(D) := \limsup_{m\to\infty} \frac{h^0(mD)}{m^n/n!}.$$

If E is an effective $\mathbb{R}$-divisor, then

$$h^0(mD - mE) \le h^0(mD) \le h^0(mD + mE) \tag{$*$}$$

for every $m > 0$; hence,

$$\operatorname{vol}(D - E) \le \operatorname{vol}(D) \le \operatorname{vol}(D + E). \tag{$**$}$$

Furthermore, if equality holds in ($*$) for every $m \gg 1$, then equality holds in ($**$). The aim of this note is to prove the converse for *big* divisors, that is, when $\operatorname{vol}(D) > 0$. Although the volume does not determine the Hilbert function, we prove that

$$\begin{aligned} \mathcal{H}(X, D) \equiv \mathcal{H}(X, D - E) \quad &\Leftrightarrow \quad \operatorname{vol}(D) = \operatorname{vol}(D - E) \quad \text{and} \\ \mathcal{H}(X, D) \equiv \mathcal{H}(X, D + E) \quad &\Leftrightarrow \quad \operatorname{vol}(D) = \operatorname{vol}(D + E). \end{aligned}$$

As a byproduct of the proof, we also obtain a characterization of such divisors E in terms of the negative part $N_\sigma(D)$ of the *Zariski–Nakayama-decomposition* (also called σ-decomposition) and of the divisorial part of the *augmented base locus* $\mathbf{B}_+^{\mathrm{div}}(D)$; see [Nak04], (4.1) and Definition 5.1 for definitions.

Another interesting consequence is that the answer depends only on the $\mathbb{R}$-linear equivalence class of D. This is obvious for $\mathbb{Z}$-linear equivalence, but it can easily happen that $D' \sim_{\mathbb{R}} D$ yet $h^0(X, mD) \ne h^0(X, mD')$ for every $m > 0$; see Example 2.6. In fact, the only relationship between $\mathcal{H}(X, D)$ and $\mathcal{H}(X, D')$ that we know of is $\operatorname{vol}(D) = \operatorname{vol}(D')$.

Our main results are the following.

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THEOREM A. *Let X be a proper, normal algebraic variety over a perfect field, D a big $\mathbb{R}$-divisor on X, and E an effective $\mathbb{R}$-divisor on X. Then the following are equivalent.*

(i) *The equality* $\operatorname{vol}(D-E)=\operatorname{vol}(D)$ *holds.*
(ii) *The negative part $N_\sigma(D)$ dominates E, that is $E \le N_\sigma(D)$.*
(iii) *The equality of Hilbert functions $h^0(mD'-mE)=h^0(mD')$ holds for every $D' \sim_{\mathbb{R}} D$ and all $m>0$.*
(iv) *The equality of Hilbert functions $h^0(mD-mE)=h^0(mD)$ holds for all $m>0$.*

Furthermore, if D is $\mathbb{R}$-Cartier and nef, then these are also equivalent to

(v) *The divisor $E=0$.*

THEOREM B. *Let X be a proper, normal algebraic variety over a perfect field, D a big $\mathbb{R}$-divisor on X, and E an effective $\mathbb{R}$-divisor on X. Then the following are equivalent.*

(i) *The equality* $\operatorname{vol}(D+E)=\operatorname{vol}(D)$ *holds.*
(ii) *The support of E is contained in $\mathbf{B}_+^{\mathrm{div}}(D)$.*
(iii) *The equality $h^0(mD'+rE)=h^0(mD')$ holds for every $D' \sim_{\mathbb{R}} D$ and all $m,r>0$.*
(iv) *The equality of Hilbert functions $h^0(mD+mE)=h^0(mD)$ holds for all $m>0$.*

Furthermore, if D is $\mathbb{R}$-Cartier and nef, then these are also equivalent to

(v) *The vanishing of the intersection number $D^{n-1}\cdot E=0$.*

Special cases of these theorems were first conjectured in connection with the numerical stability criteria for families of canonical models of varieties of general type [Kol15]. In trying to prove these, we gradually realized that the stated results hold and the general setting led to shorter proofs.

The theorems are proved in Section 2, but the necessary technical background results involving $\mathbb{R}$-divisors, the Zariski–Nakayama-decomposition and the augmented base locus on singular varieties are left to Sections 3 through 5. Much of the relevant literature works with smooth projective varieties over $\mathbb{C}$, but many of these proofs apply in more general settings. We went through them, and we state clearly which parts work for normal varieties in any characteristic. We also establish several results that show how to reduce similar types of questions to smooth and projective varieties. These should be useful in similar contexts.

2. Proofs of the Theorems

PROPOSITION 2.1. *Let X be a normal proper variety over an algebraically closed field, and D a big $\mathbb{R}$-divisor. Suppose that $D=P+N$ with* $\operatorname{vol}(P)=\operatorname{vol}(D)$ *and N effective. Then $N \le N_\sigma(D)$.*

The proof is a modification of [FL13, Prop. 5.3].

Proof of Proposition 2.1. By Corollary 3.4 we may find a projective birational model X' and $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D' and P' on X' such that for any positive real m, the pushforward of $\mathcal{O}_{X'}(mD')$ and $\mathcal{O}_{X'}(mP')$ are respectively $\mathcal{O}_X(mD)$ and $\mathcal{O}_X(mP)$, and the difference $D' - P'$ is effective. Note that D' and P' still satisfy the hypotheses of the theorem. If we prove the statement on X', then we can conclude the statement on X by pushing forward and applying Lemma 4.2. So without loss of generality we may assume that P and D, and hence N, are $\mathbb{R}$-Cartier $\mathbb{R}$-divisors and that X is projective.

If $\pi : Y \to X$ is a generically finite proper morphism from a normal projective variety Y, then

$$\pi_* N_\sigma(\pi^* D) = (\deg \pi) \cdot N_\sigma(D)$$

by Lemma 4.12(ii). Furthermore,

$$\mathrm{vol}(\pi^* D) = (\deg \pi) \cdot \mathrm{vol}(D)$$

by Theorem 3.5(ii), the homogeneity of vol, and [Kür06, Prop. 2.9(1)] (the proof there does not use the assumption that the characteristic is zero). Therefore, after passing to a nonsingular alteration (cf. [dJ96]), it is enough to consider the case where X is nonsingular and projective.

By assumption the volume of P does not change if we add a small multiple of N. Thus, by [Cut13b, Thm. 5.6] (see also [BFJ09, Thm. A] and [LM09, Cor. C]),

$$\langle P^{n-1} \rangle \cdot N = 0,$$

where $\langle P^{n-1} \rangle$ is the positive intersection product defined in [Cut13b], inspired by [BFJ09] and classical work of Matsusaka ([Mat72, p. 1031]; see also [LM75, p. 515]).

As in the proof of [BFJ09, Thm. 4.9], it follows that for any ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A on X and any small $\varepsilon > 0$, we have

$$\mathrm{Supp}(N) \subseteq \mathrm{Supp}(N_\sigma(P - \varepsilon A)).$$

(Otherwise, from $P = \frac{\varepsilon}{2} A + (\frac{\varepsilon}{2} A + P_\sigma(P - \varepsilon A)) + N_\sigma(P - \varepsilon A)$ we get $P \geq_{N_i} \frac{\varepsilon}{2} A$ for some component N_i of N, i.e., $P - \frac{\varepsilon}{2} A$ is numerically equivalent to an effective $\mathbb{R}$-divisor that does not contain N_i in its support. Using [BFJ09, Rem. 4.5], we see that $\frac{\varepsilon^{n-1}}{2^{n-1}} A^{n-1} \cdot N_i \leq \langle P^{n-1} \rangle|_{N_i} \leq \langle P^{n-1} \rangle \cdot N_i \leq \langle P^{n-1} \rangle \cdot N$, but the LHS is only zero when $N = 0$.)

In particular, Lemma 4.13 shows that $N_\sigma(P - \varepsilon A + N) = N_\sigma(P - \varepsilon A) + N$. Letting ε tend to 0 and using the continuity of σ as in Lemma 4.1(iv), we see that $N_\sigma(D) = N_\sigma(P) + N$. □

We reduce our main theorems to the case where the base field is algebraically closed.

Remark 2.2. Let K be a field, and L/K a separable field extension. Base change to L is denoted by the subscript L. If X_K is a proper, normal algebraic variety over K, then X_L is a disjoint union of proper, normal algebraic varieties over L. If $E_K \subset X_K$ is a prime divisor, then $E_L \subset X_L$ is a sum of prime divisors, each appearing with coefficient 1. Thus, if D_K is an $\mathbb{R}$-divisor on X_K, then $\lfloor D_K \rfloor_L = \lfloor D_L \rfloor$. Thus,

$$(\mathcal{O}_{X_K}(D_K))_L = \mathcal{O}_{X_L}(D_L) \quad \text{and} \quad h^0(D_K) = h^0(D_L). \tag{2.1}$$

Similarly, if D_K is a $\mathbb{Z}$-divisor, then $|D_K|_L = |D_L|$, and hence the base locus commutes with separable field extensions. Using the characterization given in Lemma 4.1(i) and Lemma 5.3, this implies that

$$N_\sigma(D_L) = (N_\sigma(D_K))_L \quad \text{and} \quad \mathbf{B}_+^{\mathrm{div}}(D_L) = (\mathbf{B}_+^{\mathrm{div}}(D_K))_L. \tag{2.2}$$

(If X_K is geometrically normal but L/K is not separable, then it can happen that $\lfloor D_K \rfloor_L \neq \lfloor D_L \rfloor$. However, (2.2) still holds.)

If Theorems A and B hold for proper, normal varieties over an algebraically closed field, then they clearly also hold for proper, normal, equidimensional schemes over an algebraically closed field. Thus, by the preceding considerations, they hold for proper, normal varieties over any perfect field.

Proof of Theorem A. By Remark 2.2 we may work over an algebraically closed field. The implications (ii) $\to$ (iii) $\to$ (iv) $\to$ (i) are immediate, whereas (i) $\to$ (ii) is Proposition 2.1. Any nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D is movable, that is, $N_\sigma(D) = 0$. Then the equivalence between (ii) and (v) is clear. □

REMARK 2.3. The work of [KL15] hints to an approach to Theorem A using the theory of Okounkov bodies.

REMARK 2.4. Related cases of Theorem A include:

(i) If D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then in (iii) we may set D' to be any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor numerically equivalent to D.

(ii) If X is nonsingular and projective over an algebraically closed field, if D is big and *movable,* and E is *pseudoeffective* (i.e., its numerical class is in the closure of the effective cone), then $\mathrm{vol}(D - E) = \mathrm{vol}(D)$ if and only if $E = 0$.

The first statement is a consequence of Lemma 4.1(iv). For the second, by [FL13, Prop. 5.3] we get

$$P_\sigma(D - E) + (N_\sigma(D - E) + E) \equiv D = P_\sigma(D) \equiv P_\sigma(D - E).$$

Consequently, $N_\sigma(D - E) + E \equiv 0$. Since the pseudoeffective cone is pointed (e.g., by [CHMS13, Lemma 2.4]), it follows that $E = 0$.

Proof of Theorem B. As in Theorem A, we may work over an algebraically closed field. The implications (iii) $\to$ (iv) $\to$ (i) are immediate. Part (ii) of Theorem A and Lemma 4.13 prove (i) $\to$ (iii).

Assume that $\mathrm{Supp}(E) \subseteq \mathbf{B}_+^{\mathrm{div}}(D)$. Let A be ample in codimension 1 (cf. Definition 4.7). By Lemma 5.3 and Lemma 5.2 we have $\mathrm{Supp}(E) \subseteq \mathrm{Supp}(N_\sigma(D - \varepsilon A))$ for arbitrarily small $\varepsilon > 0$. By Lemma 4.13, we see that $\mathrm{vol}(D + E - \varepsilon A) = \mathrm{vol}(D - \varepsilon A)$ for sufficiently small $\varepsilon > 0$. If D, E, and A are $\mathbb{R}$-Cartier, we can conclude $\mathrm{vol}(D+E) = \mathrm{vol}(D)$ by the continuity of volumes for $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. To show that $\mathrm{vol}(D+E) = \mathrm{vol}(D)$ in general, we reduce to the $\mathbb{R}$-Cartier case by applying Theorem 3.5(ii) and Corollary 3.4. Hence, (ii) $\to$ (i).

Let F be an irreducible component of E and assume that $F \not\subset \mathrm{Supp}(N_\sigma(D-\varepsilon A))$. Then by Lemma 4.9 there exists $m > 0$ such that

$$mD + F = \left(\frac{1}{2}m\varepsilon A + F\right) + \left(\frac{1}{2}m\varepsilon A + mP_\sigma(D-\varepsilon A)\right) + mN_\sigma(D-\varepsilon A)$$

is $\mathbb{R}$-linearly equivalent to an effective divisor that does not contain F in its support. In particular, $h^0(mD' + rE) \geq h^0(mD' + F) > h^0(mD')$ for some $D' \sim_{\mathbb{R}} D$ and some $r > 0$, for example, $r = \frac{1}{\mathrm{mult}_F(E)}$. Therefore, (iii) $\to$ (ii).

Suppose now that D is a big and nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Let $\pi : Y \to X$ be a proper birational morphism with Y projective. By Lemma 3.3 there exists an effective π-exceptional divisor F on Y such that $\mathrm{vol}(D+E) = \mathrm{vol}(\pi^*D + \overline{E} + F)$, where $\overline{E}$ is a divisor with $\pi_*\overline{E} = E$. We can make choices such that $\overline{E}$ and F are $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. Of course, $\mathrm{vol}(D) = \mathrm{vol}(\pi^*D)$.

If $\mathrm{vol}(D+E) = \mathrm{vol}(D)$, then $\mathrm{vol}(\pi^*D + \overline{E} + F) = \mathrm{vol}(\pi^*D)$. By [Cut13b, Thm. 5.6] we get $\langle \pi^*D^{n-1}\rangle \cdot (\overline{E} + F) = 0$. Since D is nef, we have $(\pi^*D)^{n-1} = \langle(\pi^*D)^{n-1}\rangle$ from [Cut13b, Prop. 4.11]. By the projection formula, $D^{n-1}\cdot E = 0$.

Conversely, if $D^{n-1}\cdot E = \pi^*D^{n-1}\cdot(\overline{E}+F) = 0$, then [Luo90] shows that $h^0(\pi^*D + \overline{E} + F) = h^0(\pi^*D)$ (the analogous equality also holds for multiples). The proof there is carried out with $\mathbb{Z}$-coefficients and over base fields of characteristic zero, but extends to $\mathbb{R}$-coefficients over arbitrary algebraically closed base fields. We conclude by pushing forward to X. $\square$

REMARK 2.5. As in Theorem A, if D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then in (iii) we may set D' to be any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor numerically equivalent to D. In fact, even in the $\mathbb{R}$-Weil case, we may replace $D' \sim_{\mathbb{R}} D$ with $D' - D$ being a numerically trivial $\mathbb{R}$-*Cartier* $\mathbb{R}$-divisor (cf. Lemma 4.1(iv)).

As mentioned in the Introduction, if $D' \sim_{\mathbb{R}} D$, then there is no clear connection between the Hilbert functions $\mathcal{H}(X, D)$ and $\mathcal{H}(X, D')$ other than that $\mathrm{vol}(D) = \mathrm{vol}(D')$ (cf. Theorem 3.5(iv)):

EXAMPLE 2.6. Let $S \to \mathbb{P}^1$ be a minimal ruled surface with a negative section $E \subset S$ and a positive section $C \subset S$ that is disjoint from E. Let $F_1, \ldots, F_4$ be distinct fibers. Then

$$C \sim_{\mathbb{R}} C + (F_1 - F_2) + \sqrt{2}(F_3 - F_4).$$

Note that $\lfloor mC + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4)\rfloor$ has negative intersection with E for all real $m > 0$. This implies that

$$h^0\big(S, \mathcal{O}_S\big(mC + m(F_1 - F_2) + m\sqrt{2}(F_3 - F_4)\big)\big) < h^0(S, \mathcal{O}_S(mC))$$

for every $m > 0$.

3. Weil Divisors

Let X be a normal variety over a field. The basics of the theory of Weil $\mathbb{R}$-divisors can be found in [Sch10]. An $\mathbb{R}$-*divisor* (also called Weil $\mathbb{R}$-divisor or $\mathbb{R}$-Weil $\mathbb{R}$-divisor) is an $\mathbb{R}$-linear combination of prime divisors. D is *effective,* denoted $D \geq 0$, if it is a nonnegative combination of prime divisors on X. If $D \geq E$, that is, $D - E \geq 0$, then we say that D *dominates* E. For an $\mathbb{R}$-divisor D, the rule

$$U \mapsto H^0(U, D) := \{f \in K(X)^* | (\operatorname{div}(f) + D)|_U \geq 0\} \cup \{0\}$$

defines a coherent sheaf $\mathcal{O}_X(D)$ on X. This coincides with the classical notation when D is a $\mathbb{Z}$-divisor. Note that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. If $D \geq 0$, then $\mathcal{O}_X(-D)$ is an ideal sheaf in $\mathcal{O}_X$. If M is a Cartier $\mathbb{Z}$-divisor, then $\mathcal{O}_X(D+M) \simeq \mathcal{O}_X(D) \cdot \mathcal{O}_X(M) \simeq \mathcal{O}_X(D) \otimes \mathcal{O}_X(M)$ for any $\mathbb{R}$-divisor D.

If D and D' are $\mathbb{R}$-divisors such that $D' - D = \operatorname{div}(f)$ for some $f \in K(X)$, then we say that D and D' are *linearly equivalent* and denote this relation by $D \sim D'$ or $D \sim_{\mathbb{Z}} D'$. Denote by $|D|$ the complete linear series $\{D' | D' \geq 0, D' \sim_{\mathbb{Z}} D\}$. It coincides with $|\lfloor D \rfloor| + \{D\}$, where $\{D\}$ denotes the fractional part of D. If $mD \sim mD'$ for some $m \in \mathbb{Z}^*$, then we write $D \sim_{\mathbb{Q}} D'$. If $D' - D = \sum_{i=1}^r a_i \operatorname{div}(f_i)$ for some $r \in \mathbb{N}^*$, $a_i \in \mathbb{R}$, and $f_i \in K(X)$, then we write $D \sim_{\mathbb{R}} D'$. Denote by $|D|_{\mathbb{Q}}$ and $|D|_{\mathbb{R}}$ the set of effective $\mathbb{R}$-divisors D' that are $\mathbb{Q}$-linearly and respectively $\mathbb{R}$-linearly equivalent to D. If $D \sim D'$, then $H^0(X, D) \simeq H^0(X, D')$, and if $D \sim_{\mathbb{Q}} D'$, then $H^0(X, mD) \simeq H^0(X, mD')$ for sufficiently divisible m. However, no obvious connection seems to exist between $H^0(X, D)$ and $H^0(X, D')$ if $D \sim_{\mathbb{R}} D'$.

An $\mathbb{R}$-divisor H is *ample* if $H = \sum_i a_i(H_i + \operatorname{div}(f_i))$, where $a_i \in \mathbb{R}_+$, $f_i \in K(X)$, and where H_i are effective ample Cartier $\mathbb{Z}$-divisors. Note that an ample $\mathbb{R}$-divisor is always $\mathbb{R}$-Cartier and that this definition coincides with the classical one in [Laz04, §2].

Two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors are numerically equivalent if they have the same intersection against every proper curve in X.

We review some of the basic theory of $\mathbb{R}$-divisors. Over $\mathbb{C}$, many of the results in this section appear in [Nak04, §II] or [Fuj09].

Lemma 3.1. *Let X be a normal variety, and D an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Then D is a positive $\mathbb{R}$-linear combination $\sum_i a_i D_i$ of effective Cartier divisors.*

Proof. The argument in [Fuj09, Lem. 0.14] is characteristic free. □

Lemma 3.2. *Let $\pi : Y \to X$ be a proper birational morphism of normal varieties, and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $\pi_* \mathcal{O}_Y(\pi^* D + E) = \mathcal{O}_X(D)$ for any effective π-exceptional $\mathbb{R}$-divisor E.*

Proof. The argument is similar to [Nak04, Lemma 2.11]. Let $U \subset X$ be open, and $f \in K(X)^*$. By the projection formula [Ful84, Prop. 2.3.(c)], if $\operatorname{div}_Y(f) + \pi^* D + E \geq 0$ over $\pi^{-1}U$, then $\operatorname{div}_X(f) + D \geq 0$ over U. By Lemma 3.1 we see that if $\operatorname{div}_X(f) + D \geq 0$ on U, then $\operatorname{div}_Y(f) + \pi^* D \geq 0$ on $\pi^{-1}U$. In particular, $\operatorname{div}_Y(f) + \pi^* D + E \geq 0$ on $\pi^{-1}U$. □

The following lemma can be used to reduce many questions involving the sheaves $\mathcal{O}_X(D)$ to normal *projective* varieties.

LEMMA 3.3. *Let $\pi : Y \to X$ be a proper birational morphism of normal varieties, and D_i a finite collection of $\mathbb{R}$-divisors on X. Then there are $\mathbb{R}$-divisors D_i^Y on Y such that $\pi_* D_i^Y = D_i$ for every i and*

$$\pi_*\mathcal{O}_Y\left(F + \pi^* M + \sum_i m_i D_i^Y\right) = \mathcal{O}_X\left(M + \sum_i m_i D_i\right)$$

for every $m_i \in \mathbb{R}^+$, effective π-exceptional $\mathbb{R}$-divisor F on Y, and $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on X.

Proof. We may assume that either D_i or $-D_i$ is a prime divisor for each i.

If the statement is true for $F = 0$, then it is true for every $F \geq 0$, so we assume that $F = 0$ throughout. Note that the question is local on X—if we prove the statement locally, then we can take coefficientwise maximums to glue the D_i^Y from different open subsets—so we may also assume that X is affine. Let E be an effective Weil divisor whose support is the divisorial component of the exceptional locus of π. For D an $\mathbb{R}$-divisor on X and for $\overline{D}$ an $\mathbb{R}$-divisor on Y with $\pi_*\overline{D} = D$, we have $\mathcal{O}_X(D) = \bigcup_{r\geq 0} \pi_*\mathcal{O}_Y(\overline{D} + rE)$. Then by coherence there exists r_D such that

$$\pi_*\mathcal{O}_Y(\overline{D} + rE) = \mathcal{O}_X(D) \quad \text{for all } r \geq r_D. \tag{3.1}$$

Let ϕ be a regular function on X such that $L := \mathrm{div}_X(\phi) \geq D_i$ for all i. Let $\overline{D}_i$ be $\mathbb{R}$-divisors on Y such that $\pi_*\overline{D}_i = D_i$. For any $r \geq 0$, we have $\overline{D}_i + rE \leq E_i' + \pi^* L$ for some effective π-exceptional $\mathbb{R}$-divisor E_i'. By Lemma 3.2 any global section of $\mathcal{O}_Y(\sum_i m_i \overline{D}_i + rE)$ for any $r \geq 0$ is also a global section of $\mathcal{O}_X((\sum_i m_i)L)$. Thus, the poles along E of rational functions that are sections of $\sum_i m_i \overline{D}_i + rE$ are bounded below by $-(\sum_i m_i)\pi^* L$. This implies that there exists $r > 0$ such that $H^0(Y, \sum_i m_i(\overline{D}_i + (r+t)E))$ is independent of $t \geq 0$ for each $m_i \geq 0$. In particular, it is equal to $H^0(\mathcal{O}_X(\sum_i m_i D_i))$ by (3.1). Since X is affine, this implies $\pi_*\mathcal{O}_Y(\sum_i m_i(\overline{D}_i + rE)) = \mathcal{O}_X(\sum_i m_i D_i)$. Set $D_i^Y := \overline{D}_i + rE$.

We now show that if M is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, then $\pi^* M + \sum_i m_i D_i^Y \geq 0$ if and only if $M + \sum_i m_i D_i \geq 0$. Up to replacing M by $M + \mathrm{div}_X(f)$, this completes the proof. One implication is clear by the projection formula. Assume now that $M + \sum_i m_i D_i \geq 0$. If M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then uM is a Cartier divisor for some positive integer u, and by the projection formula,

$$\begin{aligned}\pi_*\mathcal{O}_Y\left(\pi^*(uM) + \sum_i (um_i) D_i^Y\right) &= \mathcal{O}_X(uM) \otimes \pi_*\mathcal{O}_Y\left(\sum_i (um_i) D_i^Y\right)\\ &= \mathcal{O}_X\left(u\left(M + \sum_i m_i D_i\right)\right)\end{aligned}$$

for all $m_i \in \mathbb{R}^+$. Thus, if 1 is a section of $\mathcal{O}_X(u(M + \sum_i m_i D_i))$, then it is also a section of $\mathcal{O}_Y(u(\pi^* M + \sum_i m_i D_i^Y))$.

Assume now that $M = \sum_j a_j M_j$ is an $\mathbb{R}$-combination of Cartier divisors, with $M + \sum_i m_i D_i \geq 0$. Recall that either D_i or $-D_i$ is prime by assumption. As a condition on the m_i and a_j, the effectivity of $M + \sum_i m_i D_i$ is a system of linear inequalities with integer coefficients. Any of its real solutions can be approximated arbitrarily close by rational solutions. We conclude from the case where M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor by taking limits coefficient-wise. □

The following corollary allows us to reduce questions about $\mathbb{R}$-divisors to $\mathbb{R}$-Cartier $\mathbb{R}$-divisors.

COROLLARY 3.4. *Let D_i be a finite set of $\mathbb{R}$-divisors on a normal variety X. Then there exist a quasiprojective, normal variety Y, a proper birational morphism $\pi : Y \to X$, and $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D_i^Y on Y such that $\pi_* D_i^Y = D_i$ and*

$$\pi_* \mathcal{O}_Y\left(G + \pi^* M + \sum_i m_i D_i^Y\right) = \mathcal{O}_X\left(M + \sum_i m_i D_i\right)$$

for all $m_i \in \mathbb{R}^+$, all effective π-exceptional $\mathbb{R}$-divisors G on Y and all $\mathbb{R}$-Cartier $\mathbb{R}$-divisors M on X.

Proof. We may assume that D_i or $-D_i$ is a prime divisor for each i. Successively normalize the blow-up of the birational transform of each D_i, obtaining a birational morphism $f : Z \to X$ with $\mathbb{R}$-Cartier $\mathbb{R}$-divisors D_i' such that $f_* D_i' = D_i$. Let $g : Y \to Z$ be the normalized blow-up of the exceptional locus of f. Let $\pi = f \circ g$. Then $\overline{D}_i := g^* D_i'$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor with $\pi_* \overline{D}_i = D_i$. The relative $\mathcal{O}(-1)$ for g is equal to $\mathcal{O}_Y(F)$ for an effective Cartier divisor F whose support is the exceptional locus of π. As in the proof of Lemma 3.3, for $r \gg 0$, we may set $D_i^Y := \overline{D}_i + rF$. To obtain Y quasiprojective, apply Chow's lemma and normalize. □

We have defined $\operatorname{vol}(D) := \limsup_{m\to\infty} \frac{h^0(mD)}{m^n/n!}$. For $\mathbb{R}$-Cartier $\mathbb{R}$-classes on projective varieties, this definition of volume differs from the classical one (cf. [Laz04, Cor. 2.2.45]). The definitions coincide for $\mathbb{Z}$-classes, but in [Laz04] the volume of $\mathbb{Q}$-classes is defined by homogeneous extension from $\mathbb{Z}$, and for $\mathbb{R}$-classes, it is given by continuous extension from $\mathbb{Q}$. We check that the definitions in fact agree. We also check that we can replace lim sup by lim.

THEOREM 3.5. *Let D be an $\mathbb{R}$-divisor on a proper normal variety X of dimension n. Then*

(i) $\operatorname{vol}(D) = \lim_{m\to\infty} \frac{h^0(mD)}{m^n/n!}$.

(ii) *If D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then* $\operatorname{vol}(D)$ *agrees with the definition in* [Laz04, *Cor.* 2.2.45].

(iii) (*Kodaira lemma*) $\operatorname{vol}(D) > 0$ *if and only if, for every $\mathbb{R}$-divisor B, there exist $\varepsilon > 0$ and an effective $\mathbb{R}$-divisor C such that $D \sim_{\mathbb{Q}} \varepsilon \cdot B + C$.*

(iv) *If D' is an $\mathbb{R}$-divisor on X such that $D' - D$ is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then* $\operatorname{vol}(D) = \operatorname{vol}(D')$.

Most of the references used in the proof work over $\mathbb{C}$. [Cut13b, §2.2] and the references therein explain how to extend these to arbitrary fields.

Proof. We start by proving (i). By Corollary 3.4, we may assume that X is projective and D (hence, also D') and B are $\mathbb{R}$-Cartier $\mathbb{R}$-divisors. Then there exists an ample $\mathbb{Z}$-divisor H with $D \leq H$. Hence, $H^0(X, mD)$ is a graded linear series. If $\operatorname{vol}(D) = 0$, then the limit is also zero. Otherwise, by [Cut13a, Thm. 1.2] we have

$$\operatorname{vol}(D) = \lim_{m\to\infty} \frac{h^0(m \cdot m_0 D)}{(m \cdot m_0)^n/n!} < \infty, \tag{3.2}$$

where $m_0 = \gcd\{m \in \mathbb{Z} | h^0(mD) \neq 0\}$. We will finish the proof later on by showing that $m_0 = 1$.

For now we prove (ii) and (iii). Provisionally denote by $\operatorname{Vol}(D)$ the volume of the $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D in the sense of [Laz04, Cor. 2.2.45]. From (3.2) we see that vol is also homogeneous, so that for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, we have $\operatorname{vol}(D) = \operatorname{Vol}(D)$.

We first show that if $\operatorname{vol}(D) > 0$, then $\operatorname{vol}(D) = \operatorname{Vol}(D)$. By homogeneity we may assume $h^0(D) > 0$. Then $D = E + \operatorname{div}(f)$ for some effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E and for some rational function f on X. By Lemma 3.1 we have $E = \sum_i a_i E_i$ for some positive $a_i \in \mathbb{R}$ and effective Cartier $\mathbb{Z}$-divisors E_i. Then

$$\frac{1}{m^n} \operatorname{vol}\left(\sum_i \lfloor m a_i \rfloor E_i + \operatorname{div}(f^m)\right) \leq \operatorname{vol}(D) \leq \frac{1}{m^n} \operatorname{vol}\left(\sum_i \lceil m a_i \rceil E_i + \operatorname{div}(f^m)\right).$$

The LHS and RHS both converge to $\operatorname{Vol}(D)$ as m grows. Furthermore, if $\operatorname{vol}(D) > 0$, then $\sum_i \lfloor m a_i \rfloor E_i + \operatorname{div}(f^m)$ is a big Cartier $\mathbb{Z}$-divisor for large enough m; hence, it dominates some ample $\mathbb{Q}$-divisor by Kodaira's lemma (cf. [Laz04, Cor. 2.2.7]).

It remains to show that if $\operatorname{Vol}(D) > 0$, then $\operatorname{vol}(D) > 0$. First, observe that if $\operatorname{Vol}(D) > 0$, then D is big in the sense of [Laz04, §2.2.B], that is, D dominates an ample $\mathbb{R}$-divisor. Indeed, by continuity (cf. [Laz04, Cor. 2.2.45]) there exists a small ample $\mathbb{R}$-divisor H such that $D - H$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $\operatorname{Vol}(D - H) > 0$. Then the claim follows from Kodaira's lemma. We can write

$$D = \sum_i a_i (H_i + \operatorname{div}(f_i)) + \sum_j b_j E_j,$$

where H_i are ample effective $\mathbb{Z}$-divisors, f_i are rational functions, E_j are effective $\mathbb{R}$-Cartier $\mathbb{Z}$-divisors, and a_i and b_j are positive real numbers. Let F be the union of the supports of $\operatorname{div}(f_i)$. There exists a real number $N > 0$ such that $\{m a_i\} \operatorname{div}(f_i) > -N \cdot F$ for all i and all m. Furthermore, there exists a positive integer r such that for each i, the Weil divisor $r H_i - N \cdot F$ has a section given by some rational function g_i. In particular,

$$r H_i + \{m a_i\} \operatorname{div}(f_i) + \operatorname{div}(g_i) > 0.$$

Then

$$mD > \sum_i ((\lfloor ma_i \rfloor - r)H_i + \lfloor ma_i \rfloor \operatorname{div}(f_i) - \operatorname{div}(g_i)) + \sum_j \lfloor mb_j \rfloor E_j.$$

The RHS is an effective big Cartier $\mathbb{Z}$-divisor for m sufficiently large, and therefore $\operatorname{vol}(D) > 0$. The proof of (ii) is complete. We have showed that if D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and $\operatorname{Vol}(D) = \operatorname{vol}(D) > 0$, then mD is effective for m large enough. This proves that $m_0 = 1$ and completes the proof of (i).

The forward implication of (iii) follows easily from the projective and $\mathbb{R}$-Cartier case by applying Corollary 3.4. Conversely, if we choose B a big effective Cartier divisor, then we can write $\varepsilon B = \varepsilon_0 B + B'$ for some rational $\varepsilon_0 < \varepsilon$ and some effective B'. The equality $D \sim_{\mathbb{Q}} \varepsilon_0 B + B' + C$ easily implies the bigness of D.

Similarly, to show part (iv) we may apply Corollary 3.4 to reduce to the projective normal case and to assume that D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. The volume function Vol is defined on the real Néron–Severi space $N^1(X)_{\mathbb{R}}$, and then part (iv) follows. □

4. Divisorial Zariski Decompositions

Let X be a normal proper variety over a field K. Let D be a big $\mathbb{R}$-divisor. Following Nakayama [Nak04], for Γ a prime divisor on X, we define

$$\sigma_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D' \sim_{\mathbb{R}} D, D' \geq 0\},$$

where we write $D \sim_{\mathbb{R}} D'$ if there exist rational functions f_i on X and *real* numbers a_i such that $D - D' = \sum_i a_i \cdot \operatorname{div}(f_i)$. The basic properties of $\sigma_\Gamma(D)$ are studied by [Nak04] for smooth projective varieties in characteristic 0 and by [Mus13; CHMS13] for smooth projective varieties in arbitrary characteristic. We make the brief verifications necessary to extend these results to normal proper varieties as well. We start with the projective case.

LEMMA 4.1. *Let X be a normal projective variety, and D a big $\mathbb{R}$-divisor. Fix a prime divisor Γ.*

(i) *We also have $\sigma_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D' \sim_{\mathbb{Q}} D, D' \geq 0\}$ and*

$$\sigma_\Gamma(D) = \lim_{m\to\infty} \frac{1}{m} \min\{\operatorname{mult}_\Gamma D'' \mid D'' \sim_{\mathbb{Z}} mD, D'' \geq 0\}.$$

(ii) *Let A be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Then $\lim_{\varepsilon\searrow 0} \sigma_\Gamma(D + \varepsilon A) = \sigma_\Gamma(D)$.*

(iii) *The $\mathbb{R}$-divisor $F := D - \sigma_\Gamma(D)\Gamma$ has $\sigma_\Gamma(F) = 0$ and $\sigma_{\Gamma'}(F) = \sigma_{\Gamma'}(D)$ for any other prime divisor Γ'. Furthermore, the natural inclusion $H^0(X, mF) \hookrightarrow H^0(X, mD)$ is an equality for any positive real number m.*

(iv) *If L is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then $\sigma_\Gamma(D + L) = \sigma_\Gamma(D)$. The induced function $\sigma_\Gamma : N^1(X) \to \mathbb{R}$ sending a numerical class $\alpha \in N^1(X)$ to $\sigma_\Gamma(D + \alpha)$ is continuous in a sufficiently small neighborhood of* 0.

Proof. The proofs are analogous to [Nak04, Lem. III.1.4] and [Nak04, Lem. III.1.7]. □

We can usually reduce questions involving σ_Γ to the projective case by using Lemma 3.3 and the following:

LEMMA 4.2. *Let $\pi : Y \to X$ be a birational morphism of normal, proper varieties. Suppose that D is a big $\mathbb{R}$-divisor on X. Assume that one of the following holds*:

(i) *There exists a big $\mathbb{R}$-divisor L on Y with $\pi_* L = D$ such that, for every $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on X, the condition $D + M \geq 0$ holds iff $L + \pi^* M \geq 0$.*

(ii) *X and Y are projective, and there exists a big $\mathbb{R}$-divisor L on Y such that $\pi_* \mathcal{O}_Y(mL) = \mathcal{O}_X(mD)$ for all integers $m \geq 0$.*

Then, for any prime divisor Γ on X, we have $\sigma_\Gamma(D) = \sigma_{\Gamma'}(L)$ where Γ' is the birational transform of Γ on Y.

Proof. By letting M range through the $\mathbb{R}$-linearly trivial divisors on X we immediately obtain (i). Part (ii) is a consequence of Lemma 4.1(i) and the fact that π_* induces an equality of global sections for sheaves. $\square$

REMARK 4.3. Let X be a normal proper variety. Suppose that D is a big $\mathbb{R}$-Weil $\mathbb{R}$-divisor on X. Then there are at most finitely many prime divisors Γ such that $\sigma_\Gamma(D) > 0$ (since $0 \leq \sigma_\Gamma(D) \leq \operatorname{mult}_\Gamma(D')$ for any fixed effective $D' \sim_{\mathbb{R}} D$).

We can now define

$$N_\sigma(D) = \sum_{\Gamma \text{ prime divisor on } X} \sigma_\Gamma(D) \cdot \Gamma \quad \text{and} \quad P_\sigma(D) = D - N_\sigma(\Gamma). \tag{4.1}$$

We call the decomposition $D = P_\sigma(D) + N_\sigma(D)$ the *divisorial Zariski decomposition* of D.

DEFINITION 4.4. We say that a big $\mathbb{R}$-divisor D is *movable* if $N_\sigma(D) = 0$ or, equivalently, $D = P_\sigma(D)$.

REMARK 4.5. Let D be a big, movable $\mathbb{R}$-divisor on a normal proper variety X. Let $D' \sim_{\mathbb{R}} D$ with $D' \geq 0$. Then $D' = P_\sigma(D')$ is the componentwise limit of the divisors $D'_m := D' - \frac{1}{m} \min\{\operatorname{mult}_\Gamma D'' | D'' \sim_{\mathbb{Z}} mD', D'' \geq 0\}$. (This is Lemma 4.1(i) when X is projective, and we can reduce to this case via Lemma 4.2(i).) Observe that $|mD'_m|$ is a linear series without fixed divisorial components for large m. In this sense, we understand movable $\mathbb{R}$-divisors as limits of divisors moving in linear series without fixed divisorial components.

REMARK 4.6. If D is a big and nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a proper normal variety X, then D is movable. (Indeed, we can find a birational map $\pi : Y \to X$ from a projective normal variety Y, and the pullback $\pi^* D$ satisfies the hypotheses for L in Lemma 4.2(i). Applying the result of the lemma, we can conclude by Lemma 4.1(ii).)

DEFINITION 4.7. Let X be a normal variety. An $\mathbb{R}$-divisor A is *ample in codimension* 1 if there exists a closed subset $Z \subset X$ of codimension at least 2 such that $A|_{X \setminus Z}$ is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor.

It is clear that a divisor that is ample in codimension 1 is $\mathbb{R}$-linearly equivalent to an effective divisor. We will prove in Lemma 4.10(iv) that if X is a proper normal variety, then a divisor that is ample in codimension 1 on X is also big.

The following lemma shows that all normal varieties admit divisors which are ample in codimension 1.

LEMMA 4.8. *Let $\pi : Y \to X$ be a proper, generically finite, dominant morphism of normal varieties, and A an $\mathbb{R}$-divisor on Y that is ample in codimension* 1. *Then $\pi_* A$ is ample in codimension* 1.

Proof. By removing a suitable subset of codimension 2 from X we may assume that π is finite flat and A is ample on Y. Write $A \sim_{\mathbb{R}} \sum_i r_i A_i$ where A_i is ample Cartier and $r_i \in \mathbb{R}$. Note that each $\pi_* A_i$ is Cartier in codimension 1, so that by shrinking Y and X we may further assume that $\pi_* A_i$ is also Cartier. By [Ful84, Prop. 1.4.(b)], cycle pushforwards respect linear equivalence. [Gro61, Cor. 6.6.2] shows that the pushforward of an ample Cartier divisor on Y is again ample on X, and we conclude that $\pi_* A \sim_{\mathbb{R}} \sum_i r_i \pi_* A_i$ is ample. □

LEMMA 4.9. *Let X be a normal proper variety over a field, Γ a prime divisor, and A an $\mathbb{R}$-divisor that is ample in codimension* 1. *Then*

(i) *If E is an $\mathbb{R}$-divisor, then, for m sufficiently large, $E + mA \sim_{\mathbb{R}} B_m$ for some $B_m \geq 0$ with $\Gamma \not\subset \mathrm{Supp}(B_m)$.*
(ii) *If P is a big $\mathbb{R}$-divisor with $\sigma_\Gamma(P) = 0$, then $P + A \sim_{\mathbb{R}} C$ for some $C \geq 0$ with $\Gamma \not\subset \mathrm{Supp}(C)$.*

Proof. For (i), by working over the smooth locus of X we see that $E + mA$ is ample in codimension 1 for m sufficiently large, and then the statement is clear.

Let m be as in part (i) for $E = \Gamma$. By the definition of σ_Γ there exists an effective $P_m \sim_{\mathbb{R}} P$ such that $\mathrm{mult}_\Gamma(P_m) \leq \frac{1}{m}$. By (i) we have that $P_m + A$ is $\mathbb{R}$-linearly equivalent to an effective $\mathbb{R}$-divisor C without Γ in its support. □

LEMMA 4.10. *Let X be a normal proper variety. Then*

(i) *If D is a big $\mathbb{R}$-divisor, then $P_\sigma(D)$ is big and movable. If D is effective, then so is $P_\sigma(D)$.*
(ii) *If P and D are big $\mathbb{R}$-divisors with P movable and $P \leq D$, then $P \leq P_\sigma(D)$.*
(iii) *If $\pi : Y \to X$ is a proper generically finite dominant morphism of normal proper varieties and P is a big movable $\mathbb{R}$-divisor on Y, then $\pi_* P$ is also big and movable.*
(iv) *Let A be an $\mathbb{R}$-divisor that is ample in codimension* 1. *Then, for every $\mathbb{R}$-divisor E, there exists $\varepsilon_E > 0$ such that $A - \varepsilon_E E$ is big and movable.*

Proof. Part (i) is a consequence of Lemma 4.1(iii) in the projective case and can be reduced to this case in general by Lemma 4.2(i) and Lemma 3.3.

For part (ii), assume that $D = P + N$ with N effective. By Lemma 4.2(i), and Lemma 3.3 we can assume that X is projective. Let A be an effective ample divisor. For all prime divisors Γ on X, we have

$$\sigma_\Gamma(D+\varepsilon A) \le \sigma_\Gamma(P) + \sigma_\Gamma(N+\varepsilon A) = \sigma_\Gamma(N+\varepsilon A).$$

By summing over all Γ we obtain $N_\sigma(D+\varepsilon A) \le N_\sigma(N+\varepsilon A)$, and hence $P_\sigma(D+\varepsilon A) \ge P$. The continuity property in Lemma 4.1(ii) implies $P_\sigma(D) \ge P$.

In (iii), observe first that any divisor ample in codimension 1 is big. Furthermore, an $\mathbb{R}$-divisor is big if and only if it dominates some divisor ample in codimension 1. From Lemma 4.8 it follows that if P is big, then $\pi_* P$ is also big.

To settle the movability of $\pi_* P$, by Lemmas 4.1(i), 4.2(i) and Remark 4.5 it is enough to show that if V is a linear series without fixed divisorial components on Y, then $\pi_* V$ spans a linear series without fixed divisorial components on X. By Remark 2.2 we may assume that the base field is infinite. If Γ is a prime divisor on X, let Γ'_i with $1 \le i \le r$ be the divisorial components of $\pi^{-1}\Gamma$. If $\pi_* V$ spans a linear series with a fixed component Γ, then $\operatorname{mult}_\Gamma Q > \varepsilon$ for all $Q \in \pi_* V$ and for some $\varepsilon > 0$ by the finite dimensionality of V. Then V is the union of the proper subspaces $V_i = \{R \in V \,|\, \operatorname{mult}_{\Gamma_i} R > \frac{\varepsilon}{r \cdot \deg \pi}\}$. This is impossible over an infinite field.

To see (iv), consider the open subset $U = X - Z$. By taking a closure under a suitable embedding and normalizing we find a normal projective variety X' containing U as an open subset and an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A' on X' such that $A'|_U = A|_U$. Let E' be the closure in X' of $E|_U$. By the normality of X and the codimension condition on Z in the definition of ampleness in codimension 1, for all $m \ge 1$ and $l \ge 1$, we have $H^0(X', \mathcal{O}_{X'}(l(mA' - E'))) \subseteq H^0(U, \mathcal{O}_X(l(mA - E))) = H^0(X, \mathcal{O}_X(l(mA - E)))$. The bigness of $mA - E$ then follows from that of $mA' - E'$ for $m \gg 0$.

Regarding movability, by the lower convexity of N_σ it is enough to treat the case where A and E are $\mathbb{Z}$-divisors with A ample Cartier. Then $\mathcal{O}_X(mA - E) \simeq \mathcal{O}_X(-E) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for large m. In particular, the linear series $|mA - E|$ has no fixed components, and $N_\sigma(A - \frac{1}{m}E) = 0$. □

REMARK 4.11. When $\pi : Y \to X$ is a finite morphism of normal proper varieties, for every $\mathbb{R}$-divisor D on X, we can define $\pi^* D$ as the closure in Y of $\pi_U^* D_U$, where $U \subset X$ is the smooth locus, and $\pi_U : Y \times_X U \to U$ is the induced finite morphism. Since $\operatorname{codim}(X \setminus U, X) \ge 2$, we see that π^* respects linear equivalence (with $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ coefficients).

LEMMA 4.12. *Let $\pi : Y \to X$ be a generically finite morphism of normal proper varieties, and D a big $\mathbb{R}$-divisor on X. Then*

(i) *If π is finite, then $N_\sigma(\pi^* D) = \pi^* N_\sigma(D)$.*
(ii) *If π is only generically finite, but D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then $\pi_* N_\sigma(\pi^* D) = (\deg \pi) \cdot N_\sigma(D)$.*

Proof. If π is finite, then $N_\sigma(\pi^* D) \le \pi^* N_\sigma(D)$ because $H^0(Y, \pi^* D) \supseteq \pi^* H^0(X, D)$. When π is only generically finite and D is $\mathbb{R}$-Cartier, the same argument and the projection formula (cf. [Ful84, Prop. 2.3.(c)]) prove that

$\pi_* N_\sigma(\pi^* D) \le (\deg \pi) \cdot N_\sigma(D)$. On the other hand, $D = \frac{1}{\deg \pi}\pi_* P_\sigma(\pi^* D) + \frac{1}{\deg \pi}\pi_* N_\sigma(\pi^* D)$, and $\pi_* P_\sigma(\pi^* D)$ is big and movable by Lemma 4.10(iii). By Lemma 4.10(ii) it follows that $\frac{1}{\deg \pi}\pi_* P_\sigma(\pi^* D) \le P_\sigma(D)$, and hence $\frac{1}{\deg \pi}\pi_* N_\sigma(\pi^* D) \ge N_\sigma(D)$. Therefore, in both (i) and (ii),

$$\pi_* N_\sigma(\pi^* D) = (\deg \pi) \cdot N_\sigma(D).$$

When π is finite, this forces equality in $N_\sigma(\pi^* D) \le \pi^* N_\sigma(D)$. □

Lemma 4.13. *If D is a big $\mathbb{R}$-divisor on the proper normal variety X, and $E \ge 0$ with* $\mathrm{Supp}(E) \subset \mathrm{Supp}(N_\sigma(D))$, *then*

$$N_\sigma(D + E) = N_\sigma(D) + E$$

and

$$H^0(X, D) = H^0(X, D + E) = H^0(X, P_\sigma(D) + E) = H^0(X, P_\sigma(D)).$$

Proof. We argue just as in [Nak04, Lemma III.1.8] and [Nak04, Cor. III.1.9]. When X is not projective, we replace the ample A from the proof of [Nak04, Lemma III.1.8] by a divisor ample in codimension 1. □

5. Divisorial Augmented Base Locus

The augmented base locus of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a normal complex projective variety X is defined in [ELM+06, Def. 1.2] as $\mathbf{B}_+(D) = \bigcap_{D=A+E} \mathrm{Supp}(E)$, where A is an ample $\mathbb{R}$-divisor, and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. For normal proper varieties, we mimic this construction by using divisors ample in codimension 1. The resulting subset is a good analogue of the augmented base locus in codimension 1.

Definition 5.1. Let D be a big $\mathbb{R}$-divisor on a normal proper variety X. The *divisorial augmented base locus* of D is the divisorial component $\mathbf{B}_+^{\mathrm{div}}(D)$ of

$$\bigcap_{D=A+E} \mathrm{Supp}(E) \tag{5.1}$$

with the intersection being taken over all decompositions $D = A + E$ with A an $\mathbb{R}$-divisor, ample in codimension 1, and E an effective $\mathbb{R}$-divisor.

The next lemma implies that if X is projective, then $\mathbf{B}_+^{\mathrm{div}}(D)$ equals the divisorial part of $\mathbf{B}_+(D)$ and that we can also compute $\mathbf{B}_+^{\mathrm{div}}(D)$ in terms of just one divisor that is ample in codimension 1.

Lemma 5.2. *Let X be a normal proper variety. Let D be a big $\mathbb{R}$-divisor, and let A be an $\mathbb{R}$-divisor that is ample in codimension* 1 *on X. Then $\mathbf{B}_+^{\mathrm{div}}(D)$ is the divisorial component of the intersection of the supports of all $D' \in |D - \varepsilon A|_{\mathbb{R}}$ for all $\varepsilon > 0$.*

Proof. Let U denote the intersection referred to in the statement of the lemma. Its index set is a subset of the one in (5.1); therefore, $U \supseteq \mathbf{B}_+^{\text{div}}(D)$. Let now Γ be a prime divisor that is a component of the supports of all $D' \in |D - \varepsilon A|_{\mathbb{R}}$ for all sufficiently small $\varepsilon > 0$. Let $D = A' + E$ with A' ample in codimension 1 and $E \geq 0$. By Lemma 4.9(i), for all sufficiently small $\varepsilon > 0$, there exists $B_\varepsilon \sim_{\mathbb{R}} \varepsilon A$ such that $A' - B_\varepsilon \geq 0$ and $\Gamma \not\subset \text{Supp}(A' - B_\varepsilon)$. Then

$$|D - \varepsilon A| \ni D - B_\varepsilon = (A' - B_\varepsilon) + E.$$

Consequently, Γ is a component of $\text{Supp}(E)$. □

The relationship between $\mathbf{B}_+^{\text{div}}(D)$ and the Zariski decomposition is given by the following:

LEMMA 5.3. *Let X be a normal proper variety. Let D be a big $\mathbb{R}$-divisor, and let A be an $\mathbb{R}$-divisor that is ample in codimension 1 on X. Then*

$$\mathbf{B}_+^{\text{div}}(D) = \text{Supp}(N_\sigma(D - \varepsilon A))$$

for all sufficiently small $\varepsilon > 0$.

Proof. Note that since $\text{Supp}(N_\sigma(D - \varepsilon A))$ is a closed set, for any sufficiently small $\varepsilon > 0$, the sets $\text{Supp}(N_\sigma(D - \varepsilon A))$ all coincide. Thus, we may show that $\mathbf{B}_+^{\text{div}}(D)$ coincides with the intersection over all sufficiently small $\varepsilon > 0$ of the sets $\text{Supp}(N_\sigma(D - \varepsilon A))$.

By Theorem 3.5(iii) we see that $D - \varepsilon A$ is big for sufficiently small $\varepsilon > 0$. Let Γ be a prime divisor on X. Assume that $\sigma_\Gamma(D - \varepsilon A) = 0$. Lemma 4.9(ii) shows that $\Gamma \not\subset \text{Supp}(D')$ for some $D' \in |D - \frac{\varepsilon}{2}A|_{\mathbb{R}}$. Therefore, $\mathbf{B}_+^{\text{div}}(D) \subseteq \bigcap_{\varepsilon>0} \text{Supp}(N_\sigma(D - \varepsilon A))$. The reverse inclusion is straightforward from the previous lemma and the definition of $\sigma_\Gamma(D - \varepsilon A)$. □

REMARK 5.4. Inspired by [ELM+06, Lemma 1.14], we define the *divisorial restricted base locus* as

$$\mathbf{B}_-^{\text{div}}(D) := \bigcup_A \mathbf{B}_+^{\text{div}}(D + A),$$

where A ranges through all $\mathbb{R}$-divisors on X that are ample in codimension 1. We can show that if the base field K is uncountable and D is a big $\mathbb{R}$-divisor, then $\mathbf{B}_-^{\text{div}}(D) = \text{Supp}(N_\sigma(D))$.

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References

[BFJ09] S. Boucksom, C. Favre, and M. Jonsson, *Differentiability of volumes of divisors and a problem of Teissier,* J. Algebraic Geom. 18 (2009), no. 2, 279–308.

[CHMS13] P. Cascini, C. Hacon, M. Mustaţă, and K. Schwede, *On the numerical dimension of pseudo-effective divisors in positive characteristic,* 2013, arXiv:1206.6521v3 [math.AG].

[Cut13a] S. D. Cutkosky, *Multiplicities of graded families of linear series and ideals,* 2013, arXiv:1301.5613 [math.AG].

[Cut13b] ———, *Teissier's problem on inequalities of nef divisors over an arbitrary field,* 2013, arXiv:1304.1218 [math.AG].

[dJ96] A. J. de Jong, *Smoothness, semi-stability and alterations,* Publ. Math. Inst. Hautes Études Sci. 83 (1996), 51–93.

[ELM+06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, *Asymptotic invariants of base loci,* Ann. Inst. Fourier (Grenoble) 56 (2006), no. 6, 1701–1734.

[FL13] M. Fulger and B. Lehmann, *Zariski decompositions of numerical cycle classes,* 2013, arXiv:1310.0538 [math.AG].

[Fuj09] O. Fujino, *Big R-divisors,* 2009, ⟨https://www.math.kyoto-u.ac.jp/~fujino/big-r-divisor5.pdf⟩.

[Ful84] W. Fulton, *Intersection theory,* Ergeb. Math. Grenzgeb. (3), 2, Springer-Verlag, Berlin, 1984.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes,* Publ. Math. Inst. Hautes Études Sci. 8 (1961), 222.

[KL15] A. Küronya and V. Lozovanu, *Positivity of line bundles and Newton–Okounkov bodies,* 2015, in preparation.

[Kol15] J. Kollár, *Moduli of varieties of general type,* 2015, book in preparation.

[Kür06] A. Küronya, *Asymptotic cohomological functions on projective varieties,* Amer. J. Math. 128 (2006), no. 6, 1475–1519.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. Classical setting: line bundles and linear series,* Ergeb. Math. Grenzgeb. (3), 48, Springer-Verlag, Berlin, 2004.

[LM75] D. Lieberman and D. Mumford, *Matsusaka's big theorem,* Algebraic geometry—Arcata 1974, Proc. Sympos. Pure Math., 29, pp. 513–530, Amer. Math. Soc., Providence, RI, 1975.

[LM09] R. Lazarsfeld and M. Mustaţă, *Convex bodies associated to linear series,* Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 5, 783–835.

[Luo90] T. Luo, *A note on the Hodge index theorem,* Manuscripta Math. 67 (1990), no. 1, 17–20.

[Mat72] T. Matsusaka, *Polarized varieties with a given Hilbert polynomial,* Amer. J. Math. 94 (1972), 1027–1077.

[Mus13] M. Mustaţă, *The non-nef locus in positive characteristic,* A celebration of algebraic geometry, Clay Math. Proc., 18, pp. 535–551, Amer. Math. Soc., Providence, RI, 2013.

[Nak04] N. Nakayama, *Zariski-decomposition and abundance,* MSJ Mem., 14, Mathematical Society of Japan, Tokyo, 2004.

[Sch10] K. Schwede, *Generalized divisors and reflexive sheaves,* 2010, ⟨http://www.math.utah.edu/~schwede/Notes/GeneralizedDivisors.pdf⟩.

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