Michigan Math. J. 65 (2016), 353–369

A Characterization of Singular-Hyperbolicity

C. A. Morales

1. Introduction

The relationship between dominated splittings and uniform hyperbolicity was explored by Mañé in his solution of the stability conjecture for diffeomorphisms [18]. Pujals and Sambarino [22] studied it in their nowadays famous Theorem B: For C^2 surface diffeomorphisms, every compact invariant set with a dominated splitting whose periodic points are all hyperbolic saddle splits into a hyperbolic set and finitely many disjoint normally hyperbolic irrational circles. A similar relationship but between dominated splitting *with respect to the linear Poincaré flow* and uniform hyperbolicity was obtained by Aubin and Hertz [6]. Indeed, they proved that every nonsingular compact invariant set exhibiting a dominated splitting with respect to the Poincaré flow and whose periodic points are all hyperbolic saddle splits in a hyperbolic set and finitely many disjoint normally hyperbolic irrational tori. In light of these results, it is natural to think about the singular case, namely, is it possible to obtain a similar decomposition for compact invariant sets with singularities whose nonsingular points exhibit a dominated splitting with respect to the linear Poincaré flow and whose periodic points are all hyperbolic of saddle type? However, this kind of question must face the problem of a natural candidate for uniform hyperbolicity. Indeed, the *geometric Lorenz attractor* [14] is a nonhyperbolic compact invariant set of a C^∞ three-dimensional flow for which the periodic points are all hyperbolic saddle, has no irrational tori, and, nevertheless, its nonsingular points exhibit a dominated splitting with respect to the linear Poincaré flow. The notion of *singular-hyperbolicity* emerges as this candidate, the geometric Lorenz attractor as well as any robustly transitive attractor with singularities of a three-dimensional flow enjoy it [20]. It is then natural to ask if there is a relationship between dominated splittings with respect to the linear Poincaré flow and singular-hyperbolicity, namely, if for every C^2 three-dimensional flow, every compact invariant set whose nonsingular points exhibit a dominated splitting *with respect to the linear Poincaré flow* and whose periodic points are all hyperbolic saddle splits into a singular-hyperbolic set for the flow, a singular-hyperbolic set for the reversed flow, and finitely many disjoint normally hyperbolic irrational tori. In this scenario, Crovisier and Yang announced recently

Received March 2, 2015. Revision received October 30, 2015.

Partially supported by MATHAMSUB 15 MATH05-ERGOPTIM, Ergodic Optimization of Lyapunov Exponents.

in [11] that, for C^3 three-dimensional flows, every compact invariant set whose nonsingular points exhibit a dominated splitting with respect to the linear Poincaré flow, whose periodic points are all hyperbolic saddle, and whose singularities are all *Lorenz-like in general position* has either an irrational torus or a dominated splitting for the tangent flow.

In this paper we explore the relationship between linear Poincaré flow's dominated splittings and singular-hyperbolicity for C^1 three-dimensional flows. More precisely, we shall prove that every compact invariant set whose nonsingular points exhibit a dominated splitting with respect to the linear Poincaré flow, whose *ergodic measures* are all hyperbolic saddle, and whose singularities are all Lorenz-like in general position is singular-hyperbolic. In fact, these properties characterize singular-hyperbolicity in dimension three. Different characterizations can be found in [2; 3; 4; 5].

Consider a continuous flow ϕ_t of a metric space Γ, a Riemannian vector bundle $V^- \to \Gamma$ over Γ, and a one-parameter family of bundle maps $A_t : V^- \to V^-$ over ϕ_t, that is, $A(V_p^-) = V_{\phi_t(p)}$ for every $p \in \Gamma$. We denote $A_t(z) = A_t|_{V_z^-}$ for $z \in \Gamma$ and $t \in \mathbb{R}$. We say that a subbundle $E \subset V^-$ is A_t*-invariant* if $A_t(p)E_p = E_{\phi_t(p)}$ for any $p \in \Gamma$ and $t \in \mathbb{R}$. In such a case we denote by $A_t|_E$ the restriction to E, that is, $(A_t|_E)(p) = A_p(p)|_{E_p}$ for every $p \in \Gamma$ and $t \in \mathbb{R}$. The map assigning the dimension $\dim(E_p)$ of E_p to any $p \in \Gamma$ will be denoted by $\dim(E)$. Given another subbundle $F \subset V$, we write $E \subset F$ whenever $E_p \subset F_p$ for all $p \in \Gamma$.

We say that A_t is *contracting* if there are positive constants K, λ such that

$$\|A_t(p)\| \leq Ke^{-\lambda t}, \quad \forall p \in \Gamma, t \geq 0.$$

On the other hand, we say that A_t *dominates* another bundle map $B_t : V^+ \to V^+$ over ϕ_t (or that B_t is dominated by A_t) if there are positive constants K, λ satisfying

$$\|A_t(p)\| \cdot \|B_{-t}(\phi_t(p))\| \leq Ke^{-\lambda t}, \quad \forall p \in \Gamma, t \geq 0.$$

In such a case, A_t is called a *dominating direction.*

By abuse of language, we call a *flow* any C^1 vector field X with induced flow X_t of a compact connected manifold M endowed with a Riemannian structure $\|\cdot\|$. We say that $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. Unless otherwise stated, all compact invariant sets will be *nontrivial* in the sense that they do not reduce to a finite number of closed orbits. The set of singularities (i.e., zeroes of X) is denoted by $\mathrm{Sing}(X)$. We say that $\sigma \in \mathrm{Sing}(X)$ is hyperbolic if the derivative $DX(\sigma)$ has no purely imaginary eigenvalues.

For a compact invariant set Λ, we say that Λ *has a dominated splitting with respect to the tangent flow* if there is a continuous splitting $T_\Lambda M = E \oplus F$ into DX_t-invariant subbundles E, F such that $DX_t|_E$ dominates $DX_t|_F$. In such a case, we say that $DX_t|_F$ is *volume expanding* if $\dim(F) \geq 2$ and there are $K, \lambda > 0$ such that

$$|\det DX_t(p)| \geq Ke^{\lambda t}, \quad \forall p \in \Lambda, \forall t \geq 0.$$

DEFINITION 1.1. A compact invariant set Λ is *singular-hyperbolic* if every singularity in Λ is hyperbolic and if Λ has *singular-hyperbolic splitting,* that is, a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow such that $DX_t|_E$ is contracting and $DX_t|_F$ is volume expanding.

Denote by $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$ the set of regular points in Λ. Define by E^X the map assigning to $p \in M$ the subspace of T_pM generated by $X(p)$. It turns out to be a one-dimensional subbundle of TM when restricted to M^*. Define also the normal subbundle N over M^* whose fiber N_p at $p \in M^*$ is the orthogonal complement of E^X_p in T_pM. Denoting by $\pi = \pi_p : T_pM \to N_p$ the orthogonal projection, we obtain the *linear Poincaré flow* $P_t : N \to N$ defined by $P_t(p) = \pi_{X_t(p)} \circ DX_t(p)$.

DEFINITION 1.2. For a (nonnecessarily compact) invariant set $\Omega \subset M^*$, we say that Ω *has a dominated splitting with respect to the linear Poincaré flow* if there is a continuous splitting $N_\Omega = N^- \oplus N^+$ into P_t-invariant subbundles N^-, N^+ such that $P_t|_{N^-}$ dominates $P_t|_{N^+}$. The map $\dim(N^-)$ will be referred to as the *index* of splitting.

On the other hand, a Borel probability measure μ of M is *nonatomic* if it has no points with positive mass, and *supported on H* if its support $supp(\mu)$ is contained in H. Given a flow X, we say that μ is *invariant* if $\mu(X_t(A)) = \mu(A)$ for every Borel set A and every $t \in \mathbb{R}$, and *ergodic* if it is invariant and every measurable invariant set has measure 0 or 1. Classical Oseledets's theorem asserts that every invariant measure μ is equipped with a full measure set R and, for each $x \in R$, there are integers $1 \le k(x) \le \dim(M)$, real numbers $\chi_1(x) < \chi_2(x) < \cdots < \chi_{k(x)}(x)$, and a splitting $T_xM = \hat{E}^1_x \oplus \cdots \oplus \hat{E}^k_x$ depending measurably on x such that $DX_t(x)(E^i_x) = E^i_{X_t(x)}$ $(\forall \in \mathbb{R})$ and

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|DX_t(x)e^i\| = \chi_i(x), \quad \forall x \in R, \forall e^i \in \hat{E}^i_x \setminus \{0\}, \forall 1 \le i \le k(x).$$

The points of R are the *regular points,* and the numbers χ_i the *Lyapunov exponents* of μ. It turns out that one of the Lyapunov exponents is zero corresponding to the flow direction. When the remaining exponents are nonzero, the measure will be referred to as a *hyperbolic measure* of X. If additionally, there are both positive and negative Lyapunov exponents, then the measure is said to be *hyperbolic saddle.*

By a *three-dimensional flow* we mean a flow X defined on a three-dimensional compact manifold.

DEFINITION 1.3. A singularity σ of a three-dimensional flow X is *Lorenz-like* if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $DX(\sigma)$ are real satisfying $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

For all such singularities, there are a two-dimensional stable manifold $W^s(\sigma)$, a one-dimensional unstable manifold $W^u(\sigma)$, and a one-dimensional strong stable manifold $W^{ss}(\sigma) \subset W^s(\sigma)$ (cf. [15]).

DEFINITION 1.4. A Lorenz-like singularity σ is in *general position* with respect to some subset $\Lambda \subset M$ if $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.

With these definitions we can state our main result.

THEOREM 1.5. *Let Λ be a compact invariant set of a three-dimensional flow X whose singularities are all Lorenz-like in general position. Then, Λ is singular-hyperbolic if and only if Λ^* has a dominated splitting of index 1 with respect to the linear Poincaré flow and every ergodic measure supported on Λ is hyperbolic saddle.*

The basic example where the hypotheses of the theorem are fulfilled is the *geometric Lorenz attractors* [14]. An obvious consequence is the following:

COROLLARY 1.6. *Let Λ be a compact invariant set of a three-dimensional flow X whose singularities are all Lorenz-like in general position. If Λ^* has a dominated splitting of index 1 with respect to the linear Poincaré flow and Λ does not support nonatomic ergodic measures, then Λ is singular-hyperbolic.*

An example satisfying the conditions of the corollary is a generic homoclinic loop associated to a Lorenz-like singularity. It follows from [23] that the Cherry-like flows considered in [19] also satisfy these conditions.

In light of Theorem 1.5, it is natural to ask if the saddle hypothesis can be removed from its statement or not. A motivation for this question comes from Theorem 3.3 in [1], which asserts that a generic ergodic measure of a C^1 generic diffeomorphism is hyperbolic. We can give a partial positive answer for this question based on the following standard concepts. Recall that a compact invariant set Λ of a flow X is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x) = \{y \in M : y = \lim_{n\to\infty} X_{t_n}(x)$ for some sequence $t_n \to \infty\}$. We say that Λ is a *limit cycle* if it is the limit of a sequence of periodic orbits with respect to the Hausdorff topology in the set of compact subsets of M. We say that Λ is *nontrivial* if it does not reduce to a single orbit of X.

With these definitions we can state the following corollary.

COROLLARY 1.7. *Let Λ be a nontrivial compact invariant set that is either transitive or a limit cycle of a $C^{1+\alpha}$ three-dimensional flow X. Suppose that the singularities of Λ are Lorenz-like in general position. Then, Λ is singular-hyperbolic if and only if Λ^* has a dominated splitting of index 1 with respect to the linear Poincaré flow and every ergodic measure supported on Λ is hyperbolic.*[1]

This paper is organized as follows. In Section 2 we recall the extended linear Poincaré flow [16] allowing us to rule out certain noncompact situations. In Section 3 we prove Theorem 1.5 and Corollary 1.7.

[1]This corollary is also true in the C^1 topology by the recent result [17].

2. Extended Linear Poincaré Flow

In this section we describe a technique from [16] but with different notation. Recall that M denotes a compact connected Riemannian manifold. Define

$$M^1 = \{L : L \text{ is a one-dimensional subspace of } T_x M \text{ for some } x \in M\}.$$

Then, M^1 is a fiber bundle over M with projection $\beta : M^1 \to M$, $\beta(L) = x$ if and only if $L \subset T_x M$.

Define the pullback bundle $TM^1 = \beta^*(TM)$ of TM under β, that is, the vector bundle over M^1 with fiber $T_L M^1 = \{L\} \times T_{\beta(L)} M$ at $L \in M^1$.

(*Do not confound* TM^1 *with the tangent bundle of* M^1.)

In general we define

$$T_\Delta M^1 = \bigcup_{L \in \Delta} T_L M^1, \quad \forall \Delta \subset M^1.$$

The Riemannian metric $\langle \cdot, \cdot \rangle$ of M induces one in TM^1 defined by

$$\langle (L, v), (L, w) \rangle = \langle v, w \rangle, \quad \forall (L, v), (L, w) \in T_L M^1.$$

We also have the subbundle E^{X^1} of TM^1 with fiber

$$E_L^{X^1} = \{L\} \times L$$

and the *normal bundle* $N^1 = (E^{X^1})^\perp$ with fiber

$$N_L^1 = \{L\} \times L^\perp.$$

Denote by $\pi^1 : TM^1 \to N^1$ the corresponding orthogonal projection.

Every flow X induces a flow X^1 in M^1 defined by

$$X_t^1(L) = DX_t(\beta(L))L, \quad \forall L \in M^1.$$

We also define the "derivative" $DX_t^1 : TM^1 \to TM^1$ of X_t^1 with respect to the vector bundle TM^1,

$$DX_t^1(L)(L, v) = (X_t^1(L), DX_t(\beta(L))v), \quad \forall L \in M^1, (L, v) \in T_L M^1.$$

We say that $\Omega \subset M^1$ is an invariant set of X^1 if $X_t^1(\Omega) = \Omega$ for any $t \in \mathbb{R}$.

Define the *linear Poincaré*[1] *flow* $P_t^1 : N^1 \to N^1$ by

$$P_t^1(L, v) = \pi^1_{X_t^1(L)}(DX_t^1(L)(L, v)), \quad \forall L \in M^1, (L, v) \in N_L^1.$$

Given $\Lambda \subset M$ satisfying $\Lambda^* = \Lambda$ (i.e., without singularities), we define

$$\Lambda^1 = \{E_x^X : x \in \Lambda\}.$$

If Λ is invariant for X, then so does Λ^1 for X^1 (this follows because E^X is a DX_t-invariant subbundle of $T_{M^*}M$). For general sets Λ (i.e., with singularities), we define

$$\Lambda^1 = \mathrm{Cl}((\Lambda^*)^1).$$

Equivalently,

$$\Lambda^1 = \left\{ L \in M^1 : L = \lim_{n \to \infty} E_{x_n}^X \text{ for some sequence } x_n \in \Lambda^* \right\}.$$

It follows that Λ^1 is compact (resp. X^1-invariant) if and only if Λ is compact (resp. X-invariant).

Let $\Omega \subset M^1$ be an invariant set of the induced flow X^1. We say that Ω *has a dominated splitting with respect to* X^1 if there is a continuous splitting $T_\Omega M^1 = E^1 \oplus F^1$ into DX_t^1-invariant subbundles E^1, F^1 such that $DX_t^1|_{E^1}$ dominates $DX_t|_{F^1}$. We also say that Ω *has a dominated splitting with respect to the linear Poincaré*^{1} *flow* if there is a continuous splitting $N^1_\Lambda = N^{-,1} \oplus N^{+,1}$ into P_t^1-invariant subbundles $N^{-,1}$, $N^{+,1}$ such that $P_t^1|_{N^{-,1}}$ dominates $P_t^1|_{N^{+,1}}$.

Clearly, if Λ is a compact invariant set of X, then $\beta(\Lambda^1) \subset \Lambda$. Therefore, every dominated splitting $T_\Lambda M = E \oplus F$ with respect to X induces a dominated splitting $T_{\Lambda^1} M^1 = E^1 \oplus F^1$ with respect to X^1 defined by

$$E^1_L = \{L\} \times E_{\beta(L)} \quad \text{and} \quad F^1_L = \{L\} \times F_{\beta(L)} \quad \text{for } L \in \Lambda^1.$$

Similarly, if $N_{\Lambda^*} = N^- \oplus N^+$ is a dominated splitting with respect to the linear Poincaré flow, then there is an induced dominated splitting $N^1_{(\Lambda^*)^1} = N^{-,1} \oplus N^{+,1}$ with respect to the linear Poincaré^1 flow defined by

$$N^{-,1}_L = \{L\} \times N^-_{\beta(L)} \quad \text{and} \quad N^{+,1}_L = \{L\} \times N^+_{\beta(L)} \quad \text{for } L \in (\Lambda^*)^1.$$

Passing this last splitting to the closure $\mathrm{Cl}((\Lambda^*)^1) = \Lambda^1$, we obtain a dominated splitting (still denoted by) $N^1_{\Lambda^1} = N^{-,1} \oplus N^{+,1}$ with respect to the linear Poincaré^1 flow.

We will need two lemmas.

LEMMA 2.1. *Let Λ a compact invariant set of a flow X having a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow. Then, $P_t|_{N^-}$ dominates $DX_t|_{E^X}$ if and only if $P_t^1|_{N^{-,1}}$ dominates $DX_t^1|_{E^{X^1}}$.*

Proof. We only prove the direct implication since the converse one is obvious.

Suppose that $P_t|_{N^-}$ dominates $DX_t|_{E^X}$. Then fix $T > 0$ such that

$$\|P_T(p)|_{N^-_p}\| \cdot \|DX_{-T}(X_T(p))|_{E^X_{X_T p)}}\| \le \frac{1}{2}, \quad \forall p \in \Lambda^*.$$

Now take $L \in \Lambda^1$. Then, there is a sequence $p_n \in \Lambda^*$ such that $L = \lim_{n\to\infty} L_n$, where $L_n = E^X_{p_n}$. Since $p_n \in \Lambda^*$, we have that $\|P^1_T(L_n)|_{N^{-,1}_{L_n}}\| = \|P_T(p_n)|_{N^-_{p_n}}\|$ and $\|DX^1_{-T}(X^1_T(L_n))|_{E^{X^1}_{X^1_T(L_n)}}\| = \|DX_{-T}(X_T(p_n))|_{E^X_{X_T(p_n)}}\|$ so

$$\|P^1_T(L_n)|_{N^{-,1}_{L_n}}\| \cdot \|DX^1_{-T}(X^1_T(L_n))|_{E^{X^1}_{X^1_T(L_n)}}\| \le \frac{1}{2}, \quad \forall n \in \mathbb{N}.$$

Since L is arbitrary and T fixed, we can take the limit in the last inequality to obtain

$$\|P^1_T(L)|_{N^{-,1}_L}\| \cdot \|DX^1_{-T}(X^1_T(L))|_{E^{X^1}_{X^1_T(L)}}\| \le \frac{1}{2}, \quad \forall L \in \Lambda^1.$$

But Λ^1 is compact since Λ is. So, the previous inequality implies the result. □

The proof of the following lemma is similar to that of Proposition 1.1 in [12]. It can be also obtained from Lemmas 5.5 and 5.6 in [16] as in the proof of Lemma 2.12 in [13].

LEMMA 2.2. *Let Λ be a compact invariant set of a flow X. If Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that $P_t|_{N^-}$ dominates $DX_t|_{E^X}$, then* $\mathrm{Cl}(\Lambda^*)$ *has a dominated splitting $T_{\mathrm{Cl}(\Lambda^*)}M = E \oplus F$ with respect to the tangent flow such that* $\dim(E) = \dim(N^-)$ *and $E^X \subset F$. In particular, $DX_t|_E$ is contracting.*

Proof. Let $N^1_{\Lambda^1} = N^{-,1} \oplus N^{+,1}$ be the induced dominated splitting with respect to the linear Poincaré[1] flow. For all $T > 0$, we have the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^{X^1} & \hookrightarrow & N^{-,1} \oplus E^{X^1} & \xrightarrow{\pi} & N^{-,1} & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle DX^1_T} & & \Big\downarrow {\scriptstyle DX^1_T} & & \Big\downarrow {\scriptstyle P^1_T} & & \\
0 & \longrightarrow & E^{X^1} & \hookrightarrow & N^{-,1} \oplus E^{X^1} & \xrightarrow{\pi} & N^{-,1} & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
 & & \Lambda^1 & =\!= & \Lambda^1 & =\!= & \Lambda^1 & &
\end{array}
$$

of short exact sequences of Riemannian vector bundles over the homeomorphism $X^1_T : \Lambda^1 \to \Lambda^1$ with compact base Λ^1. By Lemma 2.1 we have that $P^1_t|_{N^{-,1}}$ dominates $DX^1_t|_{E^{X^1}}$. Then there is $T > 0$ such that $\|P^1_T(L)|_{N^{-,1}_L}\| < \|DX^1_T(L)|_{E^{X^1}_L}\|$ for all $L \in \Lambda^1$. By Lemma 2.18 in [15] this supplies a unique DX_T-invariant complement $E^1 \subset N^{-,1} \oplus E^{X^1}$ of E^{X^1}. It follows from this uniqueness that E^1 is DX_t-invariant. This results in a DX^1_t-invariant splitting $T_{\Lambda^1}M^1 = E^1 \oplus F^1$ where $F^1 = N^{+,1} \oplus E^{X^1}$. Clearly, $\dim(E^1) = \dim(N^{-,1})$ and $E^{X^1} \subset F^1$. As in claims 2 and 3 of [16, p. 266], we obtain that this splitting is in fact dominated for X^1.

Finally, we have by definition that $E^X_p \in \Lambda^1$ for every $p \in \Lambda^*$. Then, there are subbundles E and F of $T_{\Lambda^*}M$ satisfying

$$E^1_{E^X_p} = \{E^X_p\} \times E_p \quad \text{and} \quad F^1_{E^X_p} = \{E^X_p\} \times F_p \quad \text{for every } p \in \Lambda^*.$$

Since $\dim(E^1) = \dim(N^{-,1})$ and $E^{X^1} \subset F^1$, we have respectively that $\dim(E) = \dim(N^-)$ and $E^X \subset F$ in Λ^*. Moreover, $T_{\Lambda^*}M = E \oplus F$ is dominated with respect to X since $T_{\Lambda^1}M^1 = E^1 \oplus F^1$ does with respect to X^1. Then, we can pass $T_{\Lambda^*}M = E \oplus F$ to the closure in the standard way to obtain the desired dominated splitting $T_{\mathrm{Cl}(\Lambda^*)}M = E \oplus F$ with respect to the tangent flow. Since $E^X \subset F$, we have that $DX_t|_E$ is contracting (see Lemma 3.2 in [2]). □

Notice that the dominated splitting with respect to the tangent flow just obtained may not exist in the whole Λ.

3. Proof of Theorem 1.5 and Corollary 1.7

We break the proof of Theorem 1.5 into a sequence of lemmas.

LEMMA 3.1. *Let σ be a Lorenz-like singularity of a three-dimensional flow X. If P_t^s denotes the linear Poincaré flow of $X|_{W^s(\sigma)}$, then*

$$\lim_{t\to\infty}\|P_t^s(p)\|=0 \quad \textit{and} \quad \lim_{t\to\infty}\frac{\|P_t^s(p)\|}{\|DX_t(p)|_{E_p^X}\|}=0, \quad \forall p\in W^s(\sigma)\setminus W^{ss}(\sigma).$$

Proof. For simplicity, we assume that $X|_{W^s(\sigma)}$ is given by the linear system

$$\begin{cases}\dot{y}=\lambda_2 y,\\ \dot{z}=\lambda_3 z,\end{cases}\qquad \lambda_2<\lambda_3<0,$$

where σ is the origin $(0,0)$.

Now, take $p=(y,z)\in W^s(\sigma)\setminus W^{ss}(\sigma)$; thus, $z\neq 0$.

For any $t\in\mathbb{R}$, we have $X_t(p)=(ye^{\lambda_2 t},ze^{\lambda_3 t})$ and also

$$DX_t(p)\cdot(a,b)=(yae^{\lambda_2 t},zbe^{\lambda_3 t})$$

for any $(a,b)\in T_pW^s(\sigma)$. Hence, $X(X_t(p))=(\lambda_2 ye^{\lambda_2 t},\lambda_3 ze^{\lambda_3 t})$, and then $N_{X_t(p)}\cap T_pM$ is the straightline through $(ye^{\lambda_2 t},ze^{\lambda_3 t})$ parallel to $(-\lambda_3 ze^{\lambda_3 t}, \lambda_2 ye^{\lambda_2 t})$.

On the other hand, the angle θ between $DX_t(p)\cdot(a,b)$ and $(-\lambda_3 ze^{\lambda_3 t}, \lambda_2 ye^{\lambda_2 t})$ is given by

$$\cos\theta=\frac{\langle DX_t(p)\cdot(a,b),(-\lambda_3 ze^{\lambda_3 t},\lambda_2 ye^{\lambda_2 t})\rangle}{\|DX_t(p)\cdot(a,b)\|\cdot\|(-\lambda_3 ze^{\lambda_3 t},\lambda_2 ye^{\lambda_2 t})\|}.$$

From this and by taking (a,b) unitary we obtain

$$\begin{aligned}\|P_t^s(p)\|&=\|P_t^s(p)\cdot(a,b)\|\\&=|\cos\theta|\cdot\|DX_t(p)\cdot(a,b)\|\\&=\frac{|\langle(yae^{\lambda_2 t},zbe^{\lambda_3 t}),(-\lambda_3 ze^{\lambda_3 t},\lambda_2 ye^{\lambda_2 t})\rangle|}{\|(-\lambda_3 ze^{\lambda_3 t},\lambda_2 ye^{\lambda_2 t})\|}\\&=\frac{Ke^{(\lambda_2+\lambda_3)t}}{\sqrt{\lambda_3^2e^{2\lambda_3 t}z^2+\lambda_2^2e^{2\lambda_2 t}y^2}},\end{aligned}$$

where K depends on p, a, b only.

Then,

$$\begin{aligned}\lim_{t\to\infty}\|P_t^s(p)\|&=\lim_{t\to\infty}\frac{Ke^{(\lambda_2+\lambda_3)t}}{\sqrt{\lambda_3^2e^{2\lambda_3 t}z^2+\lambda_2^2e^{2\lambda_2 t}y^2}}\\&=\lim_{t\to\infty}\frac{Ke^{\lambda_2 t}}{\sqrt{\lambda_3^2z^2+\lambda_2^2e^{2(\lambda_2-\lambda_3)t}y^2}}=0.\end{aligned}$$

Yet,

$$\|DX_t(p)|_{E_p^X}\| = \frac{\|X(X_t(p))\|}{\|X(p)\|} = \frac{\|(\lambda_2 y e^{\lambda_2 t}, \lambda_3 z e^{\lambda_3 t})\|}{\|(\lambda_2 y, \lambda_3 z)\|} = \frac{\sqrt{\lambda_3^2 e^{2\lambda_3 t} z^2 + \lambda_2^2 e^{2\lambda_2 t} y^2}}{\sqrt{\lambda_2^2 y^2 + \lambda_3^2 z^2}},$$

and so

$$\lim_{t\to\infty} \frac{\|P_t^s(p)\|}{\|DX_t(p)|_{E_p^X}\|} = K \lim_{t\to\infty} \frac{\sqrt{\lambda_3^2 z^2 + \lambda_2^2 y^2} e^{(\lambda_2-\lambda_3)t}}{\lambda_3^2 z^2 + \lambda_2^2 e^{2(\lambda_2-\lambda_3)t} y^2} = 0.$$

□

The proof of the following lemma is similar to that of Lemma 2.1.

LEMMA 3.2. *Let Λ a compact invariant set of a flow X, and N^- be a P_t-invariant subbundle of N_{Λ^*}. Then, $P_t|_{N^-}$ is contracting if and only if there is $T > 0$ such that $\forall p \in \Lambda^*$, $\exists 0 \le t \le T$ satisfying*

$$\|P_t(p)|_{N_p^-}\| < \frac{1}{2}.$$

Likewise, $P_t|N^-$ dominates $DX_t|_{E^X}$ if and only if there is $T > 0$ such that $\forall p \in \Lambda^*$, $\exists 0 \le t \le T$ satisfying

$$\frac{\|P_t(p)|_{N_p^-}\|}{\|DX_t(p)|_{E_p^X}\|} < \frac{1}{2}.$$

By this lemma, if $P_t|_{N^-}$ *is not contracting,* then there is a sequence $p_n \in \Lambda^*$ satisfying

$$\|P_t(p_n)|_{N_{p_n}^-}\| \ge \frac{1}{2}, \quad \forall 0 \le t \le n, \forall n \in \mathbb{N}. \tag{3.1}$$

Likewise, if $P_t|N^-$ *does not dominate* $DX_t|_{E^X}$, then there is a sequence $p_n \in \Lambda^*$ satisfying

$$\frac{\|P_t(p_n)|_{N_{p_n}^-}\|}{\|DX_t(p_n)|_{E_{p_n}^X}\|} \ge \frac{1}{2}, \quad \forall 0 \le t \le n, \forall n \in \mathbb{N}. \tag{3.2}$$

Now we prove under additional conditions that any sequence p_n satisfying (3.1) or (3.2) *cannot accumulate on the stable manifold of any singularity.* More precisely, we have the following result.

LEMMA 3.3. *Let Λ be a compact invariant set of a three-dimensional flow X. Suppose that Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that* $\dim(N^-) = 1$ *and that every singularity in Λ is Lorenz-like in general position. If $p_n \in \Lambda^*$ is a sequence satisfying* (3.1) *or* (3.2), *then $p \notin W^s(\sigma)$ for every singularity $\sigma \in \Lambda$ and every accumulation point p of p_n.*

Proof. We just consider the case where p_n satisfies (3.2) since the proof for (3.1) is similar.

Without loss of generality we can assume that $p_n \to p$. First, we prove that $p \in \Lambda^*$. Otherwise, $p = \sigma$ for some $\sigma \in \mathrm{Sing}(X)$. Still without loss of generality, we can assume that $E^X_{p_n} \to L$ for some $L \in \beta^{-1}(\sigma) \cap \Lambda^1$.

On the one hand, since σ is Lorenz-like, there is a dominated splitting $T_\sigma M = E^{ss} \oplus E^{cu}$ with respect to the flow, where E^{ss} is generated by the eigenvector associated to the eigenvalue λ_2, and E^{cu} is generated by the corresponding eigenvectors of $\{\lambda_1, \lambda_3\}$. Since σ is in general position, we can prove as in Lemma 4.4 in [16] that $L \subset E^{cu}$.

On the other hand, there is a dominated splitting $N^1_{\Lambda^1} = N^{-,1} \oplus N^{+,1}$ with respect to the linear Poincaré[1] flow induced by $N_\Lambda = N^- \oplus N^+$. Since $p_n \in \Lambda^*$ for $n \in \mathbb{N}$, (3.2) implies for $L_n = E^X_{p_n}$ that

$$\frac{\|P^1_t(L_n)|_{N^{-,1}_{L_n}}\|}{\|DX^1_t(L_n)|_{E^{X^1}_{L_n}}\|} = \frac{\|P_t(p_n)|_{N^-_{p_n}}\|}{\|DX_t(p_n)|_{E^X_{p_n}}\|} \geq \frac{1}{2}, \qquad \forall 0 \leq t \leq n, \forall n \in \mathbb{N}.$$

Fixing $t \geq 0$ and taking $n \to \infty$ in this expression, we obtain

$$\frac{\|P^1_t(L)|_{N^{-,1}_L}\|}{\|DX^1_t(L)|_L\|} \geq \frac{1}{2}, \qquad \forall t \geq 0. \tag{3.3}$$

Nevertheless, $\|P^1_t(L)|_{N^{-,1}}\| = \|DX_t(\sigma)|_{E^{ss}_\sigma}\|$ and $L \subset E^c_\sigma$ (cf. Lemma 4.2 in [16]), and thus

$$\lim_{t\to\infty} \frac{\|P^1_t(L)|_{N^{-,1}_L}\|}{\|DX^1_t(L)|_L\|} = \lim_{t\to\infty} \frac{\|DX_t(\sigma)|_{E^{ss}_\sigma}\|}{\|DX_t(\sigma)|_L\|} = 0,$$

contradicting (3.3). We conclude that $p \notin \mathrm{Sing}(X)$, and hence $p \in \Lambda^*$.

Now suppose by contradiction that $p \in W^s(\sigma)$ for some $\sigma \in \mathrm{Sing}(X)$.

Since $p \in \Lambda^*$, we can fix $t \geq 0$ and take $n \to \infty$ in (3.2) to obtain

$$\frac{\|P_t(p)|_{N^-_p}\|}{\|DX_t(p)|_{E^X_p}\|} \geq \frac{1}{2}, \qquad \forall t \geq 0.$$

Since $\dim(N^-) = 1$, Proposition 2.2 in [12] implies $N^-_p = N_p \cap T_p W^s(\sigma)$, so that $P_t(p)|_{N^-_p} = P^s_t(p)$. Moreover, $p \in \Lambda^* \subset \Lambda$, and σ is in general position, so that $p \notin W^{ss}(\sigma)$. Since σ is Lorenz-like, Lemma 3.1 implies

$$\lim_{t\to\infty} \frac{\|P_t(p)|_{N^-_p}\|}{\|DX_t(p)|_{E^X_p}\|} = \lim_{t\to\infty} \frac{\|P^s_t(p)\|}{\|DX_t(p)|_{E^X_p}\|} = 0,$$

contradicting the previous inequality. This concludes the proof. □

We use Lemma 3.3 to prove the following:

LEMMA 3.4. *Let Λ be a compact invariant set of a three-dimensional flow X. Suppose that Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that* $\dim(N^-) = 1$ *and that every singularity in Λ is*

Lorenz-like in general position. If $P_t|_{N^-}$ is contracting, then $P_t|_{N^-}$ dominates $DX_t|_{E^X}$.

Proof. Suppose by contradiction that $P_t|_{N^-}$ does not dominate $DX_t|_{E^X}$. Then, by Lemma 3.2, there is a sequence $p_n \in \Lambda^*$ satisfying (3.2). Since Λ is compact, we can assume that $p_n \to p$ for some $p \in \Lambda$.

By Lemma 3.3 we have $p \notin W^s(\sigma)$ for every singularity $\sigma \in \Lambda$. However, $P_t|_{N^-}$ is contracting, so (3.2) implies that there are $K, \lambda > 0$ such that $\|DX_t(p_n)|_{E^X_{p_n}}\| \leq 2Ke^{-\lambda t}$, $\forall 0 \leq t \leq n$, $\forall n \in \mathbb{N}$. Fixing $t \geq 0$ and taking $n \to \infty$, we obtain $\|DX_t(p)|_{E^X_p}\| \leq 2Ke^{-\lambda t}$, $\forall t \geq 0$. This easily implies $p \in W^s(\sigma)$ for some singularity $\sigma \in \Lambda$, a contradiction. □

The following lemma resembles Lemma I.5 in [18].

LEMMA 3.5. *Let Λ be a compact invariant set of a three-dimensional flow X. Suppose that Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that $\dim(N^-) = 1$ and that every singularity in Λ is Lorenz-like in general position. If there is $T > 0$ such that*

$$\int \log \|P_T|_{N^-}\| \, d\mu < 0 \tag{3.4}$$

for every ergodic measure μ supported on Λ, then $P_t|_{N^-}$ is contracting.

Proof. By hypothesis each singularity $\sigma \in \Lambda$ is Lorenz-like and so with real eigenvalues λ_1, λ_2, λ_3 satisfying $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. Denote by E^{ss}_σ and E^c_σ the eigenspaces associated to the sets of eigenvalues $\{\lambda_2\}$ and $\{\lambda_1, \lambda_3\}$, respectively. By changing the metric if necessary we can assume that $T_\sigma M = E^{ss}_\sigma \oplus E^c_\sigma$ is orthogonal. Then, since every singularity is in general position, we can extend the map $\|P_T|_{N^-}\|$ continuously to Λ by assigning the value $\|DX_T(\sigma)|_{E^{ss}_\sigma}\|$ at each singularity $\sigma \in \Lambda$.

Now suppose by contradiction that $P_t|_{N_-}$ is not contracting. Then, Lemma 3.2 furnishes a sequence $p_n \in \Lambda^*$ satisfying (3.1). Since Λ is compact, we can assume that p_n converges to some point p, which by Lemma 3.3 belongs to Λ^*. Fixing $t \geq 0$ and taking $n \to \infty$ in (3.1), we obtain

$$\|P_t(p)|_{N^-_p}\| \geq \frac{1}{2}, \quad \forall t \geq 0. \tag{3.5}$$

Let δ_z be the Dirac measure supported on $\{z\}$. Define the sequence of Borel probability measures $\mu_n = \frac{1}{n}\int_0^n \delta_{X_t(p)}\, dt$ for $n \in \mathbb{N}$. We can assume that μ_n converges, with respect to the weak-* topology, to a Borel probability measure μ_∞. It is clear that μ_∞ is invariant and supported on Λ. On the other hand, the chain rule

$$P_{T+t}(x)|_{N^-_x} = (P_T(X_t(x))|_{N^-_{X_t(x)}}) \circ (P_t(x)|_{N^-_x}), \quad \forall (x,t) \in \Lambda^* \times [0,\infty[,$$

together with $\dim(N^-) = 1$, implies

$$\log \|P_T(X_t(x))|_{N^-_{X_t(x)}}\| = \log \|P_{T+t}(x)|_{N^-_x}\| - \log \|P_t(x)|_{N^-_x}\|$$

$\forall(x,t)\in\Lambda^*\times[0,\infty[$. Since $\mu_n\to\mu_\infty$, taking $x=p$, we get

$$\begin{aligned}\int \log\|P_T|_{N^-}\|\,d\mu_\infty\\ &= \lim_{n\to\infty}\frac{1}{n}\int_0^n \log\|P_T(X_t(p))|_{N^-_{X_t(p)}}\|\,dt\\ &= \lim_{n\to\infty}\frac{1}{n}\left(\int_0^n \log\|P_{T+t}(p)|_{N_p^-}\|\,dt-\int_0^n \log\|P_t(p)|_{N_p^-}\|\,dt\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\left(\int_T^{n+T} \log\|P_t(p)|_{N_p^-}\|\,dt-\int_0^n \log\|P_t(p)|_{N_p^-}\|\,dt\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\left(\int_n^{n+T} \log\|P_t(p)|_{N_p^-}\|\,dt-\int_0^T \log\|P_t(p)|_{N_p^-}\|\,dt\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\int_n^{n+T} \log\|P_t(p)|_{N_p^-}\|\,dt, \end{aligned}\tag{3.6}$$

so (3.5) implies

$$\int \log\|P_T|_{N^-}\|\,d\mu_\infty\geq -\lim_{n\to\infty}\frac{1}{n}\int_n^{n+T}\log 2\;dt=-\lim_{n\to\infty}\frac{T\log 2}{n}=0.$$

Therefore, an ergodic component μ in the ergodic decomposition of μ_∞ (cf. p. 113 in [21]) must satisfy

$$\int \log\|P_T|_{N^-}\|\,d\mu\geq 0.$$

This contradicts (3.4) and completes the proof. □

Now we apply Lemma 3.5 to prove the following:

LEMMA 3.6. *Let Λ be a compact invariant set of a three-dimensional flow X. Suppose that Λ^* has a dominated splitting $N_{\Lambda^*}=N^-\oplus N^+$ with respect to the linear Poincaré flow such that* $\dim(N^-)=1$ *and that every singularity in Λ is Lorenz-like in general position. If every ergodic measure supported on Λ is hyperbolic saddle, then $P_t|_{N^-}$ is contracting. In particular, $P_t|_{N^-}$ dominates $DX_t|_{E^X}$.*

Proof. To prove that $P_t|_{N^-}$ is contracting, by Lemma 3.5 we only need to find $T>0$ satisfying (3.4) for every ergodic measure μ supported on Λ.

Just take $T>0$ large enough satisfying

$$\|DX_T(\sigma)|_{E^{ss}_\sigma}\|<1\quad\text{for every singularity } \sigma\in\Lambda.\tag{3.7}$$

Now suppose by contradiction that, for such a T, there is an ergodic measure μ that does not satisfy (3.4), that is,

$$\int \log\|P_T|_{N^-}\|\,d\mu\geq 0.$$

Clearly, μ is nonatomic since, otherwise, $\mu = \delta_\sigma$ for some singularity $\sigma \in \Lambda$ and then

$$\log \|DX_T(\sigma)|_{E^{ss}_\sigma}\| = \int \log \|P_T|_{N^-}\| \, d\mu \geq 0,$$

contradicting (3.7). From this we conclude that $\mu(\mathrm{Sing}(X)) = 0$. But we also have that μ is hyperbolic saddle by hypothesis. Then, we can apply linear Poincaré flow's version of Oseledets's theorem (cf. Thm. 2.1 in [7] and Thm. 2.2 in [8]) to conclude that for the two Lyapunov exponents $\chi_1 < 0 < \chi_2$, there corresponds an Oseledets splitting $N_R = \hat{N}^1 \oplus \hat{N}^2$ of index 1 over the full measure set of regular points R such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P_t(x)n^i\| = \chi_i, \quad \forall x \in R, \forall n^i \in \hat{N}^i_x \setminus \{0\}, \forall 1 \leq i \leq 2.$$

Birkhoff's theorem implies

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \log \|P_T(X_t(x))|_{N^-_{X_t(x)}}\| \, dt = \int \log \|P_T|_{N^-}\| \, d\mu \quad \text{for } \mu\text{-a.e. } x,$$

and the chain rule as in (3.6) implies

$$\lim_{L\to\infty} \frac{1}{L} \int_L^{L+T} \log \|P_t(x)|_{N^-_x}\| \, dt = \lim_{L\to\infty} \frac{1}{L} \int_0^L \log \|P_T(X_t(x))|_{N^-_{X_t(x)}}\| \, dt.$$

Then, we can select $x \in \Lambda^* \cap R$ satisfying

$$\lim_{L\to\infty} \frac{1}{L} \int_L^{L+T} \log \|P_t(x)|_{N^-_x}\| \, dt \geq 0. \tag{3.8}$$

On the other hand, $x \in R$ and $\dim(\hat{N}^1) = \dim(\hat{N}^2) = 1$, so we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P_t(x)|_{\hat{N}^i}\| = \chi_i \quad \text{for } i = 1, 2.$$

These limits implies that the splitting $N_x = \hat{N}^1_x \oplus \hat{N}^2_x$ is predominated in the sense of Definition 2.1 in [16]. Since predominated splittings of prescribed index are unique (cf. Lemma 2.3 in [16]) and $N_x = N^-_x \oplus N^+_x$ is dominated (hence, predominated), we conclude that $N^-_x \oplus N^+_x = \hat{N}^1_x \oplus \hat{N}^2_x$. In particular,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P_t(x)|_{N^-_x}\| = \chi_1. \tag{3.9}$$

Now by (3.8) for a fixed $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that

$$\frac{1}{L} \int_L^{L+T} \log \|P_t(x)|_{N^-_x}\| \, dt \geq -\varepsilon, \quad \forall L \geq L_\varepsilon.$$

From this we obtain arbitrarily large values of t satisfying

$$\frac{1}{t} \log \|P_t(x)|_{N^-_x}\| \geq -\frac{\varepsilon}{T}.$$

Then, (3.9) yields $\chi_1 \geq -\frac{\varepsilon}{T}$. Since ε is arbitrary, we conclude that $\chi_1 \geq 0$, contradicting $\chi_1 < 0$. Therefore, $P_t|_{N^-}$ is contracting, and so $P_t|_{N^-}$ dominates $DX_t|_{E^X}$ by Lemma 3.4. This ends the proof. □

Proof of Theorem 1.5. Consider a three-dimensional flow X and a compact invariant set Λ of X. If Λ is singular-hyperbolic, then Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that $\dim(N^-) = 1$ (by Lemma 2.3 in [9]). Moreover, every singularity in Λ is Lorenz-like in general position [20]. It remains to prove that every nonatomic ergodic measure supported on Λ is hyperbolic saddle.

It is clear that such a measure μ (say) has a negative Lyapunov exponent χ_1 corresponding to the contracting direction E of the singular-hyperbolic splitting $E \oplus F$. To compute the other exponent χ_2, we choose a regular point $x \in \Lambda^*$ of μ. Since $E^X \subset F$ by Lemma 3.2 in [2], we have

$$|\det DX_t(x)|_{F_x}| = \|P_t(x)|_{N_x^+}\| \cdot \|DX_t(x)|_{E_x^X}\|,$$

and so

$$\begin{aligned}\chi_2 &= \lim_{t\to\infty} \frac{1}{t} \log \|P_t(x)|_{N_x^+}\| \\ &= \lim_{t\to\infty} \frac{1}{t} (\log|\det DX_t(x)|_{F_x}| - \log \|DX_t(x)|_{E_x^X}\|).\end{aligned}$$

But M is compact, so there is $L > 0$ such that $\|X(y)\| < L$ for all $y \in M$, and thus

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log \|DX_t(x)|_{E_x^X}\| &= \lim_{t\to\infty} \frac{1}{t} (\log \|X(X_t(x))\| - \log \|X(x)\|) \\ &\le \lim_{t\to\infty} \frac{1}{t} \log L = 0.\end{aligned}$$

Moreover, $DX_t|_F$ is volume expanding, so there are positive numbers K, λ such that $|\det DX_t(x)|_{F_x}| \ge Ke^{\lambda t}$, $\forall t \ge 0$, and thus

$$\lim_{t\to\infty} \frac{1}{t} \log|\det DX_t(x)|_{F_x}| \ge \lambda,$$

so that $\chi_2 \ge \lambda > 0$, and hence μ is hyperbolic saddle.

Conversely, suppose that Λ^* has a dominated splitting $N_{\Lambda^*} = N^- \oplus N^+$ with respect to the linear Poincaré flow such that $\dim(N^-) = 1$, every singularity in Λ is Lorenz-like in general position, and every ergodic measure supported on Λ is hyperbolic saddle. By Lemma 3.6 we obtain that $P_t|_{N^-}$ dominates $P_t|_{N^+}$. Then, by Lemma 2.2, $\mathrm{Cl}(\Lambda^*)$ has a dominated splitting $T_{\mathrm{Cl}(\Lambda^*)}M = E \oplus F$ with respect to the tangent flow such that $\dim(E) = 1$ (thus, $\dim(F) = 2$), $E^X \subset F$, and $DX_t|_E$ is contracting.

It remains to prove that $DX_t|_F$ is volume expanding. The proof is similar to that of Lemma 3.6. We give the details for completeness.

First, we notice that the proof of Lemma 2.2 implies $F = N^+ \oplus E^X$ over Λ^*. From this we get

$$|\det DX_t(x)|_{F_x}| = \|P_t(x)|_{N_x^+}\| \cdot \|DX_t(x)|_{E_x^X}\|, \quad \forall x \in \Lambda^*, t \ge 0. \tag{3.10}$$

Next we observe that, as in Lemma 3.5, in order to prove that $DX_t|_F$ is volume expanding, it suffices to find $T > 0$ such that

$$\int \log |\det DX_T|_F|\, d\mu > 0 \tag{3.11}$$

for every ergodic measure μ supported on Λ.

To find such a T, we first observe that $F_\sigma = E^{cu}_\sigma$ at each singularity σ in Λ, and so, there is $T > 0$ such that

$$|\det DX_T(\sigma)|_{E^{cu}_\sigma}| > 1 \quad \text{for every singularity } \sigma \in \Lambda. \tag{3.12}$$

Afterward, we suppose by contradiction that, for such a T, there is an ergodic measure μ supported on Λ that does not satisfy (3.11), that is,

$$\int \log |\det DX_T|_F|\, d\mu \le 0.$$

We have that μ is nonatomic because of (3.12) and then $\mu(\mathrm{Sing}(X)) = 0$ by ergodicity. But we also have that μ is hyperbolic saddle by hypothesis. Since $\mu(\mathrm{Sing}(X)) = 0$, we have as before that for the two Lyapunov exponents $\chi_1 < 0 < \chi_2$, there corresponds an Oseledets splitting $N_R = \hat{N}^1 \oplus \hat{N}^2$ of index 1 over the full measure set of regular points R such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P_t(x) n^i\| = \chi_i, \quad \forall x \in R, \forall n^i \in \hat{N}^i_x \setminus \{0\}, \forall 1 \le i \le 2.$$

Again, Birkhoff's theorem implies

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \log |\det DX_T(X_t(x))|_{F_{X_t(x)}}|\, dt = \int \log |\det DX_T|_F|\, d\mu$$

for μ-a.e. x, and the chain rule as in (3.6) implies

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{L} \int_L^{L+T} \log |\det DX_t(x)|_{F_x}|\, dt \\ &\quad = \lim_{L\to\infty} \frac{1}{L} \int_0^L \log |\det DX_T(X_t(x))|_{F_{X_t(x)}}|\, dt, \end{aligned}$$

so there exists $x \in \Lambda^* \cap R$ satisfying

$$\lim_{L\to\infty} \frac{1}{L} \int_L^{L+T} \log |\det DX_t(x)|_{F_x}|\, dt \le 0. \tag{3.13}$$

Arguing as before, we have $N^-_x \oplus N^+_x = \hat{N}^1_x \oplus \hat{N}^2_x$, so

$$\lim_{t\to\infty} \frac{1}{t} \log \|P_t(x)|_{N^+_x}\| = \chi_2. \tag{3.14}$$

Finally, (3.13) for a fixed $\varepsilon > 0$ provides $L_\varepsilon > 0$ such that

$$\frac{1}{L} \int_L^{L+T} \log |\det DX_t(x)|_{F_x}| \le \varepsilon, \quad \forall L \ge L_\varepsilon,$$

yielding a sequence $t_n \to \infty$ satisfying

$$\frac{1}{t_n} |\det DX_{t_n}(x)|_{F_x}| \le \frac{\varepsilon}{T}.$$

Then, (3.10) and (3.14) imply

$$\chi_2 = \lim_{n\to\infty} \frac{1}{t_n} \log \| P_{t_n}(x)|_{N_x^+} \| = \lim_{n\to\infty} \frac{1}{t_n} \log |\det DX_{t_n}(x)|_{F_x}| \le \frac{\varepsilon}{T}.$$

Since ε is arbitrary, we get $\chi_2 \le 0$, contradicting $\chi_2 > 0$. This ends the proof. □

Proof of Corollary 1.7. Let Λ be a nontrivial compact invariant set that is either transitive or a limit cycle of a $C^{1+\alpha}$ three-dimensional flow X. Suppose that the singularities of Λ are Lorenz-like in general position. By Theorem 1.5, if Λ is hyperbolic, then Λ^* has a dominated splitting of index 1 with respect to the linear Poincaré flow, and every ergodic measure supported on Λ is hyperbolic.

Conversely, suppose that Λ^* has a dominated splitting of index 1 with respect to the linear Poincaré flow and that every ergodic measure supported on Λ is hyperbolic.

Suppose that Λ supports an ergodic measure μ that is not saddle-type. Since every singularity is Lorenz-like (hence, hyperbolic of saddle type), we have that μ cannot be supported on a singularity. Then, there are points in the support of μ where X does not vanishes. On the other hand, the two Lyapunov exponents of μ are either negative or positive. Since X is $C^{1+\alpha}$, we can apply Theorem 3.1 in [10] to conclude that μ is supported on an attracting or a repelling periodic orbit. In particular, Λ has an attracting or a repelling periodic orbit. Since Λ is transitive or a limit cycle, we conclude that Λ reduces to a single orbit, contradicting that Λ is nontrivial. This contradiction shows that every ergodic measure supported on Λ is hyperbolic saddle. Hence, Λ is singular-hyperbolic by Theorem 1.5. This completes the proof. □

References

[1] F. Abdenur, C. Bonatti, and S. Crovisier, *Nonuniform hyperbolicity for C1-generic diffeomorphisms,* Israel J. Math. 183 (2011), 1–60.

[2] V. Araújo and L. Salgado, *Infinitesimal Lyapunov functions for singular flows,* Math. Z. 275 (2013), no. 3–4, 863–897.

[3] V. Araújo, A. Arbieto, and L. Salgado, *Dominated splittings for flows with singularities,* Nonlinearity 26 (2013), no. 8, 2391–2407.

[4] A. Arbieto, *Sectional Lyapunov exponents,* Proc. Amer. Math. Soc. 138 (2010), no. 9, 3171–3178.

[5] A. Arbieto and L. Salgado, *On critical orbits and sectional hyperbolicity of the nonwandering set for flows,* J. Differential Equations 250 (2011), no. 6, 2927–2939.

[6] A. Arroyo and F. Rodriguez Hertz, *Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows,* Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003), no. 5, 805–841.

[7] M. Bessa, *The Lyapunov exponents of generic zero divergence three-dimensional vector fields,* Ergodic Theory Dynam. Systems 27 (2007), no. 5, 1445–1472.

[8] M. Bessa and J. Rocha, *Contributions to the geometric and ergodic theory of conservative flows,* Ergodic Theory Dynam. Systems 33 (2013), no. 6, 1709–1731.

[9] C. Bonatti, S. Gan, and D. Yang, *Dominated chain recurrent class with singularities,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) (to appear), preprint arXiv:1106.3905v1 [math.DS] 20 Jun 2011.

[10] M. Campanino, *Two remarks on the computer study of differentiable dynamical systems,* Comm. Math. Phys. 74 (1980), no. 1, 15–20.
[11] S. Crovisier and D. Yang, *On the density of singular hyperbolic three-dimensional vector fields: a conjecture of Palis,* C. R. Math. Acad. Sci. Paris 353 (2015), no. 1, 85–88.
[12] C. I. Doering, *Persistently transitive vector fields on three-dimensional manifolds,* Dynamical systems and bifurcation theory, Rio de Janeiro, 1985, Pitman Res. Notes Math. Ser., 160, pp. 59–89, Longman Sci. Tech., Harlow, 1987.
[13] S. Gan and D. Yang, *Morse–Smale systems and horseshoes for three-dimensional singular flows,* preprint, 5 Feb 2013, arXiv:1302.0946v1 [math.DS].
[14] J. Guckenheimer and R. F. Williams, *Structural stability of Lorenz attractors,* Inst. Hautes Études Sci. Publ. Math. 50 (1979), 59–72.
[15] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Math., 583, Springer-Verlag, Berlin–New York, 1977.
[16] M. Li, S. Gan, and L. Wen, *Robustly transitive singular sets via approach of an extended linear Poincaré flow,* Discrete Contin. Dyn. Syst. 13 (2005), no. 2, 239–269.
[17] A. M. Lopez and C. A. Morales, *On ergodic measures with negative exponents,* (2015, to appear), preprint.
[18] R. Mañé, *A proof of the C^1 stability conjecture,* Inst. Hautes Études Sci. Publ. Math. 66 (1988), 161–210.
[19] C. A. Morales, *A note on periodic orbits for singular-hyperbolic flows,* Discrete Contin. Dyn. Syst. 11 (2004), no. 2–3, 615–619.
[20] C. A. Morales, M. J. Pacifico, and E. R. Pujals, *Singular hyperbolic systems,* Proc. Amer. Math. Soc. 127 (1999), no. 11, 3393–3401.
[21] D. W. Morris, *Ratner's theorems on unipotent flows,* Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 2005.
[22] E. R. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms,* Ann. of Math. (2) 151 (2000), 961–1023.
[23] R. Saghin and E. Vargas, *Invariant measures for Cherry flows,* Comm. Math. Phys. 317 (2013), no. 1, 55–67.

Instituto de Matemática
Universidade Federal do Rio de Janeiro
P.O. Box 68530
21945-970 Rio de Janeiro
Brazil

morales@impa.br