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The Motive of the Classifying Stack of the Orthogonal Group

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Abstract. We compute the motive of the classifying stack of an orthogonal group in the Grothendieck ring of stacks over a field of characteristic different from two.

1. Introduction

The Grothendieck ring of stacks over a field k has been introduced by a number of authors [1; 6; 8; 13]. Denote this ring by $\hat{K}_0(\mathrm{Var}_k)$. An algebraic group G defined over k is called special if any G-torsor over a k-variety is locally trivial in the Zariski topology. General linear, special linear, and symplectic groups are special. Special orthogonal groups are not special in dimensions greater than two. Serre [11] proved that special groups are linear and connected. Over algebraically closed fields, the special groups were classified by Grothendieck [7].

For a special group G, the motive $[G]$ is invertible in $\hat{K}_0(\mathrm{Var}_k)$, and its inverse is equal to the motive of the classifying stack BG. This naturally raises the problem of computing the motive of BG when the group G is not special. For finite group schemes, a number of examples were computed in [5]. The case of groups of positive dimension is more difficult. In [3] it was shown that $[BPGL_n] = [PGL_n]^{-1}$ for $n = 2$ or 3 with mild restrictions on the field k.

The main result of this paper, Theorem 3.7, computes the motive of the classifying stack of an orthogonal group over a field whose characteristic is not two. In odd dimensions the result is that the motive is equal to the inverse of the motive of the split special orthogonal group in the same dimension. To prove Theorem 3.7, we first compute the motive of the variety of nondegenerate quadratic forms of fixed dimension. This motive was already computed in [2], using results of [9]. Our computation is different, relying on generating function techniques. Using Theorem 3.7, we are able to compute the motives of classifying stacks of the special orthogonal groups in odd dimensions.

1.1. Notation

We will work over a base field k with $\mathrm{char}(k) \neq 2$. If n is a nonnegative integer, then we denote by $[n]_{\mathbb{L}}$ the nth Gaussian polynomial in the Lefschetz motive $\mathbb{L}$. Explicitly,

$$[n]_{\mathbb{L}} = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}.$$

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The Gaussian polynomials $[n]_{\mathbb{L}}!$ and $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$ are defined in the usual way. The class of the Grassmannian $Gr(r,n)$ in the ring $\hat{K}_0(\mathrm{Var}_k)$ is then $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$.

2. Preliminaries

2.1. The Grothendieck Ring of Stacks

Fix a ground field k. Let $K_0(\mathrm{Var}_k)$ be the Grothendieck ring of varieties over k. Its underlying Abelian group is generated by symbols $[X]$, with X a k-variety, modulo the relations $[X] = [Y]$ if X and Y are isomorphic, and

$$[X] = [X \backslash Z] + [Z]$$

if $Z \subset X$ is a closed subvariety. Cartesian product of varieties gives $K_0(\mathrm{Var}_k)$ the structure of a commutative ring with identity $1 = [\mathrm{Spec}\, k]$. The Lefschetz motive is defined to be $\mathbb{L} = [\mathbb{A}^1_k]$.

The Grothendieck ring of stacks, $\hat{K}_0(\mathrm{Var}_k)$, is the dimensional completion of $K_0(\mathrm{Var}_k)$ defined as follows [1]. Let $F^m \subset K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ be the additive subgroup generated by those $\mathbb{L}^{-d}[X]$ with $\dim X - d \leq -m$. This defines a descending filtration of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$, and $\hat{K}_0(\mathrm{Var}_k)$ is the completion with respect to this filtration.

In this paper all stacks are assumed to be Artin stacks that are locally of finite type, all of whose geometric stabilizers are linear algebraic groups. Following [1], a stack $\mathfrak{X}$ is called essentially of finite type if it admits a stratification $\mathfrak{X} = \bigcup_{i=1}^{\infty} \mathfrak{X}_i$ by finite type, locally closed substacks with $\lim_{i \to \infty} \dim \mathfrak{X}_i = -\infty$. Any stack that is essentially of finite type admits a stratification of the above type with $\mathfrak{X}_i$ a global quotient stack of a variety X_i by a general linear group GL_{n_i}. Given such a stratification, put

$$[\mathfrak{X}] = \sum_{i=1}^{\infty} \frac{[X_i]}{[GL_{n_i}]}.$$

This defines a motivic class $[\mathfrak{X}] \in \hat{K}_0(\mathrm{Var}_k)$ that is independent of the choice of stratification of $\mathfrak{X}$ [1, Lemma 2.3].

LEMMA 2.1 ([1, Lemma 2.5]). *Let $\mathfrak{X}$ be a stack that is essentially of finite type, and let $P \to \mathfrak{X}$ be a torsor for a linear algebraic group G. Then P is essentially of finite type. Moreover, if G is special, then $[P] = [\mathfrak{X}][G]$ in $\hat{K}_0(\mathrm{Var}_k)$.*

In particular, if G is special, then applying Lemma 2.1 to the universal G-torsor $\mathrm{Spec}\, k \to BG$ shows that $[BG] = [G]^{-1}$. This equality is called the universal G-torsor relation.

More generally, if X is a variety acted on by a linear algebraic group G, then the quotient stack X/G has a class in $\hat{K}_0(\mathrm{Var}_k)$. For any closed embedding $G \hookrightarrow GL_N$, there is an isomorphism of stacks $X/G \simeq (X \times_G GL_N)/GL_N$. Since GL_N

is special, Lemma 2.1 implies that

$$[X/G] = \frac{[X \times_G GL_N]}{[GL_N]} \tag{1}$$

in $\hat{K}_0(\mathrm{Var}_k)$.

2.2. Orthogonal Groups

Assume that the ground field k is not of characteristic two. Let V be a finite-dimensional vector space over k, and let $Q : V \to k$ be a quadratic form. The radical of Q is the subspace of V defined by

$$\mathrm{rad}_Q = \{v \in V \,|\, Q(v+w) = Q(v) + Q(w)\ \forall w \in V\}.$$

The rank of Q is $\dim V - \dim \mathrm{rad}_Q$. The quadratic form Q is called nondegenerate if $\mathrm{rad}_Q = \{0\}$.

Given a nondegenerate quadratic form Q, denote by $O(Q)$ its group of isometries. If the field k is algebraically closed, then there is a unique nondegenerate quadratic form on k^n up to equivalence. The corresponding orthogonal group is unique up to isomorphism. If k is not algebraically closed, then there will in general exist inequivalent nondegenerate quadratic forms on k^n, leading to different forms of orthogonal groups.

For each $n \geq 1$, there is a canonical nondegenerate split quadratic form on k^n. Explicitly,

$$Q_{2r} = x_1x_2 + \cdots + x_{2r-1}x_{2r}$$

and

$$Q_{2r+1} = x_0^2 + x_1x_2 + \cdots + x_{2r-1}x_{2r}.$$

Define $O_n = O(Q_n)$ and $SO_n = SO(Q_n)$.

3. The Motive of $BO(Q)$

3.1. Filtration of the Space of Quadratic Forms

Recall that $\mathrm{char}(k) \neq 2$. Denote by $Quad_n \simeq \mathbb{A}_k^{\binom{n+1}{2}}$ the affine space of quadratic forms on k^n. The group GL_n acts on $Quad_n$ by change of basis. For each $0 \leq r \leq n$, let $Quad_{n,\leq r} \subset Quad_n$ denote the closed subvariety of quadratic forms whose rank is at most r. This gives an increasing filtration of $Quad_n$ by closed subvarieties. Interpreted in $K_0(\mathrm{Var}_k)$, this implies the identity

$$\mathbb{L}^{\binom{n+1}{2}} = \sum_{r=0}^{n} [Quad_{n,r}] \tag{2}$$

with $Quad_{n,r}$ the subvariety of quadratic forms of rank r. Denote by $Gr(m,n)$ the Grassmannian of m-planes in k^n.

PROPOSITION 3.1. *For each* $0 \leq r \leq n$*, the map*

$$\pi : Quad_{n,r} \to Gr(n-r,n), \qquad Q \mapsto \mathrm{rad}_Q$$

is a Zariski locally trivial fibration with fibers isomorphic to $Quad_{r,r}$.

Proof. Identify $Gr(n-r,n)$ with the quotient of the variety of $(n-r)\times n$ matrices of rank $n-r$ by the left action of GL_{n-r}. Fix coordinates $x_1,\dots,x_n$ on k^n. Consider the $(n-r)$-plane $k^{n-r}\subset k^n$ with coordinates $x_1,\dots,x_{n-r}$. A Zariski open set $U\subset Gr(n-r,n)$ containing k^{n-r} is given by the $(n-r)\times n$ matrices of the form

$$\begin{pmatrix}\mathbf{1}_{n-r} & B\end{pmatrix}$$

with $\mathbf{1}_{n-r}$ the $(n-r)\times(n-r)$ identity matrix and B an arbitrary $(n-r)\times r$ matrix. The plane k^{n-r} corresponds to the matrix $B=0$. Note that

$$\begin{pmatrix}\mathbf{1}_{n-r} & B\end{pmatrix}=\begin{pmatrix}\mathbf{1}_{n-r} & 0\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}_{n-r} & B\\ 0 & \mathbf{1}_r\end{pmatrix}.$$

Let $g_B=\begin{pmatrix}\mathbf{1}_{n-r} & B\\ 0 & \mathbf{1}_r\end{pmatrix}\in GL_n$, viewed as an automorphism of k^n.

Suppose that $Q\in\pi^{-1}(U)$. Then there exists a unique matrix $B(Q)$ such that $\mathrm{rad}_Q=g_{B(Q)}(k^{n-r})\subset k^n$. The quadratic form $g_{B(Q)}\cdot Q$ is the pullback of a nondegenerate quadratic form φ_Q in the variables $x_{n-r+1},\dots,x_n$. A trivialization of π over U is then given by

$$\pi^{-1}(U)\to U\times Quad_{r,r},\qquad Q\mapsto(\mathrm{rad}_Q,\varphi_Q).$$

This argument can be repeated, replacing k^{n-r} with the $(n-r)$-plane with coordinates labeled by an $(n-r)$-element subset $I\subset\{1,\dots,n\}$. This gives a Zariski open cover of $Gr(n-r,n)$ over which π trivializes. □

COROLLARY 3.2. *The identity*

$$\mathbb{L}^{\binom{n+1}{2}}=\sum_{r=0}^{n}\begin{bmatrix}n\\ n-r\end{bmatrix}_{\mathbb{L}}[Quad_{r,r}]_{\mathbb{L}}$$

holds in the ring $K_0(\mathrm{Var}_k)$.

Proof. It follows from Proposition 3.1 that $[Quad_{n,r}]=[Gr(n-r,n)][Quad_{r,r}]$. Since $[Gr(n-r,n)]=\begin{bmatrix}n\\ n-r\end{bmatrix}_{\mathbb{L}}$, the desired identity is implied by equation (2). □

3.2. Solving the Recurrence

In this section we will solve the recurrence relation for $[Quad_{n,n}]$ given in Corollary 3.2. In fact, the motives $[Quad_{n,r}]$ were already computed in [2, Theorem 13.5], where it was shown that $[Quad_{n,r}]$ satisfies a certain three-step recurrence relation with coefficients in $\mathbb{Z}[\mathbb{L}]$. This recurrence relation, with $\mathbb{L}$ replaced by q, was previously solved in [9] to find the number of $\mathbb{F}_q$-rational points of $Quad_{n,r}$. Hence, $[Quad_{n,r}]$ is given by the same formula, with q replaced with $\mathbb{L}$. We present here an alternative computation of $[Quad_{n,n}]$ and, therefore, also $[Quad_{n,r}]$ by Proposition 3.1, using generating functions.

We form the exponential generating function for the motives $[Quad_{n,n}]$,

$$G(x)=\sum_{n\geq 0}\frac{[Quad_{n,n}]x^n}{[n]_{\mathbb{L}}!}.$$

Consider also the auxiliary generating functions

$$P_{\text{even}}(x) = \sum_{k\geq 0} \frac{x^{2k}}{[2k]_{\mathbb{L}}!} \prod_{i=1}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i})$$

and

$$P_{\text{odd}}(x) = \sum_{k\geq 0} \frac{x^{2k+1}}{[2k+1]_{\mathbb{L}}!} \prod_{i=0}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}).$$

We will show that

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x),$$

thereby solving the recurrence relation.

PROPOSITION 3.3. *Denote by* $\exp_{\mathbb{L}}(x)$ *the* $\mathbb{L}$*-deformed exponential series*:

$$\exp_{\mathbb{L}}(x) = \sum_{n\geq 0} \frac{x^n}{[n]_{\mathbb{L}}!}.$$

The following equality holds:

$$G(x) = \frac{\prod_{i\geq 1}(1 + (1-\mathbb{L})x\mathbb{L}^i)}{\exp_{\mathbb{L}}(x)}.$$

Proof. To ease notation, set $\mathcal{Q}_n = [Quad_{n,n}]$. Using Corollary 3.2, we find that

$$\begin{aligned}
G(x) &= \sum_{n\geq 0} \frac{\mathcal{Q}_n}{[n]_{\mathbb{L}}!} x^n \\
&= \sum_{n\geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \begin{bmatrix} n \\ n-r \end{bmatrix}_{\mathbb{L}} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n\geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \frac{[n]_{\mathbb{L}}!}{[n-r]_{\mathbb{L}}![r]_{\mathbb{L}}!} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n\geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{r=0}^{n-1} \frac{\mathcal{Q}_r x^r}{[r]_{\mathbb{L}}!} \frac{x^{n-r}}{[n-r]_{\mathbb{L}}!} \right) \\
&= \sum_{n\geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{n\geq 0} \sum_{r=0}^{n} \frac{\mathcal{Q}_r x^{n-r}}{[r]_{\mathbb{L}}![n-r]_{\mathbb{L}}!} + \sum_{n\geq 0} \frac{\mathcal{Q}_n x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n\geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \exp_{\mathbb{L}}(x) G(x) + G(x).
\end{aligned}$$

Hence,

$$G(x) = \frac{\sum_{n\geq 0} \mathbb{L}^{\binom{n+1}{2}} x^n / [n]_{\mathbb{L}}!}{\exp_{\mathbb{L}}(x)}.$$

Since

$$[n]_{\mathbb{L}}! = \frac{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^n)}{(1-\mathbb{L})^n},$$

we have

$$\begin{aligned}\sum_{n\geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} &= \sum_{n\geq 0} \frac{\mathbb{L}^{\binom{n+1}{2}}(1-\mathbb{L})^n x^n}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^n)} \\ &= \prod_{i\geq 1}(1+(1-\mathbb{L})x\mathbb{L}^i),\end{aligned}$$

where the second equality follows from [12, Prop. 1.8.6]. This completes the proof. ☐

It will be convenient to make the change of variables $g(x) = G(\frac{x}{1-\mathbb{L}})$.

PROPOSITION 3.4. *We have*

$$\begin{aligned}g(x) &= (1-x)\prod_{i\geq 1}(1-x^2\mathbb{L}^{2i}) \\ &= (1-x)\sum_{k\geq 0} \frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}.\end{aligned}$$

Proof. We compute

$$\begin{aligned}\exp_{\mathbb{L}}(x) &= \sum_{n\geq 0}\frac{x^n}{[n]_{\mathbb{L}}!} \\ &= \sum_{n\geq 0}\frac{x^n(1-\mathbb{L})^n}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^n)} \\ &= \frac{1}{\prod_{i\geq 0}(1-(1-\mathbb{L})x\mathbb{L}^i)},\end{aligned}$$

where the last equality is via [12, p. 74]. The first assertion now follows from Proposition 3.3. The second follows from the first by [12, Prop. 1.8.6]. ☐

Similarly, make the change of variables $p_{\text{even}}(x) = P_{\text{even}}(\frac{x}{1-\mathbb{L}})$ and $p_{\text{odd}}(x) = P_{\text{odd}}(\frac{x}{1-\mathbb{L}})$.

PROPOSITION 3.5. *We have*

$$p_{\text{even}}(x) = \sum_{k\geq 0}\frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}$$

and

$$p_{\text{odd}}(x) = \sum_{k\geq 0}\frac{(-1)^{k+1} x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}.$$

Proof. The generating function P_{even} can be rewritten as

$$P_{\text{even}}(x) = \sum_{k\geq 0} \frac{(1-\mathbb{L})^{2k} x^{2k}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})} \prod_{i=1}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}).$$

Then we have

$$\begin{aligned} p_{\text{even}}(x) &= \sum_{k\geq 0} \frac{x^{2k}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})} \prod_{i=1}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}) \\ &= \sum_{k\geq 0} \frac{x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})} \prod_{i=1}^{k} (\mathbb{L}^{2(k-i)+1} - 1) \\ &= \sum_{k\geq 0} \frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}. \end{aligned}$$

The calculation for p_{odd} is similar. □

COROLLARY 3.6. *The following identity holds in* $\hat{K}_0(\mathrm{Var}_k)$:

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x).$$

Proof. Since $(1-\mathbb{L})$ is a unit in $\hat{K}_0(\mathrm{Var}_k)$, it suffices to show that

$$g(x) = p_{\text{even}}(x) + p_{\text{odd}}(x).$$

This follows from Propositions 3.4 and 3.5. □

3.3. *The Main Theorem*

We now state the main result.

THEOREM 3.7. *Let k be a field whose characteristic is not 2, and let $n \geq 1$. For any nondegenerate quadratic form Q on k^n, the following equality holds in* $\hat{K}_0(\mathrm{Var}_k)$:

$$[BO(Q)] = \begin{cases} \mathbb{L}^{-r} \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1} & \text{if } n = 2r+1, \\ \mathbb{L}^{r} \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1} & \text{if } n = 2r. \end{cases}$$

Proof. The subvariety $Quad_{n,n} \subset Quad_n$ is stable under the action of GL_n on $Quad_n$. Pick $Q \in Quad_{n,n}$. This gives rise to an orbit morphism $GL_n \to Quad_{n,n}$. Since $\pi : GL_n \to GL_n/O(Q)$ is a uniform categorical quotient [10, Thm. 1.1], the orbit morphism factors through a unique morphism $\psi : GL_n/O(Q) \to Quad_{n,n}$. We claim that ψ is an isomorphism.

Let $\overline{k}$ be an algebraic closure of k. Base change gives a morphism

$$\overline{\pi} : GL_{n,\overline{k}} \to GL_n/O(Q) \times_k \overline{k},$$

which is a categorical quotient for the action of $O(Q)_{\overline{k}}$ on $GL_{n,\overline{k}}$. Here $GL_{n,\overline{k}}$ denotes the general linear group over $\overline{k}$, whereas $O(Q)_{\overline{k}}$ denotes orthogonal group of the quadratic form $Q \times_k \overline{k}$ on $\overline{k}^n$. The universal property of categorical quotients implies

$$GL_n/O(Q) \times_k \overline{k} \simeq GL_{n,\overline{k}}/O(Q)_{\overline{k}}.$$

Using this isomorphism and applying base change to ψ give

$$\overline{\psi} : GL_{n,\overline{k}}/O(Q)_{\overline{k}} \to Quad_{n,n} \times_k \overline{k}.$$

Since $Quad_{n,n} \times_k \overline{k}$ is homogeneous under the action of $GL_{n,\overline{k}}$ with stabilizer $O(Q)_{\overline{k}}$, the map $\overline{\psi}$ is an isomorphism. By faithfully flat descent it follows that ψ itself is an isomorphism.

Identifying $BO(Q)$ with the quotient stack $\operatorname{Spec} k/O(Q)$, equation (1) gives

$$[BO(Q)] = \left[\frac{GL_n/O(Q)}{GL_n}\right] = \frac{[GL_n/O(Q)]}{[GL_n]} = \frac{[Quad_{n,n}]}{[GL_n]}.$$

Using Corollary 3.6, we read off from P_{even} and P_{odd} the equality

$$[Quad_{n,n}] = \begin{cases} \prod_{i=0}^{r}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i}) & \text{if } n = 2r+1, \\ \prod_{i=1}^{r}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i}) & \text{if } n = 2r. \end{cases}$$

If $n = 2r+1$, then we have

$$\begin{aligned} \frac{[Quad_{2r+1,2r+1}]}{[GL_{2r+1}]} &= \frac{\prod_{i=0}^{r}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i})}{\prod_{i=0}^{2r}(\mathbb{L}^{2r+1} - \mathbb{L}^{i})} \\ &= \prod_{i=0}^{r-1}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i+1})^{-1} \\ &= \mathbb{L}^{-r}\prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1}, \end{aligned}$$

which is the desired result. The calculation for n even is analogous. □

COROLLARY 3.8. *Suppose that $n \geq 3$ is odd and let Q be a nondegenerate quadratic form on k^n. Then $[BO(Q)] = [SO_n]^{-1}$. Moreover, $[BSO(Q)] = [SO_n]^{-1}$.*

Proof. Since $n \geq 3$, the split group SO_n is semisimple. According to [1, Lemma 2.1],

$$[SO_{2r+1}] = \mathbb{L}^r \prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i}).$$

Comparing this expression with Theorem 3.7 gives the first statement. Continuing, if Q is a nondegenerate quadratic form in odd dimensions, then there is an isomorphism $O(Q) \simeq \mu_2 \times SO(Q)$. It is shown in [5, Prop. 3.2] that $[B\mu_2] = 1$. Hence,

$$[BO(Q)] = [B\mu_2 \times BSO(Q)] = [B\mu_2][BSO(Q)] = [BSO(Q)].$$

The second statement now follows from the first. □

Since $PGL_2 \simeq SO_3$ over any field, Corollary 3.8 recovers the first part of [3, Thm. A] as a special case.

It follows from Corollary 3.8 that the universal torsor relations are satisfied for split special orthogonal groups in odd dimensions. In particular, the universal $SO_{2n+1}(\mathbb{C})$-torsor relation holds. In [4, Thm. 2.2] it is shown that for any non-special connected reductive complex algebraic group G, there exists a G-torsor $P \to X$ over a variety such that $[P]$ is not equal to $[G][X]$. Therefore, the universal G-torsor relation does not imply the general G-torsor relation, answering a question posed in [1, Rem. 3.3]. In the recent paper [3] the groups $PGL_2(\mathbb{C})$ and $PGL_3(\mathbb{C})$ were also shown to answer this question.

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