Michigan Math. J. 65 (2016), 57–87

F-pure Thresholds of Homogeneous Polynomials

Daniel J. Hernández, Luis Núñez-Betancourt, Emily E. Witt, & Wenliang Zhang

Abstract. We characterize F-pure thresholds of polynomials that are homogeneous under some $\mathbb{N}$-grading and have an isolated singularity at the origin. Our description places rigid restrictions on these invariants and allows us to produce finite lists of possible values of such F-pure thresholds; these lists are often minimal, and in specific examples, may even allow us to exactly determine the value of the F-pure threshold in question. The result, when combined with other techniques, sheds further light on the relationship between F-pure and log canonical thresholds in our setting. We compute uniform bounds for the difference between F-pure and log canonical thresholds established by Mustaţă and the fourth author and examine the set of primes for which the F-pure and log canonical threshold of a polynomial must differ. Moreover, we establish a specific subcase of the ACC conjecture for F-pure thresholds and provide further supporting evidence for this conjecture.

1. Introduction

The F-pure threshold, first defined in [TW04, Def. 2.1], is a numerical invariant of singularities in positive characteristic defined via the Frobenius (or p-th power) endomorphism; though they can be defined more generally, we will only consider F-pure thresholds of polynomials over fields of prime characteristic and thus follow the treatment given in [MTW05]. The F-pure threshold of such a polynomial f, denoted $\operatorname{fpt}(f)$, is always a rational number in $(0, 1]$, with smaller values corresponding to "worse" singularities [BMS08; BMS09; B+09].

The log canonical threshold of a polynomial $f_{\mathbb{Q}}$ over $\mathbb{Q}$, denoted $\operatorname{lct}(f_{\mathbb{Q}})$, is also a numerical invariant measuring the singularities of $f_{\mathbb{Q}}$ and can be defined via integrability conditions, or, more generally, via resolution of singularities; like the F-pure threshold, $\operatorname{lct}(f_{\mathbb{Q}})$ is a rational number in $(0, 1]$; see [BL04] for more on this and on related invariants. The connections between F-pure and log canonical thresholds run deep: Since any $\frac{a}{b} \in \mathbb{Q}$ determines a well-defined element of

Received July 2, 2014. Revision received September 14, 2015.

The first author gratefully acknowledges support from the Ford Foundation (FF) through an FF Postdoctoral Fellowship. The second author thanks the National Council of Science and Technology (CONACyT) of Mexico for support through Grant #210916. The fourth author was partially supported by NSF grants DMS #1247354 and DMS #1068946, and a Nebraska EPSCoR First Award. This collaboration began during visits supported by a travel grant from the AMS Mathematical Research Communities 2010 Commutative Algebra program.

$\mathbb{F}_p$ whenever b is not a multiple of p, we can reduce the coefficients of $f_{\mathbb{Q}}$ modulo $p \gg 0$ to obtain a family of prime characteristic models f_p over $\mathbb{F}_p$, and in this setting, the various thresholds are related as follows (see, e.g., [MTW05, Thms. 3.3 and 3.4]):

$$\operatorname{fpt}(f_p) \le \operatorname{lct}(f_{\mathbb{Q}}) \quad \text{for all } p \gg 0 \text{ and } \lim_{p\to\infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f_{\mathbb{Q}}). \tag{1.1}$$

In this article, we will not need to refer to the definition of $\operatorname{lct}(f_{\mathbb{Q}})$ via resolutions of singularities and will instead take the limit appearing in (1.1) as our working definition. We note that the relations in (1.1) are just two of many deep connections between invariants of characteristic p models defined via the Frobenius endomorphism and invariants of the original characteristic zero object that are often defined via resolution of singularities. For more in this direction, see, for example, [Smi97; Har98; Smi00; HW02; HY03; Tak04; MTW05; BMS06; Sch07; STZ12; BST15].

Motivated by the behavior exhibited when $f_{\mathbb{Q}}$ defines an elliptic curve, it is conjectured that for any polynomial $f_{\mathbb{Q}}$ over $\mathbb{Q}$, there exist infinitely many primes p for which $\operatorname{fpt}(f_p)$ equals $\operatorname{lct}(f_{\mathbb{Q}})$ [MTW05, Conj. 3.6]. This conjecture, along with other characteristic zero considerations, has fueled interests in understanding various properties of $\operatorname{fpt}(f_p)$. In particular, arithmetic properties of the denominator of $\operatorname{fpt}(f_p)$ have recently been investigated, most notably by Schwede (e.g., see [Sch08, Sect. 5]), who also asked the following *Question:*[1] Assuming $\operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})$, must the denominator of $\operatorname{fpt}(f_p)$ be a multiple of p?

In this paper, we study F-pure thresholds of homogeneous[2] polynomials with an isolated singularity at the origin. In the context of F-purity, such polynomials were originally investigated by Fedder (e.g., see [Fed83, Lemma 2.3 and Thm. 2.5]), and more recently by Bhatt and Singh, who showed that if f is a (standard-graded) homogeneous polynomial $f_{\mathbb{Q}}$ over $\mathbb{Q}$ of degree n in n variables with an isolated singularity at the origin, then $\operatorname{fpt}(f_p) = 1 - \frac{A}{p}$ for some integer $0 \le A \le n-2$. They also showed that if $f_{\mathbb{Q}}$ is (standard-graded) homogeneous of arbitrary degree with an isolated singularity at the origin such that $\operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})$, then the denominator of $\operatorname{fpt}(f_p)$ is a power of p [BS15, Thm. 1.1 and Prop. 5.4].

1.1. Main Results

Our main result, Theorem 3.5, characterizes F-pure thresholds of (not necessarily standard-graded) homogeneous polynomials with an isolated singularity at the origin. Our description places rigid restrictions on these invariants and allows us to produce finite lists of possible values of such F-pure thresholds; these lists are often minimal and, in specific examples, may even allow us to exactly determine the value of the F-pure threshold in question. For examples of these phenomena, see Examples 4.7, 4.8, and 4.10. In establishing our main result, our techniques

[1] This question, and others, may be found at the following site: ⟨https://sites.google.com/site/computingfinvariantsworkshop/open-questions⟩.

[2] We emphasize that we allow for possibly non-standard $\mathbb{N}$-gradings, where $0 \in \mathbb{N}$ by convention.

involve extending methods used in [BS15] for finding lower bounds for F-pure thresholds to our broader setting. In doing so, we are forced to execute a technical study of F-pure thresholds through the framework of their base p expansions. To illustrate some features of our main result, we state a modified version in the two-variable case (for the slightly refined version, see Theorem 4.5; for the general result, see Theorem 3.5):

THEOREM A (cf. Theorems 4.5 and 3.5). *Suppose that* $f_{\mathbb{Q}} \in \mathbb{Q}[x, y]$ *is homogeneous under some* $\mathbb{N}$*-grading and has an isolated singularity at the origin. If* $p \nmid \deg f$ *and* $\operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})$, *then for some positive integer* L *not exceeding the order of* p *modulo* $\deg f$,

$$\operatorname{fpt}(f) = \frac{\lceil p^L \cdot \operatorname{lct}(f_{\mathbb{Q}}) \rceil - 1}{p^L}.$$

Although the situation in the higher-dimensional setting is more subtle, several interesting features of Theorem A persist. For example, in higher dimensions, we still have that the denominator of $\operatorname{fpt}(f_p)$ must be a power of p when it differs from $\operatorname{lct}(f_{\mathbb{Q}})$, and, further, we are able to give an upper bound for this power of p (at least, for $p \gg 0$). Note that this provides a positive answer to the question of Schwede discussed earlier. We refer the reader to Theorem 3.5 for a detailed description of F-pure thresholds in polynomial rings with arbitrarily many variables.

1.2. Further Results

Here, we summarize results describing *how much* F-pure and log canonical thresholds may differ and *how often* (e.g., for what primes) these invariants disagree.

The first such result concerns a theorem on uniform bounds for the difference between log canonical and F-pure thresholds proven by Mustaţă and the fourth author [MZ13, Cors. 3.5 and 4.5]. For homogeneous polynomials with an isolated singularity at the origin, we explicitly find an optimal choice of bounds, giving a new proof of the result in this setting.

THEOREM B (cf. Theorem 6.2). *Suppose that* $f_{\mathbb{Q}} \in \mathbb{Q}[x_1, \dots, x_n]$ *is homogeneous under some* $\mathbb{N}$*-grading and has an isolated singularity at the origin, write the rational number* $\operatorname{lct}(f_{\mathbb{Q}}) = \frac{a}{b}$ *in lowest terms, and let* ϕ *denote Euler's phi function. Then, if* $p \gg 0$, *either* $\operatorname{fpt}(f_p) = \operatorname{lct}(f_{\mathbb{Q}})$, *or*

$$\frac{1}{b \cdot p^{\phi(b)}} \leq \operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) \leq \frac{n-1}{p}.$$

For slightly more precise bounds, we refer the reader to Theorem 6.2.

Our next result, Proposition 6.7, concerns the set of primes for which the F-pure and log canonical threshold disagree and stands out from the preceding ones in that it applies to general polynomials, requiring neither the homogeneous nor the isolated singularity assumption. In this proposition, we give a simple criterion on the prime characteristic p such that, when satisfied, guarantees that the F-pure and log canonical threshold must differ; as we might expect, this condition depends on arithmetic properties of log canonical threshold. As an application of Proposition 6.7 and Theorem 3.5, we also construct a large class of polynomials over $\mathbb{Q}$ for which the density of the set of primes for which the F-pure and log canonical thresholds differ is larger than *any* prescribed bound between zero and one; we refer the reader to Example 6.8 for more details.

1.3. Results Related to the ACC Conjecture for F-pure Thresholds

Motivated by results in characteristic zero, it was conjectured in [BMS09, Conjecture 4.4] that the set of all F-pure thresholds of polynomials in a (fixed) polynomial ring satisfies the ascending chain condition (ACC), i.e., contains no strictly increasing sequences. In Proposition 7.3, we prove that a certain subset of F-pure thresholds satisfies ACC, relying on the description of F-pure thresholds from Theorem 3.5. Finally, as detailed in [BMS09, Remark 4.5],[3] the ACC conjecture for F-pure thresholds predicts that $\operatorname{fpt}(f) \le \operatorname{fpt}(f+h)$ whenever h is contained in a large enough power of $\mathfrak{m}$. In Proposition 7.4, we verify that this is indeed the case for homogeneous polynomials, and in our final result, Proposition 7.10, we obtain a stronger conclusion for homogeneous polynomials with an isolated singularity.

Theorem C (cf. Proposition 7.10). *Consider a polynomial $f \in \mathbb{F}_p[x_1, \dots, x_n]$ that is homogeneous under some $\mathbb{N}$-grading, and such that* $\deg f \ge \deg(x_1 \cdots x_n)$ *and* $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m} := (x_1, \dots, x_n)$. *Then, there exists an integer $N \ge 1$ such that for all $g \in \mathfrak{m}^N$*, $\operatorname{fpt}(f) = \operatorname{fpt}(f+g)$.

Notation. Throughout this article, we make the convention that $0 \in \mathbb{N}$. Moreover, p denotes a prime number and $\mathbb{F}_p$ denotes the field with p elements. For every ideal I of a ring of characteristic $p > 0$, and every $e \ge 1$, $I^{[p^e]}$ denotes the eth Frobenius power of I, the ideal generated by the set $\{g^{p^e} : g \in I\}$. Given a real number a, $\lceil a \rceil$ (respectively, $\lfloor a \rfloor$) denotes the least integer that is greater than or equal to (respectively, greatest integer less or equal to) a.

2. Basics of Base p Expansions

Definition 2.1. Given $\alpha \in (0, 1]$, there exist unique integers $\alpha^{(e)}$ for every $e \ge 1$ such that $0 \le \alpha^{(e)} \le p-1$, $\alpha = \sum_{e \ge 1} \alpha^{(e)} \cdot p^{-e}$, and such that the integers $\alpha^{(e)}$ are not all eventually zero. We call $\alpha^{(e)}$ the eth digit of α (base p), and we call the expression $\alpha = \sum_{e \ge 1} \alpha^{(e)} \cdot p^{-e}$ the non-terminating expansion of α (base p).

[3] The inequality $\operatorname{fpt}(f) \ge \operatorname{fpt}(f+h)$ appearing in [BMS09, Remark 4.5] is a typo (the inequality should be reversed).

Definition 2.2. Let $\alpha \in (0, 1]$, and fix $e \geq 1$. We call $\langle \alpha \rangle_e := \alpha^{(1)} \cdot p^{-1} + \cdots + \alpha^{(e)} \cdot p^{-e}$ the eth truncation of α (base p). We adopt the convention that $\langle \alpha \rangle_0 = 0$ and $\langle \alpha \rangle_\infty = \alpha$.

Notation 2.3. We adopt notation analogous to the standard decimal notation, using ":" to distinguish between consecutive digits. For example, we often write

$$\langle \alpha \rangle_e = .\,\alpha^{(1)} : \alpha^{(2)} : \cdots : \alpha^{(e)} \text{ (base } p).$$

Convention 2.4. Given a natural number $b > 0$ and an integer m, $[\![m \,\%\, b]\!]$ denotes the least *positive* residue of m modulo b. In particular, we have that $1 \leq [\![m \,\%\, b]\!] \leq b$ for all $m \in \mathbb{Z}$. Moreover, if p and b are relatively prime, $\operatorname{ord}(p, b) = \min\{k \geq 1 : [\![p^k \,\%\, b]\!] = 1\}$, which we call the *order of p modulo b*. In particular, note that $\operatorname{ord}(p, 1) = 1$.

Lemma 2.5. *Fix $\lambda \in (0, 1] \cap \mathbb{Q}$. If we write $\lambda = \frac{a}{b}$, not necessarily in lowest terms, then*

$$\lambda^{(e)} = \frac{[\![ap^{e-1} \,\%\, b]\!] \cdot p - [\![ap^e \,\%\, b]\!]}{b} \quad \textit{and} \quad \langle \lambda \rangle_e = \lambda - \frac{[\![ap^e \,\%\, b]\!]}{bp^e}.$$

Note that it is important to keep in mind Convention 2.4 when interpreting these identities.

Proof. Since $\lambda^{(e)} = p^e(\langle \lambda \rangle_e - \langle \lambda \rangle_{e-1})$, the first identity follows from the second. Setting $\delta = \lambda - \langle \lambda \rangle_e$ and multiplying both sides of the equality $\frac{a}{b} = \lambda = \langle \lambda \rangle_e + \delta$ by bp^e shows that $ap^e = bp^e \langle \lambda \rangle_e + bp^e \delta$. As $0 < \delta \leq p^{-e}$ and $p^e \langle \lambda \rangle_e \in \mathbb{N}$, it follows that $bp^e \delta$ is the least *positive* residue of ap^e modulo b. Finally, substituting $\delta = \lambda - \langle \lambda \rangle_e$ into $bp^e \delta = [\![ap^e \,\%\, b]\!]$ establishes the second identity. $\square$

We gather some of the important basic properties of base p expansions below.

Lemma 2.6. *Fix α and β in* $[0, 1]$.

(1) *$\alpha \leq \beta$ if and only if $\langle \alpha \rangle_e \leq \langle \beta \rangle_e$ for all $e \geq 1$; if $\alpha < \beta$, then these inequalities are strict for $e \gg 0$.*
(2) *If $(p^s - 1) \cdot \alpha \in \mathbb{N}$, then the base p expansion of α is periodic, with period dividing s. In particular, if $\lambda = \frac{a}{b}$ with $p \nmid b$, then the base p expansion of λ is periodic with period equal to* $\operatorname{ord}(p, b)$.
(3) *Suppose that $\lambda = \frac{a}{b}$ with $p \nmid b$. If $s = \operatorname{ord}(p, b)$, then for all $k \geq 1$, $p^{ks} \cdot \langle \lambda \rangle_{ks} = (p^{ks} - 1) \cdot \lambda$.*

Proof. (1) follows by definition; (2) follows immediately from Lemma 2.5; (3) follows from (2). $\square$

Lemma 2.7. *Consider $\alpha < \beta$ in $(0, 1]$, and set $\Delta_e := p^e \langle \beta \rangle_e - p^e \langle \alpha \rangle_e$. Note that, by Lemma 2.6, the integer $\ell = \min\{e : \Delta_e \geq 1\}$ is well-defined. Moreover, the following hold*:

(1) *The sequence $\{\Delta_e\}_{e \geq 1}$ is non-negative, non-decreasing, and unbounded.*

(2) *Suppose* $\beta = \frac{a}{b}$ *with* $p \nmid b$. *If* $s = \operatorname{ord}(p, b)$, *then* $\Delta_{\ell+s+k} \geq p^k + 1$ *for every* $k \geq 0$.

Proof. We first observe that the following recursion holds.

$$\Delta_{e+1} = p \cdot \Delta_e + \beta^{(e+1)} - \alpha^{(e+1)} \quad \text{for every } e \geq 0. \tag{2.1}$$

Setting $e = \ell$ in (2.1) and noting that $\Delta_\ell \geq 1$ shows that

$$\begin{aligned}\Delta_{\ell+1} &= p \cdot \Delta_\ell + \beta^{(\ell+1)} - \alpha^{(\ell+1)} \\ &= (p-1) \cdot \Delta_\ell + \Delta_\ell + \beta^{(\ell+1)} - \alpha^{(\ell+1)} \\ &\geq (p-1) \cdot 1 + \Delta_\ell + \beta^{(\ell+1)} - \alpha^{(\ell+1)} \\ &\geq \Delta_\ell + \beta^{(\ell+1)}.\end{aligned}$$

Furthermore, an induction on $e \geq \ell$ shows that

$$\Delta_{e+1} \geq \Delta_e + \beta^{(e+1)} \quad \text{for every } e \geq \ell. \tag{2.2}$$

Thus, $\{\Delta_e\}_{e \geq 1}$ is non-decreasing, and as we consider non-terminating expansions, $\beta^{(e)} \neq 0$ for infinitely many e, so that (2.2) also shows that $\Delta_{e+1} > \Delta_e$ for infinitely many e. We conclude that $\{\Delta_e\}_{e \geq 1}$ is unbounded, and it remains to establish (2).

By definition, $\beta^{(\ell)} - \alpha^{(\ell)} = \Delta_\ell \geq 1$, and hence $\beta^{(\ell)} \geq 1$. In fact, setting $s = \operatorname{ord}(p, b)$, Lemma 2.6 states that $\beta^{(\ell+s)} = \beta^{(\ell)} \geq 1$, and applying (2.2) with $e = \ell + s - 1$ then shows that

$$\Delta_{\ell+s} \geq \Delta_{\ell+s-1} + \beta^{(\ell+s)} \geq 2.$$

Hence, (2) holds for $k = 0$. Utilizing (2.1), an induction on k completes the proof. $\square$

3. F-pure Thresholds of Homogeneous Polynomials: A Discussion

We adopt the following convention from this point onward.

CONVENTION 3.1. Throughout this article, $\mathbb{L}$ will denote a field of characteristic $p > 0$, and $\mathfrak{m}$ will denote the ideal generated by the variables in $R = \mathbb{L}[x_1, \ldots, x_n]$. Moreover, any fixed $\mathbb{N}$-grading of R will satisfy $[R]_0 = \mathbb{L}$.

DEFINITION 3.2. Consider a polynomial $f \in \mathfrak{m}$, and for every $e \geq 1$, set

$$\nu_f(p^e) = \max\{N : f^N \notin \mathfrak{m}^{[p^e]}\}.$$

An important property of these integers is that $\{p^{-e} \cdot \nu_f(p^e)\}_{e \geq 1}$ is a non-decreasing sequence contained in the open unit interval [MTW05, Lemma 1.1 and Remark 1.2]. Consequently, the limit

$$\operatorname{fpt}(f) := \lim_{e \to \infty} \frac{\nu_f(p^e)}{p^e} \in (0, 1]$$

exists, and is called the *F-pure threshold of f*.

The following illustrates important properties of F-pure thresholds; we refer the reader to [MTW05, Prop. 1.9] or [Her12, Key Lemma 3.1] for a proof of the first, and [Her12, Cor. 4.1] for a proof of the second.

PROPOSITION 3.3. *Consider a polynomial f contained in $\mathfrak{m}$.*

(1) *The base p expansion of the F-pure threshold determines $\{\nu_f(p^e)\}_{e\geq 1}$; more precisely,*
$$\nu_f(p^e) = p^e \cdot \langle \operatorname{fpt}(f)\rangle_e \quad \textit{for every } e \geq 1.$$
(2) *The F-pure threshold is bounded above by the rational numbers determined by its trailing digits (base p); more precisely,* $\operatorname{fpt}(f)$ *is less than or equal to*
$$.\operatorname{fpt}(f)^{(s)} : \operatorname{fpt}(f)^{(s+1)} : \cdots : \operatorname{fpt}(f)^{(s+k)} : \cdots \ (\text{base } p) \quad \textit{for every } s \geq 1.$$

3.1. A Discussion of the Main Results

In this subsection, we gather the main results of this article. Note that the proofs of these results appear in Section 5.

CONVENTION 3.4. Given a polynomial f, we use $\operatorname{Jac}(f)$ to denote the ideal of R generated by the partial derivatives of f. If f is homogeneous under some $\mathbb{N}$-grading on R, each partial derivative $\partial_i(f)$ of f is also homogeneous, and if $\partial_i(f) \neq 0$, then $\deg \partial_i(f) = \deg f - \deg x_i$. Furthermore, if $p \nmid \deg f$, then Euler's relation
$$\deg f \cdot f = \sum \deg x_i \cdot x_i \cdot \partial_i(f)$$
shows that $f \in \operatorname{Jac}(f)$. Thus, if $p \nmid \deg(f)$ and $\mathbb{L}$ is perfect, the Jacobian criterion states that $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$ if and only if f has an isolated singularity at the origin.

THEOREM 3.5. *Fix an $\mathbb{N}$-grading on $\mathbb{L}[x_1, \dots, x_n]$. Consider a homogeneous polynomial f with $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$, and write $\lambda := \min\{\sum \deg x_i / \deg f, 1\} = \frac{a}{b}$ in lowest terms.*

(1) *If* $\operatorname{fpt}(f) \neq \lambda$, *then*
$$\operatorname{fpt}(f) = \lambda - \left(\frac{[\![ap^L \,\%\, b]\!] + bE}{bp^L}\right) = \langle\lambda\rangle_L - \frac{E}{p^L}$$
for some pair $(L, E) \in \mathbb{N}^2$ with $L \geq 1$ and $0 \leq E \leq n - 1 - \lceil([\![ap^L \,\%\, b]\!] + a)/b\rceil$.
(2) *If $p > (n-2) \cdot b$ and $p \nmid b$, then $1 \leq L \leq \operatorname{ord}(p, b)$; note that $\operatorname{ord}(p, 1) = 1$.*
(3) *If $p > (n-2) \cdot b$ and $p > b$, then $a < [\![ap^e \,\%\, b]\!]$ for all $1 \leq e \leq L - 1$.*
(4) *If $p > (n-1) \cdot b$, then there exists a unique pair (L, E) satisfying the conclusions of (1).*

REMARK 3.6 (bounds on E when $n = 2$). In the context of Theorem 3.5, it is easy to see that
$$1 \leq \left\lceil \frac{[\![ap^L \,\%\, b]\!] + a}{b} \right\rceil \leq 2.$$

If $n = 2$, and a and b are such that the rounded value above equals two, then the bounds on E in the theorem become $0 \leq E \leq -1$, which is impossible. In this case, the theorem allows us to conclude that $\operatorname{fpt}(f) = \lambda$. This is just one of several special consequences of the theorem when $n = 2$. For more, see Section 4.1.

We postpone the proof of Theorem 3.5 to Section 5.2. The remainder of this subsection is focused on parsing the statement of Theorem 3.5, and presenting some related results. The reader interested in seeing examples should consult Section 4.

REMARK 3.7 (two points of view). Each of the two descriptions of $\operatorname{fpt}(f)$ in Theorem 3.5, which are equivalent by Lemma 2.5, are useful in their own right. For example, the first description plays a key role in Section 4. On the other hand, the second description makes it clear that either $\operatorname{fpt}(f) = \lambda$, or $\operatorname{fpt}(f)$ is a rational number whose denominator is a power of p, and further, describes how "far" $\operatorname{fpt}(f)$ is from being a truncation of λ; these observations allow us to address the questions of Schwede and of the first author noted in the Introduction.

The second point of Theorem 3.5 also immediately gives a bound on the power of p appearing in the denominator of $\operatorname{fpt}(f)$ whenever $\operatorname{fpt}(f) \neq \lambda$ and $p \gg 0$. For emphasis, we record this bound in the following corollary.

COROLLARY 3.8. *In the context of Theorem 3.5, if* $\operatorname{fpt}(f) \neq \lambda$, *and both* $p > (n-2) \cdot b$ *and* $p \nmid b$, *then* $p^{\operatorname{ord}(p,b)} \cdot \operatorname{fpt}(f) \in \mathbb{N}$. *In particular, for all such primes,* $p^{\phi(b)} \cdot \operatorname{fpt}(f) \in \mathbb{N}$, *where* ϕ *denotes Euler's phi function.*

Using the techniques of the proof of Theorem 3.5, we can analogously find a bound for the power of p appearing in the denominator of $\operatorname{fpt}(f)$ whenever $\operatorname{fpt}(f) \neq \lambda$ and p is not large, which we record here.

COROLLARY 3.9. *In the setting of Theorem 3.5, if* $\operatorname{fpt}(f) \neq \lambda$ *and* $p \nmid b$, *then* $p^M \cdot \operatorname{fpt}(f) \in \mathbb{N}$, *where* $M := 2 \cdot \phi(b) + \lceil \log_2(n-1) \rceil$, *and* ϕ *denotes Euler's phi function.*

REMARK 3.10. We emphasize that the constant M in Corollary 3.9 depends only on the number of variables n and the quotient $\sum \deg(x_i)/\deg f = \frac{a}{b}$, but not on the particular values of $\deg x_i$ and $\deg f$; this subtle point will play a key role in Proposition 7.3.

REMARK 3.11 (towards minimal lists). For $p \gg 0$, the bounds for L and E appearing in Theorem 3.5 allows one to produce a finite list of possible values of $\operatorname{fpt}(f)$ for each class of p modulo $\deg f$. We refer the reader to Section 4 for details and examples.

The uniqueness statement in point (4) of the Theorem 3.5 need not hold in general.

EXAMPLE 3.12 (nonuniqueness in low characteristic). If $p = 2$ and $f \in \mathbb{L}[x_1, x_2, x_3]$ is any $\mathbb{L}^*$-linear combination of x_1^7, x_2^7, x_3^7, then f satisfies the hypotheses of

Theorem 3.5, under the standard grading. Using [Her15, Corollary 3.5], one can directly compute that $\operatorname{fpt}(f) = \frac{1}{4}$. On the other hand, the identities

$$\begin{aligned}\frac{1}{4} &= \frac{3}{7} - \left(\frac{[\![3 \cdot 2^2 \ \% \ 7]\!] + 7 \cdot 0}{7 \cdot 2^2}\right) = \left\langle \frac{3}{7} \right\rangle_2 \\ &= \frac{3}{7} - \left(\frac{[\![3 \cdot 2^3 \ \% \ 7]\!] + 7 \cdot 1}{7 \cdot 2^3}\right) = \left\langle \frac{3}{7} \right\rangle_3 - \frac{1}{2^3}\end{aligned}$$

show that the pairs $(L, E) = (2, 0)$ and $(L, E) = (3, 1)$ both satisfy the conclusions in Theorem 3.5. We point out that the proof of Theorem 3.5, being somewhat constructive, predicts the choice of $(L, E) = (2, 0)$, but does not "detect" the choice of $(L, E) = (3, 1)$.

Before concluding this section, we present the following related result; like Theorem 3.5 and Corollary 3.9, its proof relies heavily on Proposition 5.6. However, in contrast to these results, its focus is on showing that $\operatorname{fpt}(f) = \min\{(\sum \deg x_i)/\deg f, 1\}$ for $p \gg 0$ in a very specific setting, as opposed to describing $\operatorname{fpt}(f)$ when it differs from this value.

THEOREM 3.13. *In the context of Theorem 3.5, suppose that* $\sum \deg x_i > \deg f$, *so that* $\rho := \sum \deg x_i / \deg f$ *is greater than* 1. *If* $p > \frac{n-3}{\rho - 1}$, *then* $\operatorname{fpt}(f) = 1$.

As we see below, Theorem 3.13 need not hold in low characteristic.

EXAMPLE 3.14 (illustrating the necessity of $p \gg 0$ in Theorem 3.13). Set $f = x_1^d + \cdots + x_n^d$. If $n > d > p$, then $f \in \mathfrak{m}^{[p]}$, and hence $f^{p^{e-1}} \in \mathfrak{m}^{[p^e]}$ for all $e \geq 1$. Consequently, $\nu_f(p^e) \leq p^{e-1} - 1$, and therefore $\operatorname{fpt}(f) = \lim_{e \to \infty} p^{-e} \cdot \nu_f(p^e) \leq p^{-1}$.

4. *F*-pure Thresholds of Homogeneous Polynomials: Examples

In this section, we illustrate, via examples, how Theorem 3.5 may be used to produce "short", or even minimal, lists of possible values for F-pure thresholds. We begin with the most transparent case: If $\deg f = \sum \deg x_i$, then the statements in Theorem 3.5 become less technical. Indeed, in this case, $a = b = 1$, and hence $\operatorname{ord}(p, b) = 1 = [\![m \ \% \ b]\!]$ for every $m \in \mathbb{N}$. In this context, substituting these values into Theorem 3.5 recovers the following identity, originally discovered by Bhatt and Singh under the standard grading.

EXAMPLE 4.1 ([BS15, Thm. 1.1]). Fix an $\mathbb{N}$-grading on $\mathbb{L}[x_1, \ldots, x_n]$. Consider a homogeneous polynomial f with $d := \deg f = \sum \deg x_i$ and $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$. If $p > n - 2$ and $\operatorname{fpt}(f) \neq 1$, then

$$\operatorname{fpt}(f) = 1 - A \cdot p^{-1} \quad \text{for some integer } 1 \leq A \leq d - 2.$$

Next, we consider the situation where $\deg f = \sum \deg x_i + 1$; already, we see that this minor modification leads to a more complex statement.

COROLLARY 4.2. *Fix an* $\mathbb{N}$*-grading on* $\mathbb{L}[x_1, \ldots, x_n]$. *Consider a homogeneous polynomial* f *with* $d := \deg f = \sum \deg x_i + 1$ *and* $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$.

(1) *If* $\operatorname{fpt}(f) = 1 - \frac{1}{d}$, *then* $p \equiv 1 \bmod d$.

(2) *Suppose* $p > (n-2) \cdot d$ *and* $p > d$. *If* $\operatorname{fpt}(f) \neq 1 - \frac{1}{d}$, *then*
$$\operatorname{fpt}(f) = 1 - \frac{1}{d} - \left(A - \frac{[\![p \,\%\, d]\!]}{d}\right) \cdot p^{-1}$$
for some integer A *satisfying*
   (a) $1 \leq A \leq n-1$ *if* $p \equiv -1 \bmod d$, *and*
   (b) $1 \leq A \leq n-2$ *otherwise.*

Proof. We begin with (1): Lemma 2.5 implies that $(d^{-1})^{(1)} \leq (d^{-1})^{(s)}$ for $s \geq 1$ and hence that
$$(1 - d^{-1})^{(1)} = p - 1 - (d^{-1})^{(1)} \geq p - 1 - (d^{-1})^{(s)} = (1 - d^{-1})^{(s)} \tag{4.1}$$
for every $s \geq 1$. However, if $\operatorname{fpt}(f) = 1 - d^{-1}$, then Proposition 3.3 implies that $(1 - d^{-1})^{(1)} \leq (1 - d^{-1})^{(s)}$ for every $s \geq 1$. Consequently, equality holds throughout (4.1), and hence $(d^{-1})^{(1)} = (d^{-1})^{(s)}$ for every $s \geq 1$, which by Lemma 2.5 occurs if and only if $p \equiv 1 \bmod d$.

We now address the second point: In this setting, Theorem 3.5 states that $\operatorname{fpt}(f) \in p^{-L} \cdot \mathbb{N}$ for some integer $L \geq 1$. We will first show that L must equal one: Indeed, otherwise $L \geq 2$, which allows us to set $e = 1$ in the third point Theorem 3.5 to deduce that
$$d - 1 < [\![p(d-1) \,\%\, d]\!] = d - [\![p \,\%\, d]\!],$$
and hence that $[\![p \,\%\, d]\!] < 1$, which is impossible since $[\![p \,\%\, d]\!]$ is always a positive integer. We conclude that $L = 1$, and the reader may verify that substituting
$$L = 1, \qquad [\![p(d-1) \,\%\, d]\!] = d - [\![p \,\%\, d]\!], \quad \text{and} \quad A = E + 1$$
into Theorem 3.5 produces the desired description of $\operatorname{fpt}(f)$. □

REMARK 4.3 (a consequence of Corollary 4.2 when $n = 2$). If $n = 2$ and f satisfies the conditions in Corollary 4.2, then the bounds on A when $p \not\equiv -1 \bmod d$ become $1 \leq A \leq 0$, which is impossible. In this case, we conclude that $\operatorname{fpt}(f) = 1 - \frac{1}{d}$. This has the following interesting consequence: If $p \neq 2$ and f is homogeneous under the standard grading with $\deg(f) = 3$, then $\operatorname{fpt}(f) = \frac{2}{3}$ if $p \equiv 1 \bmod 3$, and $\operatorname{fpt}(f) = \frac{2}{3} - \frac{1}{3p}$ if $p \equiv -1 \bmod 3$.

4.1. The Two-Variable Case

We now shift our focus to the two-variable case of Theorem 3.5, motivated by the following example.

EXAMPLE 4.4. In [Har06, Cor. 3.9], Hara and Monsky independently described the possible values of $\operatorname{fpt}(f)$ whenever f is homogeneous in two variables (under the standard grading) of degree 5 with an isolated singularity at the origin over an algebraically closed field (and hence, a product of five distinct linear forms) and $p \neq 5$; we recall their computation (the description in terms of truncations is our own):

- If $p \equiv 1 \bmod 5$, then $\operatorname{fpt}(f) = \frac{2}{5}$ or $\frac{2p-2}{5p} = \langle \frac{2}{5} \rangle_1$.
- If $p \equiv 2 \bmod 5$, then $\operatorname{fpt}(f) = \frac{2p^2-3}{5p^2} = \langle \frac{2}{5} \rangle_2$ or $\frac{2p^3-1}{5p^3} = \langle \frac{2}{5} \rangle_3$.
- If $p \equiv 3 \bmod 5$, then $\operatorname{fpt}(f) = \frac{2p-1}{5p} = \langle \frac{2}{5} \rangle_1$.
- If $p \equiv 4 \bmod 5$, then $\operatorname{fpt}(f) = \frac{2}{5}$, or $\frac{2p-3}{5p} = \langle \frac{2}{5} \rangle_1$, or $\frac{2p^2-2}{5p^2} = \langle \frac{2}{5} \rangle_2$.

The methods used in [Har06] rely on so-called "syzygy gap" techniques and the geometry of $\mathbb{P}^1$, and hence differ greatly from ours. In this example, we observe the following: First, the F-pure threshold is always $\lambda = \frac{2}{5}$ or a truncation of $\frac{2}{5}$. Second, there seem to be fewer choices for the truncation point L than might be expected, given Theorem 3.5.

In this subsection, we show that the two observations from Example 4.4 hold in general in the two-variable setting. We now work in the context of Theorem 3.5 with $n = 2$ and relabel the variables so that $f \in \mathbb{L}[x, y]$. Note that if $\deg f < \deg xy$, then $\operatorname{fpt}(f) = 1$ by Theorem 3.13 (an alternate justification: this inequality is satisfied if and only if, after possibly reordering the variables, $f = x + y^m$ for some $m \geq 1$, in which case we can directly compute that $\nu_f(p^e) = p^e - 1$ and hence that $\operatorname{fpt}(f) = 1$). Thus, the interesting case here is where $\deg f \geq \deg xy$; in this case, we obtain the following result.

Theorem 4.5 (cf. Theorem 3.5). *Fix an $\mathbb{N}$-grading on $\mathbb{L}[x, y]$. Consider a homogeneous polynomial f with $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$ and* $\deg f \geq \deg xy$. *If* $\operatorname{fpt}(f) \neq \frac{\deg xy}{\deg f} = \frac{a}{b}$, *written in lowest terms, then*

$$\operatorname{fpt}(f) = \left\langle \frac{\deg xy}{\deg f} \right\rangle_L = \frac{\deg xy}{\deg f} - \frac{[\![a p^L \,\%\, b]\!]}{b \cdot p^L}$$

for some integer L satisfying the following properties:

(1) *If $p \nmid b$, then* $1 \leq L \leq \operatorname{ord}(p, b)$.
(2) *If $p > b$, then $a < [\![a p^e \,\%\, b]\!]$ for all* $1 \leq e \leq L - 1$.
(3) $1 \leq [\![a p^L \,\%\, b]\!] \leq b - a$ *for* all *possible values of p.*

Proof. Assuming that $\operatorname{fpt}(f) \neq \frac{\deg xy}{\deg f}$, the bounds for E in Theorem 3.5 become

$$0 \leq E \leq 1 - \left\lceil \frac{[\![a p^L \,\%\, b]\!] + a}{b} \right\rceil.$$

As the rounded term is always either one or two, the inequality forces it to equal one, so that $E = 0$, which shows that $\operatorname{fpt}(f)$ is a truncation of $\frac{\deg xy}{\deg f}$. Moreover, the fact that the rounded term equals one also implies that $[\![a p^L \,\%\, b]\!] + a \leq b$. $\square$

REMARK 4.6. Though the first two points in Theorem 4.5 appear in Theorem 3.5, the third condition is special to the setting of two variables. Indeed, this extra condition will be key in eliminating potential candidate F-pure thresholds. For example, this extra condition allows us to recover the data in Example 4.4. Rather than justify this claim, we present two new examples.

EXAMPLE 4.7. Let $f \in \mathbb{L}[x, y]$ be as in Theorem 4.5, with $\frac{\deg(xy)}{\deg f} = \frac{1}{3}$. For $p \geq 5$, the following hold:

- If $p \equiv 1 \bmod 3$, then $\operatorname{fpt}(f) = \frac{1}{3}$ or $\langle \frac{1}{3} \rangle_1 = \frac{1}{3} - \frac{1}{3p}$.
- If $p \equiv 2 \bmod 3$, then $\operatorname{fpt}(f) = \frac{1}{3}$, or $\langle \frac{1}{3} \rangle_1 = \frac{1}{3} - \frac{2}{3p}$, or $\langle \frac{1}{3} \rangle_2 = \frac{1}{3} - \frac{1}{3p^2}$.

In Example 4.7, the second and third points of Theorem 4.5 were uninteresting since they did not "whittle away" any of the candidate F-pure thresholds identified by the first point of Theorem 4.5. The following example is more interesting, as we will see that both of the second and third points of Theorem 4.5, along with Proposition 3.3, will be used to eliminate potential candidates.

EXAMPLE 4.8. Let $f \in \mathbb{L}[x, y]$ be as in Theorem 4.5 with $\frac{\deg(xy)}{\deg f} = \frac{2}{7}$. For $p \geq 11$, the following hold:

- If $p \equiv 1 \bmod 7$, then $\operatorname{fpt}(f) = \frac{2}{7}$ or $\langle \frac{2}{7} \rangle_1 = \frac{2}{7} - \frac{2}{7p}$.
- If $p \equiv 2 \bmod 7$, then $\operatorname{fpt}(f) = \langle \frac{2}{7} \rangle_1 = \frac{2}{7} - \frac{4}{7p}$ or $\langle \frac{2}{7} \rangle_2 = \frac{2}{7} - \frac{1}{7p^2}$.
- If $p \equiv 3 \bmod 7$, then $\operatorname{fpt}(f) = \langle \frac{2}{7} \rangle_2 = \frac{2}{7} - \frac{4}{7p^2}$, or $\langle \frac{2}{7} \rangle_3 = \frac{2}{7} - \frac{5}{7p^3}$, or $\langle \frac{2}{7} \rangle_4 = \frac{2}{7} - \frac{1}{7p^4}$.
- If $p \equiv 4 \bmod 7$, then $\operatorname{fpt}(f) = \langle \frac{2}{7} \rangle_1 = \frac{2}{7} - \frac{1}{7p}$.
- If $p \equiv 5 \bmod 7$, then $\operatorname{fpt}(f) = \langle \frac{2}{7} \rangle_1 = \frac{2}{7} - \frac{3}{7p}$ or $\langle \frac{2}{7} \rangle_2 = \frac{2}{7} - \frac{1}{7p^2}$.
- If $p \equiv 6 \bmod 7$, then $\operatorname{fpt}(f) = \frac{2}{7}$, or $\langle \frac{2}{7} \rangle_1 = \frac{2}{7} - \frac{5}{7p}$, or $\langle \frac{2}{7} \rangle_2 = \frac{2}{7} - \frac{2}{7p^2}$.

For the sake of brevity, we only indicate how to deduce the lists for $p \equiv 3 \bmod 7$ and $p \equiv 4 \bmod 7$. Similar methods can be used for the remaining cases.

($p \equiv 3 \bmod 7$). In this case, it follows from Lemma 2.5 that $(\frac{2}{7})^{(1)} = \frac{2p-6}{7}$ and $(\frac{2}{7})^{(5)} = \frac{p-3}{7}$. In light of this, the second point of Proposition 3.3, which shows that the first digit of $\operatorname{fpt}(f)$ must be the smallest digit, implies that $\operatorname{fpt}(f) \neq \frac{2}{7}$. Thus, the first point of Theorem 4.5 states that

$$\operatorname{fpt}(f) = \left\langle \frac{2}{7} \right\rangle_L \quad \text{for some } 1 \leq L \leq \operatorname{ord}(p, 7) = 6 \text{ since } p \equiv 3 \bmod 7.$$

However, since $2 \not\leq [\![2p^4 \,\%\, 7]\!] = 1$, the second point of Theorem 4.5 eliminates the possibility that $L = 5$ or 6. Moreover, since $[\![2p \,\%\, 7]\!] = 6 \not\leq 7 - 2 = 5$, the third point of Theorem 4.5 eliminates the possibility that $L = 1$. Thus, the only remaining possibilities are $\operatorname{fpt}(f) = \langle \frac{2}{7} \rangle_2$, $\langle \frac{2}{7} \rangle_3$, and $\langle \frac{2}{7} \rangle_4$.

($p \equiv 4 \bmod 7$). As before, we compute that $(\frac{2}{7})^{(1)} = \frac{2p-1}{7}$ is greater than $(\frac{2}{7})^{(2)} = \frac{p-4}{7}$, and hence it again follows the second point of Proposition 3.3 that

$\mathrm{fpt}(f) \neq \frac{2}{7}$. Consequently, the first point of Theorem 4.5 states that

$$\mathrm{fpt}(f) = \left\langle \frac{2}{7} \right\rangle_L \quad \text{for some } 1 \leq L \leq \mathrm{ord}(p,7) = 3 \text{ since } p \equiv 4 \bmod 7.$$

However, we observe that $2 \not\leq [\![2p^2 \,\%\, 7]\!] = 1$, and hence the second point of Theorem 4.5 eliminates the possibility that $L = 2$ or 3. Thus, the only remaining option is that $\mathrm{fpt}(f) = \langle \frac{2}{7} \rangle_1$.

Remark 4.9 (Minimal lists). In many cases, we are able to verify that the "whittled down" lists obtained through the application of Theorems 4.5 and 3.5 and Proposition 3.3 is, in fact, minimal. For example, every candidate listed in Example 4.4 is of the form $\mathrm{fpt}(f)$, where f varies among the polynomials $x^5 + y^5$, $x^5 + xy^4$, and $x^5 + xy^4 + 7x^2y^3$, and p varies among the primes less than or equal to 29.

We now give an extreme example of the "minimality" of the lists of candidate thresholds. Note that, in this example, the list of candidate thresholds is so small that it actually determines the precise value of $\mathrm{fpt}(f)$ for $p \gg 0$.

Example 4.10 (F-pure thresholds are precisely determined). Let $f \in \mathbb{L}[x, y]$ be as in Theorem 4.5 with $\frac{\deg(xy)}{\deg f} = \frac{3}{5}$; for example, we may take $f = x^5 + x^3y + xy^2$, under the grading given by $(\deg x, \deg y) = (1, 2)$. Using Theorem 4.5 and Proposition 3.3 in a manner analogous to that used in Example 4.8, we obtain the following complete description of $\mathrm{fpt}(f)$ for $p \geq 7$.

- If $p \equiv 1 \bmod 5$, then $\mathrm{fpt}(f) = \frac{3}{5}$.
- If $p \equiv 2 \bmod 5$, then $\mathrm{fpt}(f) = \langle \frac{3}{5} \rangle_1 = \frac{3}{5} - \frac{1}{5p}$.
- If $p \equiv 3 \bmod 5$, then $\mathrm{fpt}(f) = \langle \frac{3}{5} \rangle_2 = \frac{3}{5} - \frac{2}{5p^2}$.
- If $p \equiv 4 \bmod 5$, then $\mathrm{fpt}(f) = \langle \frac{3}{5} \rangle_1 = \frac{3}{5} - \frac{2p}{5}$.

We conclude this section with one final example illustrating "minimality". In this instance, however, we focus on the higher-dimensional case. Although the candidate list for F-pure thresholds produced by Theorem 3.5 is more complicated (due to the possibility of having a nonzero "E" term when $n > 2$), the following example shows that we can nonetheless obtain minimal lists in these cases using methods analogous to those used in the previous examples of this section.

Example 4.11 (Minimal lists for $n \geq 3$). Let $f \in \mathbb{L}[x, y, z]$ satisfy the hypotheses of Theorem 3.5 with $\frac{\deg xyz}{\deg f} = \frac{2}{3}$. Using the bounds for E and L therein, we obtain the following for $p \geq 5$:

- If $p \equiv 1 \bmod 3$, then $\mathrm{fpt}(f) = \frac{2}{3}$ or $\langle \frac{2}{3} \rangle_1 = \frac{2}{3} - \frac{2}{3p}$.
- If $p \equiv 2 \bmod 3$, then $\mathrm{fpt}(f) = \langle \frac{2}{3} \rangle_1 = \frac{2}{3} - \frac{1}{3p}$ or $\langle \frac{2}{3} \rangle_1 - \frac{1}{p} = \frac{2}{3} - \frac{4}{3p}$.

We claim that this list is minimal. In fact, if $f = x^9 + xy^4 + z^3$, homogeneous under the grading determined by $(\deg x, \deg y, \deg z) = (1, 2, 3)$, we obtain each of these possibilities as p varies. For example, if $p = 13$, then F-pure threshold equals $\frac{2}{3}$, and if $p = 7$, then it equals $\langle \frac{2}{3} \rangle_1 = \frac{4}{7}$ (and these exhaust the possibilities for primes congruent to 1 modulo 3). On the other hand, if $p = 5$, then the F-pure threshold equals $\langle \frac{2}{3} \rangle_1 = \frac{3}{5}$, and if $p = 11$, then it equals $\langle \frac{2}{3} \rangle_1 - \frac{1}{p} = \frac{6}{11}$ (and these exhaust the possibilities for primes congruent to 2 modulo 3).

5. F-pure Thresholds of Homogeneous Polynomials: Details

Here, we prove the statements referred to in Section 3; we begin with some preliminary results.

5.1. *Bounding the Defining Terms of the F-pure Threshold*

This subsection is dedicated to deriving bounds for $\nu_f(p^e)$. Our methods for deriving lower bounds are an extension of those employed by Bhatt and Singh [BS15].

LEMMA 5.1. *If $f \in \mathbb{L}[x_1, \dots, x_n]$ is homogeneous under some $\mathbb{N}$-grading, then for every $e \geq 1$, $\nu_f(p^e) \leq \lfloor (p^e - 1) \cdot \sum \deg x_i / \deg f \rfloor$. In particular,* $\operatorname{fpt}(f) \leq \min\{\sum \deg x_i / \deg f, 1\}$.

Proof. By Definition 3.2 it suffices to establish the upper bound on $\nu_f(p^e)$. However, since $f^{\nu_f(p^e)} \notin \mathfrak{m}^{[p^e]}$, there is a supporting monomial $\mu = x_1^{a_1} \cdots x_n^{a_n}$ of $f^{\nu_f(p^e)}$ not in $\mathfrak{m}^{[p^e]}$, and comparing degrees shows that

$$\nu_f(p^e) \cdot \deg f = \deg \mu = \sum a_i \cdot \deg x_i \leq (p^e - 1) \cdot \sum \deg x_i. \qquad \square$$

COROLLARY 5.2. *Let $f \in \mathbb{L}[x_1, \dots, x_n]$ be a homogeneous polynomial under some $\mathbb{N}$-grading, and write $\lambda = \min\{\sum \deg x_i / \deg f, 1\} = \frac{a}{b}$ in lowest terms. If* $\operatorname{fpt}(f) \neq \lambda$, *then $\Delta_e := p^e \langle \lambda \rangle_e - p^e \langle \operatorname{fpt}(f) \rangle_e$ defines a nonnegative, nondecreasing, and unbounded sequence. Moreover, if $p \nmid b$, then*

$$1 \leq \min\{e : \Delta_e \neq 0\} \leq \operatorname{ord}(p, b).$$

Proof. By Lemma 5.1 the assumption that $\operatorname{fpt}(f) \neq \lambda$ implies that $\operatorname{fpt}(f) < \lambda$, so that the asserted properties of $\{\Delta_e\}_e$ follow from Lemma 2.7. Setting $s := \operatorname{ord}(p, b)$, by Lemma 2.5 we have that

$$\lambda := .\overline{\lambda^{(1)} : \cdots : \lambda^{(s)}} \text{ (base } p),$$

where the bar indicates that the digits of λ begin repeating after the sth digit. More precisely, if we write $n \geq 1$ uniquely as $n = sq + r$ with $1 \leq r \leq s$, then $\lambda^{(n)} = \lambda^{(r)}$.

By means of contradiction, suppose $\Delta_s = 0$, so that $\langle \lambda \rangle_s = \langle \operatorname{fpt}(f) \rangle_s$, that is,

$$\operatorname{fpt}(f) = .\lambda^{(1)} : \cdots : \lambda^{(s)} : \operatorname{fpt}(f)^{(s+1)} : \operatorname{fpt}(f)^{(s+2)} : \cdots \text{ (base } p). \tag{5.1}$$

Since $\operatorname{fpt}(f) \leq \lambda$, comparing the tails of the expansions of $\operatorname{fpt}(f)$ and λ shows that

$$.\operatorname{fpt}(f)^{(s+1)} : \cdots : \operatorname{fpt}(f)^{(2s)} \text{ (base } p) \leq .\lambda^{(1)} : \cdots : \lambda^{(s)} \text{ (base } p).$$

On the other hand, comparing the first s digits appearing in the second point of Proposition 3.3, recalling the expansion (5.1), shows that

$$.\,\lambda^{(1)}:\cdots:\lambda^{(s)} \text{ (base } p) \le .\,\mathrm{fpt}(f)^{(s+1)}:\cdots:\mathrm{fpt}(f)^{(2s)} \text{ (base } p),$$

and thus we conclude that $\mathrm{fpt}(f)^{(s+e)} = \lambda^{(s+e)}$ for every $1 \le e \le s$, that is, $\Delta_{2s} = 0$. Finally, a repeated application of this argument shows that $\Delta_{ms} = 0$ for every $m \ge 1$, which implies that $\mathrm{fpt}(f) = \lambda$, a contradiction. $\square$

NOTATION 5.3. If R is any $\mathbb{N}$-graded ring, and M is a graded R-module, $[M]_d$ will denote the degree d component of M, and $[M]_{\le d}$ and $[M]_{\ge d}$ the obvious $[R]_0$ submodules of M. Furthermore, we use $H_M(t) := \sum_{d\in\mathbb{Z}} \dim[M]_d \cdot t^d$ to denote the Hilbert series of M.

For the remainder of this subsection, we work in the following context.

SETUP 5.4. Fix an $\mathbb{N}$-grading on $R = \mathbb{L}[x_1,\ldots,x_n]$, and consider a homogeneous polynomial $f \in \mathfrak{m}$ with $\sqrt{\mathrm{Jac}(f)} = \mathfrak{m}$. In this context, $\partial_1(f),\ldots,\partial_n(f)$ form a homogeneous system of parameters for R and hence a regular sequence. Consequently, if we set $J_k = (\partial_1(f),\ldots,\partial_k(f))$, then the sequences

$$0 \to (R/J_{k-1})(-\deg f + \deg x_k) \xrightarrow{\partial_k(f)} R/J_{k-1} \to R/J_k \to 0$$

are exact for every $1 \le k \le n$. Furthermore, using the fact that the Hilbert series is additive across short exact sequences, the well-known identities $H_R(t) = \prod_{i=1}^n 1/(1-t^{\deg x_i})$ and $H_{M(-s)}(t) = t^s H_M(t)$ imply that

$$H_{R/\mathrm{Jac}(f)}(t) = \prod_{i=1}^{n} \frac{1-t^{\deg f - \deg x_i}}{1-t^{\deg x_i}}, \tag{5.2}$$

an identity that will play a key role in what follows.

LEMMA 5.5. *In the setting of Setup 5.4, we have that* $(\mathfrak{m}^{[p^e]} \colon \mathrm{Jac}(f)) \setminus \mathfrak{m}^{[p^e]} \subseteq [R]_{\ge (p^e+1)\cdot\sum\deg x_i - n\cdot\deg f}$.

Proof. To simplify the notation, set $J = \mathrm{Jac}(f)$. By (5.2) the degree of $H_{R/J}(t)$ (a polynomial since $\sqrt{J} = \mathfrak{m}$) is $N := n\deg f - 2\sum\deg x_i$, and so $[R/J]_d = 0$ whenever $d \ge N+1$. It follows that $[R]_{\ge N+1} \subseteq J$, and to establish the claim, it suffices to show that

$$\begin{aligned}(\mathfrak{m}^{[p^e]} \colon [R]_{\ge N+1}) \setminus \mathfrak{m}^{[p^e]} \\ \subseteq [R]_{\ge (p^e-1)\cdot\sum\deg x_i - N} = [R]_{\ge (p^e+1)\cdot\sum\deg x_i - n\cdot\deg f}.\end{aligned} \tag{5.3}$$

By means of contradiction, suppose that (5.3) is false. Consequently, there exists a monomial

$$\mu = x_1^{p^e-1-s_1}\cdots x_n^{p^e-1-s_n} \in (\mathfrak{m}^{[p^e]} \colon [R]_{\ge N+1})$$

such that $\deg\mu \le (p^e-1)\cdot\deg x_i - (N+1)$. This condition implies that the monomial $\mu_\circ := x_1^{s_1}\cdots x_n^{s_n}$ is in $[R]_{\ge N+1}$, and since $\mu \in (\mathfrak{m}^{[p^e]} \colon [R]_{\ge N+1})$, it follows that $\mu\mu_\circ$ (which is apparently equal to $(x_1\cdots x_n)^{p^e-1}$) is in $\mathfrak{m}^{[p^e]}$, a contradiction. □

PROPOSITION 5.6. *In the setting of Setup 5.4, if $p \nmid (\nu_f(p^e)+1)$, then $\nu_f(p^e) \geq \lceil (p^e+1)\cdot \sum \deg x_i/\deg f - n\rceil$.*

Proof. The Leibniz rule shows that $\partial_i(\mathfrak{m}^{[p^e]}) \subseteq \mathfrak{m}^{[p^e]}$, and so differentiating $f^{\nu_f(p^e)+1} \in \mathfrak{m}^{[p^e]}$ shows that $(\nu_f(p^e)+1)\cdot f^{\nu_f(p^e)}\cdot \partial_i(f) \in \mathfrak{m}^{[p^e]}$ for all i. Our assumption that $p \nmid \nu_f(p^e)+1$ then implies that $f^{\nu_f(p^e)} \in (\mathfrak{m}^{[p^e]} \colon J)\setminus \mathfrak{m}^{[p^e]} \subseteq [R]_{\geq (p^e+1)\cdot\sum \deg x_i - n\cdot \deg f}$, where the exclusion follows by definition, and the final containment by Lemma 5.5. Therefore,

$$\deg f \cdot \nu_f(p^e) \geq (p^e+1)\cdot \sum \deg x_i - n\cdot \deg f,$$

and the claim follows. □

COROLLARY 5.7. *In the setting of Setup 5.4, write $\lambda = \min\{\sum \deg x_i/\deg f, 1\} = \frac{a}{b}$ in lowest terms. If $\mathrm{fpt}(f)^{(e)}$, the eth digit of $\mathrm{fpt}(f)$, is not equal to $p-1$, then*

$$p^e\langle\lambda\rangle_e - p^e\langle \mathrm{fpt}(f)\rangle_e \leq n - \left\lceil \frac{[\![ap^e \,\%\, b]\!] + a}{b}\right\rceil.$$

Proof. By Proposition 3.3, $\nu_f(p^e) = p^e\langle \mathrm{fpt}(f)\rangle_e \equiv \mathrm{fpt}(f)^{(e)} \bmod p$, and so the condition that $\mathrm{fpt}(f)^{(e)} \neq p-1$ is equivalent to the condition that $p \nmid (\nu_f(p^e)+1)$. In light of this, we are free to apply Proposition 5.6. In what follows, we set $\delta := (\sum \deg x_i)\cdot(\deg f)^{-1}$.

First, suppose that $\min\{\delta, 1\} = 1$, so that $a = b = 1$. Then $\lceil([\![ap^e \,\%\, b]\!] + a)\cdot b^{-1}\rceil = 2$, and so it suffices to show that $p^e\langle 1\rangle_e - p^e\langle \mathrm{fpt}(f)\rangle_e \leq n-2$. However, the assumption that $\min\{\delta,1\} = 1$ implies that $\delta \geq 1$, and Proposition 5.6 then shows that

$$\begin{aligned} p^e\cdot\langle \mathrm{fpt}(f)\rangle_e &= \nu_f(p^e) \\ &\geq \lceil (p^e+1)\cdot\delta - n\rceil \geq \lceil p^e + 1 - n\rceil \\ &= p^e - 1 + 2 - n \\ &= p^e\cdot\langle 1\rangle_e + 2 - n. \end{aligned}$$

If, instead, $\min\{\delta, 1\} = \delta$, then Proposition 5.6 once again shows that

$$\begin{aligned} p^e\langle \mathrm{fpt}(f)\rangle_e &= \nu_f(p^e) \geq \lceil (p^e+1)\cdot\delta - n\rceil \\ &= \lceil p^e\cdot\delta + \delta - n\rceil \\ &= \left\lceil p^e\cdot\left(\langle\delta\rangle_e + \frac{[\![ap^e \,\%\, b]\!]}{b\cdot p^e}\right) + \delta - n\right\rceil \\ &= p^e\cdot\langle\delta\rangle_e + \left\lceil \frac{[\![ap^e \,\%\, b]\!]}{b} + \delta\right\rceil - n, \end{aligned}$$

the second to last equality following from Lemma 2.5. □

EXAMPLE 5.8 (illustrating that Corollary 5.7 is not an equivalence). If $p = 2$ and f is any $\mathbb{L}^*$-linear combination of $x_1^{15}, \ldots, x_5^{15}$, then Corollary 5.7 states that if $\operatorname{fpt}(f)^{(e)} \neq 1$, then $\Delta_e := 2^e \langle 3^{-1} \rangle_e - 2^e \langle \operatorname{fpt}(f) \rangle_e \leq 4$. We claim that the converse fails when $e = 4$. Indeed, a direct computation using [Her15, Cor. 3.5] shows that $\operatorname{fpt}(f) = \frac{1}{8}$, and comparing the base 2 expansions of $\operatorname{fpt}(f) = \frac{1}{8}$ and $\lambda = \frac{1}{3}$ shows that $\Delta_4 = 4$, even though $\operatorname{fpt}(f)^{(4)} = 1 = p - 1$.

5.2. Proofs of the Main Results

In this subsection, we return to the statements in Section 3, whose proofs were postponed. For the benefit of the reader, we restate these results here.

THEOREM 3.5. *Fix an $\mathbb{N}$-grading on R. Consider a homogeneous polynomial f with $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$, and write $\lambda := \min\{\sum \deg x_i / \deg f, 1\} = \frac{a}{b}$ in lowest terms.*

(1) *If $\operatorname{fpt}(f) \neq \lambda$, then*

$$\operatorname{fpt}(f) = \lambda - \left(\frac{[\![ap^L \ \% \ b]\!] + b \cdot E}{b \cdot p^L} \right) = \langle \lambda \rangle_L - \frac{E}{p^L}$$

for some $(L, E) \in \mathbb{N}^2$ with $L \geq 1$ and $0 \leq E \leq n - 1 - \lceil ([\![ap^L \ \% \ b]\!] + a)/b \rceil$.

(2) *If $p > (n-2) \cdot b$ and $p \nmid b$, then $1 \leq L \leq \operatorname{ord}(p, b)$; note that $\operatorname{ord}(p, 1) = 1$.*

(3) *If $p > (n-2) \cdot b$ and $p > b$, then $a < [\![ap^e \ \% \ b]\!]$ for all $1 \leq e \leq L - 1$.*

(4) *If $p > (n-1) \cdot b$, then there exists a unique pair (L, E) satisfying the conclusions of (1).*

Proof. We begin by establishing (1): The two descriptions of $\operatorname{fpt}(f)$ are equivalent by Lemma 2.5, and so it suffices to establish the identity in terms of truncations. Setting $\Delta_e := p^e \langle \lambda \rangle_e - p^e \langle \operatorname{fpt}(f) \rangle_e$, Corollary 5.2 states that $\{\Delta_e\}_{e \geq 1}$ is a nonnegative, nondecreasing, and unbounded sequence; in particular, $\min\{e : \Delta_e \neq 0\} \geq 1$ is well defined, and we claim that

$$\ell := \min\{e : \Delta_e \neq 0\} \leq L := \max\{e : \operatorname{fpt}(f)^{(e)} \neq p - 1\},$$

the latter also being well defined. Indeed, set $\mu_e := \lceil ([\![ap^e \ \% \ b]\!] + a)/b \rceil$. Since $1 \leq \mu_e \leq 2$, the sequence $\{n - \mu_e\}_{e \geq 1}$ is bounded above by $n - 1$, and therefore $\Delta_e > n - \mu_e$ for $e \gg 0$. For such $e \gg 0$, Corollary 5.7 implies that $\operatorname{fpt}(f)^{(e)} = p - 1$, which demonstrates that L is well defined. Note that, by definition, $\Delta_\ell = \lambda^{(\ell)} - \operatorname{fpt}(f)^{(\ell)} \geq 1$, so that $\operatorname{fpt}(f)^{(\ell)} \leq \lambda^{(\ell)} - 1 \leq p - 2$; by definition of L, it follows that $\ell \leq L$.

Since $\operatorname{fpt}(f)^{(e)} = p - 1$ for $e \geq L + 1$, we have

$$\operatorname{fpt}(f) = \langle \operatorname{fpt}(f) \rangle_L + \frac{1}{p^L} = \langle \lambda \rangle_L - \frac{\Delta_L}{p^L} + \frac{1}{p^L} = \langle \lambda \rangle_L - \frac{E}{p^L}, \tag{5.4}$$

where $E := \Delta_L - 1$. In order to conclude this step of the proof, it suffices to note that

$$1 \leq \Delta_\ell \leq \Delta_L \leq n - \mu_L \leq n - 1; \tag{5.5}$$

indeed, the second bound in (5.5) follows from the fact that $L \geq \ell$, the third follows from Corollary 5.7, and the last from the bound $1 \leq \mu_e \leq 2$.

For point (2), we continue to use the notation adopted before. We begin by showing that

$$\Delta_e = 0 \quad \text{for all } 0 \le e \le L-1 \text{ whenever } p > (n-2)\cdot b. \tag{5.6}$$

Since the sequence Δ_e is nonnegative and nondecreasing, it suffices to show that $\Delta_{L-1} = 0$. Therefore, by way of contradiction, we suppose that $\Delta_{L-1} \ge 1$. By definition, $0 \le \operatorname{fpt}(f)^{(L)} \le p-2$, and hence

$$\Delta_L = p \cdot \Delta_{L-1} + \lambda^{(L)} - \operatorname{fpt}(f)^{(L)} \ge \lambda^{(L)} + 2.$$

Comparing this with (5.5) shows that $\lambda^{(L)} + 2 \le \Delta_L \le n-1$, so that

$$\lambda^{(L)} \le n-3.$$

On the other hand, if $p > (n-2)\cdot b$, then it follows from the explicit formulas in Lemma 2.5 that

$$\begin{aligned}\lambda^{(e)} &= \frac{[\![ap^{e-1} \mathbin{\%} b]\!]\cdot p - [\![ap^{e} \mathbin{\%} b]\!]}{b} \ge \frac{p-b}{b} \\ &> \frac{(n-2)\cdot b - b}{b} = n-3 \quad \text{for every } e \ge 1.\end{aligned} \tag{5.7}$$

In particular, setting $e = L$ in this identity shows that $\lambda^{(L)} > n-3$, contradicting our earlier bound.

Thus, we conclude that (5.6) holds, which, when combined with (5.5), shows that $L = \min\{e : \Delta_e \ne 0\}$. In summary, we have just shown that $L = \ell$ when $p > (n-2)\cdot b$. If we assume further that $p \nmid b$, the desired bound $L = \ell \le \operatorname{ord}(p,b)$ then follows from Corollary 5.2.

We now focus on point (3) and begin by observing that

$$\begin{aligned}&\operatorname{fpt}(f) = \lambda^{(1)} : \cdots : \lambda^{(L-1)} : \lambda^{(L)} - \Delta_L : \overline{p-1} \text{ (base } p) \\ &\quad \text{whenever } p > (n-2)\cdot b.\end{aligned} \tag{5.8}$$

Indeed, by (5.6), the first $L-1$ digits of $\operatorname{fpt}(f)$ and λ agree, whereas $\operatorname{fpt}(f)^{(e)} = p-1$ for $e \ge L+1$ by definition of L. Finally, (5.6) shows that $\Delta_L = \lambda^{(L)} - \operatorname{fpt}(f)^{(L)}$, so that $\operatorname{fpt}(f)^{(L)} = \lambda^{(L)} - \Delta_L$.

Recall that, by the second point of Proposition 3.3, the first digit of $\operatorname{fpt}(f)$ is its smallest digit, and it follows from (5.8) that $\lambda^{(1)} \le \lambda^{(e)}$ for all $1 \le e \le L$, and this inequality is strict for $e = L$. However, it follows from the explicit formulas in Lemma 2.5 that whenever $p > b$,

$$\begin{aligned}\lambda^{(1)} \le \lambda^{(e)} \quad &\Longleftrightarrow \quad a\cdot p - [\![ap \mathbin{\%} b]\!] \le [\![ap^{e-1} \mathbin{\%} b]\!]\cdot p - [\![ap^{e} \mathbin{\%} b]\!] \\ &\Longleftrightarrow \quad a \le [\![ap^{e-1} \mathbin{\%} b]\!],\end{aligned}$$

where the second equivalence relies on the fact that $p > b$. Summarizing, we have just shown that $a \le [\![ap^{e-1} \mathbin{\%} b]\!]$ for all $1 \le e \le L$ whenever $p > (n-2)\cdot b$ and $p > b$; relabeling our index, we see that

$$a \le [\![ap^{e} \mathbin{\%} b]\!] \quad \text{for all } 0 \le e \le L-1 \text{ whenever } p > (n-2)\cdot b \text{ and } p > b.$$

It remains to show that this bound is strict for $1 \le e \le L-1$. By contradiction, assume that $a = [\![ap^{e} \mathbin{\%} b]\!]$ for some such e. In this case, $a \equiv a \cdot p^e \bmod b$, and since a and b are relatively prime, we conclude that $p^e \equiv 1 \bmod b$, so that $\operatorname{ord}(p,b) \mid e$. However, by definition $1 \le e \le L-1 \le \operatorname{ord}(p,b)-1$, where the last inequality follows point (2). Thus, we have arrived at a contradiction and therefore conclude that our asserted upper bound is strict for $1 \le e \le L-1$.

To conclude our proof, it remains to establish the uniqueness statement in point (4). To this end, let (L', E') denote any pair of integers satisfying the conclusions of point (1) of this theorem; that is,

$$\operatorname{fpt}(f) = \langle\lambda\rangle_{L'} - E' \cdot p^{-L'} \quad \text{with } 1 \le E' \le n-1-\mu_{L'} \le n-2.$$

A modification of (5.7) shows that $\lambda^{(e)} > n-2$ and hence that $\lambda^{(e)} \ge E'+1$ whenever $p > (n-1)\cdot b$, and it follows that

$$\operatorname{fpt}(f) = \langle\lambda\rangle_{L'} - E' \cdot p^{-L'} = .\,\lambda^{(1)} : \cdots : \lambda^{(L'-1)} : \lambda^{(L')} - (E'+1) : \overline{p-1}$$
$$\text{whenever } p > (n-1)\cdot b.$$

The uniqueness statement then follows from comparing this expansion with (5.8) and invoking the uniqueness of nonterminating base p expansions. □

Corollary 3.9. *In the setting of Theorem 3.5, if* $\operatorname{fpt}(f) \neq \lambda$ *and* $p \nmid b$, *then* $p^M \cdot \operatorname{fpt}(f) \in \mathbb{N}$, *where* $M := 2\cdot\phi(b) + \lceil \log_2(n-1)\rceil$, *and* ϕ *denotes Euler's phi function.*

Proof. We adopt the notation used in the proof of Theorem 3.5. In particular, $\ell \le L$ and $\operatorname{fpt}(f) \in p^{-L}\cdot\mathbb{N}$. Setting $s = \operatorname{ord}(p,b)$ and $k = \lceil \log_p(n-1)\rceil$ in Lemma 2.7 shows that

$$\Delta_{\ell+s+\lceil\log_p(n-1)\rceil} \ge p^{\lceil\log_p(n-1)\rceil} + 1 \ge n. \tag{5.9}$$

By definition of L, Corollary 5.7 states that $\Delta_L \le n-1$, and since $\{\Delta_e\}_{e\ge 1}$ is nondecreasing, (5.9) then shows that L is bounded above by $\ell+s+\lceil\log_p(n-1)\rceil$. To obtain a uniform bound, note that $\ell \le s$ by Corollary 5.2, whereas $s \le \phi(b)$ by definition, and $\log_p(n-1) \le \log_2(n-1)$ since $p \ge 2$. □

Theorem 3.13. *In the context of Theorem 3.5, suppose that* $\sum \deg x_i > \deg f$, *so that* $\rho := \sum \deg x_i/\deg f$ *is greater than* 1. *If* $p > \frac{n-3}{\rho-1}$, *then* $\operatorname{fpt}(f) = 1$.

Proof. We begin with the following elementary manipulations, the first of which relies on the assumption that $\rho-1$ is positive: Isolating $n-3$ in our assumption that $p > (n-3)\cdot(\rho-1)^{-1} - 1$ shows that $(p+1)\cdot(\rho-1) > n-3$, and adding $p+1$ and subtracting n from both sides then show that $(p+1)\cdot\rho - n > p-2$; rounding up, we see that

$$\lceil (p+1)\cdot\rho - n\rceil \ge p-1. \tag{5.10}$$

Assume, by means of contradiction, that $\operatorname{fpt}(f) \neq 1$. By hypothesis, $1 = \min\{\rho, 1\}$, and Corollary 5.2 then states that $1 = \min\{e : p^e\langle 1\rangle_e - p^e\langle \operatorname{fpt}(f)\rangle_e \ge 1\}$; in particular,

$$\nu_f(p) = \operatorname{fpt}(f)^{(1)} = p\cdot\langle\operatorname{fpt}(f)\rangle_1 \le p\langle 1\rangle_1 - 1 = p-2.$$

Thus, we can apply Proposition 5.6, which, when combined with (5.10), implies that

$$\nu_f(p) \geq \lceil (p+1) \cdot \rho - n \rceil \geq p - 1.$$

We have arrived at a contradiction, and conclude that $\operatorname{fpt}(f) = 1$. □

6. Applications to Log Canonical Thresholds

Given a polynomial $f_{\mathbb{Q}}$ over $\mathbb{Q}$, we will denote its *log canonical threshold* by $\operatorname{lct}(f_{\mathbb{Q}})$. In this article, we will not need to refer to the typical definition(s) of $\operatorname{lct}(f_{\mathbb{Q}})$ (e.g., via resolution of singularities) and will instead rely on the limit in (6.1) as our definition. However, so that the reader unfamiliar with this topic may better appreciate (6.1), we present the following characterizations. In what follows, we fix $f_{\mathbb{Q}} \in \mathbb{Q}[x_1, \dots, x_n]$.

(1) If $\pi : X \to \mathbb{A}^n_{\mathbb{Q}}$ is a *log resolution* of the pair $(\mathbb{A}^n_{\mathbb{Q}}, \mathbb{V}(f_{\mathbb{Q}}))$, then $\operatorname{lct}(f_{\mathbb{Q}})$ is the supremum over all $\lambda > 0$ such that the coefficients of the divisor $K_\pi - \lambda \cdot \pi^* \operatorname{div}(f)$ are all greater than -1; here, K_π denotes the relative canonical divisor of π.
(2) For every $\lambda > 0$, consider the function $\Gamma_\lambda(f_{\mathbb{Q}}) : \mathbb{C}^n \to \mathbb{R}$ given by

$$(z_1, \dots, z_n) \mapsto |f(z_1, \dots, z_n)|^{-2\lambda},$$

where $|\cdot| \in \mathbb{R}$ denotes the norm of a complex number; note that $\Gamma_\lambda(f_{\mathbb{Q}})$ has a pole at all (complex) zeros of $f_{\mathbb{Q}}$. In this setting, $\operatorname{lct}(f_{\mathbb{Q}}) := \sup\{\lambda : \Gamma_\lambda(f_{\mathbb{Q}}) \text{ is locally } \mathbb{R}\text{-integrable}\}$, where "locally $\mathbb{R}$-integrable" here means that we identify $\mathbb{C}^n = \mathbb{R}^{2n}$, and require that this function be (Lebesgue) integrable in a neighborhood of every point in its domain.
(3) The roots of the Bernstein–Sato polynomial $b_{f_{\mathbb{Q}}}$ of $f_{\mathbb{Q}}$ are all negative rational numbers, and $-\operatorname{lct}(f_{\mathbb{Q}})$ is the largest such root [Kol97, Thm. 10.6].

For more information on these invariants, the reader is referred to the surveys [BL04; EM06]. We now recall the striking relationship between F-pure and log canonical thresholds: Though there are many results due to many authors relating characteristic zero and characteristic $p > 0$ invariants, the one most relevant to our discussion is the following theorem, which is due to Mustaţă and the fourth author.

THEOREM 6.1 ([MZ13, Cors. 3.5 and 4.5]). *Given an polynomial $f_{\mathbb{Q}}$ over $\mathbb{Q}$, there exist constants $C \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ (depending only on $f_{\mathbb{Q}}$) with the following property: For $p \gg 0$, either* $\operatorname{fpt}(f_p) = \operatorname{lct}(f_{\mathbb{Q}})$, *or*

$$\frac{1}{p^N} \leq \operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) \leq \frac{C}{p}.$$

Note that, as an immediate corollary of Theorem 6.1,

$$\operatorname{fpt}(f_p) \leq \operatorname{lct}(f_{\mathbb{Q}}) \quad \text{for all } p \gg 0 \text{ and } \lim_{p \to \infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f_{\mathbb{Q}}). \tag{6.1}$$

We point out that (6.1) (which follows from the work of Hara and Yoshida) appeared in the literature well before Theorem 6.1 (see, e.g., [MTW05, Thms. 3.3 and 3.4]).

6.1. Regarding Uniform Bounds

Though the constants $C \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ appearing in Theorem 6.1 are known to depend only on $f_{\mathbb{Q}}$, their determination is complicated (e.g., they depend on numerical invariants coming from resolution of singularities), and they therefore are not explicitly described. In Theorem 6.2, we give an alternate proof of this result for homogeneous polynomials with an isolated singularity at the origin; in the process of doing so, we also identify explicit values for C and N.

THEOREM 6.2. *If $f_{\mathbb{Q}} \in \mathbb{Q}[x_1, \dots, x_n]$ is homogeneous under some $\mathbb{N}$-grading with $\sqrt{\operatorname{Jac}(f_{\mathbb{Q}})} = \mathfrak{m}$, then $\operatorname{lct}(f_{\mathbb{Q}}) = \min\{\sum \deg x_i / \deg f, 1\}$, which we write as $\frac{a}{b}$ in lowest terms. Moreover, if $\operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})$, then*

$$\frac{b^{-1}}{p^{\operatorname{ord}(p,b)}} \leq \operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) \leq \frac{n-1-b^{-1}}{p} \quad \textit{for } p \gg 0,$$

where $\operatorname{ord}(p, b)$ denotes the order of p mod b (which by convention equals one when $b = 1$).

Proof. Since the reduction of $\partial_k(f)$ mod p equals $\partial_k(f_p)$ for large values of p, the equality $\sqrt{\operatorname{Jac}(f_{\mathbb{Q}})} = \mathfrak{m}$ reduces mod p for $p \gg 0$. Taking $p \to \infty$, it follows from Theorem 3.5 and (6.1) that $\operatorname{lct}(f_{\mathbb{Q}}) = \min\{\sum \deg x_i / \deg f, 1\}$, and in light of this, Theorem 3.5 states that

$$\operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) = \frac{[\![ap^L \,\%\, b]\!]}{b \cdot p^L} + \frac{E}{p^L}. \tag{6.2}$$

Suppose first that $\operatorname{lct}(f_{\mathbb{Q}}) \neq 1$. Having already chosen $p \gg 0$, we may further enlarge p so as to assume that b is not a power of p. It follows that $1 \leq [\![ap^L \,\%\, b]\!] \leq b - 1$, and then

$$\frac{1}{b \cdot p^L} \leq \frac{[\![ap^L \,\%\, b]\!]}{b \cdot p^L} \leq \frac{1 - b^{-1}}{p^L}.$$

In this case, Theorem 3.5 implies that $1 \leq L \leq \operatorname{ord}(p, b)$ and $0 \leq E \leq n - 2$ for $p \gg 0$, and substituting these bounds into (6.2) produces the desired bounds on the difference between the log canonical and F-pure threshold.

If, instead, $\operatorname{lct}(f_{\mathbb{Q}}) = 1$, then $a = b = 1$, so that $[\![ap^L \,\%\, b]\!] = \operatorname{ord}(p, b) = 1$. Having chosen $p \gg 0$, the bounds on L in Theorem 3.5 then imply that $L = 1$ and

$$\frac{[\![ap^L \,\%\, b]\!]}{b \cdot p^L} = \frac{1}{p}.$$

Moreover, substituting $a = b = 1$ into Theorem 3.5 also shows that $0 \leq E \leq n - 3$ for $p \gg 0$. Finally, it is left to the reader to verify that substituting these inequalities into (6.2) produces the desired bounds in each case. $\square$

REMARK 6.3 (on uniform bounds). Of course, $\operatorname{ord}(p, b) \le \phi(b)$, where ϕ denotes Euler's phi function. By enlarging p, if necessary, it follows that the lower bound in Theorem 6.2 is itself bounded below by $p^{-\phi(b)-1}$. In other words, in the language of Theorem 6.1, we may take $N = \phi(b) + 1$ and $C = n - 1 - b^{-1}$.

REMARK 6.4 (regarding sharpness). The bounds appearing in Theorem 6.2 are sharp: If $d > 2$ and $f_{\mathbb{Q}} = x_1^d + \cdots + x_d^d$, then $\operatorname{lct}(f_{\mathbb{Q}}) = 1$, and Theorem 6.2 states that

$$\frac{1}{p} \le \operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) \le \frac{d-2}{p} \tag{6.3}$$

whenever $\operatorname{fpt}(f_p) \ne 1$ and $p \gg 0$. However, it is shown in [Her15, Cor. 3.5] that

$$\operatorname{lct}(f_{\mathbb{Q}}) - \operatorname{fpt}(f_p) = 1 - \operatorname{fpt}(f_p) = \frac{[\![p \,\%\, d]\!] - 1}{p}$$

whenever $p > d$. If d is odd and $p \equiv 2 \bmod d$, then the lower bound in (6.3) is obtained, and similarly, if $p \equiv d - 1 \bmod d$, then the upper bound in (6.3) is obtained; in both these cases, Dirichlet's theorem guarantees that there are infinitely many primes satisfying these congruence relations.

6.2. *On the Size of a Set of Bad Primes*

In this subsection, we record some simple observations regarding the set of primes for which the F-pure threshold does *not* coincide with the log canonical threshold, and we begin by recalling the case of elliptic curves: Let $f_{\mathbb{Q}} \in \mathbb{Q}[x, y, z]$ be a homogeneous polynomial of degree three with $\sqrt{\operatorname{Jac}(f_{\mathbb{Q}})} = \mathfrak{m}$, so that $E := \mathbb{V}(f)$ defines an elliptic curve in $\mathbb{P}^2_{\mathbb{Q}}$. As shown in the proof of Theorem 6.2, the reductions $f_p \in \mathbb{F}_p[x, y, z]$ satisfy these same conditions for $p \gg 0$ and thus define elliptic curves $E_p = \mathbb{V}(f_p) \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ for all $p \gg 0$. Recall that the elliptic curve E_p is called *supersingular* if the natural Frobenius action on the local cohomology module $H^2_{(x,y,z)}(\mathbb{F}_p[x, y, z]/(f_p))$ is injective or, equivalently, if $(f_p)^{p-1} \notin (x^p, y^p, z^p)$ (see, e.g., [Sil09, Chaps. V.3 and V.4] for these and other characterizations of supersingularity). Using these descriptions, we can show that E_p is supersingular if and only if $\operatorname{fpt}(f_p) = 1$ [MTW05, Ex. 4.6]. In light of this, Elkies' well-known theorem on the set of supersingular primes, which states that E_p is supersingular for infinitely many primes p, can be restated as follows.

THEOREM 6.5 ([Elk87, Thm. 1]). *For $f_{\mathbb{Q}} \in \mathbb{Q}[x, y, z]$ as just described, there are infinitely many primes p such that* $\operatorname{fpt}(f_p) \ne \operatorname{lct}(f_{\mathbb{Q}})$.

Recall that given a set S of prime numbers, the density of S, $\delta(S)$, is defined as

$$\delta(S) = \lim_{n \to \infty} \frac{{}^{\#}\{p \in S : p \le n\}}{{}^{\#}\{p : p \le n\}}.$$

In the context of elliptic curves over $\mathbb{Q}$, the set of primes $\{p : \operatorname{fpt}(f_p) \ne \operatorname{lct}(f_{\mathbb{Q}})\}$, which is infinite by Elkies' result, may be quite large (i.e., have density $\frac{1}{2}$) or may be quite small (i.e., have density zero); see [MTW05, Ex. 4.6] for more information. This discussion motivates the following question.

QUESTION 6.6. For which polynomials $f_{\mathbb{Q}}$ is the set of primes $\{p : \operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})\}$ infinite? In the case that this set is infinite, what is its density?

As illustrated by the case of an elliptic curve, Question 6.6 is quite subtle, and we expect it to be quite difficult to address in general. However, as we further see, when the numerator of $\operatorname{lct}(f_{\mathbb{Q}})$ is not equal to 1, we are able to give a partial answer to this question using simple methods. Our main tool will be Proposition 3.3, which provides us with a simple criterion for disqualifying a rational number from being an F-pure threshold. We stress the fact that Proposition 6.7 is not applicable when $\operatorname{lct}(f_{\mathbb{Q}}) = 1$ and hence sheds no light on the elliptic curve case discussed.

PROPOSITION 6.7. *Let $f_{\mathbb{Q}}$ denote any polynomial over $\mathbb{Q}$, and write $\operatorname{lct}(f_{\mathbb{Q}}) = \frac{a}{b}$ in lowest terms. If $a \neq 1$, then the set of primes for which $\operatorname{lct}(f_{\mathbb{Q}})$ is not an F-pure threshold (of* any *polynomial) is infinite and contains all primes p such that $p^e \cdot a \equiv 1 \bmod b$ for some $e \geq 1$. In particular,*

$$\delta(\{p : \operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})\}) \geq \frac{1}{\phi(b)}.$$

Proof. Since a and b are relatively prime, there exists $c \in \mathbb{N}$ such that $a \cdot c \equiv 1 \bmod b$. We claim that

$$\begin{aligned}
&\{p : p \equiv c \bmod b\} \\
&\quad \subseteq \{p : p^e \cdot a \equiv 1 \bmod b \text{ for some } e \geq 1\} \\
&\quad \subseteq \{p : \operatorname{lct}(f_{\mathbb{Q}}) \text{ is not an } F\text{-pure threshold in characteristic } p > 0\}.
\end{aligned}$$

Once we establish this, the proposition will follow since $\delta(\{p : p \equiv c \bmod b\}) = \frac{1}{\phi(b)}$ by Dirichlet's theorem. By the definition of c, the first containment holds by setting $e = 1$, and so it suffices to establish the second containment. However, if $p^e \cdot a \equiv 1 \bmod b$ for some $e \geq 1$, then Lemma 2.5 shows that

$$\operatorname{lct}(f_{\mathbb{Q}})^{(e+1)} = \frac{[\![ap^e \,\%\, b]\!] \cdot p - [\![ap^{e+1} \,\%\, b]\!]}{b} = \frac{p - [\![ap^{e+1} \,\%\, b]\!]}{b}.$$

On the other hand, Lemma 2.5 also shows that

$$\operatorname{lct}(f_{\mathbb{Q}})^{(1)} = \frac{a \cdot p - [\![ap \,\%\, b]\!]}{b},$$

and since $a \geq 2$ by assumption, we see that $\operatorname{lct}(f_{\mathbb{Q}})^{(1)} > \operatorname{lct}(f_{\mathbb{Q}})^{(e)}$ for all $p \gg 0$. In light of this, the second point of Proposition 3.3, which states an F-pure threshold's first digit must be its smallest, shows that $\operatorname{lct}(f_{\mathbb{Q}})$ cannot be the F-pure threshold of *any* polynomial in characteristic $p > 0$. $\square$

We conclude this section with the following example, which follows immediately from Corollary 4.2 and which illustrates a rather large family of polynomials whose set of "bad" primes greatly exceeds the bound given by Proposition 6.7.

EXAMPLE 6.8. If $f_{\mathbb{Q}} \in \mathbb{Q}[x_1, \dots, x_{d-1}]$ is homogeneous (under the standard grading) of degree d with $\sqrt{\operatorname{Jac}(f_{\mathbb{Q}})} = \mathfrak{m}$, then $\{p : p \not\equiv 1 \bmod d\} \subseteq \{p : \operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}}) = 1 - \frac{1}{d}\}$. In particular,

$$\delta(\{p : \operatorname{fpt}(f_p) \neq \operatorname{lct}(f_{\mathbb{Q}})\}) \geq \delta(\{p : p \not\equiv 1 \bmod d\}) = 1 - \frac{1}{\phi(d)}.$$

7. Supporting Evidence for the ACC Conjecture for F-pure Thresholds

Motivated by the relationship between F-pure thresholds and log canonical thresholds, Blickle, Mustaţă, and Smith conjectured that a certain collection of F-pure thresholds satisfies the ascending chain condition. A slight variation of their conjecture is the following; recall that under Convention 3.1, $R = \mathbb{L}[x_1, \dots, x_n]$, where $\mathbb{L}$ is a field of characteristic $p > 0$.

CONJECTURE 7.1 (cf. [BMS09, Conj. 4.4]). Fix an integer $n \geq 1$.

(1) The set $\{\operatorname{fpt}(f) : f \in R\}$ satisfies the ascending chain condition (ACC); that is, it contains no strictly increasing infinite sequence.
(2) For every $f \in R$, there exists an integer N (which may depend on f) such that
$$\operatorname{fpt}(f) \leq \operatorname{fpt}(f + g) \text{ for all } g \in \mathfrak{m}^N.$$

As discussed in [BMS09, Rem. 4.5], the first conjecture implies the second by a straightforward argument using [BMS09, Cor. 3.4]. For the sake of clarity, we point out that there is a typo in the statement of [BMS09, Rem. 4.5]. Indeed, the inequality "$\operatorname{fpt}(f) \geq \operatorname{fpt}(f + g)$" appearing therein is incorrect and should be reversed (i.e., so that it will be consistent with Conjecture 7.1 (2)).

In this section, we confirm Conjecture 7.1 (1) for a restricted set of F-pure thresholds (see Proposition 7.3). Additionally, we confirm Conjecture 7.1 (2) in the case that f is homogeneous under some $\mathbb{N}$-grading (see Proposition 7.4) and establish stronger versions of this conjecture under some additional hypotheses (see Propositions 7.8 and 7.10).

7.1. *A Special Case of Conjecture 7.1 (1)*

DEFINITION 7.2. For every $\omega \in \mathbb{N}^n$, let W_ω denote the set of polynomials $f \in R$ satisfying the following conditions:

(1) $\sqrt{\operatorname{Jac}(f)} = \mathfrak{m}$.
(2) f is homogeneous under the grading determined by $(\deg x_1, \dots, \deg x_n) = \omega$.
(3) $p \nmid \deg f$ (and hence does not divide the denominator of $\min\{\sum \deg x_i / \deg f, 1\}$ in lowest terms).

Given $N \in \mathbb{N}$, set $W_{\preccurlyeq N} := \bigcup_\omega W_\omega$, where the union is taken over all $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{N}^n$ with $\omega_i \leq N$ for each $1 \leq i \leq n$.

PROPOSITION 7.3. *For every* $N \in \mathbb{N}$ *and* $\mu \in (0, 1]$, *the set*

$$\{\operatorname{fpt}(f) : f \in W_{\preccurlyeq N}\} \cap (\mu, 1]$$

is finite. In particular, this set of F-pure thresholds satisfies ACC.

Proof. Fix $f \in W_{\preccurlyeq N}$ such that $\operatorname{fpt}(f) > \mu$. By definition, there exists an $\mathbb{N}$-grading on R such that $\deg x_i \leq N$ for all $1 \leq i \leq N$ and under which f is homogeneous. Moreover, by Lemma 5.1,

$$\mu < \operatorname{fpt}(f) \leq \frac{\sum_{i=1}^{n} \deg x_i}{\deg f} \leq \frac{n \cdot N}{\deg f}.$$

Consequently, $\deg f \leq \frac{n \cdot N}{\mu}$, and it follows that

$$\lambda := \min\left\{\frac{\sum_{i=1}^{n} \deg x_i}{\deg f}, 1\right\} \subseteq S := (0, 1] \cap \left\{\frac{a}{b} \in \mathbb{Q} : b \leq \frac{n \cdot N}{\mu}\right\},$$

a finite set. We now argue that $\operatorname{fpt}(f)$ can take on only finitely many values: If $\operatorname{fpt}(f) \neq \lambda$, then by Corollary 3.9 there exists an integer M_λ, depending only on λ and n, such that $p^{M_\lambda} \cdot \operatorname{fpt}(f) \in \mathbb{N}$. If $\mathscr{M} := \max\{M_\lambda : \lambda \in S\}$, then it follows that $\operatorname{fpt}(f) \in \{a/p^{\mathscr{M}} : a \in \mathbb{N}\} \cap (0, 1]$, a finite set. □

7.2. *Regarding Conjecture 7.1 (2)*

Throughout this subsection, we fix an $\mathbb{N}$-grading on R.

PROPOSITION 7.4. *Fix* $f \in \mathfrak{m}$ *homogeneous. If* $g \in [R]_{\geq \deg f + 1}$, *then* $\operatorname{fpt}(f) \leq \operatorname{fpt}(f + g)$. *In particular, Conjecture 7.1 (2) holds for* f.

Proof. It suffices to show that $\nu_f(p^e) \leq \nu_{f+g}(p^e)$ for every $e \geq 1$; that is, setting $N := \nu_f(p^e)$, it suffices to show that $(f + g)^N \notin \mathfrak{m}^{[p^e]}$. Suppose, by way of contradiction, that $(f + g)^N = f^N + \sum_{k=1}^{N} \binom{N}{k} f^{N-k} g^k \in \mathfrak{m}^{[p^e]}$; note that, since $f^N \notin \mathfrak{m}^{[p^e]}$ by definition, each monomial summand of f^N must cancel with one of $\sum_{k=1}^{N} \binom{N}{k} f^{N-k} g^k$. However, for any monomial summand μ of any $f^{N-k} g^k$, $k \geq 1$,

$$\deg \mu \geq (N - k) \deg f + k(\deg f + 1) > N \deg f = \deg(f^N),$$

and such cancellation is impossible. □

The remainder of this article is dedicated to establishing a stronger version of Conjecture 7.1 (2).

LEMMA 7.5. *Consider* $f \in \mathfrak{m}$ *such that* $p^L \cdot \operatorname{fpt}(f) \in \mathbb{N}$ *for some* $L \in \mathbb{N}$. *If* $g \in \mathfrak{m}$, *then*

$$\operatorname{fpt}(f + g) \leq \operatorname{fpt}(f) \quad \textit{if and only if} \quad (f + g)^{p^L \cdot \operatorname{fpt}(f)} \in \mathfrak{m}^{[p^L]}.$$

Proof. If $(f + g)^{p^L \cdot \operatorname{fpt}(f)} \in \mathfrak{m}^{[p^L]}$, then $(f + g)^{p^s \cdot \operatorname{fpt}(f)} \in \mathfrak{m}^{[p^s]}$ for $s \geq L$. Consequently, $\nu_{f+g}(p^s) < p^s \cdot \operatorname{fpt}(f)$ for $s \gg 0$, and hence $\operatorname{fpt}(f + g) \leq \operatorname{fpt}(f)$. We now focus on the remaining implication.

By the hypothesis, $p^L \cdot \operatorname{fpt}(f) - 1 \in \mathbb{N}$, so that the identity $\operatorname{fpt}(f) = (p^L \cdot \operatorname{fpt}(f) - 1)/p^L + 1/p^L$ shows that

$$\langle \operatorname{fpt}(f) \rangle_L = \frac{p^L \cdot \operatorname{fpt}(f) - 1}{p^L}.$$

If $\operatorname{fpt}(f+g) \leq \operatorname{fpt}(f)$, then the preceding identity and Proposition 3.3 show that

$$\nu_{f+g}(p^L) = p^L \langle \operatorname{fpt}(f+g) \rangle_L \leq p^L \langle \operatorname{fpt}(f) \rangle_L = p^L \operatorname{fpt}(f) - 1,$$

and consequently, this bound for $\nu_{f+g}(p^L)$ shows that $(f+g)^{p^L \operatorname{fpt}(f)} \in \mathfrak{m}^{[p^L]}$. □

LEMMA 7.6. *If h is homogeneous and $h \notin \mathfrak{m}^{[p^e]}$, then* $\deg h \leq (p^e - 1) \cdot \sum_{i=1}^n \deg x_i$.

Proof. Every supporting monomial of h is of the form $x_1^{p^e - a_1} \cdots x_n^{p^e - a_n}$, where each $a_i \geq 1$. Then

$$\deg h = \sum_{i=1}^n (p^e - a_i) \deg x_i \leq (p^e - 1) \sum_{i=1}^n \deg x_i.$$ □

LEMMA 7.7. *Fix $f \in \mathfrak{m}$ homogeneous such that $\lambda := \sum \deg x_i / \deg f \leq 1$. If $(p^e - 1) \cdot \lambda \in \mathbb{N}$ and $g \in [R]_{\geq \deg f + 1}$, then $(f+g)^{p^e \langle \lambda \rangle_e} \equiv f^{p^e \langle \lambda \rangle_e}$* mod $\mathfrak{m}^{[p^e]}$.

Proof. We claim that

$$f^{p^e \langle \lambda \rangle_e - k} g^k \in \mathfrak{m}^{[p^e]} \quad \text{for all } 1 \leq k \leq p^e \langle \lambda \rangle_e. \tag{7.1}$$

Indeed, suppose that (7.1) is false. Since $g \in [R]_{\geq \deg f + 1}$ and μ is a supporting monomial of $f^{p^e \langle \lambda \rangle_e - k} g^k$, we also have that

$$\deg \mu \geq \deg f \cdot (p^e \langle \lambda \rangle_e - k) + (\deg f + 1) \cdot k = \deg f \cdot p^e \langle \lambda \rangle_e + k. \tag{7.2}$$

However, since $(p^e - 1) \cdot \lambda \in \mathbb{N}$, it follows from Lemma 2.6 that $p^e \langle \lambda \rangle_e = (p^e - 1) \cdot \lambda$. Substituting this into (7.2) shows that

$$\deg \mu \geq \deg f \cdot (p^e - 1) \cdot \lambda + k = k + (p^e - 1) \sum_{i=1}^n \deg x_i,$$

which contradicts Lemma 7.6 since $k \geq 1$. Thus, (7.1) holds, and it follows from the binomial theorem that $(f+g)^{p^e \langle \lambda \rangle_e} \equiv f^{p^e \langle \lambda \rangle_e} \bmod \mathfrak{m}^{[p^e]}$. □

We are now able to prove our first result on the local $\mathfrak{m}$-adic constancy of the F-pure threshold function $f \mapsto \operatorname{fpt}(f)$.

PROPOSITION 7.8. *Fix $f \in \mathfrak{m}$ homogeneous such that $\lambda := \sum \deg x_i / \deg f \leq 1$, and suppose that either* $\operatorname{fpt}(f) = \lambda$, *or* $\operatorname{fpt}(f) = \langle \lambda \rangle_L$ *and* $(p^L - 1) \cdot \lambda \in \mathbb{N}$ *for some $L \geq 1$. Then* $\operatorname{fpt}(f+g) = \operatorname{fpt}(f)$ *for each $g \in [R]_{\geq \deg f + 1}$.*

Proof. By Proposition 7.4 it suffices to show that $\mathrm{fpt}(f) \geq \mathrm{fpt}(f+g)$. First, say that $\mathrm{fpt}(f) = \lambda$. It is enough to show that for all $e \geq 1$, $(f+g)^{\nu_f(p^e)+1} \in \mathfrak{m}^{[p^e]}$, so that $\nu_f(p^e) \geq \nu_{f+g}(p^e)$. By the binomial theorem it suffices to show that for all $0 \leq k \leq \nu_f(p^f)+1$, $f^{\nu_f(p^e)+1-k} g^k \in \mathfrak{m}^{[p^e]}$. To this end, take any monomial μ of such an $f^{\nu_f(p^e)+1-k} g^k$. Then

$$\begin{aligned} \deg \mu &\geq (\nu_f(p^e)+1-k) \cdot \deg f + k \cdot (\deg f + 1) \\ &= (\nu_f(p^e)+1) \cdot \deg f + k \geq (\nu_f(p^e)+1) \cdot \deg f. \end{aligned} \tag{7.3}$$

By Lemma 3.3, $\nu_f(p^e) = p^e \langle \lambda \rangle_e$, and by definition, $\langle \alpha \rangle_e \geq \alpha - 1/p^e$ for all $0 < \alpha \leq 1$. Then by (7.3),

$$\begin{aligned} \deg \mu &\geq (p^e \langle \lambda \rangle_e + 1) \cdot \deg f \\ &\geq \left(p^e \left(\lambda - \frac{1}{p^e} \right) + 1 \right) \cdot \deg f \\ &= p^e \cdot \lambda \cdot \deg f \\ &= p^e \cdot \sum \deg x_i. \end{aligned}$$

We may now conclude that $\mu \in \mathfrak{m}^{[p^e]}$ by Lemma 7.6.

Now say that $\mathrm{fpt}(f) = \langle \lambda \rangle_L$ and $(p^L - 1) \cdot \lambda \in \mathbb{N}$ for some $L \leq 1$. By Lemma 7.5 it suffices to show that $(f+g)^{p^L \cdot \mathrm{fpt}(f)} \in \mathfrak{m}^{[p^L]}$. Indeed, $p^L \cdot \mathrm{fpt}(f) > p^L \langle \mathrm{fpt}(f) \rangle_L = \nu_f(p^L)$ (the equality by Proposition 3.3), so that $f^{p^L \cdot \mathrm{fpt}(f)} \in \mathfrak{m}^{[p^L]}$; thus, $(f+g)^{p^L \cdot \mathrm{fpt}(f)} \equiv f^{p^L \cdot \mathrm{fpt}(f)} \equiv 0 \bmod \mathfrak{m}^{[p^L]}$ by Lemma 7.7. □

We see that the hypotheses of Proposition 7.8 are often satisfied in Example 7.9. We also see that the statement of the proposition is sharp in the sense that there exist f and g satisfying its hypotheses such that $\mathrm{fpt}(f) = \langle \lambda \rangle_L$ for some $L \geq 1$, $(p^L - 1) \cdot \lambda \notin \mathbb{N}$, and $\mathrm{fpt}(f+g) > \mathrm{fpt}(f)$.

EXAMPLE 7.9. Let $f = x^{15} + xy^7 \in \mathbb{L}[x, y]$, which is homogeneous with $\deg f = 15$ under the grading determined by $(\deg x, \deg y) = (1, 2)$, and has an isolated singularity at the origin when $p \geq 11$. It follows from Theorem 4.5 that

$$\mathrm{fpt}(f) = \left\langle \frac{1+2}{15} \right\rangle_L = \left\langle \frac{1}{5} \right\rangle_L,$$

where $1 \leq L \leq \mathrm{ord}(p, 5) \leq 4$, or $L = \infty$ (i.e., $\mathrm{fpt}(f) = \frac{1}{5}$). Furthermore, since f is a binomial, we can use the algorithm in [Her14, Alg. 4.2], recently implemented by Sara Malec, Karl Schwede, and the third author in an upcoming `Macaulay2` package, to compute the exact value $\mathrm{fpt}(f)$ and hence the exact value of L for a fixed p. We list some of these computations in Figure 1.

We see that the hypotheses of Proposition 7.8 are often satisfied in this example, and it follows that $\mathrm{fpt}(f) = \mathrm{fpt}(f+g)$ for every $g \in [R]_{\geq 16}$ whenever either "∞" appears in the second column or "Yes" appears in the third. When $p = 17$, however, we have that $\mathrm{fpt}(f) = \langle \frac{1}{5} \rangle_1 = \frac{3}{17}$, and when $g \in \{x^{14}y, x^{12}y^2, y^8, x^{13}y^2, x^{14}y^2\} \subseteq [R]_{\geq 16}$, we may verify that $(f+g)^3 \notin \mathfrak{m}^{[17]}$,

p	L	$(p^L-1)\cdot\frac{1}{5}\in\mathbb{N}$?
11	1	Yes
13	1	No
17	1	No
19	2	Yes
23	4	Yes
29	∞	–
31	1	Yes

p	L	$(p^L-1)\cdot\frac{1}{5}\in\mathbb{N}$?
37	4	Yes
41	1	Yes
43	∞	–
47	1	No
53	4	Yes
59	2	Yes
61	1	Yes

p	L	$(p^L-1)\cdot\frac{1}{5}\in\mathbb{N}$?
67	1	No
71	∞	–
73	3	No
79	2	Yes
83	2	No
97	1	No
101	1	Yes

Figure 1 Some data on F-pure thresholds of $f=x^{15}+xy^7\in\mathbb{L}[x,y]$

so that it follows from Lemma 7.5 that $\operatorname{fpt}(f+g)>\frac{3}{17}$. For another example of this behavior, it can be computed that when $p=47$ and $g\in\{x^{12}y^2, x^{10}y^3, x^8y^4, x^4y^6, x^9y^4, x^{10}y^4\}$, $\operatorname{fpt}(f+g)>\operatorname{fpt}(f)$.

We further show that the F-pure threshold function $f\mapsto\operatorname{fpt}(f)$ is locally constant in the $\mathfrak{m}$-adic topology at all homogeneous polynomials with an isolated singularity at the origin.

PROPOSITION 7.10. *Suppose that $f\in\mathbb{L}[x_1,\dots,x_n]$ is homogeneous under some $\mathbb{N}$-grading such that $\sqrt{\operatorname{Jac}(f)}=(x_1,\dots,x_n)$ and* $\deg f\geq\sum\deg x_i$. *Then* $\operatorname{fpt}(f+g)=\operatorname{fpt}(f)$ *for each $g\in[R]_{\geq n\deg f-\sum\deg x_i+1}$.*

Proof. Let $\lambda=\sum\deg x_i/\deg f$. If $\operatorname{fpt}(f)=\lambda$, then Proposition 7.8 implies that $\operatorname{fpt}(f)=\operatorname{fpt}(f+g)$. For the remainder of this proof, we will assume that $\operatorname{fpt}(f)\neq\lambda$. By Proposition 7.4 it suffices to show that $\operatorname{fpt}(f)\geq\operatorname{fpt}(f+g)$. Since $\operatorname{fpt}(f)=\langle\lambda\rangle_L-E/p^L$ for some integers $E\geq 0$ and $L\geq 1$ by Theorem 3.5, it suffices to show that $(f+g)^{p^L\operatorname{fpt}(f)}\in\mathfrak{m}^{[p^L]}$ by Lemma 7.5.

To this end, note that

$$\operatorname{fpt}(f)=\langle\lambda\rangle_L-\frac{E}{p^L}\geq\lambda-\frac{1}{p^L}-\frac{E}{p^L}\geq\lambda-\frac{n-1}{p^L}, \tag{7.4}$$

where the first inequality follows from Lemma 2.6, and the second from our bounds on E. Suppose, by way of contradiction, that $(f+g)^{p^L \cdot \mathrm{fpt}(f)} \notin \mathfrak{m}^{[p^L]}$. Since

$$(f+g)^{p^L \cdot \mathrm{fpt}(f)} = f^{p^L \cdot \mathrm{fpt}(f)} + \sum_{k=1}^{p^L \cdot \mathrm{fpt}(f)} \binom{p^L \cdot \mathrm{fpt}(f)}{k} f^{p^L \cdot \mathrm{fpt}(f) - k} g^k,$$

the inequality $p^L \cdot \mathrm{fpt}(f) > p^L \langle \mathrm{fpt}(f) \rangle_L = \nu_f(p^L)$ implies that $f^{p^L \cdot \mathrm{fpt}(f)} \in \mathfrak{m}^{[p^L]}$, and so there must exist $1 \le k \le p^L \cdot \mathrm{fpt}(f)$ for which $f^{p^L \cdot \mathrm{fpt}(f) - k} g^k \notin \mathfrak{m}^{[p^L]}$. We will now show, as in the proof of Lemma 7.7, that this is impossible for degree reasons. Indeed, for such a k, there exists a supporting monomial μ of $f^{p^L \mathrm{fpt}(f) - k} g^k$ not contained in $\mathfrak{m}^{[p^L]}$, so that $\deg \mu \le (p^L - 1) \cdot \sum \deg x_i$ by Lemma 7.6. However, since $g \in [R]_{\ge n \cdot \deg f - \sum \deg x_i + 1}$,

$$\deg \mu \ge \deg f \cdot (p^L \cdot \mathrm{fpt}(f) - k) + k \cdot \left(n \cdot \deg f - \sum \deg x_i + 1 \right). \tag{7.5}$$

The derivative with respect to k of the right-hand side of (7.5) is $(n-1) \deg f - \sum \deg x_i + 1$, which is always nonnegative by our assumption that $\deg f \ge \sum \deg x_i$. Thus, the right-hand side of (7.5) is increasing with respect to k, and since $k \ge 1$,

$$\begin{aligned} \deg \mu &\ge \deg f \cdot (p^L \cdot \mathrm{fpt}(f) - 1) + \left(n \cdot \deg f - \sum \deg x_i + 1 \right) \\ &\ge \deg f \cdot (p^L \cdot \lambda - n) + \left(n \cdot \deg f - \sum \deg x_i + 1 \right) \\ &= p^L \cdot \deg f \cdot \lambda - \sum \deg x_i + 1 \\ &= (p^L - 1) \cdot \sum \deg x_i + 1, \end{aligned}$$

where the second inequality is a consequence of (7.4). Thus, we have arrived at a contradiction, and we conclude that $(f+g)^{p^L \cdot \mathrm{fpt}(f)} \in \mathfrak{m}^{[p^L]}$, completing the proof. □

Acknowledgments. We are indebted to Bhargav Bhatt and Anurag Singh; the latter shared ideas on their work during the Midwest Commutative Algebra and Geometry Conference at Purdue University in 2011 that would eventually form a foundation of our approach. We also thank Benjamin Weiss and Karen Smith for comments on an earlier draft. Finally, much of the authors' collaborations took place at the University of Michigan, the University of Minnesota, and the Mathematical Sciences Research Institute; we thank these institutions for their hospitality. The authors are grateful to the anonymous referee for useful comments.

References

[BS15] B. Bhatt and A. K. Singh, *The F-pure threshold of a Calabi–Yau hypersurface,* Math. Ann. 362 (2015), no. 1–2, 551–567.

[BL04] M. Blickle and R. Lazarsfeld, *An informal introduction to multiplier ideals,* Trends in commutative algebra, Math. Sci. Res. Inst. Publ., 51, pp. 87–114, Cambridge Univ. Press, Cambridge, 2004.

[BMS08] M. Blickle, M. Mustaţă, and K. E. Smith, *Discreteness and rationality of F-thresholds,* Michigan Math. J. 57 (2008), 43–61. Special volume in honor of Melvin Hochster.

[BMS09] ______, *F-thresholds of hypersurfaces,* Trans. Amer. Math. Soc. 361 (2009), no. 12, 6549–6565.

[BST15] M. Blickle, K. Schwede, and K. Tucker, *F-singularities via alterations,* Amer. J. Math. 137 (2015), no. 1, 61–109.

[B+09] M. Blickle, K. Schwede, S. Takagi, and W. Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties,* Math. Ann. 347 (2010), no. 4, 917–949.

[BMS06] N. Budur, M. Mustaţă, and M. Saito, *Roots of Bernstein–Sato polynomials for monomial ideals: a positive characteristic approach,* Math. Res. Lett. 13 (2006), no. 1, 125–142.

[EM06] L. Ein and M. Mustaţă, *Invariants of singularities of pairs,* International congress of mathematicians, Vol. II, pp. 583–602, Eur. Math. Soc., Zürich, 2006.

[Elk87] N. D. Elkies, *The existence of infinitely many supersingular primes for every elliptic curve over* $\mathbf{Q}$*,* Invent. Math. 89 (1987), no. 3, 561–567.

[Fed83] R. Fedder, *F-purity and rational singularity,* Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480.

[Har98] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius maps,* Amer. J. Math. 120 (1998), no. 5, 981–996.

[Har06] ______, *F-pure thresholds and F-jumping exponents in dimension two,* Math. Res. Lett. 13 (2006), no. 5–6, 747–760. With an appendix by Paul Monsky.

[HW02] N. Hara and K.-I. Watanabe, *F-regular and F-pure rings vs. log terminal and log canonical singularities,* J. Algebraic Geom. 11 (2002), no. 2, 363–392.

[HY03] N. Hara and K.-I. Yoshida, *A generalization of tight closure and multiplier ideals,* Trans. Amer. Math. Soc. 355 (2003), no. 8, 3143–3174 (electronic).

[Her12] D. J. Hernández, *F-purity of hypersurfaces,* Math. Res. Lett. 19 (2012), no. 2, 1–13.

[Her14] ______, *F-pure thresholds of binomial hypersurfaces,* Proc. Amer. Math. Soc. 142 (2014), no. 7, 2227–2242.

[Her15] ______, *F-invariants of diagonal hypersurfaces,* Proc. Amer. Math. Soc. 143 (2015), no. 1, 87–104.

[Kol97] J. Kollár, *Singularities of pairs,* Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, pp. 221–287, Amer. Math. Soc., Providence, RI, 1997.

[MTW05] M. Mustaţă, S. Takagi, and K.-I. Watanabe, *F-thresholds and Bernstein–Sato polynomials,* European congress of mathematics, pp. 341–364, Eur. Math. Soc., Zürich, 2005.

[MZ13] M. Mustaţă and W. Zhang, *Estimates for F-jumping numbers and bounds for Hartshorne–Speiser–Lyubeznik numbers,* Nagoya Math. J. 210 (2013), 133–160.

[Sch07] K. Schwede, *A simple characterization of Du Bois singularities,* Compos. Math. 143 (2007), no. 4, 813–828.

[Sch08] ______, *Generalized test ideals, sharp F-purity, and sharp test elements,* Math. Res. Lett. 15 (2008), no. 6, 1251–1261.

[STZ12] K. Schwede, K. Tucker, and W. Zhang, *Test ideals via a single alteration and discreteness and rationality of F-jumping numbers,* Math. Res. Lett. 19 (2012), no. 1, 191–197.

[Sil09] J. H. Silverman, *The arithmetic of elliptic curves,* second edition, Grad. Texts in Math., 106, Springer, Dordrecht, 2009.

[Smi97] K. E. Smith, *F-rational rings have rational singularities,* Amer. J. Math. 119 (1997), no. 1, 159–180.

[Smi00] ______, *The multiplier ideal is a universal test ideal,* Comm. Algebra 28 (2000), no. 12, 5915–5929. Special issue in honor of Robin Hartshorne.

[Tak04] S. Takagi, *F-singularities of pairs and inversion of adjunction of arbitrary codimension,* Invent. Math. 157 (2004), no. 1, 123–146.

[TW04] S. Takagi and K.-I. Watanabe, *On F-pure thresholds,* J. Algebra 282 (2004), no. 1, 278–297.

D. J. Hernández
Department of Mathematics
University of Utah
Salt Lake City, UT 84112
USA

dhernan@math.utah.edu

L. Núñez-Betancourt
Department of Mathematics
University of Virginia
Charlottesville, VA 22904
USA

lcn8m@virginia.edu

E. E. Witt
Department of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA

ewitt@umn.edu

W. Zhang
Department of Mathematics,
Statistics, and Computer Science
University of Illinois at Chicago
851 S. Morgan St.
Chicago, IL 60607
USA

wlzhang@uic.edu