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On IHS Fourfolds with $b_2 = 23$

Grzegorz Kapustka

(*with an appendix written jointly with* Michał Kapustka)

Abstract. The present work is concerned with the study of four-dimensional irreducible holomorphic symplectic manifolds with second Betti number 23. We describe their birational geometry and their relations to EPW sextics.

1. Introduction

By an *irreducible holomorphic symplectic* (*IHS*) *fourfold* we mean (see [B1]) a four-dimensional simply connected Kähler manifold with trivial canonical bundle that admits a unique (up to a constant) closed nondegenerate holomorphic 2-form and is not a product of two manifolds. These manifolds are among the building blocks of Kähler fourfolds with trivial first Chern class [B1, Thm. 2]. In the case of four-dimensional examples their second Betti number b_2 is bounded, and $3 \leq b_2 \leq 8$ or $b_2 = 23$ (see [Gu]). There are however only two known families of IHSs in this dimension, one with $b_2 = 7$ and the other with $b_2 = 23$ [B1]. The first is the deformation of the Hilbert scheme of two points on a K3 surface, and the second is the deformation of the Hilbert scheme of three points that sum to 0 on an Abelian surface.

In this paper we address the problem of classification of IHS fourfolds X with $b_2 = 23$. This program was initiated by O'Grady, whose purpose is to prove that IHS fourfolds that are numerically equivalent to the Hilbert scheme of two points on a K3 surface are deformation equivalent to this Hilbert scheme (are of Type $\mathrm{K3}^{[2]}$).

It is known from [V] and [Gu] that for IHS fourfolds with $b_2 = 23$, the cup product induces an isomorphism

$$\mathrm{Sym}^2 H^2(X, \mathbb{Q}) \simeq H^4(X, \mathbb{Q}) \tag{1.1}$$

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and that $H^3(X,\mathbb{Q})=0$. By [F] the Hodge diamond admits additional symmetries, and by [S] it has the following shape:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 232 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & & 1 & & & &
\end{array}
$$

Recall that for an IHS fourfold X, we can find a (Fujiki) constant c such that for $\alpha \in H^2(X,\mathbb{Z})$, we have $cq(\alpha)^2 = \int \alpha^4$ where q is a primitive integral quadric form called the Beauville–Bogomolov form defining a lattice structure on $H^2(X,\mathbb{Z})$ called the Beauville–Bogomolov (for short, B–B) lattice.

In order to classify IHS fourfolds with $b_2 = 23$, we have to find the possible lattices and the possible Fujiki invariants for the given lattice. Next, for a fixed Fujiki invariant and B–B lattice, find all deformation families of IHS manifolds with the given numerical data. Note that the lattices for the known examples are even but not unimodular.

The plan of the paper is the following. We show that each ample divisor on an IHS fourfold X with $b_2(X) = 23$ has self-intersection that is an integer of the form $12k^2$ for some $k \in \mathbb{N}$. Next, we study the case where X admits a divisor H with $H^4 = 12$, that is, the minimal possible self-intersection. The first possibility to consider is when H defines a birational morphism $\varphi_{|H|}\colon X \to \mathbb{P}^5$ into a hypersurface of degree 12. Recall that the ideal of the conductor of $\varphi_{|H|}$ then defines a scheme structure C on the singular locus of the image $\varphi_{|H|}(X) \subset \mathbb{P}^5$. It is known that $C \subset \mathbb{P}^5$ is Cohen–Macaulay of pure dimension 3.

Recall that an *EPW sextic* $S_A \subset \mathbb{P}^5 =: \mathbb{P}(W)$ is a special sextic hypersurface defined as the determinant of the morphism

$$A \otimes \mathcal{O}_{\mathbb{P}^5} \to \Omega^2_{\mathbb{P}^5}(3) \subset \mathbb{P}(W) \times \bigwedge^3 W \tag{1.2}$$

corresponding to the choice of a 10-dimensional Lagrangian $A \subset \bigwedge^3 W$ with respect to the natural symmetric form (as in [EPW, Ex. 9.3]). Furthermore, following O'Grady, we denote

$$\Theta_A = \{V \in G(3,W) \mid V \in G(3,W) \cap \mathbb{P}(A) \subset \mathbb{P}(\bigwedge^3 W)\}. \tag{1.3}$$

The set Θ_A is empty for a generic choice of A and generally measures how singular the EPW sextic is. Recall that EPW sextics were also constructed by O'Grady [O1] as quotients by an involution of an IHS fourfold deformation equivalent to $\mathrm{Hilb}^2(S)$ where S is a K3 surface that admits a polarization of degree 12. Our main result is the following:

THEOREM 1.1. *Suppose that an IHS fourfold X with $b_2 = 23$ admits an ample divisor with $H^4 = 12$ such that H defines a birational morphism $\varphi_{|H|}$. Then there is a unique sextic containing the singular scheme $C \subset \mathbb{P}^5$ of $\varphi_{|H|}(X) \subset \mathbb{P}^5$ defined before. Moreover, this sextic is an EPW sextic that we denote by S_A (we call it the EPW hypersurface adjoint to the image $\varphi_{|H|}(X) \subset \mathbb{P}^5$).*

When H is fixed, we denote $\varphi := \varphi_{|H|}$ and $X' = \varphi(X) \subset \mathbb{P}^5$. Our approach to the study of the embedding $C \subset \mathbb{P}^5$ is to use the methods of homological algebra described in [EFS; EPW]. In Section 4 we show that the unique adjoint EPW sextic S_A obtained in Theorem 1.1 has to be special.

PROPOSITION 1.2. *Suppose that an IHS fourfold X with $b_2 = 23$ admits an ample divisor with $H^4 = 12$ such that H defines a birational morphism $\varphi_{|H|}$. Then the sextic S_A adjoint to the image $X' \subset \mathbb{P}^5$ is an EPW sextic that is not generic. More precisely, if we denote by Θ_A the set defined by (1.3), then $\Theta_A \neq \emptyset$.*

This proposition suggests in fact that the morphism $\varphi_{|H|}$ is never birational. Indeed, Proposition 1.2 implies that for a fixed sextic S_A that is adjoint to the image of an IHS manifold, there is an at least one-dimensional family of polarized IHS fourfolds X such that S_A is the adjoint hypersurface to $\varphi_{|H|}(X) \subset \mathbb{P}^5$.

The idea of the proof of the proposition is the following: Suppose that S_A with $\Theta_A = \emptyset$ is the adjoint hypersurface to $X' \subset \mathbb{P}^5$. Then we show that S_A is normal, and we construct a natural desingularization $\pi: V \to S_A$ described in Section 4.1. We obtain a contradiction by considering the pull-back $\pi^*(X' \cap S_A)$ on V using the knowledge of the Picard group of V and the natural duality of V. In the Appendix we present technical results used in the proofs concerning the geometry of the orbits of the natural PGL(6) action on $\mathbb{P}(\bigwedge^3 \mathbb{C}^6)$.

This work is motivated by the study of the following question of Beauville [B, p. 4].

PROBLEM 1.3. Is each IHS fourfold with $b_2 = 23$ deformation equivalent to $\mathrm{Hilb}^2(S)$ where S is a K3 surface?

More precisely, we are motivated by the special case of this question, called the O'Grady conjecture [O]: Show that if an IHS fourfold X is numerically equivalent to $S^{[2]}$ where S is a K3 surface (i.e., the Fujiki invariant c is 3, and $(H^2(X,\mathbb{Z}), q)$ is isometric to $U^3 \oplus E_8^2 \oplus \langle -2 \rangle$ with the standard notation), then it is deformation equivalent to it.

If an IHS fourfold X satisfies the assumptions of the O'Grady conjecture, then we have $b_2(X) = 23$, and it is proven in [O] that X is either of type $\mathrm{K3}^{[2]}$ or is deformation equivalent to a polarized manifold (X_0, H_0) (satisfying the conditions of [O6, Claim 4.4]) such that $\varphi_{|H_0|}$ is a birational map whose image is a hypersurface of degree $6 \leq d \leq 12$. O'Grady conjectured that the latter case cannot happen. In [K] we showed that $d \geq 9$ and that $|H_0|$ has at most three isolated base points. The case where $\varphi_{|H_0|}$ is a birational morphism is where the method of [K] cannot work; see also [O6, Claim 4.9]. Applications of our results to the O'Grady conjecture in the case where $\varphi_{|H_0|}$ is a birational morphism will be discussed in Section 5.

2. Preliminaries

It was shown in [Hu] that there are a finite number of deformation types of hyper-Kähler manifolds with fixed form $H^2(X, Z) \ni \alpha \mapsto \int \alpha^2 c_2 \in \mathbb{Z}$. In a similar way we obtain the following:

PROPOSITION 2.1. *Let X be an IHS fourfold with $b_2 = 23$. The Fujiki constant of X is an integer of the form $3n^2$ for some $n \in \mathbb{N}$. In particular, the minimal degree of the self-intersection H^4 of an ample divisor $H \subset X$ is 12, and in this case $h^0(\mathcal{O}_X(H)) = 6$.*

Proof. First, from the Hirzebruch–Riemann–Roch theorem for IHS fourfolds we infer that

$$h^0(\mathcal{O}_X(H)) = \chi(\mathcal{O}_X(H)) = \frac{1}{24}H^4 + \frac{1}{24}c_2(X)H^2 + \chi(\mathcal{O}_X). \tag{2.1}$$

Next, by the formula of Hitchin and Sawon we deduce that

$$(c_2(X) \cdot \alpha^2)^2 = 192 \int \sqrt{\hat{A}(X)} \cdot \int \alpha^4$$

for any class $\alpha \in H^2(X, \mathbb{R})$, where the $\hat{A}$-genus in our case is just the Todd genus of X.

We claim that $\int \sqrt{\hat{A}(X)}$ is independent of X with $b_2(X) = 23$. Indeed, by the Riemann–Roch formula, as in [HS], we have

$$\sqrt{\hat{A}}(X) = \frac{1}{2}\hat{A}_2(X) - \frac{1}{8}\hat{A}_1^2(X),$$

where $\hat{A}_1(X) = \frac{1}{12}c_2$ and $\hat{A}_2(X) = \frac{1}{720}(3c_2^2 - c_4)$. It remains to show that $c_2^2(X) = 828$. But this follows from the fact that $c_4 = 324$ and $\hat{A}_2 = 3$. This proves the claim.

We also deduce that $(H^2.c_2(X))^2/H^4 = 300$, so $\sqrt{300H^4} \in \mathbb{N}$. It follows that $H^4 = 3k^2$. On the other hand, from (2.1) we deduce that $k^2/8 + 10k/8 \in \mathbb{N}$, and thus k is even.

Let us now take an element $\alpha \in H^2(X, \mathbb{Z})$ with positive square. Then there exist a deformation Y of X and a Gauss–Manin deformation $\beta \in H^{1,1}(Y, \mathbb{Z})$ of α such that $\pm\beta$ is ample (Huybrehts projectivity criterion). In particular, we infer $\alpha^4 = 12m^2$, where $m \in \mathbb{Z}$ with α of positive square; so also for all $\alpha \in H^2(X, \mathbb{Z})$. We conclude that the Fujiki constant is of the form $3n^2$. □

REMARK 2.2. For an IHS manifold X with $b_2(X) = 23$ to admit an ample divisor with $H^4 = 12$, there are two possibilities:

- The Fujiki invariant is 3, the B–B lattice is even, and there exists $h \in H^2(X, \mathbb{Z})$ with $(h, h) = 2$.

• The Fujiki invariant is 12, and there exists $h \in H^2(X, \mathbb{Z})$ with $(h, h) = 1$.

It is a natural problem to decide whether the latter case can occur.

3. The Proof of Theorem 1.1

The idea of the proof of our theorem is to construct a quadratic symmetric sheaf $\mathcal{F}$ on the unique sextic S_A containing the scheme $C \subset \mathbb{P}^5$ (we know that the sextic is unique from [K]). We extract $\mathcal{F}$ from a natural resolution of $\varphi_*(\mathcal{O}_X(2))$. The first step will be to find a "symmetric" resolution of $\varphi_*(\mathcal{O}_X(2))$. The second is to restrict this resolution in order to find the equation of the adjoint sextic.

We find that $\varphi \colon X \to X' \subset \mathbb{P}^5 = \mathbb{P}(W)$ is a birational morphism and a finite map onto a hypersurface of degree 12. Let us consider the Beilinson monad $\mathcal{M}$ applied to $\varphi_*(\mathcal{O}_X(2))$. This is the following complex:

$$\cdots \to \bigoplus_{j=0}^{5} H^j(\varphi_*(\mathcal{O}_X(2+e-j))) \otimes \Omega^{j-e}_{\mathbb{P}^5}(j-e) \to \cdots$$

(see [EFS] and [DE]). We have $H^j(\varphi_*(\mathcal{O}_X(2-k))) = H^j(\mathcal{O}_X(2-k))$ since φ is finite. Let us write the monad $\mathcal{M}$ in the following form:

$$\begin{matrix} H^4(\mathcal{O}_X(-3)) & H^4(\mathcal{O}_X(-2)) & H^4(\mathcal{O}_X(-1)) & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & H^0(\mathcal{O}_X(1)) & H^0(\mathcal{O}_X(2)) \end{matrix}$$

From [EFS, Cor. 6.2] the maps in the last row correspond to the natural multiplication map $W \otimes H^0(\mathcal{O}(k)) \to H^0(\mathcal{O}(k+1))$. Since by a result of Guan [Gu] we have $\operatorname{Sym}^2 H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_X(2))$, the maps in the last row correspond to the maps in the Beilinson monad of $\mathcal{O}_{\mathbb{P}^5}(2)$. Moreover, we denote by A a vector space such that $A^\vee \oplus \operatorname{Sym}^3 H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_X(3))$. Then analogously the natural complex

$$0 \to \Omega^3_{\mathbb{P}^5}(3) \to \Omega^2_{\mathbb{P}^5}(2) \otimes W \to \Omega^1_{\mathbb{P}^5}(1) \otimes \operatorname{Sym}^2 W \to \mathcal{O} \otimes \operatorname{Sym}^3 W$$

is exact and is a free resolution of $\mathcal{O}_{\mathbb{P}^5}(3)$. Its Serre dual can be seen as a part of the first row of the monad.

We claim that our Beilinson monad is cohomologous to the following (cf. [CS]):

$$\begin{matrix} \Omega^5_{\mathbb{P}^5}(5) \otimes A \oplus \mathcal{O}_{\mathbb{P}^5}(-4) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega^2_{\mathbb{P}^5}(2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{O}_{\mathbb{P}^5}(2) \end{matrix}$$

Let us consider the complex $\mathcal{T}$ constructed from the bottom row of $\mathcal{M}$,

$$\mathcal{T}\colon 0 \to \Omega^2_{\mathbb{P}^5}(2) \to \Omega^1_{\mathbb{P}^5}(1) \otimes H^0(\mathcal{O}_X(1)) \to \mathcal{O}_{\mathbb{P}^5} \otimes H^0(\mathcal{O}_X(2)) \to 0.$$

It is naturally a subcomplex of $\mathcal{M}$ such that the quotient complex is denoted by $\mathcal{M}'$. We have an exact sequence of complexes

$$0 \to \mathcal{T} \to \mathcal{M} \to \mathcal{M}' \to 0. \tag{3.1}$$

Denote now by $\mathcal{N}$ the complex obtained by replacing the bottom row of $\mathcal{M}$ by $\mathcal{O}_{\mathbb{P}^5}(2)$, that is,

$$\Omega^5_{\mathbb{P}^5}(5) \otimes H^4(\mathcal{O}_X(-3)) \to \mathcal{O}_{\mathbb{P}^5}(2) \oplus \Omega^2_{\mathbb{P}^5}(2) \oplus \Omega^4_{\mathbb{P}^5}(4) \otimes H^4(\mathcal{O}_X(-2))$$
$$\to \Omega^3_{\mathbb{P}^5}(3) \otimes H^4(\mathcal{O}_X(-1)) \to \Omega^2_{\mathbb{P}^5}(2).$$

This complex also maps surjectively onto $\mathcal{M}'$ with kernel $\mathcal{K}$; we thus obtain another exact sequence of complexes:

$$0 \to \mathcal{K} \to \mathcal{N} \to \mathcal{M}' \to 0. \tag{3.2}$$

From the long exact homology sequence associated with (3.1) we infer that the only nonzero homology spaces are

$$H_1(\mathcal{M}') \to \mathcal{O}_{\mathbb{P}^5}(2) \to \varphi_*\mathcal{O}_X(2) \to H_0(\mathcal{M}').$$

Looking now at the second sequence (3.2), we infer that the only nonzero homology of $\mathcal{N}$ is in degree 0. We deduce from the 5-lemma that this term is isomorphic to $\varphi_*\mathcal{O}_X(2)$. We treat the upper row of our monad similarly and deduce our claim.

So from [EFS, Thm. 6.1] we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{F} \Omega^2_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5} \to \varphi_*(\mathcal{O}_X) \to 0, \tag{3.3}$$

where A is the 10-dimensional vector space, dual to the quotient of $H^0(\mathcal{O}_X(3))$ by the cubics of $\mathbb{P}^5$.

We shall show that the sheaf $\varphi_*(\mathcal{O}_X)$ is symmetric, so that we can apply the results of [EPW] and find that we can choose the map F as symmetric as possible.

LEMMA 3.1. *There exists a symmetric isomorphism*

$$a\colon \varphi_*(\mathcal{O}_X(3)) \to \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5}).$$

Proof. By relative duality, $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \omega_{X'}) = \varphi_*(\mathcal{O}_X(3))$. Now applying the functor $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \cdot)$ to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}(6) \to \omega_{X'} \to 0,$$

we obtain

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \omega_{X'}) \to \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_{\mathbb{P}^5}(-6))$$
$$\xrightarrow{k} \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \mathcal{O}_{\mathbb{P}^5}(6)),$$

where k is locally given by multiplication by the equation of $X' \subset \mathbb{P}^5$, so it is 0. From the projection formula we obtain an isomorphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(-3)), \omega_{X'}) \to \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5}).$$

To see that a is symmetric, we repeat the arguments from [CS, §2]. First, we get

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\mathcal{O}_{X'}(3), \mathcal{O}_{X'}(3)) = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^5}}(\mathcal{O}_{X'}, \mathcal{O}_{X'}) = \mathbb{C}.$$

Next, $a' = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^5}}(a, \mathcal{O}_{\mathbb{P}^5}) = \lambda a$; but $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^5}}(a', \mathcal{O}_{\mathbb{P}^5}) = a$, thus $\lambda^2 = 1$, so $\lambda = \pm 1$. If $\lambda = -1$, then $\varphi_*(\mathcal{O}_X(3))$ is skew-symmetric, so arguing as in [CS, §2], we find that the hypersurface $X' \subset \mathbb{P}^5$ is nonreduced; this is a contradiction. It follows that $\lambda = 1$, so a is symmetric. □

Since $S^2(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5})$ is a sum of line bundles, we deduce that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(S^2(\mathcal{O}_{\mathbb{P}^5}(-3) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5}), \mathcal{O}_{\mathbb{P}^5}) = 0.$$

Thus, we deduce as in the proof of [EPW, Thm. 9.2] that there is no obstruction for a^{-1} to be a chain map, so we can find a map ψ that closes the following diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_{\mathbb{P}^5}(-3) \oplus \Omega^3_{\mathbb{P}^5}(3) & \xrightarrow{F^*} & \mathcal{O}_{\mathbb{P}^5}(3) \oplus A^\vee \otimes \mathcal{O}_{\mathbb{P}^5} & \to & \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^5}}(\varphi_*(\mathcal{O}_X(3)), \mathcal{O}_{\mathbb{P}^5}) & \to 0 \\
& \psi^* \downarrow & & \downarrow \psi & & a^{-1} \downarrow & \\
0 \to & \mathcal{O}_{\mathbb{P}^5}(-3) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5} & \xrightarrow{F} & \mathcal{O}_{\mathbb{P}^5}(3) \oplus \Omega^2_{\mathbb{P}^5}(3) & \to & \varphi_*(\mathcal{O}_X(3)) & \to 0
\end{array} \tag{3.4}$$

Now arguing again as in the proof of [EPW, §5], we can choose a chain map such that ψF^* is a symmetric map.

Our aim now is to make the second step: extract from F a map f whose determinant gives the adjoint sextic. We first show that the resolution of the ideal of the conductor of φ is obtained by restricting the resolution in (3.4).

Recall that the conductor of the finite map $\varphi \colon X \to X'$ is the annihilator of the $\mathcal{O}_{X'}$-module $\varphi_*(\mathcal{O}_X)/\mathcal{O}_{X'}$ and is isomorphic to the sheaf $\mathcal{H}om(\varphi_*(\mathcal{O}_X), \mathcal{O}_{X'})$. From [H, 7.2 page 249] we deduce that $\varphi^! \omega_{X'} = \omega_X$, so

$$\varphi_* \mathcal{O}_X = \varphi_*(\varphi^! \omega_{X'}) = \mathcal{H}om_{\mathcal{O}_{X'}}(\varphi_*(\omega_X), \mathcal{O}_{X'}(6)).$$

On the other hand, ω_X is trivial, so the conductor $\mathcal{C}$ is isomorphic to $\varphi_*(\mathcal{O}_X(-6))$. The inclusion $\mathcal{C} \subset \mathcal{O}_{X'}$ can be lifted to a map of complexes:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-12) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5}(-9) & \xrightarrow{F} & \mathcal{O}_{\mathbb{P}^5}(-6) \oplus \Omega^2_{\mathbb{P}^5}(-6) & \longrightarrow & \mathcal{C} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-12) & \xrightarrow{\det F} & \mathcal{O}_{\mathbb{P}^5} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow 0
\end{array}$$

By the mapping cone construction [E, Prop. 6.15] we obtain the nonminimal resolution

$$\begin{aligned}
0 &\to \mathcal{O}_{\mathbb{P}^5}(-12) \oplus A \otimes \mathcal{O}_{\mathbb{P}^5}(-9) \\
&\to \mathcal{O}_{\mathbb{P}^5}(-6) \oplus \Omega^2_{\mathbb{P}^5}(-6) \oplus \mathcal{O}_{\mathbb{P}^5}(-12) \to \mathcal{I}_{C|\mathbb{P}^5} \to 0,
\end{aligned}$$

where $C \subset X' \subset \mathbb{P}^5$ is the subscheme defined by the conductor $\mathcal{C}$. It follows that the following map b is given by restriction of F:

$$0 \to 10\mathcal{O}_{\mathbb{P}^5}(-9) \xrightarrow{b} \Omega^2_{\mathbb{P}^5}(-6) \oplus \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{I}_{C|\mathbb{P}^5} \to 0. \tag{3.5}$$

Recall that C is supported on the singular locus of X'; moreover, it is locally Cohen–Macaulay and has pure dimension 3 and degree 36 (see [K]).

Consider the part of b given by

$$A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega^2_{\mathbb{P}^5}. \tag{3.6}$$

The determinant of this map gives the unique sextic $S_A \subset \mathbb{P}^5$ containing C. Indeed, taking the long exact sequence associated to (3.5) tensorized by $\mathcal{O}_{\mathbb{P}^5}(6)$, we see that the unique sextic containing $C \subset \mathbb{P}^5$ is the image of $H^0(\mathcal{O}_{\mathbb{P}^5}) \subset H^0(\mathcal{O}_{\mathbb{P}^5} \oplus \Omega^2_{\mathbb{P}^5})$.

Since there is no nonzero map $\mathcal{O}_{\mathbb{P}^5}(3) \to \Omega^2_{\mathbb{P}^5}(3)$, the restriction of the diagram (3.4) gives

$$\begin{tikzcd}
\Omega^3_{\mathbb{P}^5}(3) \arrow[r, "f^*"] \arrow[d, "\rho^*"'] & A^\vee \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[d, "\rho"'] \\
A \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r, "f"] & \Omega^2_{\mathbb{P}^5}(3)
\end{tikzcd}$$

where ρf^* is a symmetric map, which is the restriction of ψF^* to $\Omega^3_{\mathbb{P}^5}(3)$. We saw in (3.6) that $\det f$ gives the equation of the adjoint sextic. The cokernels $\mathcal{F}$ and $\mathcal{F}^*$ of f and f^* are sheaves supported on the adjoint sextic. We complete the diagram such that

$$\begin{tikzcd}
0 \arrow[r] & \Omega^3_{\mathbb{P}^5}(3) \arrow[r, "f^*"] \arrow[d, "\rho^*"'] & A^\vee \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r] \arrow[d, "\rho"'] & \mathcal{F}^* \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & A \otimes \mathcal{O}_{\mathbb{P}^5} \arrow[r, "f"] & \Omega^2_{\mathbb{P}^5}(3) \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Since ρf^* is symmetric, we infer that $\mathcal{F}$ is a symmetric sheaf supported on the adjoint sextic $\det f$ with resolution

$$0 \to A \otimes \mathcal{O} \xrightarrow{f} \Omega^2_{\mathbb{P}^5}(3) \to \mathcal{F} \to 0,$$

and thus the adjoint sextic is an EPW sextic (see [EPW, §9.3]).

4. The Proof of Proposition 1.2

For contradiction, suppose that a sextic S_A with $\Theta_A = \emptyset$ can be the adjoint hypersurface, which is the image of an IHS fourfold. Recall that such a generic EPW sextic is singular along a surface of degree 40 with A_1 singularities along this surface. The idea of the proof of our proposition is to construct a resolution V of singularities of S_A and then to consider the pull-back of the fourfold $\varphi_{|H|}(X) \subset \mathbb{P}^5$ on V. We obtain a contradiction by considering the natural duality of V.

We shall first construct the desingularization V in Section 4.1. In Section 4.2 we describe the duality on V. The proof of our proposition is given in Section 4.3.

4.1. Desingularization of EPW Sextic

Let us construct V. First, consider

$$O_2 = \{[\alpha \wedge \omega] \in \mathbb{P}(\textstyle\bigwedge^3 W) \mid \alpha \in W, \omega \in \textstyle\bigwedge^2 W\} \subset \mathbb{P}(\textstyle\bigwedge^3 W),$$

the closure of the second orbit of the action of $\mathrm{PGL}(W)$ on $\mathbb{P}(\bigwedge^3 W)$ (see the Appendix). Note that O_2 is singular along $G(3, W)$; moreover, we have the diagram

$$\mathbb{P}(W) \overset{\pi_1}{\dashleftarrow} O_2 \overset{\pi_2}{\dashrightarrow} \mathbb{P}(W^\vee) \tag{4.1}$$

such that $\pi_1([\alpha \wedge v]) = [\alpha] \in \mathbb{P}(W)$ and $\pi_2([\alpha \wedge v]) = [\alpha \wedge v \wedge v] \in \mathbb{P}(W^\vee)$ for $\alpha \in W$, $\omega \in \bigwedge^2 W$. The maps π_1 and π_2 are rational and defined outside $G(3, W) \subset O_2$ by Lemma A.1.

Now the wedge product $\bigwedge^3 W \oplus \bigwedge^3 W \to \bigwedge^6 W = \mathbb{C}$ induces a skew-symmetric form on $\bigwedge^3 W$. We consider a maximal 10-dimensional Lagrangian subspace $A \subset \bigwedge^3 W$ isotropic with respect to this form. For a fixed A, we define the manifold

$$V' := \mathbb{P}(A) \cap O_2.$$

Proposition 4.1. *The image $\pi_1(V')$ is the EPW sextic S_A associated to A. Moreover, if $\Theta_A = \emptyset$, then V' is a smooth Calabi–Yau fourfold.*

Proof. In order to find the image $\pi_1(V')$, we consider a natural desingularization of O_2 that is the projectivization of the vector bundle $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$. Comparing the following construction with the definition of the EPW sextic given in the introduction (see (1.2)), we deduce the first part of the statement.

Let us describe this desingularization. From [EPW, Thm. 9.2] we can see that $\binom{f^*}{\rho^\vee}$ defines an embedding of $\Omega^3_{\mathbb{P}^5}(3)$ as a symplectic subbundle of $(A \oplus A^\vee) \otimes \mathcal{O}_{\mathbb{P}^5} = \bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}^5}$. On the other hand, from [O1, §5.2] we deduce that we can look at $\bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}^5}$ as a symplectic vector bundle with the symplectic form induced from the wedge product $\bigwedge^3 W \oplus \bigwedge^3 W \to \bigwedge^6 W = \mathbb{C}$ such that the fiber of the subbundle $\Omega^3_{\mathbb{P}^5}(3)$ over $v \in \mathbb{P}^5$ corresponds to the 10-dimensional linear space

$$F_v = \{[v \wedge \gamma] \in \mathbb{P}(\textstyle\bigwedge^3 W) \mid \gamma \in \textstyle\bigwedge^2 W\} \subset \mathbb{P}(\textstyle\bigwedge^3 W).$$

Then ρ^* is given by the considered embedding composed with the quotient map

$$\textstyle\bigwedge^3 W \otimes \mathcal{O}_{\mathbb{P}(W)} \to (\bigwedge^3 W/A) \otimes \mathcal{O}_{\mathbb{P}(W)},$$

where A is a Lagrangian subspace of $\bigwedge^3 W$ (there is a canonical isomorphism $\bigwedge^3 W/A = A^\vee$). More precisely, we have a diagram

$$\begin{tikzcd} \mathbb{P}(\bigwedge^3 W) \supset O_2 \arrow[rr, dotted, "\pi_1"] & & \mathbb{P}^5 \\ & & \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3)) \arrow[ull, "\alpha"] \arrow[u, "\pi"'] \end{tikzcd}$$

such that the image of α is the variety O_2. The first part follows.

Let us prove the second part. The smoothness follows from Proposition A.5. Indeed, suppose that V' is singular at a point p. Then $\mathbb{P}(A)$ intersects the tangent space to O_2 in a nontransversal way along a five-dimensional isotropic subspace Z. By Proposition A.5 the space Z has to cut $G(3, W)$, a contradiction.

Let us find the canonical bundle of V'. Observe that α is given by the complete linear system of the *line bundle* $\mathcal{T} := \mathcal{O}_{\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))}(-1)$. Denote

$$V := \alpha^{-1}(\mathbb{P}(A) \cap O_2).$$

Since V is smooth, V' is isomorphic to V. We find the canonical divisor of V using the adjunction formula and the knowledge of the canonical divisor of $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$. The dimension of the cohomology group $h^1(\mathcal{O}_{V'})$ is found by using the Lefschetz hyperplane theorem. □

REMARK 4.2. Alternatively, for the proof of the last proposition, we can use the results from [O1] to prove that V is the blow-up of the quotient of X_A by an antisymplectic involution. We infer in this way that V is a smooth Calabi–Yau fourfold with Picard group of rank 2.

REMARK 4.3. Denote by E the exceptional divisor of α. It maps to $G(3, W) \subset O_2$ such that the fiber over a point $U \in G(3, W)$ is a projective plane that maps under π to $\mathbb{P}(U) \subset \mathbb{P}(W)$. Moreover, E is isomorphic to the projectivization of the tautological bundle on $G(3, W)$. By Lemma A.6 we deduce that the pull-back $(\alpha \circ \pi_2)^*(H_2)$ is a Cartier divisor in the linear system $|2T - H|$ on $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$. Moreover, E is the base locus of $|2T - H|$ such that, after blowing-up $E \subset \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$, this linear system become base-point-free and factorizes through π_2.

The idea of the proof of Proposition 1.2 is by contradiction. Denote by H the pull-back by $\pi: V \to \mathbb{P}(W)$ of the hyperplane section in $\mathbb{P}(W)$ and $T \in |T|$. From Proposition A.2 and the Lefshetz theorem (or from Remark 4.2) the divisors H and T generate $\mathrm{Pic}(V)$. First, we need the following:

PROPOSITION 4.4. *There exists a divisor $D \subset V$ in the linear system $|3H + T|$ that projects under π to C.*

Proof. We shall show that D is given by the vanishing of a section of the vector bundle $10\mathcal{O}_{\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))}(-1) \oplus (\mathcal{O}_{\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))}(1) \otimes \mathcal{O}_{\mathbb{P}^5}(3))$ on $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$. Recall that the sequence (3.5) defines a codimension 1 subscheme $C \subset S_A$. Let us apply Kempf's idea and pull back $b^\vee$ (where b is defined by (3.5)) by $p\colon \mathbb{P}(10\mathcal{O}_{\mathbb{P}^5}) \to \mathbb{P}^5$. Then, as in [L, Appendix B], we obtain a diagram

$$\begin{tikzcd} p^*\Omega^3_{\mathbb{P}^5}(3) \oplus p^*\mathcal{O}_{\mathbb{P}^5}(-3) \arrow[r, "p^*b^\vee"] \arrow[dr, "v"] & p^*(10\mathcal{O}_{\mathbb{P}^5}) \arrow[d] \\ & \mathcal{O}_{\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})}(1) \end{tikzcd} \tag{4.2}$$

We see that the degeneracy locus of $b^\vee$ can be seen on $\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})$ as the degeneracy of v and thus as the zero section of

$$(\mathcal{O}_{(10\mathcal{O}_{\mathbb{P}^5})}(1) \otimes p^*\mathcal{O}_{\mathbb{P}^5}(3)) \oplus (\mathcal{O}_{(10\mathcal{O}_{\mathbb{P}^5})}(1) \otimes p^*\Omega^2_{\mathbb{P}^5}(3)).$$

Finally, note that the zero scheme of the bundle $\mathcal{O}_{(10\mathcal{O}_{\mathbb{P}^5})}(1) \otimes p^*(\Omega^2_{\mathbb{P}^5}(3))$ defines V set-theoretically and that the restrictions $\mathcal{O}_{(10\mathcal{O}_{\mathbb{P}^5})}(1)|_V$ and $\mathcal{O}_{\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))}(-1)|_V$ are equal. □

Finally, we shall translate geometrical properties of the map $\varphi\colon X \to X' \subset \mathbb{P}^5$ into geometrical properties of the adjoint EPW sextic. Let us also consider the subschemes $N_r \subset X'$ defined by $\mathit{Fitt}_r^{X'}(\varphi_*(\mathcal{O}_X))$ (e.g., $N_1 = C$). Recall that from the results of [MP, §4] the scheme N_2 has a symmetric presentation matrix and is of codimension ≤ 3 if it is nonempty. Moreover, N_2 is supported on points where C is not a locally complete intersection (see [MP, p. 131]). Denote by M_r the degeneracy locus of rank $\leq 10 - r$ of the map

$$A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega^2_{\mathbb{P}^5}.$$

LEMMA 4.5. *The subschemes N_2 and M_2 of $\mathbb{P}^5$ are equal, and the radicals of the schemes N_r and M_r are equal for $r \geq 2$. Moreover, suppose that $p \in M_k - M_{k+1}$. Then for $k \geq 1$, the dimension of the intersection $F_p \cap \mathbb{P}(A)$ is $k-1$.*

Proof. This is an analogous statement to the rank condition (see [CS, Rem. 2.8]). We claim that locally the map F can be seen as a symmetric map. Indeed, in the diagram (3.4) using alternating homotopies as in [EPW, p. 447], we have the freedom of choice of the map ψ. In particular, restricting to an affine neighborhood, we can assume that the matrix $A := F\psi$ is symmetric and that ψ is an isomorphism. Note that the matrix B consisting of the last nine columns of A and the matrix B' which is the last nine rows of B have maximal degeneracy loci defining locally the scheme C and the sextic S_A, respectively (see (3.5)). Since we know that X' has a nonsingular normalization, we can conclude with [KU, Prop. 3.6(3)].

For the second part, we use [KU, Lem. 2.8]. It follows from the proof of Proposition 4.4 that the dimension of the fiber $V \cap \pi^{-1}(p)$ is equal to $k-1$. We conclude by observing that the map α does not contract curves on $\pi^{-1}(p)$. □

4.2. The Duality

Since we have a second fibration π_2 of the variety O_2, it is natural to consider the following picture:

$$\begin{tikzcd}
\mathbb{P}(W) & & O_2 \subset \mathbb{P}(\bigwedge^3 W) \arrow[ll, dotted, "\pi_1"'] \arrow[rr, dotted, "\pi_2"] & & \mathbb{P}(W^\vee) \\
\mathbb{P}(\Omega^3_{\mathbb{P}(W)}(3)) \arrow[u, "\pi"] \arrow[urr, "\alpha"'] & & & & \mathbb{P}(\Omega^3_{\mathbb{P}(W^\vee)}(3)) \arrow[u, "\pi'"'] \arrow[ull, "\alpha'"]
\end{tikzcd} \tag{4.3}$$

Denote by F'_v the closure of the fiber of π_2 and by $\pi_2(V') = S'_A \subset \mathbb{P}(W^\vee)$ the corresponding EPW sextic constructed from A. Denote $\mathcal{O}_{V'}(H_2) := \pi_2^*(\mathcal{O}_{\mathbb{P}(W^\vee)}(1))$. Without lost of generality, we can denote $\mathcal{O}_{V'}(H) := \pi_1^*(\mathcal{O}_{\mathbb{P}(W)}(1))$ (we identify it with the divisor $\pi^*(\mathcal{O}_{\mathbb{P}(W)}(1))$ on V).

LEMMA 4.6. *Assume that the sextic $S_A \subset \mathbb{P}(W)$ is integral. Then $S'_A \subset \mathbb{P}(W^\vee)$ is integral and dual to S_A.*

Proof. It follows from the definition of π_1 and π_2 that $\pi_2(F_v)$ is a hyperplane in $\mathbb{P}(W^\vee)$ that is dual to $v \in \mathbb{P}(W)$. Next, it follows from the description in [O2, Cor. 1.5(2)] of the tangent space T to S_A at a smooth point that there is a point $w \in S'_A$ such that $\pi_1(F'_w) = T$. $\square$

REMARK 4.7. As remarked by O'Grady [O2, §1.3], the map $\pi_2|_{F_v}$ is given by the linear system of Plücker quadrics defining $F_v \cap G(3, W) = G(2, 5) \subset \mathbb{P}^9$. Thus, the fibers of $\pi_2|_{F_v}$ are five-dimensional linear spaces spanned by $G(2, 4) \subset G(2, 5) \subset \mathbb{P}^9$.

4.3. The Proof of (1.2)

The aim of this section is to prove that an EPW sextic S_A constructed by choosing $\mathbb{P}(A)$ disjoint from $G(3, W)$ (i.e., with $\Theta_A = \emptyset$) cannot be the adjoint hypersurface of a birational image of an IHS manifold with $b_2 = 23$.

For contradiction, suppose that S_A can be such a hypersurface. Then for the corresponding Lagrangian space A with $\Theta_A = \emptyset$, the variety $V' = O_2 \cap \mathbb{P}(A)$ is isomorphic to $V = \alpha^{-1}(V')$. Thus, let us identify $V = V'$.

From [O3, Claim 3.7] we deduce that there are only a finite number of planes on V contracted to points by π_1 and there are no higher-dimensional contracted linear spaces. Denote by E and E_2 the exceptional loci of $\pi_1|_V$ and $\pi_2|_V$, respectively, by T the restriction of the hyperplane in $\mathbb{P}(\bigwedge^3 W)$, and by H and H_2 the pull-backs by π_1 and π_2, respectively, of the hyperplane sections in $\mathbb{P}(W)$ and $\mathbb{P}(W^\vee)$. By [O2, Prop. 1.9] the singular locus of S_A is a surface G of degree 40 that is smooth outside the image of the contracted planes. Moreover, S_A has ODP singularities along the smooth locus of G. Hence, E and E_2 are reduced.

Using Proposition A.2 and the Lefschetz theorem [RS, Thm. 1], which works when V is smooth and omits the singular locus of O_2, we deduce that the Picard group of V has rank 2 and is generated by the restrictions of H and H_2.

LEMMA 4.8. *In $CH^1(V) = \mathrm{Pic}(V)$ we have the equalities*

$$H_2 = 5H - E \quad \textit{and} \quad H + H_2 = 2T.$$

Proof. The first equality follows from [Dol, §1.2.2] (see Corrolary A.7), and the second from Lemma A.6. □

Now, using Proposition 4.4, we find a divisor $D \subset V$ in the linear system $|3H+T|$ such that $p(D) = C$. It follows from Lemma 4.5 that $D - E$ is an effective divisor D_1. Let $l \subset V$ be a line contracted by π_2 (such lines cover E_2). Since $l.T = 1$, from Lemma 4.8 we obtain $l.H = 2$. It follows that

$$l.(D - E) = l.(3H + T - E) = l.(T + H_2 - 2H) = -3$$

in $CH^1(V)$. Since D_1 is effective, we infer that $l \subset D_1$; thus, $E_2 \subset D_1$, and $D_1 - E_2$ is effective (E_2 is reduced). We obtain the following equalities in $CH^1(V)$:

$$D_1 - E_2 = T - 4H_2 - H = -3H_2 - T.$$

This is a contradiction because $-3H_2 - T$ cannot be effective.

5. On the O'Grady Conjecture

The aim of this section is to apply the results from the previous sections to prove some special cases of Conjecture 5.3 of O'Grady. In fact, we shall generalize the results of Proposition 1.2 to a special class of IHS fourfolds with $b_2 = 23$ satisfying an additional condition **O** described in the next subsection.

*5.1. IHS Fourfolds with $b_2 = 23$ Satisfying Condition **O***

Let (X, H) be a polarized IHS fourfold with $b_2 = 23$ such that $H^4 = 12$. Consider the following definition.

DEFINITION 5.1. We say that (X, H) satisfies *condition* **O** if for all $D_1, D_2, D_3 \in |H|$ that are independent, the intersection $D_1 \cap D_2 \cap D_3$ is a curve.

Intuitively, condition **O** says that the image $\varphi_{|H|}(X) \subset \mathbb{P}^5$ does not contain planes. Note that this is one of the conditions from [O6, Claim 4.4]. Moreover, each IHS manifold numerically equivalent to $\mathrm{Hilb}^2(S)$, where S is a K3 surface, can be deformed to one that satisfies condition **O**. Motivated by this, we can state the following:

PROBLEM 5.2. Is each IHS fourfold with $b_2 = 23$ deformation equivalent to a polarized IHS fourfold (X_0, H_0) satisfying condition **O** such that $H_0^4 = 12$?

Note that if we find such a deformation, we can repeat the arguments from [O] in order to show that either $\varphi_{|H_0|}$ is the double cover of an EPW sextic (thus, X_0 is of type $\mathrm{K3}^{[2]}$) or X_0 is birational to a hypersurface of degree $12 \geq d \geq 7$, or a $4:1$ morphism to a cubic hypersurface with isolated singularities, or a 3-to-1 morphism to a normal quartic hypersurface, or $\dim \varphi_{|H_0|}(X_0) \leq 3$. It is a natural geometric problem to decide which one of these cases can occur.

5.2. *The O'Grady Conjecture*

In this section we discuss the following conjecture of O'Grady.

CONJECTURE 5.3. *If an IHS fourfold X is numerically equivalent to* $\mathrm{Hilb}^2(S)$ *where S is a K3 surface* (*i.e., $c=3$ and $(H^2(X,\mathbb{Z}),q)$ is isometric to $U^3 \oplus E_8^2 \oplus \langle -2\rangle$ with the standard notation*), *then it is deformation equivalent to it.*

Let X be an IHS manifold numerically equivalent to $S^{[2]}$ where S is a K3 surface. Consider $\mathcal{M}'_X$, a connected component of the moduli space of marked IHS fourfolds deformation equivalent to X, and the surjective period map

$$P\colon \mathcal{M}'_X \to \Omega_L.$$

Then choose an appropriate $\rho \in \Omega_L$ such that $P^{-1}(\rho)$ is an IHS manifold X deformation equivalent to X_0 and $\mathrm{Pic}(X_0) = \mathbb{Z}H_0$ where H_0 is an ample divisor with $H_0^4 = 12$. The special choice of ρ requires X_0 to satisfy condition **O** and additional conditions that are described in [O6, Claim 4.4]. For such (X_0, H_0), O'Grady proved that the linear system $|H_0|$ gives a map $\varphi_{|H_0|}$ of degree ≤ 2 that is either birational onto its image or a special double cover of an EPW sextic. Since this double cover is deformation equivalent to $\mathrm{Hilb}^2(S)$ where S is a K3 surface, his conjecture follows if we prove that $\deg \varphi_{|H_0|} \neq 1$.

If we suppose that $\deg \varphi_{|H_0|} = 1$ (i.e., $\varphi_{|H_0|}$ is a birational map), then O'Grady remarked that the image of $\varphi_{|H_0|}$ is a hypersurface of degree $6 \leq d \leq 12$. In [K] we showed that $d \geq 9$ and $|H_0|$ has at most three isolated base points. Note that if $|H_0|$ has one isolated point, then the scheme defined by the ideal of the conductor of $\varphi_{|H_0|}$ is contained in a unique quintic (containing the singular locus of $\varphi_{|H_0|}$). There is a lot of geometry appearing as discussed in [G].

In this work we consider the case $d = 12$ (i.e., $|H_0|$ has no base points); this is the case where the method of [K] does not work and also the most difficult one from the point of view of O'Grady (see [O6, Claim 4.9]). Then the image of $\varphi_{|H_0|}$ is a nonnormal degree 12 hypersurface $\varphi_{|H_0|}(X') \subset \mathbb{P}(W)$. Our idea is to study the adjoint hypersurface S_A to $X' \subset \mathbb{P}(W)$. We know that it is an EPW sextic, so we can use the classification of such sextics given in [O2; O3; O4], and [IM] in order to describe S_A more precisely. Recall that for S_A, the set Θ_A (defined in (1.3)) is empty for a generic choice of A and if $\Theta_A \neq \emptyset$, then it measures how singular the EPW sextic is. For special A, all the values $0 \leq \dim \Theta_A \leq 6$ can be obtained.

Recall again that each numerical $(\mathrm{K3})^{[2]}$ can be deformed to a polarized IHS fourfold (X_0, H_0) that satisfies condition **O**. Our main result of this section is the following:

PROPOSITION 5.4. *Suppose that a hypersurface $X' \subset \mathbb{P}^5$ of degree 12 is the birational image of a polarized IHS manifold (X, H) with $b_2 = 23$ such that $H^4 = 12$ satisfying $\boldsymbol{O}$ through a morphism given by the complete linear system $|H|$. Let $S_A \subset \mathbb{P}^5$ be the adjoint EPW sextic to the image $X' \subset \mathbb{P}^5$. Then, for S_A, we have either $\dim \Theta_A = 1$, or S_A is the double determinantal cubic, or S_A has a nonreduced linear component.*

The idea of the proof is as follows: We separately treat the cases where $\dim \Theta_A = 0$, $\dim \Theta_A \geq 2$, and $\dim \Theta_A = 1$ (in Sections 5.4, 5.5, and 5.6, respectively). The case $\dim \Theta_A = 0$ is similar to $\Theta_A = \emptyset$. In the other cases, for each point $U \in \Theta_A$, we consider the plane $\mathbb{P}(U) \subset \mathbb{P}^5$ contained in S_A such that S_A is singular along $\mathbb{P}(U)$. Then we consider after O'Grady (see [O4]) the sets $\mathcal{C}_{U,A} \subset \mathbb{P}(U)$ defined further in (5.1). Each $\mathcal{C}_{U,A} \subset \mathbb{P}(U)$ is either the whole plane or the support of some sextic curve $C_{U,A}$. We show that $\mathcal{C}_{U,A}$ has to be contained in X' and thus cannot be a plane (by condition $\mathbf{O}$). We also show that $\mathcal{C}_{U,A}$ must have degree ≤ 3 (see Lemma 5.15). Checking case by case, we exclude all the possibilities with $\dim \Theta_A \geq 2$ except where either S_A is the double determinantal cubic and X' has generically tacnodes along $S_A \cap X'$, or S_A is reducible and equal to $2H_0 + Q$ where H_0 is a hyperplane and Q a quartic such that $H_0 \cap Q$ supports the scheme C defined by the conductor. In particular, in the second case, X' has triple points along C that are not ordinary triple points (see the end of Section 5.5 for a precise description). A new idea is needed to conclude in those cases.

We believe that by the methods of this paper we can also exclude the case $\dim \Theta_A = 1$, but the problem becomes more technical, and we only show that the Lagrangian subspace $A \subset \bigwedge^3 W$ defining S_A cannot be generic in the set of Lagrangian A with $\dim \Theta_A = 1$ (see Section 5.6). Before proving Proposition 5.4, we first introduce some technical results.

5.3. Preliminary Results

For $U \in G(3, W)$, we see that $\pi(\alpha^{-1}(U)) = \mathbb{P}(U) \subset \mathbb{P}(W)$ is the corresponding plane contained in S_A. Let us consider, after O'Grady, the set

$$\mathcal{C}_{U,A} := \{[v] \in \mathbb{P}(U) \mid \dim(F_v \cap \mathbb{P}(A)) \geq 1\}, \tag{5.1}$$

where F_v is the linear space being the closure of the fiber of the map $\pi_1 \colon O_2 \dashrightarrow \mathbb{P}(W)$ at the point $[v]$. There is a natural scheme structure $C_{U,A}$ on $\mathcal{C}_{U,A}$ described in [O4, §3.1] such that $C_{U,A}$ is either a sextic curve or the whole plane $\mathbb{P}(U)$.

PROPOSITION 5.5. *The set $\mathcal{C}_{U,A}$ is contained in $X' \subset \mathbb{P}(W)$. In particular, $\mathcal{C}_{U,A}$ is never equal to $\mathbb{P}(U)$ if (X, H) satisfies condition $\boldsymbol{O}$.*

Proof. First, over the points from the set $\mathcal{C}_{U,A}$, the map

$$A \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \xrightarrow{f} \Omega^2_{\mathbb{P}^5}$$

has corank ≥ 2; so $\mathcal{C}_{U,A} \subset M_2$. But from Lemma 4.5 we have $N_2 = M_2$, and thus

$$C_{U,A} \subset M_2 = N_2 \subset X'.$$

Finally, it follows from condition **O** that $X' \subset \mathbb{P}(W)$ cannot contain any plane. $\square$

DEFINITION 5.6. Recall that O'Grady defined, for $A \in \mathbb{LG}(10, \bigwedge^3 W)$ and $U \in \Theta_A$, the set $\mathcal{B}(U, A)$ of $v \in \mathbb{P}(U)$ such that either

(1) there exists $U' \in (\Theta_A - \{U\})$ such that $v \in \mathbb{P}(U')$, or

(2) $\dim(\mathbb{P}(A) \cap F_v \cap T_U) \geq 1$,

where T_U is the projective tangent space to $G(3, W)$ at U.

LEMMA 5.7. *The curve $C_{U,A} \subset \mathbb{P}(U)$ can have only isolated singularities outside $\mathcal{B}(U, A)$. If $\mathbb{P}(U) \neq \mathcal{C}_{U,A}$, then $\mathcal{B}(U, A) \subset \operatorname{sing} C_{U,A}$. Moreover, if $U_1, U_2 \in \Theta_A$, then $\mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$ intersect as planes in $\mathbb{P}^5$ at the point of intersection $\mathcal{C}_{U_1,A} \cap \mathcal{C}_{U_2,A}$.*

Proof. This is proved in [O4, Cor. 3.2.7]. $\square$

We have the following description of the EPW sextic S_A in the case $\dim \Theta_A = 0$.

LEMMA 5.8. *If $\dim \Theta_A = 0$, then S_A is normal. Moreover, $V = \alpha^{-1}(V')$ and $V' = \mathbb{P}(A) \cap O_2$ are irreducible.*

Proof. Since S_A is locally a complete intersection, the normality of S_A follows from the Serre criterion if S_A is nonsingular in codimension 1. On the other hand, it follows from [O2, §1.3] that S_A is only singular along the sum of the planes $\mathbb{P}(U)$ for $U \in \Theta_A$ and along the set $\mathcal{D}$ such that, for $v \in \mathcal{D}$, we have

$$F_v \cap \mathbb{P}(A) \cap G(3, W) = \emptyset \quad \text{and} \quad \dim(F_v \cap \mathbb{P}(A)) \geq 1.$$

From [O2, Prop. 1.9] we infer that $\mathcal{D}$ is a surface.

Since the intersection of $\mathbb{P}(A)$ with the tangent to O_2 at P is five-dimensional isotropic, we deduce from Proposition A.5 that $\mathbb{P}(A) \cap O_2$ is smooth at

$$P \in (F_v \cap \mathbb{P}(A)) - G(3, W)$$

when $F_v \cap \mathbb{P}(A) \cap G(3, W) = \emptyset$. Thus, we have to show that the dimension of the exceptional set of $\pi\colon V \to S_A$ that maps to $\mathcal{G} := (\bigcup_{U \in \Theta_A} \mathbb{P}(U))_{\mathrm{red}}$ is smaller than 4. From the fact that Θ_A is a finite set it is enough to consider the exceptional set above $\mathcal{C}_{U_0,A} \subset \mathbb{P}(U_0)$ for a fixed $U_0 \in \Theta_A$. Since Θ_A is finite, the fiber $\alpha(\pi^{-1}(v)) \subset F_v$ for a given $v \in \mathcal{C}_{U_0,A}$ intersects $G(2, 5) = G(3, W) \cap F_v$ in a finite number of points. Since the dimension of $G(3, 5) \subset \mathbb{P}^9$ is 6, we infer $\dim \pi^{-1}(v) \leq 3$ for all $v \in \mathcal{C}_{U_0,A}$ and $\dim \pi^{-1}(v) \leq 2$ for a generic $v \in \mathcal{C}_{U_0,A}$. It follows that V' and V are irreducible. $\square$

The map $V \xrightarrow{\alpha} V' = \mathbb{P}(A) \cap O_2$ is an isomorphism outside $\alpha^{-1}(G(3, W))$. Thus, from the previous proof we deduce that if $\dim \Theta_A = 0$, then V can only be singular at points that map to a curve $C_{U,A}$ for some $U \in \Theta_A$.

PROPOSITION 5.9. *If $\dim \Theta_A = 0$, then the varieties $V = \alpha^{-1}(V')$ and $V' = \mathbb{P}(A) \cap O_2$ are nonsingular in codimension 1. Moreover, V is normal.*

Proof. Note that V is locally a complete intersection; thus, it is enough to show the first part. Our aim is to show that the singular points of $\mathbb{P}(A) \cap O_2$ are contained in the sum of the tangent spaces to $G(3, W)$ at points from Θ_A. Next, we show that the intersection of $\mathbb{P}(A) \cap O_2$ with those tangent spaces is of codimension 2.

We need to consider points in the preimage

$$B' := \pi^{-1}(\mathcal{C}_{U,A}).$$

Denote by B an irreducible component of B'. Suppose that for a given U_0, this set is three-dimensional; then either there is a one-parameter family of planes parameterized by $\mathcal{C}_{U_0,A}$, or there is a three-dimensional linear space (i.e., $\mathbb{P}^3$) mapping to a point on $\mathcal{C}_{U_0,A}$. Let us consider the first case; the other is treated similarly.

Suppose that V is singular along B. Then at each $p \in \alpha(B) - G(3, W)$, the space $\mathbb{P}(A)$ does not intersect transversally the tangent plane to O_2. By Proposition A.5 the intersection $\mathbb{P}(A) \cap F_v \cap F'_w$, where $v = \pi_1(p)$ and $w = \pi_2(p)$, contains the line $[p, U'] \subset F_v$ where U' is one of the finite number of points (at most two) in $\mathbb{P}(A) \cap G(3, W) \cap F_v \cap F'_w$.

We claim that for a generic choice of $p \in B$, the line $[p, U']$ is contained in the tangent space to $G(2, 5) \subset F_v$ at U (i.e., $B \subset T_U$). Since Θ_A is finite, for a generic choice of $p \in B$, the line $[p, U']$ with $U' \in \mathbb{P}(A) \cap G(3, W)$ intersects $G(3, W)$ in one point. From Remark 4.7 the line is contained in a five-dimensional linear space $L_p = F_v \cap F'_w$ such that $L_p \cap G(2, 5)$ is a quadric. Since this line intersects $G(3, W)$ in one point, it has to be tangent to $G(2, 5)$. The claim follows.

Let T_U be the projective tangent space to $U \in G(3, W)$, and let $M_U := \mathbb{P}(A) \cap T_U$. The following is a nice exercise.

LEMMA 5.10. *The intersection $K_U := T_U \cap G(3, W)$ can be seen as the set of planes in $\mathbb{P}(W)$ that intersect the plane $\mathbb{P}(U)$ along a line. In particular, K_U has dimension 5 and is a cone over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. The sum of the linear spaces $F_v \cap T_U$ for $v \in \mathbb{P}(U)$ is a cone over the determinantal cubic E_U. Moreover, K_U is the singular set of E_U.*

First, we have $\dim M_U \leq 4$ since otherwise we infer

$$\dim \Theta_A \geq \dim(M_U \cap K_U) \geq 1,$$

contrary to $\dim \Theta_A = 0$. So we have three possibilities: $\dim M_U = 2$, 3, or 4. Note that $F_v \cap T_U$ is the tangent space to $G(2, 5)$ at U and so has dimension 6. If $\dim M_U = 4$, then each linear space $F_v \cap T_U$ for $v \in \mathbb{P}(U)$ intersects $\mathbb{P}(A)$ along a linear space of dimension at least 1 (because such an intersection contains $F_v \cap M_U$). Thus, $\mathcal{C}_{U,A} = \mathbb{P}(U)$, contrary to Proposition 5.5.

So we can assume that $\dim M_U \leq 3$. We saw before that the generic fiber of $\pi|_B \colon B \to C_{U,A}$ is a plane contained in T_U. Since these fibers are contained in M_U and disjoint outside U, we obtain a contradiction. $\square$

Finally, we will use several times the following:

PROPOSITION 5.11. *Suppose that the set of points $v \in \mathbb{P}(W)$ with $\dim(F_v \cap \mathbb{P}(A)) \geq 2$ is a curve $C \subset \mathbb{P}(W)$. Then the tangent space $T_{v_0} \subset \mathbb{P}(W)$ to C at v_0 is perpendicular to the linear space spanned by the image $\pi_2(\mathbb{P}(A) \cap F_v) \subset \mathbb{P}(W^\vee)$.*

Proof. Denote, after O'Grady,

$$\tilde{\Delta}(0) := \{(A, v) \in LG(10, \bigwedge^3 W) : \dim(F_v \cap \mathbb{P}(A)) = 2\}.$$

It was observed by O'Grady that $\tilde{\Delta}(0)$ is smooth and is an open subset of $\tilde{\Delta}$ where we have $\dim(F_v \cap \mathbb{P}(A)) \geq 2$. We know from [O3, Prop. 2.3] the description of the tangent space to $\tilde{\Delta}$. In particular $T_{v_0} = \operatorname{Ker} \tau_K^{v_0}$, where $K := \mathbb{P}(A) \cap F_{v_0}$, in the notation of [O3, eq. (2.1.11)]. It remains to show that the linear space spanned by $\pi_2(K)$ is perpendicular to $\operatorname{Ker} \tau_K^{v_0}$. To see this, note that $\pi_2|_{\mathbb{P}(K)}$ is given by the system of Plücker quadrics $\phi_v^{v_0}$ and use [O3, eq. (2.1.11)]. $\square$

5.4. *The Case Where* $\dim \Theta_A = 0$

The aim of this section is to study the case $\dim \Theta_A = 0$ in Proposition 5.4 by showing that an EPW sextic S_A with $\dim \Theta_A = 0$ cannot be the adjoint hypersurface to the birational image of a polarized IHS fourfold (X, H) with $b_2(X) = 23$ and $H^4 = 12$ satisfying condition **O**.

The closure in V of the exceptional set of the restriction of the morphism π:

$$V \dashrightarrow S_A - \left(\bigcup_{U \in \Theta_A} \mathbb{P}(U) \right)$$

is a reduced Weil divisor E_G that maps to the surface $\operatorname{supp} N_2$. We also have exceptional sets of π over points from $\bigcup_{U \in \Theta_A} \mathbb{P}(U)$. Since $O_2 \cap \mathbb{P}(A)$ is irreducible, we deduce that there are two kinds of irreducible components of the exceptional set of π: either

- ♣ one-parameter families of planes such that the image through π is a curve C_0 that is a component of $C_{U,A} \subset \mathbb{P}(U)$, or
- ♠ three-dimensional linear spaces E_i for $i = 1, \dots, s$ mapping to points in $\mathcal{C}_{U,A} \subset S_A$ for some $U \in G(3, W)$.

We believe that such exceptional sets cannot exist. However, we only prove that the first type of exceptional set cannot occur (this is enough to complete the proof of Proposition 5.4). For this, we need to better understand the duality between S_A and S'_A. It would be nice to find a simpler proof of the following:

LEMMA 5.12. *The morphism π has no exceptional set as in ♣.*

Proof. Suppose that such an exceptional set $G' \subset V$ exists. Denote by

$$G \subset \mathbb{P}(A) \cap O_2$$

the image of G' under α such that each fiber $G \supset G_v = G \cap F_v$ is a plane and G maps to a curve $C_0 \in \mathbb{P}(U_0)$ (which is a component of $C_{U_0,A}$).

We claim that G_v intersects T_{U_0} along a line contained in the determinantal cubic $E_{U_0} \subset T_{U_0}$. Indeed, from the proof of Proposition 5.9 it follows that the generic fiber G_v cannot be contained in T_{U_0}. Next, from [O4, Prop. 3.2.6 (3)] we infer that G_v intersects the tangent space $T_{U_0} \cap F_v$ only at U_0 and is disjoint from Θ_A; thus, C_0 has a node at v. The claim follows since the nodes on C_0 are at isolated points. We also deduce that C_0 is a triple component of $C_{U,A}$, so it is either a multiple conic or a line.

We infer that $G \cap T_{U_0}$ has dimension ≥ 2. From the proof of Proposition 5.9 we know that $\dim(T_{U_0} \cap \mathbb{P}(A)) \leq 3$, so either $G \cap T_{U_0}$ is a plane, or $\dim(T_{U_0} \cap \mathbb{P}(A)) = 3$.

Let us show that the second case cannot happen. Suppose that $\dim(T_{U_0} \cap \mathbb{P}(A)) = 3$. Since $G_v \cap T_{U_0}$ is a line contained in the cubic E_{U_0}, we infer that $G \cap T_{U_0}$ is a cone over a cubic curve $\mathcal{A}$ (which is a section of E_{U_0}). Denote by N a generic hyperplane section of G. Note that N is smooth because it maps under π_1 to a smooth curve with linear spaces as fibers. It follows that N is the projection of a rational normal scroll that has $\mathcal{A}$ as $\mathbb{P}^2$ section. This is only possible when $\mathcal{A}$ is reducible, but then N should be reducible, a contradiction. We deduce that $G \cap T_{U_0}$ is a plane.

Claim 5.13. *The support of the curve C_0 cannot be a line.*

Proof. Suppose the contrary and fix a $v \in C_0$. Since the morphism $\pi_2|_{G_v}$ is given by a linear subsystem of conics with base point $G_v \cap G(3, W)$, is birational, and contracts the line $G_v \cap T_{U_0}$ to a point, we deduce that $\pi_2(G_v)$ is a surface that is an irreducible quadric cone $Q_v \subset \mathbb{P}^5$ tangent to $\mathbb{P}(U_0^\vee)$ along a line with vertex at the image of the contracted line (because the image of a line passing through U_0 on G_v is a line passing through the image of $G_v \cap T_{U_0}$). Consider the rational scroll N and denote by f a generic fiber of $\pi_1|_N$ and by c_0 the section $T_{U_0} \cap N$. We saw that c_0 is a line (since $G \cap T_{U_0}$ is a plane). We have $H|_N = f$ and $H_2|_N = a.f + b.c_0$ for some $a, b \in \mathbb{Z}$.

We have two possibilities: $\pi_2(G)$ is either a quadric surface or a threefold. Let us treat the first case. Suppose that Q_{v_1} and Q_{v_2} are equal for $v_1 \neq v_2$. Since C_0 is a line, $H|_{c_0}$ has degree 1. Next, from $2T = H + H_2$ and $\pi_2(c_0) \subset \mathbb{P}(U_0^\vee) \cap Q_{v_0}$ we infer that $H_2|_{c_0}$ has degree $2 \deg c_0 - 1 \leq 2$. So c_0 is a line. It follows that $N \subset \mathbb{P}(A)$ is embedded by $c_0 + (e+1)f$ where $c_0^2 = -e$ on N. Observe that $\pi_2|_N$ has connected linear fibers that are linear sections of the spaces F'_v. On the other hand, $\pi_2(N) = \pi_2(G) = Q_v$ so $2 = (H_2|_N)^2$ because $\pi_2|_N$ is birational. So using $2T = H + H_2$, we infer $H_2 = 2c_0 + (2e+1)f$, contradicting $4(2e+1) = (H_2|_N)^2$.

It follows that the dimension of $\pi_2(G)$ is 3 and $\pi_2|_N$ is birational. One should have in mind that $\pi_2|_G$ is an isomorphism outside the singular locus

$$\mathbb{G} = G' \cup \bigcup_{U \in \Theta_A} \mathbb{P}(U^\vee)$$

of S'_A. From Proposition 5.11 the tangent line $T_r C_0$ to C_0 at $r \in C_0$ is projectively dual to the space $\mathbb{P}^3_r$ spanned by $\pi(G_r) = Q_r$. We have assumed that C_0 is a line, so the image of $\pi_2(G)$ is a projective space, which we denote by $\mathbb{P}$. Since the double point locus of S'_A is of codimension 2, we infer that $\pi_2|_G$ is birational.

Consider the locus $\mathbb{G}'$ of points $p \in \mathbb{P}$ such that there are two different $v_1, v_2 \in C_0$ with $p \in Q_{v_1} \cap Q_{v_2}$ and $\mathbb{G}' \subset \mathbb{G}$. We shall obtain a contradiction by proving that $\mathbb{G}' = \mathbb{P}$. Fix a generic $v_0 \in C_0$; it is enough to prove that $Q_{v_0} \subset \mathbb{G}'$. When $v \in C_0$ varies, the center of the cone Q_v moves along a curve in $\mathbb{P}(U_0^\vee) \subset \mathbb{P}$ such that Q_v is tangent to $\mathbb{P}(U_0^\vee)$. We conclude by observing that such quadrics cannot be in the same pencil determined by a common quartic curve. $\square$

We deduce that C_0 is a triple conic and $T_{U_0} \cap \mathbb{P}(A)$ is a plane. Consider again the ruled surface N such that c_0 is a line and $N \subset \mathbb{P}(A)$ is embedded by $c_0 + (e+1)f$ for some $e \in \mathbb{Z}$. Then $H|_N = 2f$ so $H_2|_N = 2c_0 + 2e.f$. On the other hand, using again Proposition 5.11, we see that $\pi_2(G) \subset \mathbb{P}(W^\vee)$ is contained in a quadric hypersurface $\mathcal{Q}$ of rank 3. More precisely, $\mathcal{Q}$ is a cone, with a plane $\mathbb{P}(U_0^\vee)$ as vertex, over a conic *curve* $\mathcal{W}$ such that $\mathcal{Q}$ is covered by projective spaces $\mathbb{P}^3_r$ dual to the tangent lines to C_0. It follows that $\pi_2|_G$ is an isomorphism outside $G \cap T_{U_0}$. Consider the pull-back by $\pi_2|_N$ of a generic hyperplane containing $\mathbb{P}(U_0^\vee)$. Since the intersection of the hyperplane with $\mathcal{Q}$ are two projective spaces, the class of the pull-back $H_2|_N$ is $a.c_0 + 2.f$. Using $2T = H + H_2$ (see Lemma 4.8), we compute that $a = 2$ and $e = 1$; thus, N is the blow-up of $\mathbb{P}^2$ in one point with c_0 as exceptional line. Moreover, $\pi_2|_N$ contracts c_0 and maps N to a projective plane. We infer that $\pi_2(N)$ intersects $\mathbb{P}(U_0^\vee)$ at only one point, which is the image of c_0. It also follows that $\pi_2(N)$ is either the second Veronese embedding of $\mathbb{P}^2$ or a smooth central projection of this second Veronese (because $\pi_2(N)$ can be singular only at one point). Consider the curve D_0 that is the generic fiber of the projection of $\pi_2(N)$ with center $\mathbb{P}(U_0^\vee)$ to the curve $\mathcal{W}$. The curve D_0 can be seen as the intersection $\pi_2(N) \cap \mathbb{P}^3_v$ for some generic $v \in C_0$. Since there are no lines or degree 3 curve contained in the projection of the double Veronese and a hyperplane section intersects $\pi_2(N)$ along a degree 4 curve, we deduce that D_0 is an irreducible plane conic. We obtain a contradiction since a smooth conic $D_0 \subset Q_v = \pi(G_v) \subset \mathbb{P}^3_v$ cannot contain the center of the cone Q_v. $\square$

We can now return to the proof of Proposition 5.4. We showed that the exceptional locus of π consists of three-dimensional linear spaces E_i for $i = 1, \dots, s$ mapping to points in some $\mathcal{C}_{U,A} \subset S_A$ for some $U \in G(3, W)$. To obtain a contradiction, we proceed as in the general case. By [Dol, §1.2.2] the rational map between the sextic S_A and its dual S'_A is given by the partial derivatives of the sextic s_A defining S_A. The composition

$$V \xrightarrow{\pi} S_A \dashrightarrow S'_A \subset \mathbb{P}(W^\vee)$$

is given by the linear system induced by the pull-back of quintics that are the partial derivatives of s_A on V. On the other hand, by Remark 4.3, each such generic quintic q' corresponds to an irreducible Cartier divisor $Q' \in |2T - H|$ on V. The divisor Q' coincides with the proper transform of the zero locus $\{q' = 0\} \cap S_A$ on V (they are equal on an open subset of Q'). Recall that S_A has ordinary double points along a generic point of $\operatorname{supp} N_2$. It follows from Lemma 5.12 that

$$\pi^*(Q') = E_G + \sum a_i E_i + B,$$

where $a_i \geq 0$, $B \in |2T - H|$ is an effective Cartier divisor on the normal variety V, E_i are exceptional divisors mapping to points on the singular locus $C \subset X'$, and E_G is the exceptional divisor over $\operatorname{supp} N_2$. We infer that $E_G + \sum a_i E_i$ is a Cartier divisor in the linear system $|6H - 2T|$.

By Proposition 4.4 we find, as in the general case, a divisor $D \subset V$ in the linear system $|3H - T|$ that maps to $C \subset S_A$. From Proposition 4.5 we deduce that D is decomposable such that $D - E_G$ is an effective Weil divisor. We infer that

$$D - \left(E_G + \sum a_i E_i\right)$$

is a Cartier divisor in the linear system $|3T - 3H|$ and denote it by D'. Since the Weil divisor E_i intersects $\alpha^{-1}(U)$ in isolated points, we infer that D' restricts to an effective curve on the plane $\alpha^{-1}(U)$, where $U \in \Theta_A$ is fixed. On the other hand, $\mathcal{O}_V(T)|_{\alpha^{-1}(U)} = \mathcal{O}_{\alpha^{-1}(U)}$ and

$$\mathcal{O}_V(H)|_{\alpha^{-1}(U)} = \mathcal{O}_{\alpha^{-1}(U)}(1).$$

Thus, the restriction of a divisor from $|3T - 3H|$ cannot be an effective curve on $\mathbb{P}(U)$ (see [KM, Prop. 1.35(1)]). It follows that D contains $\alpha^{-1}(U)$, so X' contains $\mathbb{P}(U)$, a contradiction by Proposition 5.5. It follows that the adjoint sextic S_A has $\dim \Theta_A \geq 1$.

5.5. *The Case Where* $\dim \Theta_A \geq 2$

In this section we consider adjoint EPW sextics with $\dim \Theta_A \geq 2$. We show that such a sextic has to be very special as described in Proposition 5.4.

5.5.1. $\dim \Theta_A \geq 3$. We show first that $\dim \Theta_A \geq 3$ cannot happen. Choose an irreducible component Θ'_A of Θ_A. Denote by

$$\mathcal{G} = (\pi(\alpha^{-1}(\Theta'_A)))_{\text{red}}$$

the reduced sum of the planes $\mathbb{P}(U)$ for $U \in \Theta'_A$.

LEMMA 5.14. *If* Θ'_A *has dimension* k *and* $\mathcal{G}$ *has dimension* $\leq k+1$, *then there is a point* $U \in \Theta'_A$ *such that* $\mathcal{C}_{U,A}$ *is a plane.*

Proof. First, $(\alpha^{-1}(\Theta'_A))_{\text{red}}$ is irreducible of dimension $k+2$, so the image $\mathcal{G}$ is irreducible. Suppose it has dimension $\leq k+1$ and all the $\mathcal{C}_{U,A}$ are curves (outside these curves, the fibers of π are points). Then there exists an open set $\mathcal{U} \subset (\alpha^{-1}(\Theta'_A))_{\text{red}}$ such that $\pi|_{\mathcal{U}}$ is $1:1$ onto a proper subset of $\mathcal{G}$, a contradiction since $(\alpha^{-1}(\Theta'_A))_{\text{red}}$ is irreducible. □

Since $\dim \mathcal{G} \leq \dim S_A \leq 4$ and $\dim \Theta_A \geq 3$, we infer that $X' \subset \mathbb{P}(W)$ has to contain a plane, contrary to condition **O**.

5.5.2. $\dim \Theta_A = 2$. The strategy in this case is to show that in many cases the support $\mathcal{C}_{U,A} \subset \mathbb{P}(U)$ has degree ≥ 4. Then we apply several times the following:

LEMMA 5.15. *If $\mathbb{P}(U) \cap X' \subset \mathbb{P}(W)$ has dimension 1, then it supports a cubic curve.*

Proof. If $\dim \Theta_A \leq 1$, then the assertion is a consequence of Proposition 4.4. If $\dim \Theta_A = 2$, then similar arguments apply: For a fixed $U \in \Theta'_A$, the plane $\alpha^{-1}(U) \subset \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$ is a plane that maps under π to $\mathbb{P}(U)$. On the other hand, $\alpha^{-1}(U)$ is contained in $\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})$ such that $\pi^*(\mathcal{O}_{\mathbb{P}^5}(1))$ is equal to the pull-back of $\mathcal{O}_{\mathbb{P}^5}(1)$ on $\mathbb{P}(10\mathcal{O}_{\mathbb{P}^5})$ and

$$\mathcal{O}_{(10\mathcal{O}_{\mathbb{P}^5})}(1)|_{\alpha^{-1}(U)} = \mathcal{O}_{(\Omega^3_{\mathbb{P}^5}(3))}(-1)|_{\alpha^{-1}(U)}.$$

Thus, we can conclude as in Proposition 4.4. □

O'Grady observed also that we can apply the Morin theorem [M]. Indeed, if Θ'_A is an irreducible component of Θ_A of dimension ≥ 1, then it parameterizes mutually intersecting planes in $\mathbb{P}(W)$. By the Morin theorem, Θ'_A is then a linear section of one of the following sets:

(1) $\mathbb{P}^3$ embedded in $G(3, W) \subset \mathbb{P}(\bigwedge^3 W)$ by the double Veronese embedding,
(2) $G(2, 5) \subset F_v \subset G(3, W)$ embedded by the closures of fibers of π_1,
(3) $G(2, 5) \subset F'_v \subset G(3, W)$ embedded by the closures of fibers of π_2,
(4) $T_P \cap G(3, W)$ where T_P is the projective tangent space at P to $G(3, W) \subset \mathbb{P}(\bigwedge^3 W)$,
(5) $\mathbb{P}^2$ embedded in $G(3, W) \subset \mathbb{P}(\bigwedge^3 W)$ by the triple Veronese embedding.

In order to complete the proof of Proposition 5.4, we check case by case the possible two-dimensional irreducible components Θ'_A of Θ_A and find that either:

(I) the adjoint EPW sextic S_A is a double determinantal cubic, or
(II) the EPW sextic $S_A \subset \mathbb{P}(W)$ has a nonreduced component supported on a hyperplane.

In case (I), Θ'_A is the third Veronese embedding of $\mathbb{P}^2$ in $G(3, W) \subset \mathbb{P}(\bigwedge^3 W)$. Case (II) happens, for example, when Θ'_A is a plane. Note that by Lemma 5.14 we can assume that $\mathcal{G}$ is a hypersurface of degree ≤ 3 (because $\mathcal{G}$ is a nonreduced component of S_A). Let us study using Lemma 5.15 each case of the Morin theorem separately.

Case (1) From Lemma 5.14 we deduce that Θ'_A is a hyperplane section of the double Veronese embedding of $\mathbb{P}^3$ (this is the only possibility because there are no planes contained in this double Veronese). It follows from [O2, Claim 1.14] that $\mathcal{G} = (\pi(\alpha^{-1}(\Theta'_A)))_{\text{red}}$ is a smooth quadric, and we have the following:

- from [O5, Prop. 2.1] it follows that $\mathcal{G}$ has multiplicity 2 in the EPW sextic S_A (thus, S_A can be written in the form $2\mathcal{G} + R$ where R is a quadric),
- $R \cap \mathcal{G}$ is contained in the sum of $\mathcal{C}_{U,A}$ for $U \in \Theta'_A$ (because the sextic can be more singular only along such curves),

- the restriction of $\pi : (\alpha^{-1}(\Theta'_A))_{\mathrm{red}} \to \mathcal{G}$ is the blow-up of a plane F contained in $\mathcal{G} \cap R$ $(\alpha^{-1}(\Theta'_A) \to \Theta'_A$ is the restriction of $\mathbb{P}(\Omega^1_{\mathbb{P}^3}(2)) \to \mathbb{P}^3$, and $\pi|_{\alpha^{-1}(\Theta'_A)}$ is given by the system $\mathcal{O}_{\alpha^{-1}(\Theta'_A)}(1))$.

Since the curves $\mathcal{C}_{U,A}$ cover F, we have $F \subset X'$. Since each curve $\mathcal{C}_{U,A}$ is contained in X', this contradicts condition **O**.

Case (2) The planes parameterized by Θ'_A contain the point v and are defined by a line $l_p \subset G(2, V/[v])$. Using [O2, Prop. 2.31], we deduce that Θ'_A is either

(a) a plane or $\Theta'_A \subset G(2, T) \subset G(2, 5)$ where $T \in G(4, 5)$, or
(b) Θ'_A is a linear section of $G(2, 5)$ which is a del Pezzo surface, or
(c) there is a line $l_0 \subset \mathbb{P}(V/[v])$ that intersects all the lines $\mathbb{P}(V/[v])$ parameterized by Θ'_A.

We shall treat each case separately.

Assume (a); then the planes parameterized by Θ'_A cover a hyperplane. This hyperplane has to be a multiple component of S_A, so we are in case (II).

Assume (b), so that Θ'_A is a linear section of $G(2, 5) \subset F_v$. Then Θ'_A is a possibly singular del Pezzo surface D_5 of degree 5 (observe that D_5 cannot be reduced if it has one component because of the degree). Then the sum of the planes parameterized by Θ'_A is a cone over a cubic hypersurface; denote it by **Q**. More precisely, these planes are spanned by the lines corresponding to points on $D_5 \subset G(2, 5)$ (the sum of these lines is a cubic threefold, denote it by $\mathbf{Q}' \subset \mathbb{P}(V/[v])$). It follows that the corresponding EPW sextic is a double cubic. Since $\dim(\mathbb{P}(A) \cap F_v) = 5$, it follows from [O4, Prop. 3.1.2] and [O4, Claim 3.2.2] that v is a point of multiplicity 6 on $C_{U,A}$ for $U \in D_5$. Thus, $C_{U,A}$ is a sum of multiple lines passing through v (if it is the whole plane,then we obtain a contradiction).

Let us now identify the sets $\mathcal{B}(U, A)$ in order to prove that $C_{U,A}$ has to be reduced for a generic $U \in D_5$. Let us fix such a generic point U of D_5; then $\mathbb{P}(A) \cap T_{U,G(3,W)}$ has dimension 2. Moreover, $\dim(F_v \cap \mathbb{P}(A) \cap T_{U,G(3,W)}) = 2$ because this space contains the tangent space to the del Pezzo surface $D_5 \subset F_v$ and is contained in the previous intersection. It also follows that the set of $w \in \mathbb{P}(U)$ such that

$$\dim(\mathbb{P}(A) \cap F_w \cap T_{U,G(3,W)}) \geq 1$$

is the singleton $\{v\}$. Since D_5 is irreducible of dimension 2, we infer that U does not belong to any line on $D_5 \subset \mathbb{P}^5$ (such lines cannot cover the whole D_5). Thus, for $U' \in D_5 - \{U\}$, we have $\mathbb{P}(U') \cap \mathbb{P}(U) = \{v\}$.

So $\mathcal{B}(U, A)$ is the sum of the intersections $\mathbb{P}(U) \cap \mathbb{P}(V_0)$, where $V_0 \in \Theta_A - D_5$, and $\{v\}$.

For a fixed V_0, $\mathbb{P}(V_0)$ intersects $\mathbb{P}(U)$ outside v (because $F_v \cap G(3, W) = G(2, 5)$) and from Lemma 5.7 in one point (since $\mathcal{C}_{U,A}$ is a sum of lines passing through v). Since the plane $\mathbb{P}(V_0)$ has to be contained in our cubic hypersurface S, the set $\mathcal{C}_{V_0,A}$ must be the whole $\mathbb{P}(V_0)$.

It follows that $C_{U,A}$ is a reduced sum of six lines for a generic choice of $U \in D_5$. We deduce that for each such V_0, we have $\mathcal{C}_{V_0,A} = \mathbb{P}(V_0)$, contradicting condition **O**.

Assume (c); then Θ'_A is a linear section of the cone with vertex U_0 over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2$. The planes parameterized by points in Θ'_A are spanned by the point v and a line in $\mathbb{P}(V/[v])$. More precisely, the line in $\mathbb{P}(V/[v])$ is described as follows: the first factor of $\mathbb{P}^1 \times \mathbb{P}^2$ corresponds to a choice of a point on the line l_0, and the second factor corresponds to a choice of a plane containing $l_0 \subset \mathbb{P}^4$; finally, the directrix of our cone with vertex U_0 gives a choice of a line on this plane passing through our point.

We will obtain a contradiction by showing that $\mathbb{P}(U_0)$ must be contained in X'. Thus, it is enough to show that the sum of the curves $\mathcal{C}_{U,A}$ for $U \in \Theta'_A$ covers the line l_0. By Lemma 5.7 it is enough to prove that for each point of l_0, there are at least two lines parameterized by Θ'_A that contain this point.

If Θ'_A contains U_0, then it is a cone, and we obtain a contradiction unless Θ'_A is a plane spanned by U_0 and a line contained in the second factor of $\mathbb{P}^1 \times \mathbb{P}^2$. Indeed, the planes in $\mathbb{P}(W)$ parameterized by the point from Θ'_A intersect in this case along a line spanned by v and the fixed point from l_0 and cover a hyperplane.

If Θ'_A does not contain the vertex U_0, then we obtain a contradiction similarly unless the image of the projection

$$\mathbb{P}^1 \times \mathbb{P}^2 \supset \Theta'_A \to \mathbb{P}^1$$

is a point. Suppose that the image of this projection is a point, which we denote by Q_0. Then Θ'_A is a plane. Next, the planes parameterized by Θ'_A pass through a line l (determined by v and Q_0) and cover a hyperplane H_0 that is a nonreduced component of S_A, so we are in case (II).

Case (3) Suppose that $G(2,5)$ is equal to $F_v \cap G(3,W)$ for some $v \in W$. This embedding is given by choosing a point $L \in G(5,W)$ that gives a natural embedding $G(3,L) \subset G(3,W)$. In this case the sum of the planes corresponding to points in Θ'_A is contained in the hyperplane $\mathbb{P}(L) \subset \mathbb{P}(W)$. By Lemma 5.14 we can assume that this sum covers $\mathbb{P}(L)$. It follows from [O2, Cor. 1.5] that S_A has a nonreduced linear component; so we are in case (II).

Case (4) Then from Lemma 5.10 the component Θ'_A is a two-dimensional linear section of the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^9$ with vertex U_0. It is useful to have in mind the description of the family of planes parameterized by $\Theta'_A \subset \mathbb{P}^2 \times \mathbb{P}^2$:

LEMMA 5.16. *Geometrically, the first factor of* $\mathbb{P}^2 \times \mathbb{P}^2$ *corresponds to a choice of a line in* $\mathbb{P}(U_0)$, *and the second factor to the choice of a* $\mathbb{P}^3$ *containing* $\mathbb{P}(U_0)$. *The directrix of the cone corresponds to planes containing the fixed line in a fixed* $\mathbb{P}^3$.

Suppose first that Θ'_A contains the vertex of the cone $U_0 \in G(3,W)$. Then the plane $\mathbb{P}(U_0)$ is covered by the intersection with other planes corresponding to points from Θ'_A unless Θ'_A maps to a point under the projection $\mathbb{P}^2 \times \mathbb{P}^2 \supset \Theta'_A \to \mathbb{P}^2$. Thus, in the first case, we obtain a contradiction from Proposition 5.5. But in the second case we see that Θ'_A is a plane; then we are in Case (2) described before.

We can assume that Θ'_A does not contain the vertex of the cone, so we can use [O2, Prop. 2.33]. We want to obtain a contradiction by showing that $\mathbb{P}(U_0) \subset X'$. For this, it is enough to see that the sum of the curves $C_{U,A}$ for $U \in \Theta'_A$ contains $\mathbb{P}(U_0)$. Consider the projections to the factors $\mathbb{P}^2 \leftarrow \Theta'_A \rightarrow \mathbb{P}^2$ (recall that $\Theta'_A \subset \mathbb{P}^2 \times \mathbb{P}^2$). Since by Lemma 5.7 the intersection of two planes $\mathbb{P}(U)$ and $\mathbb{P}(V)$ is contained in the curve $C_{U,A}$, we obtain a contradiction when the images of both projections have dimension ≥ 1. The remaining case is $\Theta'_A = v \times \mathbb{P}^2$, where v corresponds to a fixed line in $\mathbb{P}(U_0)$. But then we are in Case (2).

Case (5) We assume that Θ'_A is the triple Veronese embedding of $\mathbb{P}^2$. Then from [O2, Claim 1.16] we know that $\mathcal{G} = (\pi(\alpha^{-1}(\Theta'_A)))_{\mathrm{red}}$ is the secant cubic of the Veronese surface in $\mathbb{P}^5$. It follows from [O4, §4.4] that for all $U \in \Theta'_A$, the set $C_{U,A}$ is a triple smooth conic. Consider the restriction $\mathcal{E}_\Theta \rightarrow \Theta'_A$ of the tautological bundle on $G(3, W)$. In this case we obtain $\mathcal{E}_\Theta = S^2\Omega^1_{\mathbb{P}^2}(1)$ and the following diagram:

$$\begin{array}{c} \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3)) \supset \mathbb{P}(S^2\Omega^1_{\mathbb{P}^2}(1)) \xrightarrow{f} \Theta'_A \subset \mathbb{P}(\bigwedge^3 W) \\ \downarrow \pi \\ \mathbb{P}^5 \supset \mathcal{G} \end{array}$$

The system of quadrics containing the Veronese surface gives the Cremona transformation

$$\begin{array}{ccc} \mathbb{P}^5 & \dashleftarrow & \mathbb{P}^5 \\ & \nwarrow^{c_1} \quad \nearrow_{c_2} & \\ & \mathbb{K} & \end{array} \tag{5.2}$$

where c_1 and c_2 are the blow-ups of the Veronese surface $V_i \subset \mathbb{P}^5$ for $i = 1, 2$, respectively. Then the exceptional divisor $\mathbb{E}$ of c_1 maps under c_2 to the determinantal cubic singular along V_2. Moreover, the exceptional divisor F of the induced map $\mathbb{E} \rightarrow \mathcal{G}$ is naturally isomorphic to the projective bundle $\mathbb{P}(\Omega^1_{V_2}(1))$. We also see that $\pi|_{\mathbb{P}(S^2\Omega^1_{\mathbb{P}^2}(1))}$ can be seen as the blow-up of $\mathcal{G}$ along its singular locus, and thus we can identify it with $c_2|_{\mathbb{E}}$.

We deduce from the diagram (5.2) that $(2H - F) = 2B$ on $\mathbb{P}(S^2\Omega^1_{\mathbb{P}^2}(1))$ where B (resp. H) is the pull-back of the hyperplane from $\mathbb{P}^2 = \Theta'_A$ (resp. $\mathbb{P}^5$). The linear system $|3H + T|$ can be seen on $\mathbb{E}$ as $|3H + 3B|$. By Proposition 4.5 we infer that $3H + 3B - F$ is effective, so it is an element of $|H + 5B|$.

We can go in the other direction: choose an element from $|H + 5B|$, map it to $\mathcal{G}$, and choose a hypersurface of degree 12 singular along the image. Since the conductor locus is nonreduced, the singularities of this hypersurface have to have generically tacnodes (see [Re, §4.4]) along the intersection with S_A. This can lead to a possible counterexample to the O'Grady conjecture.

REMARK 5.17. Let us describe more precisely the EPW sextic S_A in the missing cases where Θ'_A is a plane. First, observe that if Θ'_A is a plane, then it is contained in the tangent space to $G(2, 5) \subset F_v$ at one of its points; we can thus assume that we are in case (c). In this case, S_A is singular along a hyperplane H_0, which is a multiple component such that there is a line $l \subset H_0$ contained in all the planes $\mathbb{P}(U)$ for $U \in \Theta'_A$. By Lemma 5.7 the line $l \subset H_0$ is also contained in all the curves $C_{U,A}$ for $U \in \Theta'_A$. Moreover, the divisor $D \in |3H + T|$ from Proposition 4.4 intersects $\mathcal{G} = \alpha^{-1}(\Theta'_A)_{\mathrm{red}}$ (this is just the blow-up of H_0 along l) along a divisor in the system $|4H - 2E| + E$. So there is a quartic on H_0 singular along l that defines set-theoretically the intersection of H_0 with the scheme C defined by the conductor. So we can describe the situation (in the generic case) as follows: the EPW sextic is decomposable $2H_0 \cup Q$ such that Q is a quartic intersecting the hyperplane H_0 along a quartic. This quartic is singular along l. Moreover, the intersection $H_0 \cap Q$ supports the singular locus of $C \subset X' = \varphi(X) \subset \mathbb{P}^5$. Since C has multiplicity 3 at a generic point of the image, the hypersurface $X' \subset \mathbb{P}^5$ has multiplicity 3 along C, and the singularities along C are worse than ordinary triple points (see [Re, §4.4]).

5.6. The Case Where $\dim \Theta_A = 1$

The aim of this section is to show that the adjoint EPW sextic from Theorem 1.1 cannot correspond to a generic A with Θ_A of dimension 1, that is, such that Θ_A is a line (with some more conditions). Following [O2, §2], we set

$$\mathcal{G} = \Big(\bigcup_{P \in \Theta_A} \mathbb{P}(P) \Big)_{\mathrm{red}},$$

and we denote by $\mathcal{E}_{\Theta_A} \to \Theta_A \subset G(3, W)$ the restriction of the tautological bundle from $G(3, W)$ and by $f_{\Theta_A} : \mathbb{P}(\mathcal{E}_{\Theta_A}) \to R_{\Theta_A}$ the tautological surjective map. Observe that there is a natural embedding of $\mathbb{P}(\mathcal{E}_{\Theta_A})$ in $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$ (in fact, into the exceptional set $E \subset \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$ described in Remark 4.3). The divisor $D \in |3H + T|$ (that maps to the conductor locus $C \subset \mathbb{P}(W)$) intersects $\mathbb{P}(\mathcal{E}_{\Theta_A})$ along an effective divisor D' that we shall analyze.

Suppose that Θ'_A is an irreducible component of Θ_A. O'Grady applied the Morin theorem to show that $1 \le \deg(\Theta'_A) \le 9$. He also presented in [O2, Table 2] a precise description of the corresponding curves and of the corresponding three-dimensional sets $\mathcal{G}$.

If $\deg \Theta_A = 1$, then Θ_A is a line, which we denote by t. Then the variety $\mathcal{G}$ is a three-dimensional linear space containing a line l such that the exceptional divisor E' of f_Θ (in fact, f_Θ is the blow-up along l) maps to l. We compute that on $\mathbb{P}(\mathcal{E}_\Theta)$ we have $T = H - E'$, so that $D' = 4H - E'$. Since the planes $\mathbb{P}(P) \subset \mathbb{P}(W)$ contain l and $C_{P,A} \subset \mathbb{P}(P)$ cannot be a plane, we deduce that the image of D' on $\mathcal{G}$ is an irreducible quartic containing l or a sum of two quadrics (if there is a plane component, then we obtain a contradiction with **O** because this component has to be contained in $X' \subset \mathbb{P}(W)$).

On the other hand, let us analyze the reduced sum $\mathcal{Z} \subset \mathcal{G}$ of the curves $C_{P,A} \subset \mathbb{P}(P)$ for $P \in \Theta_A$. As observed before, we have $\mathcal{Z} \subset \operatorname{supp} D'$. Observe that generically $C_{P,A}$ is a sum of a reduced quartic and a double line l, so we obtain a contradiction in this case. The problem is the special choices of A. There are a lot of possibilities; we hope to consider them in a future work.

Appendix

Let W be a six-dimensional vector space. The exterior product defines a symplectic form on the 20-dimensional vector space $\bigwedge^3 W$. The natural action of $\mathrm{PGL}(W)$ on $\mathbb{P}(\bigwedge^3 W)$ has four orbits $\mathbb{P}(\bigwedge^3 W) \setminus O_1$, $O_1 \setminus O_2$, $O_2 \setminus O_3$, and O_3, where $O_1 \supset O_2 \supset O_3$ are subvarieties of dimensions 18, 14, and 9. Moreover, it is known that $O_3 = G(3, W)$, O_1 is a quartic described in [Don, Lem. 3.6], and O_2 (resp. O_3) is the singular locus of O_1 (resp. O_2). In this paper we are only interested in the orbits $O_3 \subset O_2$.

The locus $O_2 \subset \mathbb{P}(\bigwedge^3 W)$ can be seen as the set of points lying on more than one chord of $G(3, W) \subset \mathbb{P}(\bigwedge^3 W)$ (see [Don, Lem. 3.3]) or as the union of all spaces spanned by some $G(3, N)$ for $N \subset W$ of dimension 5, which is equal to the union of all spaces spanned by some flag variety $F(p, 3, N)$ for some $p \in W$. With this interpretation, we get a description of O_2 as the set of 3-forms

$$\{[\alpha \wedge \omega] \in \mathbb{P}(\textstyle\bigwedge^3 W) \mid \alpha \in W, \omega \in \textstyle\bigwedge^2 W\}.$$

It follows that there are two natural fibrations of $\pi_1, \pi_2 \colon O_2 \setminus O_3 \to \mathbb{P}^5$ such that the closures of the fibers are nine-dimensional linear spaces. More precisely, π_1 is defined as the map

$$O_2 \setminus O_3 \ni [\alpha \wedge \omega] \mapsto [\alpha] \in \mathbb{P}(W),$$

and π_2 is the map

$$O_2 \setminus O_3 \ni [\alpha \wedge \omega] \mapsto [\alpha \wedge \omega \wedge \omega] \in \mathbb{P}(W^\vee).$$

Lemma A.1. *The maps π_1 and π_2 are well defined on $O_2 \setminus O_3$.*

Proof. Assume that $[\alpha_1 \wedge \omega_1] = [\alpha_2 \wedge \omega_2] \in O_2 \setminus O_3$ for some $\alpha_1, \alpha_2 \in V$ and $\omega_1, \omega_2 \in \bigwedge^2 W$. We need to show that $[\alpha_1] = [\alpha_2]$ and

$$[\alpha_1 \wedge \omega_1 \wedge \omega_1] = [\alpha_2 \wedge \omega_2 \wedge \omega_2].$$

Observe that under our assumption we have $\alpha_1 \wedge \alpha_2 \wedge \omega_2 = 0$, but $\alpha_2 \wedge \omega_2$ is not a simple form; hence, $\alpha_1 \wedge \alpha_2 = 0$, and the first part of the assertion follows. We infer the second part since

$$\begin{aligned}[\alpha_2 \wedge \omega_2 \wedge \omega_2] &= [\alpha_1 \wedge \omega_1 \wedge \omega_2] = [\alpha_1 \wedge \omega_2 \wedge \omega_1] \\ &= [\alpha_2 \wedge \omega_2 \wedge \omega_1] = [\alpha_1 \wedge \omega_1 \wedge \omega_1].\end{aligned}$$ □

Proposition A.2. *The divisor class group of O_2 has rank 2 and is generated by the closures of the pull-backs of the hyperplane sections by π_1 and π_2; denote them by H and H_2.*

Proof. First, the Picard group of the projectivized vector bundle

$$\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3)) \subset \mathbb{P}(\textstyle\bigwedge^3 W) \times \mathbb{P}^5$$

has rank 2 and is generated by H and T, the pull-backs of hyperplanes from $\mathbb{P}(W)$ and $\mathbb{P}(\bigwedge^3 W)$, respectively. So it is enough to consider the map

$$\alpha \colon \mathbb{P}(\Omega^3_{\mathbb{P}^5}(3)) \to O_2 \subset \mathbb{P}(\textstyle\bigwedge^3 W)$$

given by the linear system of the big divisor T. By [RS, Thm. 1] the divisor class group of $O_2 \subset \mathbb{P}^{19}$ is isomorphic to the divisor class group of its generic codimension 10 linear section O_2'. Since O_2' is smooth, the latter is equal to the Picard group of O_2'. On the other hand, α restricted to the preimage O_2'' of O_2' is an isomorphism. Since O_2'' is the intersection of ten generic big divisors from the system $|H|$, we deduce from the generalized Lefschetz theorem [RS, Thm. 6] that the Picard group of O_2'' is isomorphic to the Picard group of $\mathbb{P}(\Omega^3_{\mathbb{P}^5}(3))$. □

Let us describe the projective tangent space to O_2 at a point $p \in O_2 \setminus O_3$. Denote first by $F_p = \overline{\pi_1^{-1}(\pi_1(p))}$ and $F_p' = \overline{\pi_2^{-1}(\pi_2(p))}$ the fibers of π_i for $i = 1, 2$.

LEMMA A.3. *Let $p = [\alpha \wedge \omega] \in O_2 \setminus O_3$, where $\alpha \in W$ and $\omega \in \bigwedge^2 W$. Then the projective tangent space $T_p O_2$ is the linear space spanned by the two fibers F_p and F_p', passing through p, and by the linear space*

$$\Pi = \{[\gamma \wedge \omega] \in \mathbb{P}(\textstyle\bigwedge^3 W) \mid \gamma \in W\}.$$

Proof. It is clear that all three linear spaces are contained in O_2 and pass through p. It follows that they span a subspace of the tangent space $T_p O_2$. Recall that O_2 is of dimension 14 and the intersection $F_p \cap F_p'$ is a $\mathbb{P}^5$. It follows that the two fibers span a hyperplane in $T_p O_2$. It is hence enough to prove that Π is not contained in the span of the two fibers. To do so, denote by Σ_p the hyperplane

$$\mathbb{P}(\{\beta \in \textstyle\bigwedge^3 W \mid \beta \wedge \alpha \wedge \omega = 0\}).$$

Clearly, $F_p \cap F_p' \subset \Sigma_p$, whereas $\Pi \not\subseteq \Sigma$ since there exists $\gamma \in W$ such that $\gamma \wedge \alpha \wedge \omega \wedge \omega \neq 0$. □

REMARK A.4. Observe that $\Sigma_p \cap T_p O_2$ is the $\mathbb{P}^{13}$ spanned by the two fibers.

PROPOSITION A.5. *Let T_p be the projective tangent space to O_2 at a smooth point $p \in O_2$. Then there are no five-dimensional isotropic subspaces $K \subset T_p O_2$ such that $p \in K$ and*

$$K \cap F_p \cap F_p' \cap O_3 = \emptyset.$$

Proof. Let K be an isotropic subspace of $T_p O_2$, and let L be a Lagrangian (maximal isotropic) subspace of $T_p O_2$ containing K. Then, since $p \in K \subset L$, we have $L \subset \Sigma_p$, where Σ_p is as in the proof of Lemma A.3. By Remark A.4 we get $K \subset L \subset \mathbb{P}(U_1) + \mathbb{P}(U_2)$. We observe that the projectivized support S of the intersection form on the latter $\mathbb{P}^{13}$ has dimension 7 and is disjoint from $F_p \cap F_p'$. It follows that $\dim(L \cap S) = 3$, $\dim(L) = 9$, and $F_p \cap F_p' \subset L$. It is easy to see that $F_p \cap F_p' \cap O_3$ is a quadric hypersurface in $F_p \cap F_p'$. It follows that any five-dimensional subspace of L meets $F_p \cap F_p' \cap O_3$ since it meets $F_p \cap F_p'$ in a line. □

LEMMA A.6. *Let us keep the previous notation. Then the linear system $|H + H_2|$ is given by the restrictions of quadrics to $O_2 \subset \mathbb{P}(\bigwedge^3 W)$.*

Proof. Let $v \in W^\vee$ and $\gamma \in W = (W^\vee)^\vee$ correspond to the hyperplanes $L_1 \subset \mathbb{P}(W)$ and $L_2 \subset \mathbb{P}(W^\vee)$, respectively. Consider the quadric form

$$Q\colon \bigwedge^3 W : \omega \mapsto \omega(v) \wedge \omega \wedge \gamma \in \bigwedge^6 W = \mathbb{C}.$$

It is enough to prove that $Q^{-1}(0) \cap O_2 = \pi_1^{-1}(L_1) \cup \pi_2^{-1}(L_2)$, and this has to be checked only outside $G(3, W) \subset O_2$.

- We first prove the inclusion $\supseteq$. Take $\omega \in \pi_1^{-1}(L_1)$. Then there exists $\alpha \in H$ such that $\alpha \wedge \omega = 0$. We then observe that since $\alpha \in H$, it follows that $\alpha \wedge \omega(v) = 0$. The inclusion of the second component follows by duality.
- Let us pass to the inclusion $\subseteq$. Take

$$\omega \in O_2 \setminus (\pi_1^{-1}(L_1) \cup \pi_2^{-1}(L_2) \cup G(3, W)).$$

Then ω may be written in the form $\alpha \wedge \beta$ with $\beta \in \bigwedge^2 W$ such that $\alpha \wedge \beta^2 \wedge w$ and $v(\alpha)$ are nonzero. The value of the quadric on ω is then the product of these nonzero values.

□

Denote by G (resp. G') the singular locus of the EPW sextic $S_A \subset \mathbb{P}(W)$ (resp. $S'_A \subset \mathbb{P}(W^\vee)$). It is known (see [EPW]) that S_A has A_1 singularities along G and that $G \subset \mathbb{P}^5$ is a smooth surface of degree 40. It follows that the G is scheme-theoretically defined by the six quintics that are the partial derivatives of the sextic S_A. Denote by $E, E_2 \subset V' := O_2 \cap \mathbb{P}(A)$ (where $A \subset \bigwedge^3 W$ is a 10-dimensional Lagrangian subspace) the exceptional loci of π_i for $i = 1, 2$ and by abusing notation H the restrictions of H to $V' \subset O_2$.

Corollary A.7. *The morphism $\pi_1\colon V' \to S_A$ is the blow-up of $G \subset S_A$. Moreover, the birational map $\pi_2\colon V' \to S'_A$ is given by the linear system $|5H - E|$.*

We also obtain the following corollary (note that it can also be proved using the methods from [W]).

Corollary A.8. *The degree of $O_2 \subset \mathbb{P}^{19}$ is* 42.

Proof. We have to compute $(6H - E)^4/16$. Thus, it is enough to prove that $H^4 = 6$, $H^3E = 0$, $H^2E^2 = -80$, $HE^3 = -480$, and $E^4 = -1344$. First, by the adjunction formula, $E^2H^2 = K_E H^2$, $E^3 H = K_E^2 H$, and $E^4 = K_E^3$. Now from [O, §4] we deduce that $p\colon E = \mathbb{P}(T_G) \to G$. Thus, $K_E = -2\psi$ where ψ is the tautological divisor. Finally, we need the equality

$$\psi^2 - 3\psi \cdot H + c_2(p^*(T_G)) = 0. \qquad \square$$

Since $E = 2(3H - T)$ (see Lemma 4.8) is even in the Picard group of V', there exists a double cover of $\overline{X} \to V'$ ramified along E (we can take the double cover ramified along E_2). The strict transform of E on $\overline{X}$ can be blown down so that the image is the irreducible symplectic manifold X constructed by O'Grady.

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G. Kapustka
Institute of Mathematics of the Polish Academy of Sciences
ul. Śniadeckich 8
P.O. Box 21
00-656 Warszawa
Poland

Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland

grzegorz.kapustka@uj.edu.pl

M. Kapustka
Department of Mathematics and Informatics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland

michal.kapustka@uj.edu.pl