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A Topological Characterization of the Underlying Spaces of Complete R-Trees

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Abstract. We prove that a topological space (P, τ) admits a compatible metric d such that (P, d) is a complete R-tree if and only if P is a topological R-tree (i.e. metrizable, locally path-connected, and uniquely arcwise connected) and also *locally interval compact.* The latter notion means that each point $x \in P$ has a closed neighborhood $\overline{U}$ such that $\overline{U} \cap \alpha$ is compact for each closed half interval $\alpha \subset P$. For topological R-trees, the property "locally interval compact" is strictly stronger than topological completeness.

1. Introduction

An *R-tree* (P, d) is a uniquely arcwise connected metric space such that for each pair of points $\{x, y\} \subset P$, the arc $([x, y], d) \subset P$ from x to y is isometric to the Euclidean segment $[0, d(x, y)]$. R-trees have received considerable attention as objects of study in their own right, and R-trees also play a prominent role in geometric group theory, notably in the study of group actions on spaces of nonpositive curvature [1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 14; 15; 16; 17; 18; 20; 21; 22; 23; 24; 25; 26; 29; 30].

However, the following fundamental question has apparently escaped collective inquiry: Which topological spaces (P, τ) underly the complete R-trees?

To answer this question, observe that open metric balls in the metric R-tree (P, d) are path connected and hence (P, τ) is metrizable, uniquely arcwise connected, and locally path connected, that is, R-trees are *topological R-trees.* Thanks to a result of John Mayer and Lex Oversteegen [27], the converse is also true: each topological R-tree (P, τ) is the underlying space of some R-tree (P, d). (A preprint of the author contains an alternate shorter proof [13].)

For the metric R-tree (P, d) to be complete, it is of course necessary that (P, τ) is topologically complete, but somewhat surprisingly, this is not sufficient. Example 1, the planar subspace $([0, 1] \times \{0\}) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, \frac{1}{n}])$, shows it is *false* that a topologically complete topological R-tree (P, τ) is necessarily the underlying space of some complete R-tree (P, d).

As mentioned in the abstract, to strengthen topological completeness and ensure that the topological R-tree (P, τ) is the underlying space of a complete metric R-tree, it is precisely adequate to demand that (P, τ) has the following extra property:

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DEFINITION 1. The space (P, τ) is *locally interval compact* if for each $x \in P$, there exists an open set $U \subset P$ such that $x \in U$ and $\alpha \cap \overline{U}$ is compact for all closed subspaces $\alpha \subset P$ such that α is homeomorphic to $[0, 1)$.

We also establish that a metric R-tree (P, d) is locally interval compact if and only if (P, d) is open in its metric completion, and in turn such spaces precisely underly complete R-trees. With the exception of the reference to [27], this paper is self contained, and the main result is the following.

THEOREM 1. *Suppose (P, τ) is a topological space. The following are equivalent*:

(1) *There exists a compatible metric d such that (P, d) is a complete R-tree.*
(2) *There exists a compatible metric d such that (P, d) is an R-tree and such that (P, d) is an open subspace of its metric completion $\overline{(P, d)}$.*
(3) *P is metrizable, locally path connected, uniquely arcwise connected, and locally interval compact.*

2. Preliminaries, Examples, Remarks, and Lemmas

An *arc* is a single point or a space homeomorphic to $[0, 1]$. A p-based topological R-tree $(P, \tau, p, \leq, \hat{})$ is a metrizable, uniquely arcwise connected, locally path connected space with $p \in P$ and $[x, y] \subset P$ denoting the unique arc from x to y. The space P enjoys both the associative binary operation $\hat{}$ such that $[p, x \hat{} y] = [p, x] \cap [p, y]$ and the partial order $\leq$ such that $y \leq x$ iff $y \in [p, x]$. Notationally, we may suppress $\leq$ and $\hat{}$ if it is understood that p is the basepoint, and τ can be replaced by d or D if P is equipped with the particular metric d or D. A metric space (P, d) is complete if each Cauchy sequence has a limit, and we remind the reader that every metric space can be embedded as a dense subspace of a complete metric space [28], uniquely up to isometry.

EXAMPLE 1. Let P denote the planar subspace $([0, 1] \times \{0\}) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, \frac{1}{n}))$. Note that P is not the underlying space of a complete R-tree since the half open intervals $\{\frac{1}{n}\} \times [0, \frac{1}{n})$ would be forced to have infinite geometric length, violating the topological fact that $x_n \to 0$ if $x_n \in \{\frac{1}{n}\} \times [0, \frac{1}{n})$. Note that P is a G_δ subspace of the plane (the intersection of countably many open planar sets), and hence P is topologically complete.

The following fact follows easily from the algebraic properties of $(P, \hat{}, \leq)$.

LEMMA 1. *Suppose $(P, p, \tau \leq, \hat{})$ is a p-based topological R-tree and $[p, z] \cap [x, y] = \emptyset$. Then $x \hat{} z = y \hat{} z$.*

Proof. Note that $a \hat{} b \leq b$ since $a \hat{} b \in [p, b]$ and $a \leq b \Rightarrow a \hat{} b = a$ since $[p, a] \cap [p, b] = [p, a] = [p, a \hat{} b]$. Note $\{x \hat{} z, x \hat{} y\} \subset [p, x]$ and $x \hat{} z < x \hat{} y$ (since otherwise we obtain the contradiction $x \hat{} z \in [p, z] \cap [x \hat{} y, x] \subset [p, z] \cap [x, y]$). By a symmetric argument we conclude $y \hat{} z < y \hat{} x$. Thus, $\{x \hat{} z, y \hat{} z\} \subset [p, x \hat{} y]$.

Note that $y \hat{} z \in [p, x] \cap [p, z]$ and thus $y \hat{} z \leq x \hat{} z$. By a symmetric argument, $x \hat{} z \leq y \hat{} z$, and thus $x \hat{} z = y \hat{} z$. □

The following lemma is also a consequence of the fact that the metric completion of an R-tree is an R-tree [19; 8].

LEMMA 2. *Suppose $(P, d, p, \leq, \hat{})$ is an incomplete p-based R-tree with metric completion $\overline{(P, d, p, \leq, \hat{})}$. Suppose $y \in \partial P = \overline{(P, d)} \setminus P$. There exists an order-preserving isometric embedding $h : [0, d(x, y)) \to (P, d)$ such that $h(0) = p$ and $y = \lim_{t \to d(x,y)} h(t)$. In particular, since the compactum $h([0, d(x, y)])$ is closed in the metric space $\overline{(P, d)}$, $h([0, d(x, y)) = P \cap h([0, d(x, y)])$ is a closed subspace of P.*

Proof. Obtain a sequence $z_n \in P$ with $d(z_n, y) \to 0$. For each $N \in \{1, 2, 3, \ldots\}$, obtain $M_N > N$ such that $[p, z_N] \cap [z_m, z_n] = \emptyset$ if $M_N \leq m \leq n$. Define $y_N = z_N \hat{} z_{M_N}$ and note that by Lemma 1 $y_N = z_N \hat{} z_m \hat{} z_n = z_N \hat{} z_m$ if $M_N \leq m \leq n$. Note that $y_n \to y$ and by construction there exists a subsequence $y_{k_1} < y_{k_2} \ldots$. Let $h : [0, d(x, y)) \to \bigcup_{k=1}^{\infty} [p, y_{n_k}] \subset P$ be the natural isometry mapping $[d(p, y_{k_n}), d(p, y_{k_{n+1}})]$ onto $[y_{k_n}, y_{k_{n+1}}] \subset P$. By construction, h is continuously extendable at $d(p, y)$. □

The following lemma establishes that locally interval compact R-trees are open subspaces of their metric completions.

LEMMA 3. *Suppose that (P, d, p) is a p-based incomplete R-tree and $\partial P = \overline{(P, d, p)} \setminus P$ is not a closed subspace of the metric completion $\overline{(P, d, p)}$. Then P is not locally interval compact.*

Proof. Obtain $x \in P \cap \overline{\partial P}$. Suppose $\varepsilon > 0$. Obtain $y \in \partial P$ such that $d(x, y) < \varepsilon$. Obtain by Lemma 2 an isometric embedding $[0, d(p, y)] \to \overline{P}$ such that $0 \mapsto p$, $d(p, y) \mapsto y$, and $[0, d(x, y))$ is order isometric to a closed subspace $\alpha \subset P$. Let $\delta = \varepsilon - d(x, y)$. Obtain $z \in \alpha$ with $d(z, y) < \delta$. Note that if $z < w$ and $w \in \alpha$, then $d(w, x) = d(w, z) + d(z, x) < (\varepsilon - d(x, y)) + d(x, y) < \varepsilon$. Thus, $[z, y)$ is a closed subspace of P, $[z, y)$ is homeomorphic to $[0, 1)$, $[z, y) \subset \overline{B(x, \varepsilon)}$, and $[z, y)$ is not compact. □

REMARK 1. If (P, d, p) is a p-based R tree and $\alpha \subset P$ is homeomorphic to $[0, 1)$, then (α, d) is isometric to a unique finite Euclidean half open interval $[0, R)$ for some $R > 0$ or the infinite ray $[0, \infty)$. If α is closed in P and (α, d) is isometric to the finite interval $[0, R)$, then the preimage of the sequence $R - \frac{1}{n}$ shows that (P, d, p) is incomplete.

The following easy lemma is used in the proof of Lemma 6.

LEMMA 4. *Suppose that (X, D) is a metric space and $A \subset X$ and 2^X denotes the collection of compact subsets of X with the Hausdorff distance. Define $L : 2^X \to$*

$[0,\infty)$ as $L(C)=\inf_{(c,a)\in C\times A} D(c,a)$. Then L is continuous. If $(P,d,p,\leq,\hat{})$ is an R-tree, then λ is continuous if $\lambda : P\to 2^P$ is defined as $\lambda(x)=[p,x]$.

Proof. By definition the Hausdorff distance $H(C,B)$ [28] between compacta $\{B,C\}\subset X$ satisfies $0\leq H(B,C)<\varepsilon$ iff for each $b\in B$, there exists $c\in C$ with $D(b,c)<\varepsilon$ and for each $c\in C$, there exists $b\in B$ with $D(b,c)<\varepsilon$. If $b\in B$ and $c\in C$ with $D(b,c)<\varepsilon$, then $|L(C)-L(B)|<\varepsilon$, and in particular L is continuous. If $\{x,y\}\subset P$ with $d(x,y)<\varepsilon$, then $H([p,x],[p,y])=d(x,y)<\varepsilon$, and in particular λ is continuous. □

The following lemma and its proof also appear in another preprint of the author [13].

LEMMA 5. *Suppose that $(P,p,\tau,\leq,\hat{})$ is a p-based topological R-tree. Suppose that the continuous function $l:P\to[0,\infty)$ satisfies $x<y\Rightarrow l(x)<l(y)$. Define $d:P\times P\to[0,\infty)$ as $d(x,y)=l(x)+l(y)-2l(x\hat{}y)$. Then d is a metric on the set P, inclusion $\kappa:(P,\tau)\to(P,d)$ is a continuous bijection, each arc $\kappa[x,y]\subset(P,d)$ is isometric to the Euclidean segment $[0,d(x,y)]$, and $d(x,x\hat{}x_m)\to 0\Rightarrow x\hat{}x_m\to x$ in (P,τ).*

Proof. Note that $d(x,x)=0$ since $x\hat{}x=x$ and $y\neq x\Rightarrow x\hat{}y<x$ or $x\hat{}y<y$ and hence $d(x,y)>0$. $d(x,y)=d(y,x)$ since $x\hat{}y=y\hat{}x$. Note that $0\leq 2(l(y)-l(x\hat{}y))$ since $x\hat{}y\leq y$. Note that $d(x,z)\leq d(x,y)+d(y,z)$ iff $-2l(x\hat{}z)\leq 2l(y)-2l(x\hat{}y)-2l(x\hat{}z)$ iff $0\leq 2(l(y)-l(x\hat{}y))$. The latter holds since $x\hat{}y\leq y$. Thus, d is a metric on the set P.

If $x_m\to x$ in (P,τ), then $x\hat{}x_m\to x$ in (P,τ). Thus, since l is continuous at x, $l(x)-l(x_m)\to 0$ and $l(x)-l(x\hat{}x_m)\to 0$. Hence, $(l(x)-l(x\hat{}x_m))+(l(x_m)-l(x))+(l(x)-l(x\hat{}x_m))=d(x,x_m)\to 0$.

Note that if $\{w,z\}\subset(P,\tau)$ then $w\leq z$ iff $w=z\hat{}w$, and hence by definition, $d(w,z)=l(z)-l(w)$. Thus, if $\{x,y\}\subset(P,\tau)$, then the natural homeomorphism $h_{x,y}:\kappa[x\hat{}y,x]\to[0,l(x)-l(x\hat{}y)]$ (defined as $h_{x,y}(z)=l(z)-l(x\hat{}y)$) is an isometry onto the Euclidean segment since $w<u<z\Rightarrow d(z,w)=l(z)-l(w)=(l(z)-l(u))+(l(u)-l(w))=d(z,u)+d(u,z)$. Pasting at 0 ($h^{-1}_{y,x}$ union the reverse of $h^{-1}_{x,y}$) yields the natural isometry $[l(x\hat{}y)-l(x),l(y)-l(x\hat{}y)]\to\kappa[x,y]$.

Suppose $d(x,x\hat{}x_m)\to 0$. Then $\{x\hat{}x_m\}$ is a sequence in the (metrizable) compact arc $[p,x]\subset(P,\tau)$. Since κ is continuous at y, if $y\in[p,x]\subset(P,\tau)$ is a subsequential limit of $\{x\hat{}x_m\}$, then $y=\kappa(y)=x$. Hence, $x\hat{}x_m\to x$ in (P,τ). □

The standard fact that a space U is topologically complete if U is an open subspace of some complete metric space (X,d) is often established [28] via a closed embedding $\phi:U\to X\times R$ with $u\mapsto(u,\frac{1}{\partial(u,\partial U)})$. For several reasons, this proof does not work "off the shelf" when trying to obtain a complete R-tree metric for a connected open subspace $P\subset Q$ of a complete R-tree (Q,D). Instead, we build a strictly increasing length function $l:P\to[0,\infty)$ such that $l(x_n)\to\infty$ if

$x_n \to \partial P$, apply Lemma 5, and verify completeness of the metric and continuity of the inverse mapping.

LEMMA 6. *Suppose that* (Q, D) *is a complete metric space, suppose that the subspace* $P \subset Q$ *is open, nonempty, and dense, and suppose tthat he metric space* $(P, D, p, \leq, \hat{})$ *is a* p*-based R-tree. There exists a topologically compatible metric* d *on* P *such that* (P, d, p) *is a complete R-tree.*

Proof. Let $\partial P = Q \setminus P$. Define $L : P \to [0, \infty)$ as $L(x) = \inf\{D(y, z) \mid y \in [p, x]$ and $z \in \partial P\}$. Note that $L > 0$ since $[p, x]$ is compact and ∂P is closed. Note that $y \leq x \Rightarrow L(y) \geq L(x)$ since $[p, y] \subset [p, x]$. Define $l : P \to [0, \infty)$ as $l(x) = D(p, x) + \frac{1}{L(x)}$. Note that l is continuous since D is continuous and by Remark 4 L is continuous. Observe that $\{x, y\} \subset P$ and $x < y \Rightarrow D(p, x) < D(p, y)$ (since (P, D) is an R-tree) and $\frac{1}{L(x)} \leq \frac{1}{L(y)}$ since $L(y) \geq L(x)$, and hence $l(x) < l(y)$. Thus, applying Lemma 5, the metric $d(x, y) = l(x) + l(y) - 2l(x \hat{} y)$ ensures that the inclusion $\kappa : (P, D) \to (P, d)$ is a continuous bijection, and $\kappa[x, y] \subset (P, d)$ is isometric to the Euclidean segment $[0, d(x, y)]$. By definition, $D(x, y) = d(x, y) - l(x) - l(y) \leq d(x, y)$. Hence, κ is a homeomorphism. Thus, (P, d) is uniquely arcwise connected, and hence (P, d) is an R-tree.

Observe that for real numbers, if $0 < t < s$, then $1 < \frac{1}{t} - \frac{1}{s}$ iff $st < s - t$.

To obtain a contradiction, suppose that (P, d) is incomplete. Let $\overline{(P, d)}$ denote the metric completion of (P, d). By Lemma 2 obtain $y \in \overline{(P, d)} \setminus P$, and an isometric embedding $h : [0, d(p, y)] \to \overline{(P, d)}$, so that $h(0) = p$, $h(d(p, y)) = y$ and $h|[0, d(p, y))$ is an order-preserving embedding into P. Let $y_m = h(\frac{d(p,y)m}{m+1})$. Note that $\{y_m\}$ is Cauchy in (P, d) and hence $\{y_m\}$ is Cauchy in (P, D) since $D \leq d$.

Note that for all $m \geq 1$ and $k \geq 1$, $0 < L(y_m) \leq D(y_m, y_{m+k})$ since $[p, y_m] \subset [p, y_{m+k}]$. Thus, since $\{y_m\}$ is Cauchy in (P, D), the sequence $L(y_m) \to 0$. Hence (applying the continuity of $\times : R \times R \to R$ and $- : R \times R \to R$ (familiar multiplication and substraction of real numbers)), for each $M \geq 1$, we obtain $N_M > M$ so that $L(y_M) \times L(y_n) < L(y_M) - L(y_n)$. Thus, if $n \geq N_M > M$, then $y_M = y_M \hat{} y_n$, and hence $d(y_n, y_M) = D(y_n, y_M) + (\frac{1}{L(y_n)} - \frac{1}{L(y_M)}) \geq (\frac{1}{L(y_n)} - \frac{1}{L(y_M)}) > 1$, contradicting the fact that $\{y_m\}$ is Cauchy in (P, d). □

3. Proof of Theorem 1

For $3 \Rightarrow 2$, suppose that (P, τ) is a locally interval complete topological R-tree. Obtain by [27] a topologically compatible metric d such that (P, d) is an R-tree. If $(P, d) = \overline{(P, d)}$, then note that $\overline{(P, d)}$ is open in $\overline{(P, d)}$. If $(P, d) \neq \overline{(P, d)}$, then Lemma 3 ensures that P is open in $\overline{(P, d)}$. For $2 \Rightarrow 1$, suppose that (P, d) is an R-tree, open in its metric completion $\overline{(P, d)}$. Apply Lemma 6. For $1 \Rightarrow 3$, suppose that (P, d) is a complete R-tree. Note that, by definition, (P, d) is metrizable and uniquely arcwise connected, and (P, d) is locally path connected since open metric balls are path-connected. Recall Remark 1 and observe that the bounded open metric balls of radius 1 establish that (P, d) is locally interval compact.

References

[1] A. G. Aksoy and M. A. Khamsi, *A selection theorem in metric trees,* Proc. Amer. Math. Soc. 134 (2006), no. 10, 2957–2966.

[2] J. A. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space,* Geom. Topol. 10 (2006), 1523–1578.

[3] V. N. Berestovskii and C. P. Plaut, *Covering R-trees, R-free groups, and dendrites,* Adv. Math. 224 (2010), no. 5, 1765–1783.

[4] M. Bestvina, *R-trees in topology, geometry, and group theory,* Handbook of geometric topology, pp. 55–91, North-Holland, Amsterdam, 2002.

[5] B. H. Bowditch and J. Crisp, *Archimedean actions on median pretrees,* Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 3, 383–400.

[6] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature,* Springer-Verlag, Berlin, 1999.

[7] J. W. Cannon, *The theory of negatively curved spaces and groups,* Ergodic theory, symbolic dynamics, and hyperbolic spaces, Oxford Sci. Publ., pp. 315–369, Oxford Univ. Press, New York, 1991.

[8] I. Chiswell, *Introduction to* Λ*-trees,* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[9] I. Chiswell, T. W. Müller, and J.-C. Schlage-Puchta, *Completeness and compactness criteria for R-trees,* preprint.

[10] J. J. Dijkstra and K. I. S. Valkenburg, *The instability of nonseparable complete Erdős spaces and representations in R-trees,* Fund. Math. 207 (2010), no. 3, 197–210.

[11] A. Dranishnikov and M. Zarichnyi, *Universal spaces for asymptotic dimension,* Topology Appl. 140 (2004), no. 2–3, 203–225.

[12] C. Drutu and M. V. Sapir, *Groups acting on tree-graded spaces and splittings of relatively hyperbolic groups,* Adv. Math. 217 (2008), no. 3, 1313–1367.

[13] P. Fabel, *A short proof characterizing topologically the underlying spaces of R-trees,* preprint.

[14] H. Fischer and A. Zastrow, *Combinatorial R-trees as generalized Cayley graphs for fundamental groups of one-dimensional spaces,* Geom. Dedicata 163 (2013), 19–43.

[15] M. Gromov, *Hyperbolic groups,* Essays in group theory, Math. Sci. Res. Inst. Publ., 8, pp. 75–263, Springer, New York, 1987.

[16] V. Guirardel and A. Ivanov, *Non-nesting actions of Polish groups on real trees,* J. Pure Appl. Algebra 214 (2010), no. 11, 2074–2077.

[17] M. Hamann, *On the tree-likeness of hyperbolic spaces,* preprint.

[18] B. Hughes, *Trees and ultrametric spaces: a categorical equivalence,* Adv. Math. 189 (2004), no. 1, 148–191.

[19] W. Imrich, *On metric properties of tree-like spaces,* Contributions to graph theory and its applications (Internat. Colloq., Oberhof, 1977), pp. 129–156, Tech. Hochschule Ilmenau, Ilmenau, 1977.

[20] I. Kapovich and N. Benakli, *Boundaries of hyperbolic groups,* Combinatorial and geometric group theory, Contemp. Math., 296, pp. 39–93, Amer. Math. Soc., Providence, RI, 2002.

[21] I. Kapovich and M. Lustig, *Stabilizers of R-trees with free isometric actions of* F_N*,* J. Group Theory 14 (2011), no. 5, 673–694.

[22] W. A. Kirk, *Hyperconvexity of R-trees,* Fund. Math. 156 (1998), no. 1, 67–72.

[23] ———, *Fixed point theorems in* CAT(0) *spaces and R-trees,* Fixed Point Theory Appl. 4 (2004), 309–316.

[24] G. Levitt, *Non-nesting actions on real trees,* Bull. Lond. Math. Soc. 30 (1998), no. 1, 46–54.

[25] J. C. Mayer, L. K. Mohler, L. G. Oversteegen, and E. D. Tymchatyn, *Characterization of separable metric R-trees,* Proc. Amer. Math. Soc. 115 (1992), no. 1, 257–264.

[26] J. C. Mayer, J. Nikiel, and L. G. Oversteegen, *Universal spaces for R-trees,* Trans. Amer. Math. Soc. 334 (1992), no. 1, 411–432.

[27] J. C. Mayer and L. G. Oversteegen, *A topological characterization of R-trees,* Trans. Amer. Math. Soc. 320 (1990), no. 1, 395–415.

[28] J. R. Munkres, *Topology: a first course,* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[29] F. Paulin, *The Gromov topology on R-trees,* Topology Appl. 32 (1989), no. 3, 197–221.

[30] K. Ruane, CAT(0) *groups with specified boundary,* Algebr. Geom. Topol. 6 (2006), 633–649.

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