

# The Affine Automorphism Group of $\mathbb{A}^3$ is Not a Maximal Subgroup of the Tame Automorphism Group

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ABSTRACT. We construct explicitly a family of proper subgroups of the tame automorphism group of affine three-space (in any characteristic) that are generated by the affine subgroup and a nonaffine tame automorphism. One important corollary is the titular result that settles negatively the open question (in characteristic zero) of whether the affine subgroup is a maximal subgroup of the tame automorphism group. We also prove that all groups of this family have the structure of an amalgamated free product of the affine group and a finite group over their intersection.

## 1. Introduction

Throughout,  $\mathbb{K}$  denotes a field of any characteristic. We denote by  $GA_n(\mathbb{K})$  the group of polynomial automorphisms of  $\mathbb{A}_{\mathbb{K}}^n$ . We consider  $\text{Aff}_n(\mathbb{K})$  (resp.  $\text{BA}_n(\mathbb{K})$ , resp.  $\text{TA}_n(\mathbb{K})$ ), the subgroup of  $GA_n(\mathbb{K})$  of affine (resp. triangular, resp. tame) automorphisms (see Section 2 or [4] for precise definitions). In this paper we are interested with the question of finding proper intermediate subgroups between  $\text{Aff}_n(\mathbb{K})$  and  $\text{TA}_n(\mathbb{K})$ .

If  $n = 2$ , then it is well known that such intermediate subgroups exist. The classical Jung–van der Kulk theorem [5; 6] states that  $GA_2(\mathbb{K}) = \text{TA}_2(\mathbb{K})$  and, moreover,  $GA_2(\mathbb{K})$  is the amalgamated free product of  $\text{Aff}_2(\mathbb{K})$  and  $\text{BA}_2(\mathbb{K})$  along their intersection. Using this structure theorem, we can uniquely define the height of any automorphism  $\phi \in GA_2(\mathbb{K})$  as the maximum of the degrees of the triangular automorphisms in any reduced decomposition of  $\phi$ . Let  $H_d$  denote the set of all automorphisms of height at most  $d$ . Then we have that  $\text{Aff}_2(\mathbb{K}) = H_1 \subset H_2 \subset H_3 \subset \cdots \subset \text{TA}_2(\mathbb{K})$  is an ascending sequence of (proper) subgroups of  $\text{TA}_2(\mathbb{K})$ . In particular, for all  $\beta \in \text{BA}_2(\mathbb{K}) \setminus \text{Aff}_2(\mathbb{K})$ ,  $\langle \text{Aff}_2(\mathbb{K}), \beta \rangle$  is a proper subgroup of  $\text{TA}_2(\mathbb{K})$ .

In the case that  $n > 2$  and  $\mathbb{K}$  has positive characteristic, it is also known that there are many intermediate subgroups between  $\text{Aff}_n(\mathbb{K})$  and  $\text{TA}_n(\mathbb{K})$  (see, e.g., [3]). However, in characteristic zero, the question is much more nuanced.<sup>1</sup> The first partial results in this direction concern subgroups of the form  $\langle \text{Aff}_n(\mathbb{K}), \beta \rangle$

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<sup>1</sup>Recently, Wright [8] showed that in characteristic zero,  $\text{TA}_3(\mathbb{K})$  is an amalgamated free product of three subgroups along their pairwise intersection, which implies a much weaker structure on  $\text{TA}_3(\mathbb{K})$ . Unlike in dimension two, we no longer have a reasonably unique representation of every tame automorphism.

for a single automorphism  $\beta \in \mathrm{GA}_n(\mathbb{K}) \setminus \mathrm{Aff}_n(\mathbb{K})$ . In 1997, Derksen gave an elementary proof (unpublished, but see [4], Theorem 5.2.1 for a proof) that the triangular automorphism  $\sigma := (x_1 + x_2^2, x_2, \dots, x_n) \in \mathrm{BA}_n(\mathbb{K})$ , along with the affine subgroup, generates the entire tame group (when  $\mathrm{char}(\mathbb{K}) = 0$ ); that is,  $\langle \mathrm{Aff}_n(\mathbb{K}), \sigma \rangle = \mathrm{TA}_n(\mathbb{K})$ . This motivated the definition of *co-tame* automorphisms as follows:

**DEFINITION 1.** An automorphism  $\phi \in \mathrm{GA}_n(\mathbb{K})$  is called *co-tame* if  $\langle \mathrm{Aff}_n(\mathbb{K}), \phi \rangle \supset \mathrm{TA}_n(\mathbb{K})$ .

One can naturally ask:

**QUESTION 1.** Let  $n \geq 3$ , and let  $\mathbb{K}$  be a field of characteristic zero. Is every automorphism in  $\mathrm{GA}_n(\mathbb{K}) \setminus \mathrm{Aff}_n(\mathbb{K})$  co-tame?

Note that this is intimately related to the question of finding intermediate subgroups, as an example of an automorphism  $\beta$  that is tame but not co-tame would provide an intermediate subgroup  $\mathrm{Aff}_n(\mathbb{K}) \subset \langle \mathrm{Aff}_n(\mathbb{K}), \beta \rangle \subset \mathrm{TA}_n(\mathbb{K})$ .

In 2004, Bodnarchuk [1] generalized Derksen's result in the following way: if  $\mathbb{K}$  has characteristic zero, then all nonaffine triangular and bitriangular automorphisms (i.e., elements of the form  $\beta_1\alpha\beta_2$  for some  $\beta_1, \beta_2 \in \mathrm{BA}_n(\mathbb{K})$  and  $\alpha \in \mathrm{Aff}_n(\mathbb{K})$ ) are co-tame. Interestingly, the first author [2] recently showed that certain wild (i.e., not tame) automorphisms, including the famous Nagata automorphism, are co-tame.

In this paper, we provide a negative answer to Question 1 when  $n = 3$  by constructing an automorphism that is tame but not co-tame. More precisely, fix an integer  $N \geq 1$  and consider the automorphisms  $\beta = (x + y^2(y + z^2)^2, y + z^2, z) \in \mathrm{BA}_3(\mathbb{K})$ ,  $\pi = (y, x, z) \in \mathrm{Aff}_3(\mathbb{K})$ , and  $\theta_N = (\pi\beta)^N \pi (\pi\beta)^{-N} \in \mathrm{TA}_3(\mathbb{K})$ . We prove the following result (without any assumption about the characteristic of  $\mathbb{K}$ ).

**MAIN THEOREM.** For all integers  $N \geq 3$ , the automorphism  $\theta_N$  is not co-tame. In other words,  $\langle \mathrm{Aff}_3(\mathbb{K}), \theta_N \rangle$  is a proper subgroup of  $\mathrm{TA}_3(\mathbb{K})$ . Moreover, this group is the amalgamated free product of  $\mathrm{Aff}_3(\mathbb{K})$  and  $\langle \mathcal{C}, \theta_N \rangle$  along their intersection  $\mathcal{C}$  where  $\mathcal{C} = \{\alpha \in \mathrm{Aff}_3(\mathbb{K}) \mid \alpha\theta_N = \theta_N\alpha\}$  is a finite cyclic group.

**REMARK 1.**  $\theta_1$  is co-tame by the aforementioned result of Bodnarchuk.

The Main Theorem immediately implies the result in the title of this paper:

**COROLLARY 1.** For any field  $\mathbb{K}$ ,  $\mathrm{Aff}_3(\mathbb{K})$  is not a maximal subgroup of  $\mathrm{TA}_3(\mathbb{K})$ .

In Section 2, we describe some general, commonly used definitions. We make some specific notations and definitions in Section 3, which are necessary to state our key technical result (Theorem 3). This statement of Theorem 3 and the proofs of its consequences (including the main theorem) comprise Section 4. The proof of Theorem 3 is quite technical and is deferred to the ultimate section.

## 2. General Definitions

### 2.1. Degrees

Let  $n \geq 1$  be an integer. We denote by  $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$  the polynomial algebra in  $n$  commutative variables  $\mathbf{x} = \{x_1, \dots, x_n\}$ . We write  $\mathbf{x}^v = x_1^{i_1} \cdots x_n^{i_n}$  for any  $v = (i_1, \dots, i_n) \in \mathbb{N}^n$ . For a given  $P \in \mathbb{K}[\mathbf{x}]$ , we denote by  $\text{supp}(P) \subset \mathbb{N}^n$  the *support* of  $P$ , that is, the set of  $n$ -tuples  $v \in \mathbb{N}^n$  such that the coefficient of  $\mathbf{x}^v$  in  $P$  is nonzero.

The main technical tool in this paper is the use of various degree functions. Here, we mean “degree function” in a little more generality than most authors, so we give a precise definition. Typically, the codomain of a degree function is the natural numbers or the integers; we instead consider any totally ordered commutative monoid  $M$ , and set  $\overline{M} = M \cup -\infty$  with the convention that  $-\infty + n = -\infty$  and  $-\infty < n$  for all  $n \in M$ .

DEFINITION 2. Let  $A$  be a  $\mathbb{K}$ -domain, and  $M$  a totally ordered commutative monoid. A map  $\text{deg} : A \rightarrow \overline{M}$  is called a *degree function* provided that

- (1)  $\text{deg}(f) = -\infty$  if and only if  $f = 0$ ,
- (2)  $\text{deg}(fg) = \text{deg}(f) + \text{deg}(g)$  for all  $f, g \in A$ ,
- (3)  $\text{deg}(f + g) \leq \max\{\text{deg}(f), \text{deg}(g)\}$  for all  $f, g \in A$ .

Two easy consequences of the definition are that  $\text{deg}(c) = 0$  for any  $c \in \mathbb{K}^*$  and that if  $\text{deg}(f) \neq \text{deg}(g)$ , then equality holds in property (3).

The two families of degree functions that we will use are *weighted degree* and *lexicographic degree*, the latter of which takes values in  $\mathbb{N}^n$ .

- For any  $w \in \mathbb{N}^n \setminus \{\mathbf{0}\}$ , we denote the  $w$ -*weighted degree* of  $P$  by

$$\text{deg}_w(P) = \max_{v \in \text{supp}(P)} \{v \cdot w\},$$

where  $(\cdot)$  denotes the scalar product in  $\mathbb{R}^n$ . The case  $w = (1, \dots, 1)$  corresponds to the usual notion of the total degree of a polynomial.

- For an integer  $1 \leq i \leq n$ , let  $\geq_i$  denote the  $i$ th cyclic lexicographic ordering of  $\mathbb{N}^n$ , that is, the standard basis vectors are ordered by

$$e_i >_i e_{i+1} >_i \cdots >_i e_n >_i e_1 >_i \cdots >_i e_{i-1}.$$

Letting  $\max_i$  denote the maximum with respect to this ordering, we define the  $i$ th *lexicographic degree* of  $P$  to be

$$\text{ldeg}_i(P) = \max_i(\text{supp}(P)).$$

EXAMPLE 1. Let  $P = x + y^2(y + z^2)^2$ . Then we have

$$\begin{aligned} \text{deg}_{(4,1,0)}(P) &= 4, & \text{ldeg}_1(P) &= (1, 0, 0), \\ \text{deg}_{(4,0,1)}(P) &= 4, & \text{ldeg}_2(P) &= (0, 4, 0), \\ \text{deg}_{(8,2,1)}(P) &= 8, & \text{ldeg}_3(P) &= (0, 2, 4). \end{aligned}$$

2.2. Polynomial Automorphisms

We adopt the following standard notations of polynomial automorphism groups:

- $\text{MA}_n(\mathbb{K})$  denotes the monoid of polynomial endomorphisms, that is, the set  $\mathbb{K}[\mathbf{x}]^n$  with the composition

$$(\phi_1, \dots, \phi_n)(\psi_1, \dots, \psi_n) = (\phi_1(\psi_1, \dots, \psi_n), \dots, \phi_n(\psi_1, \dots, \psi_n)).$$

- $\text{GA}_n(\mathbb{K})$  is the group of polynomial automorphisms (or the *general automorphism group*), defined to be the group of invertible elements of  $\text{MA}_n(\mathbb{K})$ .
- The *affine subgroup* of  $\text{GA}_n(\mathbb{K})$  is

$$\text{Aff}_n(\mathbb{K}) = \{(\phi_1, \dots, \phi_n) \in \text{GA}_n(\mathbb{K}) \mid \deg_{(1, \dots, 1)}(\phi_i) = 1 \text{ for each } 1 \leq i \leq n\}.$$

- The *triangular subgroup* of  $\text{GA}_n(\mathbb{K})$  is

$$\text{BA}_n(\mathbb{K}) = \{(\phi_1, \dots, \phi_n) \in \text{GA}_n(\mathbb{K}) \mid \phi_i \in \mathbb{K}^* x_i + \mathbb{K}[x_{i+1}, \dots, x_n] \text{ for each } 1 \leq i \leq n\}.$$

- The *tame subgroup* is  $\text{TA}_n(\mathbb{K}) = \langle \text{Aff}_n(\mathbb{K}), \text{BA}_n(\mathbb{K}) \rangle$ . It is well known to be the entire group  $\text{GA}_n(\mathbb{K})$  for  $n = 1, 2$ , whereas Shestakov and Umirbaev [7] famously showed that it is a proper subgroup when  $n = 3$  and  $\text{char}(\mathbb{K}) = 0$ . Whether it is a proper subgroup or not is a well-known, quite difficult open question in higher dimensions and/or positive characteristic.

The group  $\text{GA}_n(\mathbb{K})$  is isomorphic to the group of automorphisms of  $\text{Spec } \mathbb{K}[\mathbf{x}]$  over  $\text{Spec } \mathbb{K}$  and is antiisomorphic to the group of  $\mathbb{K}$ -automorphisms of  $\mathbb{K}[\mathbf{x}]$ . We freely abuse this correspondence, and, given  $\phi \in \text{GA}_n(\mathbb{K})$  and  $P \in \mathbb{K}[\mathbf{x}]$ , we denote by  $(P)\phi \in \mathbb{K}[\mathbf{x}]$  the image of  $P$  by the  $\mathbb{K}$ -automorphism of  $\mathbb{K}[\mathbf{x}]$  corresponding to  $\phi$ . By writing the automorphism on the right, the expected composition holds, namely  $(P)\phi\psi = ((P)\phi)\psi$  for  $P \in \mathbb{K}[\mathbf{x}]$  and  $\phi, \psi \in \text{GA}_n(\mathbb{K})$ . We refer the reader to [4] for a comprehensive reference on polynomial automorphisms.

We make one elementary observation on how we can compute the degree of the image of a polynomial under an automorphism. We will use this frequently and without further mention.

LEMMA 2. *Let  $\gamma \in \text{GA}_n(\mathbb{K})$ , and let  $P \in \mathbb{K}[\mathbf{x}]$ . Let  $\deg : \mathbb{K}[\mathbf{x}] \rightarrow \overline{M}$  denote a degree function (for some totally ordered commutative monoid  $M$ ), and let  $m \in M$ . If  $\deg((\mathbf{x}^v)\gamma) \leq m$  for all  $v \in \text{supp}(P)$ , then  $\deg((P)\gamma) \leq m$ . Moreover, if  $\deg((\mathbf{x}^v)\gamma) = m$  for a unique  $v \in \text{supp}(P)$ , then  $\deg((P)\gamma) = m$ .*

3. Notations

For the remainder of this paper, we restrict our attention to dimension 3. For convenience, we set  $\mathbf{x} = \{x, y, z\}$  instead of  $\{x_1, x_2, x_3\}$ . We denote by  $\mathcal{A} = \text{Aff}_3(\mathbb{K})$  (resp.  $\mathcal{B} = \text{BA}_3(\mathbb{K})$ ) the subgroup of affine (resp. triangular) automorphisms.

We fix also an integer  $N \geq 3$ , and we consider the following automorphisms:

$$\begin{aligned} \beta &= (x + y^2(y + z^2)^2, y + z^2, z) \in \mathcal{B}, \\ \pi &= (y, x, z) \in \mathcal{A}, \\ \theta &= \theta_N = (\pi\beta)^N \pi (\pi\beta)^{-N} \in \text{TA}_3(\mathbb{K}). \end{aligned}$$

REMARK 2. We will repeatedly make use of the fact that  $\pi$  and  $\theta$  are involutions, that is,  $\theta^2 = \pi^2 = \text{id}$ .

### 3.1. Some Sets of Polynomials

Let  $m \geq 1$  and  $n \geq 0$  be integers. The following technical definitions will play a crucial role in our methods (see Figures 1 and 2):

$$P_{m,n} = \{(i, j, k) \in \mathbb{N}^3 \mid 4i + j \leq 4m, 4i + k \leq 4m + n, 8i + 2j + k \leq 8m + n\},$$

$$\mathcal{P}_{m,n} = \{P \in \mathbb{K}[\mathbf{x}] \mid \text{supp}(P) \subset P_{m,n}\}$$

$$= \{P \in \mathbb{K}[\mathbf{x}] \mid \text{deg}_{(4,1,0)}(P) \leq 4m, \text{deg}_{(4,0,1)}(P) \leq 4m + n, \text{deg}_{(8,2,1)}(P) \leq 8m + n\},$$

$$\mathcal{P}_{m,n}^* = \{P \in \mathcal{P}_{m,n} \mid \text{ldeg}_2(P) = (0, 4m, n), \text{ldeg}_3(P) = (0, 2m, 4m + n)\},$$

$$Q_{m,n} = \{(i, j, k) \in \mathbb{N}^3 \mid i + j \leq m, 3i + 3j + k \leq 3m + n\},$$

$$\mathcal{Q}_{m,n} = \{P \in \mathbb{K}[\mathbf{x}] \mid \text{supp}(P) \subset Q_{m,n}\}$$

$$= \{P \in \mathbb{K}[\mathbf{x}] \mid \text{deg}_{(1,1,0)}(P) \leq m, \text{deg}_{(3,3,1)}(P) \leq 3m + n\},$$

$$\mathcal{Q}_{m,n}^* = \{P \in \mathcal{Q}_{m,n} \mid \text{ldeg}_2(P) = (0, m, n)\}.$$

Moreover, we consider

$$\mathcal{P}^* = \bigcup_{m \geq 1, n \geq 0} \mathcal{P}_{m,n}^* \quad \text{and} \quad \mathcal{Q}^* = \bigcup_{m \geq 1, n \geq 0} \mathcal{Q}_{m,n}^*.$$

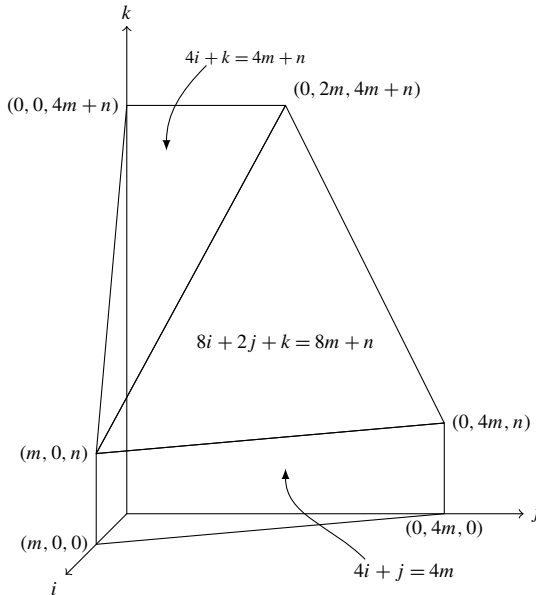


Figure 1  $P_{m,n}$

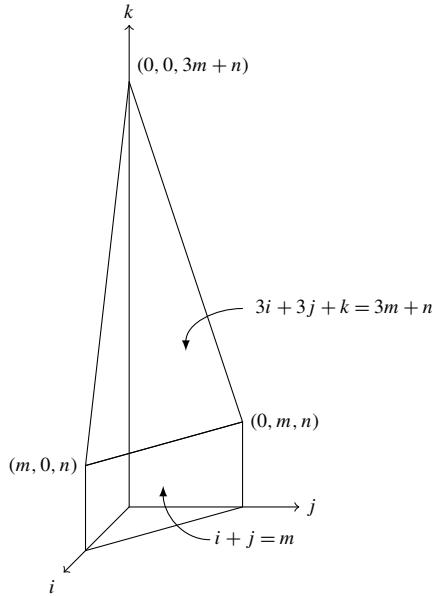


Figure 2  $Q_{m,n}$

For all integers  $m \geq 1$  and  $n \geq 0$ , we can easily check that  $\mathcal{P}_{m,n}^* \subset \mathcal{Q}_{4m,n}^*$ , and thus  $\mathcal{P}^* \subset \mathcal{Q}^*$ .

EXAMPLE 2.  $(x^m z^n)\beta \in \mathcal{P}_{m,n}^*$  for all  $m \geq 1, n \geq 0$ .

REMARK 3. These definitions are variations of a standard tool for studying polynomials, namely the Newton polytope. The Newton polytope of a polynomial  $P$  is defined as  $\text{New}(P) = \text{conv}(\text{supp}(P) \cup \{\mathbf{0}\})$  (here,  $\text{conv}$  denotes the convex hull in  $\mathbb{R}^3$ ). We can view  $P_{m,n}$  as an enlargement of  $\text{New}((x^m z^n)\beta)$ ; in particular, we have

$$P_{m,n} = \{a - b \mid a \in \text{New}((x^m z^n)\beta), b \in (\mathbb{R}_{\geq 0})^3\} \cap \mathbb{N}^3.$$

### 3.2. Some Subgroups of the Affine Group

We consider the following nested sequence of subgroups of the affine group:

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{A} = \text{Aff}_3(\mathbb{K}), \\ \mathcal{A}_1 &= \mathcal{A} \cap \mathcal{B} = \text{Aff}_3(\mathbb{K}) \cap \text{BA}_3(\mathbb{K}), \\ \mathcal{A}_2 &= \{(u^8 x + by + cz + d, u^2 y, uz) \mid u \in \mathbb{K}^*, b, c, d \in \mathbb{K}\}, \\ \mathcal{A}_3 &= \{(u^8 x + cz + d, u^2 y, uz) \mid u \in \mathbb{K}^*, c, d \in \mathbb{K}\}, \\ \mathcal{A}_4 &= \{(u^2 x, u^2 y, uz) \mid u \in \mathbb{K}^*, u^6 = 1\}. \end{aligned}$$

If we set  $\mathcal{C} = \{\alpha \in \mathcal{A} \mid \alpha\theta = \theta\alpha\}$ , then we have  $\mathcal{A}_4 \subset \mathcal{C}$  since for every element  $\alpha \in \mathcal{A}_4$ , it is easy to check that  $\alpha\pi = \pi\alpha$  and  $\alpha\beta = \beta\alpha$ . The opposite inclusion  $\mathcal{C} \subset \mathcal{A}_4$  is a consequence of our main result.

### 4. Main Results

Using results from Section 5, we prove our main technical result.

**THEOREM 3.** *The set  $\mathcal{P}^*$  is stable under the action of the automorphisms  $\pi\beta$ ,  $\pi\beta^{-1}$ , and  $(\pi\beta^{-1})^3\alpha\pi(\pi\beta)^3$  for any  $\alpha \in \mathcal{A} \setminus \mathcal{A}_4$ .*

*Proof.* If  $\gamma \in \{\beta, \beta^{-1}\}$ , then  $(\mathcal{P}^*)\pi\gamma \subset \mathcal{P}^*$  by Proposition 10. To show that  $(\pi\beta^{-1})^3\alpha\pi(\pi\beta)^3$  preserves  $\mathcal{P}^*$ , we separately consider the four cases  $\alpha \in \mathcal{A}_{i-1} \setminus \mathcal{A}_i$  ( $i \in \{1, 2, 3, 4\}$ ). In particular, we have:

- (1) If  $\alpha \in \mathcal{A}_0 \setminus \mathcal{A}_1$ , then  $(\mathcal{P}^*)\alpha\beta \subset \mathcal{Q}^*$  (Proposition 12).
- (2) If  $\alpha \in \mathcal{A}_1 \setminus \mathcal{A}_2$ , then  $(\mathcal{P}^*)\pi\beta^{-1}\alpha\beta \subset \mathcal{Q}^*$  (Proposition 14).
- (3) If  $\alpha \in \mathcal{A}_2 \setminus \mathcal{A}_3$ , then  $(\mathcal{P}^*)\pi\beta^{-1}\alpha\beta\pi\beta \subset \mathcal{Q}^*$  (Proposition 15).
- (4) If  $\alpha \in \mathcal{A}_3 \setminus \mathcal{A}_4$ , then  $(\mathcal{P}^*)(\pi\beta^{-1})^2\alpha\beta\pi\beta \subset \mathcal{Q}^*$  (Proposition 16).

But by Proposition 10,  $(\mathcal{Q}^*)\pi\beta \subset \mathcal{P}^*$ , so we can simply apply  $\pi\beta$  once more to end up in  $\mathcal{P}^*$ . Thus,  $(\mathcal{P}^*)(\pi\beta^{-1})^3\alpha\pi(\pi\beta)^3 \subset \mathcal{P}^*$ . □

**REMARK 4.** If  $\alpha \in \mathcal{A}_4$ , then  $(\pi\beta^{-1})^3\alpha\pi(\pi\beta)^3 = \pi\alpha$ , which does not preserve  $\mathcal{P}^*$  in general. However, since  $\mathcal{A}_4 \subset \mathcal{C}$ , we will take advantage of the commutativity of these elements to push them out of the way.

**COROLLARY 4.** *Let  $r \geq 1$  be an integer. Let  $\alpha_0, \dots, \alpha_r \in \mathcal{A}$ , and set  $\phi = \alpha_0\theta\alpha_1 \cdots \theta\alpha_r$ . If  $\alpha_1, \dots, \alpha_{r-1} \in \mathcal{A} \setminus \mathcal{A}_4$ , then there exist  $\alpha, \alpha' \in \mathcal{A}$  such that  $(y)\alpha\phi\alpha' \in \mathcal{P}^*$ .*

*Proof.* We set  $\theta' = \theta\pi = (\pi\beta)^N(\pi\beta^{-1})^N$ ,  $\alpha = \alpha_0^{-1}$ ,  $\alpha' = \alpha_r^{-1}\pi$ , and  $\alpha'_i = \pi\alpha_i\pi$  for  $1 \leq i \leq r - 1$ . Note that since  $\mathcal{A}_4$  is fixed under conjugation by  $\pi$ ,  $\alpha'_1, \dots, \alpha'_{r-1} \notin \mathcal{A}_4$ . Then we have

$$\begin{aligned} \alpha\phi\alpha' &= \theta'\alpha'_1\pi\theta'\alpha'_2\pi \cdots \theta'\alpha'_{r-1}\pi\theta' \\ &= (\pi\beta)(\pi\beta)^{N-1} \left( \prod_{i=1}^{r-1} (\pi\beta^{-1})^N \alpha'_i \pi (\pi\beta)^N \right) (\pi\beta^{-1})^N. \end{aligned}$$

Since  $(y)\pi\beta \in \mathcal{P}^*_{1,0} \subset \mathcal{P}^*$ , we deduce  $(y)\alpha\phi\alpha' \in \mathcal{P}^*$  by Theorem 3. □

**COROLLARY 5.** *Let  $\phi \in \langle \mathcal{A}, \theta \rangle \setminus \mathcal{A}$ . Then there exist  $\alpha, \alpha' \in \mathcal{A}$  such that  $(y)\alpha\phi\alpha' \in \mathcal{P}^*$ .*

*Proof.* Since  $\theta^{-1} = \theta$  and  $\phi \in \langle \mathcal{A}, \theta \rangle \setminus \mathcal{A}$ , we can write  $\phi = \alpha_0\theta\alpha_1\theta \cdots \alpha_{r-1}\theta\alpha_r$  for some  $\alpha_0, \dots, \alpha_r \in \mathcal{A}$  with  $r \geq 1$ . If some  $\alpha_i \in \mathcal{C}$  ( $1 \leq i \leq r - 1$ ), then we can

shorten our sequence. Thus, we may assume that  $\alpha_1, \dots, \alpha_{r-1} \in \mathcal{A} \setminus \mathcal{C} \subset \mathcal{A} \setminus \mathcal{A}_4$ , and hence the result follows from Corollary 4.  $\square$

**COROLLARY 6.**  $\mathcal{C} = \mathcal{A}_4$ . In particular,  $\mathcal{C}$  is a finite cyclic group of order 1, 2, 3, or 6.

*Proof.* We noted above that  $\mathcal{A}_4 \subset \mathcal{C}$ , so we are left to prove the opposite containment. Suppose for contradiction that there exists  $\rho \in \mathcal{C} \setminus \mathcal{A}_4$ . By Corollary 4 (with  $r = 2, \alpha_0 = \alpha_2 = \text{id}$ , and  $\alpha_1 = \rho$ ), there exist  $\alpha, \alpha' \in \mathcal{A}$  such that  $(y)\alpha\theta\rho\theta\alpha' \in \mathcal{P}^*$ . Since  $\rho$  commutes with the involution  $\theta$ , we have  $\alpha\theta\rho\theta\alpha' = \alpha\rho\alpha' \in \mathcal{A}$ . This is a contradiction since for any  $\gamma \in \mathcal{A}$ ,  $\text{ldeg}_2((y)\gamma) \leq_2 (0, 1, 0) <_2 (0, 4m, n)$  for any  $m \geq 1, n \geq 0$  and thus  $(y)\gamma \notin \mathcal{P}^*$ .  $\square$

**COROLLARY 7.** Let  $\phi \in \text{TA}_3(\mathbb{K})$  be a tame automorphism with  $\text{deg}(x)\phi \leq 5$ ,  $\text{deg}(y)\phi \leq 5$ , and  $\text{deg}(z)\phi \leq 5$ . If  $\phi \notin \mathcal{A}$ , then  $\phi \notin \langle \mathcal{A}, \theta \rangle$ . In particular,  $\langle \mathcal{A}, \theta \rangle$  is a proper subgroup of  $\mathcal{T}$ .

*Proof.* Suppose for contradiction that  $\phi \in \langle \mathcal{A}, \theta \rangle$ . Applying Corollary 5 to  $\phi$ , there exist  $\alpha, \alpha' \in \mathcal{A}$  such that  $(y)\alpha\phi\alpha' \in \mathcal{P}^*$ . By assumption,  $\text{deg}_{(1,1,1)}((y)\alpha\phi\alpha') \leq 5$  (since  $\alpha, \alpha'$  are affine). But if  $P = (y)\alpha\phi\alpha' \in \mathcal{P}^*$ , then there must exist integers  $m \geq 1$  and  $n \geq 0$  such that  $P \in \mathcal{P}_{m,n}^*$ , in which case  $(0, 2m, 4m + n) \in \text{supp}(P)$ . Thus,  $5 \geq \text{deg}_{(1,1,1)}(P) \geq 6m + n \geq 6$ , a contradiction. So we must have  $\phi \notin \langle \mathcal{A}, \theta \rangle$ .  $\square$

**EXAMPLE 3.**  $(x + y^2, y, z) \in \text{TA}_3(\mathbb{K}) \setminus \langle \mathcal{A}, \theta \rangle$ .

**COROLLARY 8.** The group  $\langle \mathcal{A}, \theta \rangle$  is the amalgamated free product of  $\mathcal{A}$  and  $\langle \mathcal{C}, \theta \rangle$  along their intersection  $\mathcal{C}$ .

*Proof.* Let  $r \geq 1$  be an integer. Let  $\alpha_1, \dots, \alpha_{r-1} \in \mathcal{A} \setminus \langle \mathcal{C}, \theta \rangle$  and  $\rho_1, \dots, \rho_r \in \langle \mathcal{C}, \theta \rangle \setminus \mathcal{A}$ . Set  $\phi := \rho_1\alpha_1 \cdots \alpha_{r-1}\rho_r$ . Since it is clear that  $\mathcal{A}$  and  $\langle \mathcal{C}, \theta \rangle$  generate  $\langle \mathcal{A}, \theta \rangle$ , it suffices to check that  $\phi \notin \mathcal{C}$ . Using the fact that  $\theta$  is an involution, we can write  $\rho_i = \theta c_i$  for some  $c_i \in \mathcal{C} = \mathcal{A}_4$  for each  $1 \leq i \leq r$ . By Corollary 4 (with  $\alpha'_0 = \text{id}, \alpha'_i = c_i\alpha_i \in \mathcal{A} \setminus \mathcal{C} = \mathcal{A} \setminus \mathcal{A}_4$  for all  $1 \leq i \leq r - 1$  and  $\alpha'_r = c_r$ ) there exist  $\alpha, \alpha' \in \mathcal{A}$  such that  $(y)\alpha\phi\alpha' \in \mathcal{P}^*$ . As in the proof of Corollary 6, this implies  $\phi \notin \mathcal{A}$  and hence  $\phi \notin \mathcal{C}$ , as required.  $\square$

The Main Theorem in the [Introduction](#) is a direct consequence of these last two corollaries.

### 5. Proofs of the Five Propositions

The technical details necessary to prove Theorem 3, namely Propositions 10, 12, 14, 15, and 16, are contained in this section. The basic idea is to understand the actions of various automorphisms on  $\mathcal{P}^*$  and  $\mathcal{Q}^*$ . First, we show in Section 5.1 that the map  $\pi\beta$  behaves very nicely in this respect (in particular, it preserves  $\mathcal{P}^*$ ). The subsequent two sections study how affine automorphisms affect things. The



essential idea is that an affine map can distort  $\mathcal{P}^*$  some, but this can be rectified by subsequent applications of  $\pi$  and/or  $\beta$ . For technical reasons, we treat triangular affine maps separately in the final section, and nontriangular affine maps in Section 5.2.

5.1. From  $\mathcal{Q}_{m,n}^*$  to  $\mathcal{P}_{m,n}^*$

DEFINITION 3. Let  $\gamma \in \text{GA}_3(\mathbb{K})$ , and set  $w_1 = (4, 1, 0)$ ,  $w_2 = (4, 0, 1)$ , and  $w_3 = (8, 2, 1)$ .  $\gamma$  is called  $\beta$ -shaped if  $\deg_{w_i}(f)\gamma = \deg_{w_i}(f)\beta$  and  $\text{ldeg}_i(f)\gamma = \text{ldeg}_i(f)\beta$  for all  $i = 1, 2, 3$  and  $f \in \{x, y, z\}$ .

EXAMPLE 4.  $\beta$  and  $\beta^{-1} = (x - y^2(y - z^2)^2, y - z^2, z)$  are both  $\beta$ -shaped.

LEMMA 9. Let  $m \geq 1$  and  $n \geq 0$  be integers. If  $\gamma \in \mathcal{G}$  is  $\beta$ -shaped, then  $(\mathcal{Q}_{m,n}^*)\pi\gamma \subset \mathcal{P}_{m,n}^*$ .

Proof. Let  $P \in \mathcal{Q}_{m,n}^*$ , and write  $\gamma = (X, Y, Z)$ . Then since  $\gamma$  is  $\beta$ -shaped, the degrees of  $X, Y$ , and  $Z$  are given in Table 1.

Since  $P \in \mathcal{Q}_{m,n}^*$ , for all  $v = (i, j, k) \in \text{supp}(P)$ , we have  $i + j \leq m$ ,  $3i + 3j + k \leq 3m + n$ , and  $(\mathbf{x}^v)\pi\gamma = Y^i X^j Z^k$ . We deduce:

$$\begin{aligned} \deg_{(4,1,0)}((\mathbf{x}^v)\pi\gamma) &= i + 4j \leq 4(i + j) \leq 4m, \\ \deg_{(4,0,1)}((\mathbf{x}^v)\pi\gamma) &= 2i + 4j + k \leq (i + j) + (3i + 3j + k) \\ &\leq m + 3m + n = 4m + n, \\ \deg_{(8,2,1)}((\mathbf{x}^v)\pi\gamma) &= 2i + 8j + k \leq 5(i + j) + (3i + 3j + k) \\ &\leq 5m + 3m + n = 8m + n, \\ \text{ldeg}_2((\mathbf{x}^v)\pi\gamma) &= (0, i + 4j, k) \leq_2 (0, 4m, n), \\ \text{ldeg}_3((\mathbf{x}^v)\pi\gamma) &= (0, 2j, 2i + 4j + k) \leq_3 (0, 2m, 4m + n). \end{aligned}$$

We note that each of these last two inequalities is an equality if and only if  $(i, j, k) = (0, m, n)$  which belongs to  $\text{supp}(P)$ . Thus  $(P)\pi\gamma \in \mathcal{P}_{m,n}^*$ . □

Applying this to  $\beta$  and  $\beta^{-1}$  and recalling that  $\mathcal{P}^* \subset \mathcal{Q}^*$ , we have the following:

PROPOSITION 10. If  $\gamma \in \{\beta, \beta^{-1}\}$ , then  $(\mathcal{Q}^*)\pi\gamma \subset \mathcal{P}^*$  and  $(\mathcal{P}^*)\pi\gamma \subset \mathcal{P}^*$ .

Table 1

	$\deg_{(4,1,0)}$	$\deg_{(4,0,1)}$	$\deg_{(8,2,1)}$	$\text{ldeg}_1$	$\text{ldeg}_2$	$\text{ldeg}_3$
$X$	4	4	8	(1, 0, 0)	(0, 4, 0)	(0, 2, 4)
$Y$	1	2	2	(0, 1, 0)	(0, 1, 0)	(0, 0, 2)
$Z$	0	1	1	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)

## 5.2. The Nontriangular Case

The following technical lemma is necessary to prove Proposition 12.

LEMMA 11. Let  $(a, b, c) \in \mathbb{N}^3 \setminus \{(0, 0, 0)\}$ . We set  $f(v) = ai + bj + ck$  for  $v = (i, j, k) \in P_{m,n}$ . Set  $m' = \max\{f(v) \mid v \in P_{m,n}\}$ .

- (1) If  $b > \max\{\frac{a}{4}, 2c\}$  and  $c \neq 0$ , then  $f(v) = m'$  if and only if  $v = (0, 4m, n)$ .
- (2) If  $b > \frac{a}{4}$  and  $c = 0$ , then  $f(v) = m'$  if and only if  $v = (0, 4m, d)$  with  $0 \leq d \leq n$ .
- (3) If  $c > \max\{\frac{b}{2}, \frac{a-2b}{4}\}$  and  $b \neq 0$ , then  $f(v) = m'$  if and only if  $v = (0, 2m, 4m + n)$ .
- (4) If  $c > \frac{a}{4}$  and  $b = 0$ , then  $f(v) = m'$  if and only if  $v = (0, d, 4m + n)$  with  $0 \leq d \leq 2m$ .
- (5) If  $c = \frac{a-2b}{4} > \frac{b}{2}$ , then  $f(v) = m'$  if and only if  $v = (m - d, 2d, 4d + n)$  with  $0 \leq d \leq m$ .

*Proof.* Let  $\text{conv}$  denote the convex hull in  $\mathbb{R}^3$ . Define  $S_1, S_2 \in P_{m,n}$  by

$$\begin{aligned} S_1 &= \{(0, 0, 0), (m, 0, 0), (0, 4m, 0), (m, 0, n), \\ &\quad (0, 4m, n), (0, 2m, 4m + n), (0, 0, 4m + n)\}, \\ S_2 &= \{(m, 0, n), (0, 4m, n), (0, 2m, 4m + n)\}. \end{aligned}$$

Note that  $S_2 \subset S_1$ . It is easy to check (see Figure 1) that  $\text{conv } P_{m,n} = \text{conv } S_1$  (and in fact,  $P_{m,n} = (\text{conv } S_1) \cap \mathbb{N}^3$ ). Then, since  $f$  is a linear form and  $a, b, c \geq 0$ , we have

$$m' = \max\{f(v) \mid v \in \text{conv } P_{m,n}\} = \max\{f(v) \mid v \in S_1\} = \max\{f(v) \mid v \in S_2\}.$$

We deduce

$$m' = \max\{am + cn, 4bm + cn, 2bm + 4cm + cn\} = m(\max\{a, 4b, 2b + 4c\}) + cn.$$

Cases (1) and (2): If  $b > \max\{\frac{a}{4}, 2c\}$ , then  $m' = 4bm + cn$ , and  $E := \{v \in \mathbb{R}^3 \mid f(v) = m'\}$  is a plane that contains  $(0, 4m, n)$  and neither  $(m, 0, n)$  nor  $(0, 2m, 4m + n)$ . Since  $\text{conv } P_{m,n} = \text{conv } S_1$  and  $E$  is not parallel to any face of  $\text{conv } S_1$ , we have that  $E \cap \text{conv } P_{m,n}$  must be either a single point or an edge in  $S_1$ ; thus, we must either have  $E \cap P_{m,n} = \{(0, 4m, n)\}$  or  $E \cap P_{m,n} = \{(0, 4m, d) \mid 0 \leq d \leq n\}$ . It is easy to check that the former case happens precisely when  $c \neq 0$  and the latter when  $c = 0$ .

Cases (3) and (4): If  $c > \max\{\frac{b}{2}, \frac{a-2b}{4}\}$ , then  $m' = 2bm + 4cm + cn$ , and  $E := \{v \in \mathbb{R}^3 \mid f(v) = m'\}$  is a plane that contains  $(0, 2m, 4m + n)$  and neither  $(m, 0, n)$  nor  $(0, 4m, n)$ . Since  $\text{conv } P_{m,n} = \text{conv } S_1$  and  $E$  is not parallel to any face of  $\text{conv } S_1$ , we have that  $E \cap \text{conv } P_{m,n}$  must be either a single point in  $S_1$  or a line segment connecting two points in  $S_1$ ; thus, we must either have  $E \cap P_{m,n} = \{(0, 2m, 4m + n)\}$  or  $E \cap P_{m,n} = \{(0, d, 4m + n) \mid 0 \leq d \leq 2m\}$ . It is easy to check that the former case happens precisely when  $b \neq 0$  and the latter when  $b = 0$ .

Case (5): If  $c = \frac{a-2b}{4} > \frac{b}{2}$ , then  $m' = ma + cn$ , and  $E := \{v \in \mathbb{R}^3 \mid f(v) = m'\}$  is a plane containing  $(m, 0, n)$  and  $(0, 2m, 4m + n)$  but not  $(m, 0, n)$ . Then

$E \cap \text{conv } P_{m,n}$  is the line segment from  $(m, 0, n)$  to  $(0, 2m, 4m + n)$ , giving the result. □

DEFINITION 4. A degree function  $\text{deg}$  is called  $\beta$ -lexicographic if, writing  $\beta = (X, Y, Z)$ ,  $\text{deg}(X) > \text{deg}(Y) > \text{deg}(Z)$ .

Note that any  $(w_1, w_2, w_3)$ -weighted degree satisfying  $w_i \geq 0$  and  $(w_2, w_3) \neq (0, 0)$  is  $\beta$ -lexicographic.

EXAMPLE 5. The three lexicographic degrees, as well as the weighted degrees for the weights  $(4, 1, 0)$ ,  $(4, 0, 1)$ ,  $(8, 2, 1)$ ,  $(1, 1, 0)$ , and  $(3, 3, 1)$ , are all  $\beta$ -lexicographic, as is the usual degree (i.e., the  $(1, 1, 1)$ -weighted degree).

PROPOSITION 12. Let  $m \geq 1$  and  $n \geq 0$  be integers. Let  $\alpha \in \mathcal{A}_0 \setminus \mathcal{A}_1 = \mathcal{A} \setminus \mathcal{B}$ . Then there exist  $m' \geq m, n' \geq 0$  such that  $(\mathcal{P}_{m,n}^*)\alpha\beta \subset \mathcal{Q}_{m',n'}^*$ .

Proof. Let  $\text{deg}$  be a  $\beta$ -lexicographic degree function. We observe that for any fixed  $v = (i, j, k) \in \mathbb{N}^3$ ,  $\text{deg}(\mathbf{x}^v)\alpha\beta = \text{deg}(X^{i'}Y^{j'}Z^{k'})$  for some triple  $(i', j', k') \in \mathbb{N}^3$ . Moreover, since  $\alpha \in \mathcal{A}$  and  $\alpha \notin \mathcal{B}$ , it must be the case that  $(i', j', k')$  is one of the following 15 types of triples:  $(i + j + k, 0, 0)$ ,  $(i + j, k, 0)$ ,  $(i + j, 0, k)$ ,  $(i + k, j, 0)$ ,  $(i + k, 0, j)$ ,  $(j + k, i, 0)$ ,  $(j + k, 0, i)$ ,  $(i, k, j)$ ,  $(j, i, k)$ ,  $(j, k, i)$ ,  $(k, i, j)$ ,  $(k, j, i)$ ,  $(i, j + k, 0)$ ,  $(j, i + k, 0)$ ,  $(k, i + j, 0)$ . We alert the reader that the type of the triple  $(i', j', k')$  is determined only by  $\alpha$  and is independent of  $(i, j, k)$  and the choice of  $\beta$ -lexicographic degree function  $\text{deg}$ . We have

$$\begin{aligned} \text{deg}_{(1,1,0)}(X^{i'}Y^{j'}Z^{k'}) &= 4i' + j', \\ \text{deg}_{(3,3,1)}(X^{i'}Y^{j'}Z^{k'}) &= 3(4i' + j') + k', \\ \text{ldeg}_2(X^{i'}Y^{j'}Z^{k'}) &= (0, 4i' + j', k'). \end{aligned}$$

We can write  $4i' + j' = ai + bj + ck = f(i, j, k)$  and  $3(4i' + j') + k' = a'i + b'j + c'k = (3a + \varepsilon)i + (3b + \mu)j + (3c + \nu)k = g(i, j, k)$  where  $a, b, c \in \{0, 1, 4\}$  and  $a' = 3a + \varepsilon, b' = 3b + \mu$  and  $c' = 3c + \nu$  where  $\varepsilon, \mu, \nu \in \{0, 1\}$  (since  $k' \in \{0, i, j, k\}$ ). We note that  $a', b',$  and  $c'$  must all be nonzero. We set

$$\begin{aligned} m' &= \max\{f(v) \mid v \in P_{m,n}\}, \\ n' &= \max\{g(v) \mid v \in P_{m,n}\} - 3m'. \end{aligned}$$

Now, we consider  $P \in \mathcal{P}_{m,n}^*$ . We recall that this implies

$$\{(0, 4m, n), (0, 2m, 4m + 2n)\} \subset \text{supp}(P) \subset P_{m,n}.$$

Setting  $Q = (P)\alpha\beta$ , we will show that  $Q \in \mathcal{Q}_{m',n'}^*$ . Note that

$$\text{deg}(Q) = \max_{(i,j,k) \in \text{supp}(P)} \text{deg}(X^{i'}Y^{j'}Z^{k'}),$$

where  $(i', j', k')$  is one of the same 15 triples where  $(i, j, k) \in \text{supp}(P)$ . Since  $\text{supp}(P) \subset P_{m,n}$ , our definitions of  $m'$  and  $n'$  immediately imply  $\text{deg}_{(1,1,0)}(Q) \leq m'$  and  $\text{deg}_{(3,3,1)}(Q) \leq 3m' + n'$ . It remains to check that

$\text{ldeg}_2(Q) = (0, m', n')$ ; to do so, we show that there exists a unique  $v = (i, j, k) \in \text{supp}(P)$  such that  $\text{ldeg}_2(X^i Y^j Z^k) = (0, m', n')$ . Equivalently, we show that there exists a unique  $v = (i, j, k) \in \text{supp}(P)$  such that  $f(v) = m'$  and  $g(v) = 3m' + n'$ .

Case (1): Suppose  $(i', j', k') \in \{(i + j, k, 0), (i + j, 0, k), (j, i, k), (j, k, i), (j, i + k, 0)\}$ . In this case,  $b = 4$  (since  $j$  appears in the first component) and  $c \leq 1$  (since  $k$  does not appear in the first component). We deduce  $b > \max\{\frac{a}{4}, 2c\}$  and  $b' > \max\{\frac{a'}{4}, 2c'\}$ . Note that  $c' \neq 0$ , so applying the preceding lemma (Case (1)) to  $g$  yields that  $v = (0, 4m, n)$  is the unique element in  $P_{m,n}$ , hence in  $\text{supp}(P)$ , such that  $g(v) = 3m' + n'$ . But the lemma (Case (1) or (2)) applied to  $f$  implies  $f(0, 4m, n) = m'$ , as required.

Case (2): Suppose  $(i', j', k') \in \{(i + j + k, 0, 0), (i + k, j, 0), (j + k, i, 0), (j + k, 0, i), (k, j, i), (k, i + j, 0), (i + k, 0, j), (k, i, j)\}$ . In this case,  $c = 4$  (since  $k$  appears in the first component). We deduce  $c > \max\{\frac{b}{2}, \frac{a-2b}{4}\}$  and  $c' > \max\{\frac{b'}{2}, \frac{a'-2b'}{4}\}$ . Since  $b' \neq 0$ , the preceding lemma (Case 3) applied to  $g$  gives that  $v = (0, 2m, 4m + n)$  is the unique element in  $P_{m,n}$ , hence in  $\text{supp}(P)$ , such that  $g(v) = 3m' + n'$ . But the lemma (Case (3) or (4)) applied to  $f$  implies  $f(0, 2m, 4m + n) = m'$ , as required.

Case (3): Suppose  $(i', j', k') \in \{(i, j + k, 0), (i, k, j)\}$ . In this case,  $a' = 12$  and  $c' = 3$  (with  $b' \in \{1, 3\}$ ). We deduce  $c' > \max\{\frac{b'}{2}, \frac{a'-2b'}{4}\}$ . The lemma (Case (3)) applied to  $g$  yields that  $v = (0, 2m, 4m + n)$  is the unique element in  $P_{m,n}$ , hence in  $\text{supp}(P)$ , such that  $g(v) = 3m' + n'$ . But the lemma applied to  $f$  (Case (3) or (5)) implies  $f(0, 2m, 4m + n) = m'$ , as required. □

### 5.3. The Triangular Case

Once again, we begin with a technical lemma, which will aid in the proof of Proposition 14.

LEMMA 13. *Let  $m \geq 1$  and  $n \geq 0$  be integers. Let  $\gamma = (X, Y, Z) \in \mathcal{B}$  be such that  $\text{ldeg}_2(X) = (0, b, c)$  where  $b \geq 1$  and  $c \geq 0$ ,  $\text{deg}_{(3,3,1)}(X) = 3b + c$  and  $\text{deg}_{(3,3,1)}(Y) = 3$ . Then  $(\mathcal{P}_{m,n}^*)\pi\gamma \subset \mathcal{Q}_{m',n'}^*$  where  $m' = 4bm \geq m$  and  $n' = 4cm + n$ .*

*Proof.* Note that the assumptions (particularly,  $\gamma \in \mathcal{B}$ ) immediately imply

$$\begin{aligned} \text{deg}_{(1,1,0)}(X) &= b, & \text{deg}_{(1,1,0)}(Y) &= 1, & \text{deg}_{(1,1,0)}(Z) &= 0, \\ \text{deg}_{(3,3,1)}(X) &= 3b + c, & \text{deg}_{(3,3,1)}(Y) &= 3, & \text{deg}_{(3,3,1)}(Z) &= 1, \\ \text{ldeg}_2(X) &= (0, b, c), & \text{ldeg}_2(Y) &= (0, 1, 0), & \text{ldeg}_2(Z) &= (0, 0, 1). \end{aligned}$$

Let  $v = (i, j, k) \in P_{m,n}$ . Noting that  $(\mathbf{x}^v)\pi\gamma = Y^i X^j Z^k$ , we compute

$$\begin{aligned} \text{deg}_{(1,1,0)}((\mathbf{x}^v)\pi\gamma) &= i + bj \\ &\leq b(4i + j) \\ &\leq 4bm = m', \end{aligned}$$

$$\begin{aligned} \deg_{(3,3,1)}((\mathbf{x})^v \pi \gamma) &= 3i + (3b + c)j + k \\ &\leq (3b + c - 2)(4i + j) + (8i + 2j + k) \\ &\leq (3b + c - 2)(4m) + (8m + n) \\ &= 3m' + n', \\ \text{ldeg}_2((\mathbf{x})^v \pi \gamma) &= (0, i + bj, cj + k) \\ &\leq_2 (0, b(4i + j), n') \\ &\leq_2 (0, m', n'). \end{aligned}$$

Moreover, equality is attained in the last case if and only if  $(i, j, k) = (0, 4m, n)$ , which is in the support of every element of  $\mathcal{P}_{m,n}^*$ . Thus, we see that  $(\mathcal{P}_{m,n}^*)\pi\gamma \subset \mathcal{Q}_{m',n'}^*$ , as desired.  $\square$

PROPOSITION 14. *If  $\alpha \in \mathcal{A}_1 \setminus \mathcal{A}_2$ , then  $(\mathcal{P}^*)\pi\beta^{-1}\alpha\beta \subset \mathcal{Q}^*$ .*

*Proof.* We write  $\alpha = (a_1x + b_1y + c_1z + d_1, b_2y + c_2z + d_2, c_3z + d_3)$  with  $a_1, b_2, c_3 \in \mathbb{K}^*$  such that  $(a_1, b_2) \neq (c_3^8, c_3^2)$  and  $b_1, c_1, d_1, c_2, d_2, d_3 \in \mathbb{K}$ . Set  $\gamma = \beta^{-1}\alpha\beta$ , and write  $\gamma = (X, Y, Z)$ . We will show that we can apply Lemma 13 to  $\gamma$ . A direct computation shows

$$\begin{aligned} Z &= c_3z + d_3, \\ Y &= b_2y + ez^2 + fz + g, \\ X &= a_1x + F_4y^4 + F_3y^3 + F_2y^2 + F_1y + F_0, \end{aligned}$$

where

$$\begin{aligned} F_4 &= a_1 - b_2^4, & F_3 &= 2(a_1z^2 - b_2^3(Z_2 + Z_3)), \\ F_2 &= a_1z^4 - b_2^2(Z_2^2 + 4Z_2Z_3 + Z_3^2), & F_1 &= b_1 - 2b_2(Z_2 + Z_3)Z_2Z_3, \\ F_0 &= Z_1 - Z_2^2Z_3^2, \\ e &= b_2 - c_3^2, & f &= c_2 - 2c_3d_3, & g &= d_2 - d_3^2 \end{aligned}$$

and

$$Z_1 = b_1z^2 + c_1z + d_1, \quad Z_2 = b_2z^2 + c_2z + d_2, \quad Z_3 = ez^2 + fz + g.$$

We easily check that  $\deg_{(3,3,1)}(Y) = 3$ . Since  $Z_1, Z_2$ , and  $Z_3$  are polynomials in  $z$  of degree  $\leq 2$ , the support of  $X$  contains  $(1, 0, 0)$  and some points  $(0, j, k)$  such that  $2j + k \leq 8$ . Write  $X = a_1x + \sum_{2j+k \leq 8} \gamma_{j,k}y^jz^k$  for some  $\gamma_{j,k} \in \mathbb{K}$ . Then the previous direct computation shows  $\gamma_{4,0} = a_1 - b_2^4$ , whereas  $\gamma_{3,2} = 2(a_1 - 2b_2^4 + b_2^3c_3^2)$ . Since  $(a_1, b_2) \neq (c_3^8, c_3^2)$ , we must have  $(\gamma_{4,0}, \gamma_{3,2}) \neq (0, 0)$ . Thus, we see that  $\text{ldeg}_2(X) = (0, b, c)$  for some  $b \geq 3$ , and  $\deg_{(3,3,1)}(X) = 3b + c$ . Now the result follows immediately from Lemma 13.  $\square$

PROPOSITION 15. *If  $\alpha \in \mathcal{A}_2 \setminus \mathcal{A}_3$ , then  $(\mathcal{P}^*)\pi\beta^{-1}\alpha\beta\pi\beta \subset \mathcal{Q}^*$ .*

*Proof.* Since  $\alpha \in \mathcal{A}_2$ , we can write  $\alpha = (u^8x + b_1y + c_1z + d_1, u^2y, uz) \in \mathcal{A}_2 \setminus \mathcal{A}_3$  for some  $u, b_1 \in \mathbb{K}^*$  and  $c_1, d_1 \in \mathbb{K}$ . A direct computation shows that

$$\pi\beta^{-1}\alpha\beta\pi\beta = (u^2X, u^8Y + b_1X + b_1z^2 + c_1z + d_1, uz),$$

where  $X = (x)\beta = x + y^2(y + z^2)^2$  and  $Y = (y)\beta = y + z^2$ . Since  $b_1$  is nonzero, the rest of the proof proceeds exactly along the lines of the case  $(i', j', k') = (i + j, 0, k)$  (Case (1)) in Proposition 12.  $\square$

PROPOSITION 16. *If  $\alpha \in \mathcal{A}_3 \setminus \mathcal{A}_4$ , then  $(\mathcal{P}^*)(\pi\beta^{-1})^2\alpha\beta\pi\beta \subset \mathcal{Q}^*$ .*

*Proof.* Since  $\alpha \in \mathcal{A}_3$ , we can write  $\alpha = (u^8x + c_1z + d_1, u^2y, uz)$  for some  $u \in \mathbb{K}^*$  and  $c_1, d_1 \in \mathbb{K}$ . We compute

$$\gamma := \pi\beta^{-1}\alpha\beta\pi = (u^2x, u^8y + c_1z + d_1, uz).$$

Clearly,  $\gamma \in \mathcal{A}_1$ , and one easily checks that  $\gamma \notin \mathcal{A}_2$ . Then Proposition 14 implies that  $(\mathcal{P}^*)(\pi\beta^{-1})^2\alpha\beta\pi\beta = (\mathcal{P}^*)\pi\beta^{-1}\gamma\beta \subset \mathcal{Q}^*$ .  $\square$

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