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Brauer Groups of Quot Schemes

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Abstract. Let X be an irreducible smooth complex projective curve. Let $\mathcal{Q}(r,d)$ be the Quot scheme parameterizing all coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$. There are natural morphisms $\mathcal{Q}(r,d) \longrightarrow \mathrm{Sym}^d(X)$ and $\mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X)$. We prove that both these morphisms induce isomorphism of Brauer groups if $d \geq 2$. Consequently, the Brauer group of $\mathcal{Q}(r,d)$ is identified with the Brauer group of $\mathrm{Pic}^d(X)$ if $d \geq 2$.

1. Introduction

Let X be an irreducible smooth projective curve defined over $\mathbb{C}$. For any integer $r \geq 1$, consider the trivial holomorphic vector bundle $\mathcal{O}_X^{\oplus r}$ on X. For any $d \geq 0$, let $\mathcal{Q}(r,d)$ denote the Quot scheme that parameterizes all torsion quotients of degree d of the $\mathcal{O}_X$-module $\mathcal{O}_X^{\oplus r}$. This $\mathcal{Q}(r,d)$ is an irreducible smooth complex projective variety of dimension rd.

For every $Q \in \mathcal{Q}(r,d)$, we have a corresponding short exact sequence

$$0 \longrightarrow \mathcal{F}(Q) \xrightarrow{\rho} \mathcal{O}_X^{\oplus r} \longrightarrow Q \longrightarrow 0.$$

The pairs $(\mathcal{O}_X^{\oplus r})^* = \mathcal{O}_X^{\oplus r} \xrightarrow{\rho^*} \mathcal{F}(Q)^*$ are vortices of a particular numerical type. The Quot scheme $\mathcal{Q}(r,d)$ is a moduli space of vortices of a particular numerical type (see [BDW; Ba; BR], and references therein).

Sending such Q to the scheme theoretic support of the quotient for the homomorphism

$$\bigwedge^r \mathcal{F}(Q) \longrightarrow \bigwedge^r \mathcal{O}_X^{\oplus r}$$

induced by the inclusion $\mathcal{F}(Q) \longrightarrow \mathcal{O}_X^{\oplus r}$, we get a morphism

$$\varphi : \mathcal{Q}(r,d) \longrightarrow \mathrm{Sym}^d(X).$$

Sending any $Q \in \mathcal{Q}(r,d)$ to the holomorphic line bundle $\bigwedge^r \mathcal{F}(Q)^*$, we get a morphism

$$\varphi' : \mathcal{Q}(r,d) \longrightarrow \mathcal{Q}(1,d) = \mathrm{Pic}^d(X).$$

On the other hand, we have the morphism

$$\xi_d : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X)$$

that sends any $(x_1, \dots, x_d)$ to the holomorphic line bundle $\mathcal{O}_X(\sum_{i=1}^d x_i)$. Note that $\varphi' = \xi_d \circ \varphi$.

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The cohomological Brauer group of a smooth complex projective variety M will be denoted by $\mathrm{Br}'(M)$. A theorem of Gabber says that $\mathrm{Br}'(M)$ coincides with the Brauer group of M (see [dJ]).

Our aim here is to prove the following:

Theorem 1.1. *For the morphisms φ and ξ_d, the pullback homomorphisms of Brauer groups*

$$\varphi* : \mathrm{Br}'(\mathrm{Sym}^d(X)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r,d)) \quad \textit{and}$$
$$\xi_d^* : \mathrm{Br}'(\mathrm{Pic}^d(X)) \longrightarrow \mathrm{Br}'(\mathrm{Sym}^d(X))$$

are isomorphisms if $d \geq 2$.

Theorem 1.1 is proved in Lemma 4.2 and Lemma 6.1.

If $\mathrm{genus}(X) = 1$, then $\mathrm{Sym}^d(X)$ is a projective bundle over X, and hence $\mathrm{Br}'(\mathrm{Sym}^d(X)) = 0$. If $\mathrm{genus}(X) = 0$, then $\mathrm{Br}'(\mathrm{Sym}^d(X)) = 0$ because $\mathrm{Sym}^d(X) = \mathbb{CP}^d$. Therefore, Theorem 1.1 has the following corollary:

Corollary 1.2. *If* $\mathrm{genus}(X) \leq 1$, *then* $\mathrm{Br}'(\mathcal{Q}(r,d)) = 0$.

Since $\mathcal{Q}(r, 1)$ is a projective bundle over X, it follows that $\mathrm{Br}'(\mathcal{Q}(r,d)) = 0$. Note that $\mathrm{Br}'(\mathrm{Pic}^d(X))$ is nonzero if $\mathrm{genus}(X) > 1$, whereas $\mathrm{Br}'(\mathrm{Sym}^1(X)) = 0$.

Fixing a point $x_0 \in X$, construct an embedding

$$\delta : \mathcal{Q}(r,d) \longrightarrow \mathcal{Q}(r,d+r)$$

by sending any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to $\mathcal{F} \otimes \mathcal{O}_X(-x_0)$.

The following is proved in Corollary 6.2:

Proposition 1.3. *The pullback homomorphism for δ*

$$\delta^* : \mathrm{Br}'(\mathcal{Q}(r,d+r)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r,d))$$

is an isomorphism if $d \geq 2$.

Now assume that r, $\mathrm{genus}(X) \geq 2$; if $r = 2$, then also assume that $\mathrm{genus}(X) \geq 3$. Iterating the morphism δ, we get an ind-scheme. This ind-scheme has the cohomology isomorphic to the moduli stack; see [Dh, Theorem 4.5] or [Ne, Chapter 4]. Using Proposition 1.3 and Theorem 1.1, we can now describe the cohomological Brauer group of the moduli stack of rank r and degree d bundles. Further, we deduce that the cohomological Brauer group of the moduli stack of vector bundles on X of rank r and fixed determinant vanishes. This result was proved earlier in [BH, Theorem 5.2]. Using this vanishing result, we can deduce that the cohomological Brauer group of the moduli space of stable vector bundles on X of rank r and fixed determinant of degree d is a cyclic group of order g.c.d.(r,d). This result was proved earlier in [BBGN].

2. Cohomological Brauer Group

Let M be an irreducible smooth projective variety defined over $\mathbb{C}$. Let $\mathcal{O}_M^*$ denote the multiplicative sheaf on M of holomorphic functions with values in $\mathbb{C} \setminus \{0\}$. The *cohomological Brauer group* $\mathrm{Br}'(M)$ is the torsion subgroup of the cohomology group $H^2(M, \mathcal{O}_M^*)$.

Let $\mathcal{O}_M$ denote the sheaf of holomorphic functions on M. Consider the short exact sequence of sheaves on M

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \stackrel{\exp}{\longrightarrow} \mathcal{O}_M^* \longrightarrow 0,$$

where the homomorphism $\mathbb{Z} \longrightarrow \mathcal{O}_M$ sends any integer n to the constant function $2\pi\sqrt{-1} \cdot n$. Let

$$\mathrm{Pic}(M) = H^1(M, \mathcal{O}_M^*) \stackrel{c}{\longrightarrow} H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathcal{O}_M) \tag{2.1}$$

be the corresponding long exact sequence of cohomology groups. The homomorphism c in (2.1) sends a holomorphic line bundle to its first Chern class. The image $c(\mathrm{Pic}(M))$ coincides with the Néron–Severi group

$$\mathrm{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z}).$$

Define the subgroup

$$A := H^2(M, \mathbb{Z})/c(\mathrm{Pic}(M)) = H^2(M, \mathbb{Z})/\,\mathrm{NS}(M) \subset H^2(M, \mathcal{O}_M) \tag{2.2}$$

(see (2.1)). Let

$$H^3(M, \mathbb{Z})_{\mathrm{tor}} \subset H^3(M, \mathbb{Z})$$

be the torsion part.

Proposition 2.1 [Sco]. *There is a natural short exact sequence*

$$0 \longrightarrow A \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Br}'(M) \longrightarrow H^3(M, \mathbb{Z})_{\mathrm{tor}} \longrightarrow 0,$$

where A is defined in (2.2).

See [Sco, p. 878, Proposition 1.1] for a proof of Proposition 2.1.

3. The Cohomology of Symmetric Products

Let X be an irreducible smooth complex projective curve. The genus of X will be denoted by g. For any positive integer d, let P_d be the group of all permutations of $\{1, \dots, d\}$. By $\mathrm{Sym}^d(X)$ we will denote the quotient of X^d for the natural action of P_d on it. So $\mathrm{Sym}^d(X)$ parameterizes all formal sums of the form $\sum_{x \in X} n_x \cdot x$, where n_x are nonnegative integers with $\sum_{x \in X} n_x = d$. In other words, $\mathrm{Sym}^d(X)$ parameterizes all effective divisors on X of degree d. This $\mathrm{Sym}^d(X)$ is an irreducible smooth complex projective variety of complex dimension d. Let

$$q_d : X^d \longrightarrow \mathrm{Sym}^d(X) = X^d/P_d \tag{3.1}$$

be the quotient map.

Let $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ be a symplectic basis for $H^1(X, \mathbb{Z})$ chosen so that $\alpha_i \cdot \alpha_{i+g} = 1$ for $i \le g$ and $\alpha_i \cdot \alpha_j = 0$ if $|i - j| \ne g$. The oriented generator of

$H^2(X, \mathbb{Z})$ will be denoted by ω. For $i \in [1, 2g]$ and $j \in [1, d]$, we have the cohomology classes

$$\lambda_i^j := 1 \otimes \cdots \otimes \alpha_i \otimes \cdots \otimes 1 \in H^1(X^n, \mathbb{Z}) \tag{3.2}$$

and

$$\eta^j := 1 \otimes \cdots \otimes \omega \otimes \cdots \otimes 1 \in H^2(X^n, \mathbb{Z}), \tag{3.3}$$

where both α_i and ω are at the jth position.

Theorem 3.1 [Ma]. *For the morphism q_d in (3.1), the pullback homomorphism*

$$q_d^* : H^*(\mathrm{Sym}^d(X), \mathbb{Z}) \longrightarrow H^*(X^d, \mathbb{Z})$$

is injective. Further, the image of q_d^* is generated, as a $\mathbb{Z}$-algebra, by

$$\lambda_i = \sum_{j=1}^{d} \lambda_i^j, \quad 1 \le i \le 2g, \quad \textit{and} \quad \eta = \sum_{j=1}^{d} \eta^j.$$

See [Ma, p. 325, (6.3)] and [Ma, p. 326, (7.1)] for Theorem 3.1.

There is a universal divisor D^{univ} on $\mathrm{Sym}^d(X) \times X$, which consists of all $(z, x) \in \mathrm{Sym}^d(X) \times X$ such that x is in the support of z. We wish to describe the class of this divisor in $H^2(\mathrm{Sym}^d(X) \times X, \mathbb{Z})$. In view of the first part of Theorem 3.1, the algebra $H^*(\mathrm{Sym}^d(X) \times X, \mathbb{Z})$ is considered as a subalgebra of $H^*(X^{d+1}, \mathbb{Z})$.

For $i \in [1, d+1]$, let $\pi_i : X^{d+1} \longrightarrow X$ be the projection to the ith factor. For any integer $k \in [1, d]$, consider the closed immersion

$$\iota_k : X^d \hookrightarrow X^{d+1} m$$

which is uniquely determined by

$$\pi_i \circ \iota_k = \begin{cases} \pi_i' & \text{if } i \neq d+1, \\ \pi_k' & \text{if } i = d+1, \end{cases}$$

where π_j' is the projection of X^d to the jth factor. In other words, $i_k(x_1, \dots, x_k, \dots, x_d) = i_k(x_1, \dots, x_k, \dots, x_d, x_k)$. The divisor on X^{d+1} given by the image of ι_k will be denoted by D_k.

The divisor D_k is closely related to the universal divisor D^{univ} defined before. To see this, consider the projection

$$q_d \times \mathrm{Id}_X : X^{d+1} = X^d \times X^d \longrightarrow \mathrm{Sym}^d(X) \times X,$$

where q_d is constructed in (3.1). The image $(q_d \times \mathrm{Id}_X)(D_k)$ is independent of the choice of k and coincides with D^{univ}. This implies that D^{univ} is irreducible.

The classes

$$\lambda_i^j \cup \lambda_{i'}^{j'}, \quad i \neq i', 1 \le j < j' \le d+1,$$

and

$$\eta^j, \quad 1 \le j \le d+1,$$

constructed as in (3.2) and (3.3) for $d+1$, together give a basis for $H^2(X^{d+1}, \mathbb{Z})$. We have the dual basis for $H^{2d}(X^{d+1}, \mathbb{Z})$ given by

$$\eta^{j\vee} = \omega \otimes \cdots \otimes \omega \otimes 1 \otimes \omega \otimes \cdots \otimes \omega$$

and

$$(\lambda_i^j \cup \lambda_{i'}^{j'})^\vee = \omega \otimes \cdots \otimes \omega \otimes \widetilde{\alpha}_i \otimes \omega \otimes \cdots \omega \otimes \widetilde{\alpha}_{i'} \otimes \omega \otimes \cdots \otimes \omega,$$

where $\widetilde{\alpha}_i$ (respectively, $\widetilde{\alpha}_{i'}$) is the class with $\widetilde{\alpha}_i \cup \alpha_i = \omega$ (respectively, $\widetilde{\alpha}_{i'} \cup \alpha_{i'} = \omega$). Now

$$\int_{D_k} \eta^{j\vee} = \int_{X^d} \iota_k^* \eta^{j\vee} = \begin{cases} 1, & j = k, \\ 1, & j = d+1, \\ 0 & \text{otherwise,} \end{cases}$$

whereas

$$\int_{D_k} (\lambda_i^j \cup \lambda_{i'}^{j'})^\vee = \int_X \widetilde{\alpha}_i \cup \widetilde{\alpha}_{i'}$$

if $j' = d+1$ and $j = k$, and

$$\int_{D_k} (\lambda_i^j \cup \lambda_{i'}^{j'})^\vee = 0$$

otherwise. So the class of D_k is

$$\eta^k + \eta^{d+1} + \sum_{i=1}^{g} \lambda_i^k \cup \lambda_{i+g}^{d+1} - \sum_{i=g+1}^{2g} \lambda_i^k \cup \lambda_{i-g}^{d+1}.$$

By the Künneth formula we have

$$\begin{aligned} &H^2(\mathrm{Sym}^d(X) \times X, \mathbb{Z}) \\ &\quad \cong (H^2(\mathrm{Sym}^d(X), \mathbb{Z}) \otimes H^0(X, \mathbb{Z})) \\ &\qquad \oplus (H^0(\mathrm{Sym}^d(X), \mathbb{Z}) \otimes H^2(X, \mathbb{Z})) \oplus (H^1(\mathrm{Sym}^d(X), \mathbb{Z}) \otimes H^1(X, \mathbb{Z})). \end{aligned}$$

Using Theorem 3.1 (3.1), we have a basis for $H^2(\mathrm{Sym}^d(X) \times X, \mathbb{Z})$ consisting of

$$\eta \otimes 1_X, \qquad \{(\lambda_i \cup \lambda_j) \otimes 1_X\}_{i,j=1}^{2g}, \qquad 1_{\mathrm{Sym}^d(X)} \otimes \omega, \qquad \{\lambda_i \otimes \alpha_j\}_{i,j=1}^{2g}.$$

From the previous computations it follows that the class of D^{univ} is

$$[D^{\mathrm{univ}}] = \eta \otimes 1 + d(1_{\mathrm{Sym}^d(X)} \otimes \omega) + \sum_{i=1}^{g} \lambda_i \otimes \alpha_{i+g} - \sum_{i=g+1}^{2g} \lambda_i \otimes \alpha_{i-g}. \quad (3.4)$$

PROPOSITION 3.2.

(1) *For a fixed point* $x_0 \in X$, *consider the inclusion*

$$\iota_{x_0} : \mathrm{Sym}^d(X) \hookrightarrow \mathrm{Sym}^d(X) \times X$$

defined by $z \longmapsto (z, x_0)$. *The cohomology class* $\iota_{x_0}^*[D^{\mathrm{univ}}]$ *is* η.

(2) *The slant product of* $[D^{\mathrm{univ}}]$ *with* $\alpha_i^\vee$ *produces the class* λ_i *in* $H^1(\mathrm{Sym}^d(X), \mathbb{Z})$.

Proof. These follow from (3.4). □

4. Cohomological Brauer Group of the Symmetric Product

Recall that X denotes a smooth projective curve. Fix a point $x_0 \in X$. For any $d \geq 1$, let

$$f_d : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Sym}^{d+1}(X) \tag{4.1}$$

be the morphism defined by $\sum_{x\in X} n_x \cdot x \longmapsto x_0 + \sum_{x\in X} n_x \cdot x$. Let

$$f_d^* : \mathrm{Br}'(\mathrm{Sym}^{d+1}(X)) \longrightarrow \mathrm{Br}'(\mathrm{Sym}^d(X)) \tag{4.2}$$

be the pullback homomorphism for f_d in (4.1).

Lemma 4.1. *For every $d \geq 2$, the homomorphism f_d^* in (4.2) is an isomorphism.*

Proof. For every positive integer d, the cohomology group $H^*(\mathrm{Sym}^d(X), \mathbb{Z})$ is torsionfree by Theorem 3.1. Therefore, from Proposition 2.1 we conclude that

$$\mathrm{Br}'(\mathrm{Sym}^d(X)) \cong (H^2(\mathrm{Sym}^d(X), \mathbb{Z})/\,\mathrm{NS}(\mathrm{Sym}^d(X))) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}). \tag{4.3}$$

From Theorem 3.1,

$$H^2(\mathrm{Sym}^d(X), \mathbb{Z}) = \left(\bigwedge^2 H^1(X, \mathbb{Z})\right) \oplus H^2(X, \mathbb{Z}). \tag{4.4}$$

Let

$$f_d' : H^2(\mathrm{Sym}^{d+1}(X), \mathbb{Z}) \longrightarrow H^2(\mathrm{Sym}^d(X), \mathbb{Z})$$

be the homomorphism that sends a cohomology class to its pullback by the map f_d in (4.1). It is evident that in terms of the isomorphism in (4.4), this homomorphism f_d' coincides with the identity map of $(\bigwedge^2 H^1(X, \mathbb{Z})) \oplus H^2(X, \mathbb{Z})$.

The isomorphism in (4.4) is clearly compatible with the Hodge decompositions. Since f_d' coincides with the identity map of $(\bigwedge^2 H^1(X, \mathbb{Z})) \oplus H^2(X, \mathbb{Z})$, we now conclude that f_d' takes $\mathrm{NS}(\mathrm{Sym}^{d+1}(X))$ isomorphically to $\mathrm{NS}(\mathrm{Sym}^d(X))$. Therefore, the lemma follows from (4.3). □

For any positive integer d, let

$$\xi_d : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^d(X) \tag{4.5}$$

be the morphism defined by $\sum_{x\in X} n_x \cdot x \longmapsto \mathcal{O}_X(\sum_{x\in X} n_x \cdot x)$. Let

$$\xi_d^* : \mathrm{Br}'(\mathrm{Pic}^d(X)) \longrightarrow \mathrm{Br}'(\mathrm{Sym}^d(X)) \tag{4.6}$$

be the pullback homomorphism corresponding to ξ_d.

Lemma 4.2. *For any $d \geq 2$, the homomorphism ξ_d^* in (4.6) is an isomorphism.*

Proof. Let

$$\eta_d : \mathrm{Pic}^d(X) \longrightarrow \mathrm{Pic}^{d+1}(X)$$

be the isomorphism defined by $L \longmapsto L \otimes \mathcal{O}_X(x_0)$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^d(X) & \xrightarrow{f_d} & \mathrm{Sym}^{d+1}(X) \\ \downarrow \xi_d & & \downarrow \xi_{d+1} \\ \mathrm{Pic}^d(X) & \xrightarrow{\eta_d} & \mathrm{Pic}^{d+1}(X) \end{array}$$

where f_d and ξ_d are constructed in (4.1) and (4.5), respectively, and η_d is defined above. Let

$$\begin{array}{ccc} \mathrm{Br}'(\mathrm{Pic}^{d+1}(X)) & \xrightarrow{\eta_d^*} & \mathrm{Br}'(\mathrm{Pic}^d(X)) \\ \downarrow \xi_{d+1}^* & & \downarrow \xi_d^* \\ \mathrm{Br}'(\mathrm{Sym}^{d+1}(X)) & \xrightarrow{f_d^*} & \mathrm{Br}'(\mathrm{Sym}^d(X)) \end{array} \tag{4.7}$$

be the corresponding commutative diagram of homomorphisms of cohomological Brauer groups. From Lemma 4.1 we know that f_d^* is an isomorphism for $d \geq 2$. The homomorphism η_d^* is an isomorphism because the map η_d is an isomorphism. Therefore, from the commutativity of (4.7) we conclude that the homomorphism ξ_d^* is an isomorphism if ξ_{d+1}^* is an isomorphism. Consequently, it suffices to prove the lemma for all d sufficiently large.

As before, the genus of X is denoted by g. Take any $d > 2g$. Note that for any line bundle L on X of degree d, using Serre duality, we have

$$H^1(X, L) = H^0(X, K_X \otimes L^\vee)^\vee = 0 \tag{4.8}$$

because degree$(K_X \otimes L^\vee) = 2g - 2 - d < 0$.

Take a Poincaré line bundle $\mathcal{L} \longrightarrow X \times \mathrm{Pic}^d(X)$. From (4.8) it follows that the direct image

$$\mathrm{pr}_* \mathcal{L} \longrightarrow \mathrm{Pic}^d(X)$$

is locally free of rank $d - g + 1$, where pr is the natural projection of $X \times \mathrm{Pic}^d(X)$ to $\mathrm{Pic}^d(X)$. The projective bundle $\mathbb{P}(\mathrm{pr}_* \mathcal{L})$, that parameterizes the lines in the fibers of the holomorphic vector bundle $\mathrm{pr}_* \mathcal{L}$, is independent of the choice of the Poincaré line bundle $\mathcal{L}$. Indeed, this follows from the fact that any two choices of the Poincaré line bundle over $X \times \mathrm{Pic}^d(X)$ differ by tensoring with a line bundle pulled back from $\mathrm{Pic}^d(X)$ [ACGH, p. 166]. The total space of $\mathbb{P}(\mathrm{pr}_* \mathcal{L})$ is identified with $\mathrm{Sym}^d(X)$ by sending a section to the divisor on X given by the section; see [Scb]. This identification between $\mathrm{Sym}^d(X)$ and $\mathbb{P}(\mathrm{pr}_* \mathcal{L})$ takes the map ξ_d in (4.5) to the natural projection of $\mathbb{P}(\mathrm{pr}_* \mathcal{L})$ to $\mathrm{Pic}^d(X)$.

Since $\mathbb{P}(\mathrm{pr}_* \mathcal{L})$ is the projectivization of a vector bundle, the natural projection

$$\mathbb{P}(\mathrm{pr}_* \mathcal{L}) \longrightarrow \mathrm{Pic}^d(X)$$

induces an isomorphism of cohomological Bauer groups [Ga, p. 193]. Consequently, the homomorphism

$$\xi_d^* : \mathrm{Br}'(\mathrm{Pic}^d(X)) \longrightarrow \mathrm{Br}'(\mathrm{Sym}^d(X))$$

defined in (4.6) is an isomorphism if $d > 2g$. We noted earlier that it is enough to prove the lemma for all d sufficiently large. Therefore, the proof of the lemma is now complete. □

5. The Cohomology of the Quot Scheme

For integers $r \geq 1$ and d, denote by $\mathcal{Q}(r,d)$ the Quot scheme parameterizing all coherent subsheaves

$$\mathcal{F} \hookrightarrow \mathcal{O}_X^{\oplus r},$$

where $\mathcal{F}$ is of rank r and degree $-d$. Note that there is no such subsheaf if $d < 0$. If $d = 0$, then $\mathcal{F} = \mathcal{O}_X^{\oplus r}$. If $d = 1$, then $\mathcal{Q}(r,d) = X \times \mathbb{CP}^{r-1}$. We assume that $d \geq 1$.

We will now recall from [Bi] a few facts about the Białynicki-Birula decomposition of $\mathcal{Q}(r,d)$. Using the natural action of $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ on $\mathcal{O}_X$, we get an action of $\mathbb{G}_m^r$ on $\mathcal{O}_X^{\oplus r}$. This action produces an action of $\mathbb{G}_m^r$ on $\mathcal{Q}(r,d)$. The fixed points of this torus action correspond to subsheaves of $\mathcal{O}_X^{\oplus r}$ that decompose into compatible direct sums

$$\bigoplus_{i=1}^{r} \mathcal{L}_i \hookrightarrow \mathcal{O}_X^{\oplus r},$$

where $\mathcal{L}_i \hookrightarrow \mathcal{O}_X$ is a subsheaf of rank one. Let D_i be the effective divisor given by the inclusion of $\mathcal{L}_i$ in $\mathcal{O}_X$. In particular, we have $\mathcal{L}_i = \mathcal{O}_X(-D_i)$.

We use the convention that $\mathrm{Sym}^0(X)$ is a single point. Using this notation, we have

$$(D_1, \dots, D_r) \in \mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_r}(X),$$

where $m_i = \mathrm{degree}(D_i)$. Conversely, if $(D_1', \dots, D_r') \in \mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_r}(X)$, then the point of $\mathcal{Q}(r,d)$ representing the subsheaf

$$\bigoplus_{i=1}^{r} \mathcal{O}_X(-D_i') \subset \mathcal{O}_X^{\oplus r}$$

is fixed by the action of $\mathbb{G}_m^r$ on $\mathcal{Q}(r,d)$.

For $k \geq 1$, denote by $\mathbf{Part}_r^k$ the set of partitions of k of length r. So

$$\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{Part}_r^k$$

if and only if m_j are nonnegative integers with

$$\sum_{j=1}^{r} m_j = k.$$

For $\mathbf{m} \in \mathbf{Part}_r^d$, define

$$d_{\mathbf{m}} := \sum_{i=1}^{r} (i-1) m_i. \tag{5.1}$$

The connected components of the fixed point locus for the action of $\mathbb{G}_m^r$ on $\mathcal{Q}(r,d)$ are in bijection with the elements of $\mathbf{Part}_r^d$. The component corresponding to the partition $\mathbf{m} = (m_1, \dots, m_r)$ is the product

$$\mathrm{Sym}^{\mathbf{m}}(X) := \mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_r}(X).$$

It is possible (see [Bi, p. 3]) to choose a one-parameter subgroup $\mathbb{G}_m \longrightarrow \mathbb{G}_m^r$ given by $z \mapsto (z^{\lambda_1}, z^{\lambda_2}, \dots, z^{\lambda_r})$ so that the following two hold:

(1) The fixed point locus under the induced action of $\mathbb{G}_m$ is the same as the fixed point locus under the action of $\mathbb{G}_m^r$.
(2) The integers $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ are increasing.

For this action of $\mathbb{G}_m$ on $\mathcal{Q}(r, d)$, define

$$\mathrm{Sym}^{\mathbf{m}}(X)^+ := \left\{x \in \mathcal{Q}(r, d) \;\middle|\; \lim_{t \to 0} t.x \in \mathrm{Sym}^{\mathbf{m}}(X)\right\},$$

where $\mathbf{m} \in \mathbf{Part}_r^k$. This stratification of $\mathcal{Q}(r, d)$ gives us a decomposition of the Poincaré polynomial of $\mathcal{Q}(r, d)$. Further, the morphism

$$\mathrm{Sym}^{\mathbf{m}}(X)^+ \longrightarrow \mathrm{Sym}^{\mathbf{m}}(X) \tag{5.2}$$

that sends a point to its limit is a fiber bundle with fiber $\mathbb{A}^{d_{\mathbf{m}}}$ (see [Bb] and [Bi]), where $d_{\mathbf{m}}$ is defined in (5.1).

This gives

$$\dim \mathrm{Sym}^{\mathbf{m}}(X)^+ = \dim \mathrm{Sym}^{\mathbf{m}}(X) + d_{\mathbf{m}} = d + d_{\mathbf{m}} \tag{5.3}$$

(see [Bi]).

THEOREM 5.1. *For $i \geq 1$,*

$$H^i(\mathcal{Q}(r, d), \mathbb{Z}) \cong \bigoplus_{\substack{\mathbf{m} \in \mathbf{Part}_r^d \\ j + 2d_{\mathbf{m}} = i}} H^j(\mathrm{Sym}^{m_1}(X) \times \cdots \times \mathrm{Sym}^{m_r}(X), \mathbb{Z}).$$

Proof. See [Bi] and [BGL, p. 649, Remark]. □

We will construct some cohomology classes in $H^2(\mathcal{Q}(r, d), \mathbb{Z})$. There is a universal vector bundle $\mathcal{F}^{\mathrm{univ}}$ on $\mathcal{Q}(r, d) \times X$. Fix a point $x_0 \in X$. Let

$$i_x : \mathcal{Q}(r, d) \longrightarrow \mathcal{Q}(r, d) \times X$$

be the embedding defined by $z \longmapsto (z, x)$.

Let

$$c := i_x^* c_1(\mathcal{F}^{\mathrm{univ}}) \in H^2(\mathcal{Q}(r, d), \mathbb{Z}) \tag{5.4}$$

be the pullback. This cohomology class c is clearly independent of x.

We can produce cohomology classes

$$\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{2g} \in H^1(\mathcal{Q}(r, d), \mathbb{Z})$$

by taking the slant product of $c_1(\mathcal{F}^{\mathrm{univ}})$ with the elements of a basis $\{\alpha_1, \ldots, \alpha_{2g}\}$ for $H^1(X, \mathbb{Z})$. Finally, there is a cohomology class $\gamma_2 \in H^2(\mathcal{Q}(r, d), \mathbb{Z})$ obtained by taking the slant product of $c_2(\mathcal{F}^{\mathrm{univ}})$ with the fundamental class of X.

REMARK 5.2. We will see in the next proposition that the cohomology of $\mathcal{Q}(r, d)$ has no torsion. The class $c_2(\mathcal{F}^{\mathrm{univ}})$ is a (p, p)-class and so is the fundamental class of X. It follows that the class γ_2 is in the Néron–Severi subgroup of $\mathcal{Q}(r, d)$ since the slant product of two (p, p) classes is in fact (p, p).

Proposition 5.3. *Suppose that $d \geq 2$. Then the classes*

$$c, \gamma_2, \overline{\alpha}_i \cup \overline{\alpha}_j, \quad 1 \leq i < j \leq 2g,$$

generate $H^2(\mathcal{Q}(r,d), \mathbb{Z})$. In fact, $H^2(\mathcal{Q}(r,d), \mathbb{Z})$ is torsionfree, and these classes form a basis of the $\mathbb{Z}$-module $H^2(\mathcal{Q}(r,d), \mathbb{Z})$.

Proof. Using Theorem 5.1 and Theorem 3.1, it follows that $H^2(\mathcal{Q}(r,d), \mathbb{Z})$ is torsionfree of rank

$$\binom{2g}{2} + 2.$$

Hence, it suffices to show the stated classes generate the second cohomology group.

The torus action on $\mathcal{Q}(r,d)$ induces a Białynicki-Birula stratification on this variety, as described before. Using (5.3), the cell of largest dimension in the Białynicki-Birula decomposition is the cell corresponding to the partition

$$\mathbf{m}_1 = (0, 0, 0, \dots, d),$$

and the second largest cell corresponds to the partition

$$\mathbf{m}_2 = (0, 0, \dots, 0, 1, d-1).$$

It follows that $\mathrm{Sym}^{\mathbf{m}_1}(X)^+$ is an open dense subscheme of $\mathcal{Q}(r,d)$. Let $D := \mathcal{Q}(r,d) \setminus \mathrm{Sym}^{\mathbf{m}_1}(X)^+$ be the complement. Using (5.2), we have

$$H^2(\mathrm{Sym}^{\mathbf{m}_1}(X)^+, \mathbb{Z}) = H^2(\mathrm{Sym}^{\mathbf{m}_1}(X), \mathbb{Z}).$$

Further, by a dimension calculation (5.3) and a Gysin sequence,

$$H^0(D, \mathbb{Z}) \cong H^0(\mathrm{Sym}^{\mathbf{m}_2}(X), \mathbb{Z}).$$

Let

$$\iota : \mathrm{Sym}^{\mathbf{m}_1}(X) \hookrightarrow \mathcal{Q}(r,d)$$

be the inclusion map.

The Gysin sequence for the decomposition $\mathcal{Q}(r,d) = \mathrm{Sym}^{\mathbf{m}_1}(X)^+ \coprod D$ now reads:

$$\cdots \longrightarrow H^0(\mathrm{Sym}^{\mathbf{m}_2}(X), \mathbb{Z}) \xrightarrow{f_*} H^2(\mathcal{Q}(r,d), \mathbb{Z})$$
$$\xrightarrow{\iota^*} H^2(\mathrm{Sym}^{\mathbf{m}_1}(X), \mathbb{Z}) \longrightarrow \cdots,$$

where

$$f : \mathrm{Sym}^{d-1}(X) \times X \longrightarrow \mathcal{Q}(r,d) \tag{5.5}$$

is the embedding. From [CS], this sequence splits, or in other words, the Białynicki-Birula stratification is integrally perfect.

To complete the proof, it suffices to verify the following two statements:

(S1) The classes $\iota^*(\overline{\alpha}_i \cup \overline{\alpha}_j)$, $1 \leq i < j \leq 2g$, and $\iota^*(c)$ generate $H^2(\mathrm{Sym}^{\mathbf{m}_1}(X), \mathbb{Z})$.

(S2) The class γ_2 generates the image of f_*.

For (S1), observe that

$$\iota^*(\mathcal{F}^{\text{univ}}) = j_z^* \mathcal{O}_{\text{Sym}^d(X)\times X}(-D^{\text{univ}}) \oplus \mathcal{O}^{r-1}_{\text{Sym}^d(X)},$$

where $j_z : \text{Sym}^d(X) \longrightarrow \text{Sym}^d(X) \times X$ is the embedding defined by $z \longmapsto (z, x)$, and D^{univ} is the universal divisor on $\text{Sym}^d(X) \times X$. From Proposition 3.2 and Theorem 3.1 it follows that the classes

$$\iota^*(c), \qquad \iota^*(\overline{\alpha}_i \cup \overline{\alpha}_j), \quad 1 \leq i < j \leq 2g,$$

give a basis for $H^2(\text{Sym}^{\mathbf{m}_1}(X), \mathbb{Z})$. Further, $\gamma_2 \in \text{kernel}(\iota^*)$.

For (S2), we assume that $r = 2$ for simplicity. The proof in the case of higher rank is obtained by adding some trivial summands to the argument below.

As noted before, we have a split exact sequence:

$$0 \longrightarrow H^0(\text{Sym}^{d-1}(X) \times X, \mathbb{Z}) \xrightarrow{f_*} H^2(\mathcal{Q}(r,d), \mathbb{Z})$$
$$\xrightarrow{i^*} H^2(\text{Sym}^d(X), \mathbb{Z}) \longrightarrow 0.$$

Fix some quotient

$$q : \mathcal{O}_X \longrightarrow Q \longrightarrow 0$$

of degree $d-1$ and also fix some quotient

$$q' : \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

of degree 1, where $p \in X$ is a point not in the support of Q.

This gives us a point $z \in \text{Sym}^{d-1}(X) \times X$. We can expand this to a morphism

$$F : \mathbb{A}^1 \longrightarrow (\text{Sym}^{d-1}(X) \times X)^+$$

by considering the family of quotients

$$F(t) := \begin{pmatrix} qf & 0 \\ tq' & q' \end{pmatrix} : \mathcal{O}_X^{\oplus 2} \longrightarrow Q \oplus \mathcal{O}_p \longrightarrow 0.$$

Taking the closure of $F(\mathbb{A}^1)$ in $\mathcal{Q}(2,d)$, we obtain an inclusion

$$F : \mathbb{P}^1 \hookrightarrow \mathcal{Q}(2,d).$$

Since $\dim \mathcal{Q}(2,d) = 2d$, this gives a cohomology class

$$[\mathbb{P}^1] \in H^{4d-2}(\mathcal{Q}(2,d), \mathbb{Z}).$$

Let

$$\mathcal{W} \longrightarrow \mathbb{P}^1 \times X$$

be the restriction of the universal vector bundle $\mathcal{F}^{\text{univ}} \longrightarrow \mathcal{Q}(2,d) \times X$. It fits in the short exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{O}^{\oplus 2}_{\mathbb{P}^1 \times X} \longrightarrow \widetilde{Q} := (\mathcal{O}_{\mathbb{P}^1} \boxtimes Q) \oplus (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_p) \longrightarrow 0. \tag{5.6}$$

Note that the Chern character

$$\text{Ch}(\widetilde{Q}) = d\omega_X + \omega_X \cup \omega_{\mathbb{P}^1}, \tag{5.7}$$

where ω_X and $\omega_{\mathbb{P}^1}$ are the fundamental classes of X and $\mathbb{P}^1$, respectively. In particular, $c_1(\widetilde{Q}) = \omega_X$. Therefore, the slant product of $c_1(\widetilde{Q})$ with elements of $H^1(X, \mathbb{Z})$ vanish. We have

$$c_2(\widetilde{Q}) = \omega_X \cup \omega_{\mathbb{P}^1}.$$

Its slant product with X is then just $\omega_{\mathbb{P}^1}$. Therefore,

$$(F^*\gamma_2) \cup [\mathbb{P}^1] = \int_{\mathbb{P}^1} \gamma_2 = 1.$$

So the cohomology classes described in the statement of the proposition give a basis for the vector space $H^2(\mathcal{Q}(r,d), \mathbb{Q})$.

We will prove the following statements:

$$(F^*c) \cup [\mathbb{P}^1] = 0, \tag{5.8}$$

$$\alpha_i \cup [\mathbb{P}^1] = 0, \tag{5.9}$$

$$f_*([\mathrm{Sym}^{d-1}(X) \times X]) \cup [\mathbb{P}^1] = 1, \tag{5.10}$$

$$(F^*\gamma_2) \cup [\mathbb{P}^1] = \textstyle\int_{\mathbb{P}^1} \gamma_2 = 1. \tag{5.11}$$

The map f is defined in (5.5).

We first show that these statements complete the proof. For that, it is sufficient to observe that they imply that both

$$f_*([\mathrm{Sym}^{d-1}(X) \times X]) \quad \text{and} \quad \gamma_2$$

are dual to $[\mathbb{P}^1]$ and hence must be equal.

To prove (5.8), consider (5.6). Choose a point $x \in X$ away from the support of $Q \oplus \mathcal{O}_p$ and restrict $\mathcal{W}$ to $\mathbb{P}^1 \times \{x\}$. From (5.7) it follows that the first Chern class of this restriction vanishes. The first Chern class of this restriction clearly coincides with $(F^*c) \cup [\mathbb{P}^1]$.

The left-hand side of (5.9) is clearly the slant product of $c_1(\mathcal{W})$ with α_i. We noted before that this slant product vanishes.

Now, (5.10) is clear from the construction of the morphism F from $\mathbb{P}^1$. Finally, (5.11) has already been proved. □

6. The Cohomological Brauer Group

For integers $r \geq 1$ and $d \geq 0$, take any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ lying in $\mathcal{Q}(r,d)$. Taking the rth exterior power, we get a subsheaf $\bigwedge^r \mathcal{F} \subset \bigwedge^r \mathcal{O}_X^{\oplus r} = \mathcal{O}_X$. Let

$$\varphi : \mathcal{Q}(r,d) \longrightarrow \mathrm{Sym}^d(X) \tag{6.1}$$

be the morphism that sends any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to the scheme theoretic support of the quotient $\mathcal{O}_X / \bigwedge^r \mathcal{F}$. Let

$$\varphi^* : \mathrm{Br}'(\mathrm{Sym}^d(X)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r,d)) \tag{6.2}$$

be the pullback homomorphism using φ.

Lemma 6.1. *The homomorphism φ^* in (6.2) is an isomorphism.*

Proof. Note that $\mathrm{Br}'(\mathcal{Q}(r,d)) = \mathrm{Br}'(\mathrm{Sym}^d(X)) = 0$ if $d \leq 1$. Therefore, we assume that $d \geq 2$.

The cohomology group $H^3(\mathcal{Q}(r,d),\mathbb{Z})$ is torsionfree. Indeed, this follows from Theorem 5.1 and the fact that $H^*(\mathrm{Sym}^n(X),\mathbb{Z})$ is torsionfree [Ma, p. 329, (12.3)]. Therefore, Proposition 2.1 says that

$$\mathrm{Br}'(\mathcal{Q}(r,d)) = (H^2(\mathcal{Q}(r,d),\mathbb{Z})/\,\mathrm{NS}(\mathcal{Q}(r,d))) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}). \tag{6.3}$$

Let

$$\varphi' : H^2(\mathrm{Sym}^d(X),\mathbb{Z}) \longrightarrow H^2(\mathcal{Q}(r,d),\mathbb{Z})$$

be the pullback homomorphism using φ in (6.1). Recall from Theorem 3.1 the description of $H^2(\mathrm{Sym}^d(X),\mathbb{Z})$. From Proposition 5.3 we conclude that φ' is injective, and

$$H^2(\mathcal{Q}(r,d),\mathbb{Z}) = \mathrm{image}(\varphi') \oplus \mathbb{Z}\cdot\gamma_2, \tag{6.4}$$

where γ_2 is the cohomology class in Proposition 5.3. Take any point

$$y := (y_1,\ldots,y_d) \in \mathrm{Sym}^d(X)$$

such that all y_i are distinct. Then $\varphi^{-1}(y)$ is a product of copies of $\mathbb{CP}^{r-1}$, and hence

$$H^1(\varphi^{-1}(y),\mathbb{Z}) = 0.$$

From this it follows that the image of the cup product

$$H^1(\mathcal{Q}(r,d),\mathbb{Z}) \otimes H^1(\mathcal{Q}(r,d),\mathbb{Z}) \longrightarrow H^2(\mathcal{Q}(r,d),\mathbb{Z})$$

is in the image φ'. If the point $x \in X$ in (5.4) is different from all y_i, then the restriction of the universal vector bundle $\mathcal{F}^{\mathrm{univ}}$ (see (5.4)) to $\varphi^{-1}(y)$ is the trivial vector bundle of rank r. From this it follows that c is in the image of φ'.

From (6.4) it follows immediately that

$$\mathrm{NS}(\mathcal{Q}(r,d)) = \varphi'(\mathrm{NS}(\mathrm{Sym}^d(X))) \oplus \mathbb{Z}\cdot\gamma_2.$$

In view of (6.3), from this we conclude that φ^* in (6.2) is an isomorphism if $d \geq 2$. □

As before, fix a point $x_0 \in X$. Let

$$\delta : \mathcal{Q}(r,d) \longrightarrow \mathcal{Q}(r,d+r) \tag{6.5}$$

be the morphism that sends any $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ represented by a point of $\mathcal{Q}(r,d)$ to the point representing the subsheaf $\mathcal{F} \otimes \mathcal{O}_X(-x_0) \subset \mathcal{O}_X^{\oplus r}$. Let

$$\delta^* : \mathrm{Br}'(\mathcal{Q}(r,d+r)) \longrightarrow \mathrm{Br}'(\mathcal{Q}(r,d)) \tag{6.6}$$

be the pullback homomorphism by δ.

COROLLARY 6.2. *For any $d \geq 2$, the homomorphism δ^* in (6.6) is an isomorphism.*

Proof. As in (6.1), define

$$\psi : \mathcal{Q}(r, d+r) \longrightarrow \mathrm{Sym}^{d+r}(X)$$

to be the morphism that sends any subsheaf $\mathcal{F} \subset \mathcal{O}_X^{\oplus r}$ to the scheme theoretic support of the corresponding quotient $(\bigwedge^r \mathcal{O}_X^{\oplus r})/(\bigwedge^r \mathcal{F})$. Let

$$h : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Sym}^{d+r}(X)$$

be the morphism defined by $\sum_{x\in X} n_x \cdot x \longmapsto r \cdot x_0 + \sum_{x\in X} n_x \cdot x$. The diagram of morphisms

$$\begin{array}{ccc} \mathcal{Q}(r,d) & \xrightarrow{\delta} & \mathcal{Q}(r,d+r) \\ \downarrow \varphi & & \downarrow \psi \\ \mathrm{Sym}^d(X) & \xrightarrow{h} & \mathrm{Sym}^{d+r}(X) \end{array}$$

is commutative, where φ and δ are defined in (6.1) and (6.5), respectively. Consider the corresponding commutative diagram

$$\begin{array}{ccc} \mathrm{Br}'(\mathrm{Sym}^{d+r}(X)) & \xrightarrow{h^*} & \mathrm{Br}'(\mathrm{Sym}^d(X)) \\ \downarrow \psi^* & & \downarrow \varphi^* \\ \mathrm{Br}'(\mathcal{Q}(r,d+r)) & \xrightarrow{\delta^*} & \mathrm{Br}'(\mathcal{Q}(r,d)) \end{array}$$

of homomorphisms. If $d \geq 2$, from Lemma 6.1 we know that ψ^* and φ^* are isomorphisms, whereas Lemma 4.1 implies that h^* is an isomorphism. Therefore, the homomorphism δ^* is an isomorphism. □

Remark 6.3. We are grateful to an unknown referee for this comment. We give here an alternative proof of the fact that the pullback map φ^* induces an isomorphism on cohomology. Consider the big cell of the Białynicki-Birula decomposition described before. It corresponds to the partition

$$\mathbf{m}_1 = (0, 0, 0, \ldots, d).$$

We have a Zariski locally trivial fibration

$$\rho : \mathrm{Sym}^{\mathbf{m}_1}(X)^+ \longrightarrow \mathrm{Sym}^{\mathbf{m}_1}(X)$$

with fiber $\mathbb{A}^n$, see [BB, p. 492]. We claim that we have an induced isomorphism

$$\mathrm{Br}'(\mathrm{Sym}^{\mathbf{m}_1}(X)^+) \cong \mathrm{Br}'(\mathrm{Sym}^{\mathbf{m}_1}(X)).$$

To see this, we will use the exact sequence in Proposition 2.1, which is valid for noncompact spaces; see [Sco, p. 878]. The morphism ρ induces an isomorphism in cohomology groups since it has contractible fibers. Although the morphism ρ may not be a vector bundle, the Néron–Severi groups of the two varieties agree under the identification of cohomology groups as before; see [Fu, p. 22, Proposition 1.9]. It follows now from Proposition 2.1 and Lemma 5 that ρ^* induces an isomorphism on cohomological Brauer groups.

The morphism $\varphi : \mathrm{Sym}^{\mathbf{m}_1}(X) \longrightarrow \mathrm{Sym}^d(X)$ is an isomorphism. So we have a diagram

$$\begin{array}{ccc}
\mathrm{Br}'(\mathrm{Sym}^{\mathbf{m}_1}(X)^+) & \xleftarrow{\ \iota^*\ } & \mathrm{Br}'(\mathcal{Q}(r,d)) \\
\uparrow{\scriptstyle \varphi^*} & \nearrow{\scriptstyle \varphi^*} & \\
\mathrm{Br}'(\mathrm{Sym}^d(X)), & &
\end{array}$$

ι is the composition $\mathrm{Sym}^{\mathbf{m}_1}(X)^+ \hookrightarrow \mathrm{Sym}^{\mathbf{m}_1}(X) \longrightarrow \mathcal{Q}(r,d)$; we note that the homomorphism φ^* in the left is an isomorphism. The map i^* is injective by [Mi, IV, Corollary 2.6]. We can now deduce that φ^* is an isomorphism.

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