New Examples of Proper Holomorphic Maps among Symmetric Domains

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1. Introduction

Let $\Omega_{r,s}$ be a bounded symmetric domain of type I defined by

$$\Omega_{r,s} = \{Z \in M(r,s,\mathbb{C}) : I_{r,r} - ZZ^* > 0\}.$$

Here we denote by >0 the positive definiteness of square matrices, by $M(r,s,\mathbb{C})$ the set of $r \times s$ complex matrices, and by $I_{r,r}$ the $r \times r$ identity matrix. Let $D_{r,s}$ be a generalized ball defined by

$$D_{r,s} = \{[z_1, \ldots, z_{r+s}] \in \mathbb{P}^{r+s-1} : |z_1|^2 + \cdots + |z_r|^2 > |z_{r+1}|^2 + \cdots + |z_{r+s}|^2\}.$$

DEFINITION 1.1. (1) Let $f, g : \Omega_1 \to \Omega_2$ be holomorphic maps between domains Ω_1, Ω_2. We say that f and g are equivalent if and only if $f = A \circ g \circ B$ for some $B \in \mathrm{Aut}(\Omega_1)$ and $A \in \mathrm{Aut}(\Omega_2)$.

(2) Let $g_1, g_2 : \mathbb{P}^n \to \mathbb{P}^N$ be rational maps. We say g_1 and g_2 are rationally equivalent if there is a rational map $g : \mathbb{P}^n \to \mathbb{P}^N$ such that g is a common extension of g_1 and g_2.

The aim of this paper is presenting a simple way to generate proper monomial rational maps between generalized balls and via the relations between generalized balls and bounded symmetric domains of type I given in [4], giving new examples of proper holomorphic maps between bounded symmetric domains of type I.

Consider a proper rational map $g : D_{r,s} \to D_{r',s'}$. In homogeneous coordinate, put $g([z_1, \ldots, z_{r+s}]) = [g_1, \ldots, g_{r'+s'}]$. Suppose that g_i are monomials in $z_1, \ldots, z_{r+s}$ for each i, $1 \le i \le r' + s'$. Then we can define the homogeneous polynomial $P : \mathbb{R}^{r+s} \to \mathbb{R}$ satisfying

$$P(|z_1|^2, \ldots, |z_{r+s}|^2) = \sum_{k=1}^{r'} |g_k|^2 - \sum_{k=r'+1}^{r'+s'} |g_k|^2. \tag{1.1}$$

Since g is proper, $P(x) = 0$ whenever $\sum_{j=1}^{r} x_j = \sum_{j=r+1}^{r+s} x_j$. Hence, P should be of the form

$$\left(\sum_{j=1}^{r} x_j - \sum_{j=r+1}^{r+s} x_j\right)^m Q_P(x) \tag{1.2}$$

for some positive integer m and homogeneous polynomial $Q_P(x)$.

THEOREM 1.2. *Let* $g : D_{r,r} \to D_{r+1,r+1}$ $(r \geq 2)$ *be a proper monomial rational map. Then g is rationally equivalent to one of the following up to automorphisms of* $D_{r,r}$ *and* $D_{r+1,r+1}$:

(1) *In case of* $\deg(g) = 1$: $g([z_1, \dots, z_{2r}]) = [z_1, \dots, z_r, \phi(z), z_{r+1}, \dots, z_{2r}, \phi(z)]$, *where* $\phi(z)$ *is a degree one homogeneous polynomial in* $z_1, \dots, z_{2r}$;

(2) *In case of* $\deg(g) = 2$:
   (a) $g([z_1, z_2, z_3, z_4]) = [z_1^2, z_1 z_2, z_2 z_3, z_3^2, z_1 z_4, z_3 z_4]$,
   (b) $g([z_1, z_2, z_3, z_4]) = [z_1^2, \sqrt{2} z_1 z_2, z_2^2, z_3^2, \sqrt{2} z_3 z_4, z_4^2]$;

(3) *In case of* $\deg(g) \geq 3$; *if* $Q_P(x)$ *has degree* 1 *or the coefficients of the polynomial* $Q_P(x)$ *are nonnegative, then there is no proper monomial rational map.*

The condition in Theorem 1.2 on Q_P are due to combinatorial method counting monomials in expansion of a multiplied polynomial.

The method to characterize proper monomial rational maps originally comes from D'Angelo [1]. He studied proper monomial holomorphic maps from the unit ball to the higher-dimensional unit ball via characterizing the polynomials that can be obtained by taking Euclidean norm on proper maps. By characterizing these polynomials he obtained a complete list of proper monomial holomorphic maps from the two-dimensional unit ball to the four-dimensional unit ball. In this paper, we modify this polynomial, which is appropriate to proper monomial rational maps between generalized balls and characterize the polynomial by counting the number of monomials in the polynomial.

For bounded symmetric domains of rank at least 2, properties of proper holomorphic maps are deeply related to special kind of totally geodesic subspaces of given domains, which are called *invariantly geodesic subspaces*. These are totally geodesic submanifolds with respect to the Bergman metric that are still totally geodesic under the action of automorphisms of the compact dual of an ambient domain. Invariantly geodesic subspaces first appeared in [3] as far as the author knows. These subspaces play important roles to characterize proper holomorphic maps between bounded symmetric domains. Mok and Tsai [3; 6] proved that proper holomorphic maps between irreducible bounded symmetric domains preserve the maximal characteristic subspaces which are also invariantly geodesic subspaces. Based on [3; 6], the rigidity of irreducible bounded symmetric domains have been developed and incorporated by Tu [7; 8] and Ng [4; 5]. In particular, Ng [4] found that generalized balls in the projective spaces parameterize the maximal invariantly geodesic subspaces of bounded symmetric domains of type I, and we use this relation to find several examples of proper holomorphic maps between bounded symmetric domains of type I.

Consider the subspaces in $\Omega_{r,s}$ of the form

$$L_{[A,B]} = \{Z \in \Omega_{r,s} : \ AZ = B\},$$

where $A \in M(1, r, \mathbb{C})$ and $B \in M(1, s, \mathbb{C})$ satisfy $[A, B] \in D_{r,s}$, which are totally geodesic under the action of $\mathrm{SL}(r + s, \mathbb{C})$. These are the maximal invariantly geodesic subspaces. For $X = [A, B] \in D_{r,s}$, denote $X^{\#} = L_X$.

For a proper holomorphic map $f : \Omega_{r,r} \to \Omega_{r+1,r+1}$ $(r \geq 2)$ that preserves the maximal invariantly geodesic subspaces, there is a proper holomorphic map $g : D_{r,r} \to D_{r+1,r+1}$ such that $f(X^\#) \subset g(X)^\#$ for generic $X \in D_{r,r}$.

THEOREM 1.3. *Let* $f : \Omega_{r,r} \to \Omega_{r+1,r+1}$ $(r \geq 2)$ *be a proper holomorphic map. Suppose that* f *preserves the maximal invariantly geodesic subspaces and an induced proper holomorphic map* $g : D_{r,r} \to D_{r+1,r+1}$ *satisfies the conditions in Theorem 1.2. Then* f *is equivalent to one of the following:*

(1)
$$f(Z) = \begin{pmatrix} Z & 0 \\ 0 & h(Z) \end{pmatrix} \quad \text{for } Z \in \Omega_{r,r}$$

and for some holomorphic map $h : \Omega_{r,r} \to \Delta = \{z \in \mathbb{C} : |z| < 1\}$.

(2)
$$f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & z_1 z_2 & z_2 \\ z_1 z_3 & z_2 z_3 & z_4 \\ z_3 & z_4 & 0 \end{pmatrix} \quad \text{for } \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}.$$

(3)
$$f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right) = \begin{pmatrix} z_1^2 & \sqrt{2}z_1 z_2 & z_2^2 \\ \sqrt{2}z_1 z_3 & z_1 z_4 + z_2 z_3 & \sqrt{2}z_2 z_4 \\ z_3^2 & \sqrt{2}z_3 z_4 & z_4^2 \end{pmatrix}.$$

Here is the outline of the paper. Section 2 introduces some basic terminology, well-known facts, and the invariantly geodesic subspaces. In Section 3, we modify D'Angelo's method to proper monomial maps between generalized balls and classify the maps that are needed to sort proper holomorphic maps between bounded symmetric domains of type I. We count the number of monomials in a homogeneous polynomial that is multiplied by two homogeneous polynomials. In Section 4, we present a way to generate proper holomorphic maps from $\Omega_{r,s}$ to $\Omega_{r',s'}$ and prove Theorem 1.3. Furthermore, we give more interesting examples.

2. Preliminaries

2.1. Basic Facts and Terminology

First, we introduce terminology and some facts. For more detail, see [4; 3]. Let $G_{r,s}$ be the Grassmannian of r-planes in $(r + s)$-dimensional complex vector space $\mathbb{C}^{r+s}$, which is the compact dual of $\Omega_{r,s}$. For $X \in M(r, r + s, \mathbb{C})$ of rank r, denote by $[X]$ the r-plane in $\mathbb{C}^n$ that is generated by row vectors of X. For each element Z in $\Omega_{r,s}$, there corresponds the r-plane $[I_{r,r}, Z] \in G_{r,s}$. This is the Borel embedding of $\Omega_{r,s}$ into $G_{r,s}$. It is clear that $SL(r + s, \mathbb{C})$ acts holomorphically and transitively $G_{r,s}$. Denote by $SU(r, s)$ the subgroup of $SL(r + s, \mathbb{C})$ satisfying $M\begin{pmatrix} -I_{r,r} & 0 \\ 0 & I_{s,s} \end{pmatrix} M^* = \begin{pmatrix} -I_{r,r} & 0 \\ 0 & I_{s,s} \end{pmatrix}$ for all $M \in SU(r, s)$. Then $SU(r, s)$ is the automorphism group of $\Omega_{r,s}$. If we put $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in M(r, r, \mathbb{C})$, $B \in M(r, s, \mathbb{C})$, $C \in M(s, r, \mathbb{C})$, and $D \in M(s, s, \mathbb{C})$, then M acts on $\Omega_{r,s}$ by

$Z \mapsto (A + ZC)^{-1}(B + ZD)$. From now on, if we write $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SU}(r, s)$, then, without ambiguity, A, B, C, D are block matrices of the indicated form.

2.2. Invariantly Geodesic Subspaces in $\Omega_{r,s}$

Consider a complex submanifold S in $\Omega_{r,s}$. For every $g \in \mathrm{SL}(r + s, \mathbb{C})$ such that $g(S) \cap \Omega_{r,s} \neq \emptyset$, if the submanifold $g(S) \cap \Omega_{r,s}$ is totally geodesic in $\Omega_{r,s}$ with respect to the Bergman metric of $\Omega_{r,s}$, then S is called *invariantly geodesic subspace* of $\Omega_{r,s}$. In particular, for $W \in \Omega_{r',s'}$ with $r' \leq r$ and $s' \leq s$, the image of the embedding $\mathrm{i}: W \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & W \end{smallmatrix}\right) \in \Omega_{r,s}$ is an invariantly geodesic subspace of $\Omega_{r,s}$. The totally geodesic subspaces that are equivalent under the action of $\mathrm{SU}(r, s)$ to $\mathrm{i}(\Omega_{r,s})$ in $\Omega_{r,s}$ are called (r', s')-*subspaces* of $\Omega_{r,s}$. Among these (r', s')-subspaces, the maximal invariantly geodesic subspaces are parameterized by the generalized ball in $\mathbb{P}^{r+s-1}$.

PROPOSITION 2.1 [4]. *The subspaces of the form*

$$L = \{Z \in \Omega_{r,s}: \ AZ = B\}, \tag{2.1}$$

where $A \in M(1, r, \mathbb{C})$ and $B \in M(1, s, \mathbb{C})$ satisfy $[A, B] \in D_{r,s}$ are $(r - 1, s)$-subspaces.

For example, in case of invariantly geodesic subspaces

$$\left\{\begin{pmatrix} 0 \\ W \end{pmatrix} \in \Omega_{r,s}: \ W \in \Omega_{r-1,s}\right\},$$

$A = (1, 0, \ldots, 0) \in M(1, r, \mathbb{C})$ and $B = (0, \ldots, 0) \in M(1, s, \mathbb{C})$.

For $\Omega_{r,s}$ and $D_{r,s}$, consider the two surjective maps

$$\phi: \mathbb{P}^{r-1} \times \Omega_{r,s} \to \Omega_{r,s}, \qquad ([X], Z) \mapsto Z, \tag{2.2}$$

$$\psi: \mathbb{P}^{r-1} \times \Omega_{r,s} \to D_{r,s}, \qquad ([X], Z) \mapsto [X, XZ]. \tag{2.3}$$

For $Z \in \Omega_{r,s}$, denote $Z^{\#} = \psi(\phi^{-1}(Z)) \subset D_{r,s}$. Similarly for $X \in D_{r,s}$, denote $X^{\#} = \phi(\psi^{-1}(X)) \subset \Omega_{r,s}$. The subspaces $Z^{\#}$ and $X^{\#}$ are called fibral images of Z and X, respectively. Then for $Z \in \Omega_{r,s}$ and $X = [A, B] \in D_{r,s}$ where $A \in M(1, r, \mathbb{C})$ and $B \in M(1, s, \mathbb{C})$,

$$Z^{\#} = \{[A, AZ] \in D_{r,s}: \ [A] \in \mathbb{P}^{r-1}\} \cong \mathbb{P}^{r-1}, \tag{2.4}$$

$$X^{\#} = \{Z \in \Omega_{r,s}: \ AZ = B\} \cong (r - 1, s)\text{-subspace}. \tag{2.5}$$

PROPOSITION 2.2 (cf. [4]). *Let $f: \Omega_{r,r} \to \Omega_{r+1,r+1}$ be a proper holomorphic map. Suppose that there is a meromorphic map $g: D_{r,r} \to D_{r+1,r+1}$ such that $f(X^{\#}) \subset g(X)^{\#}$ for generic point $X \in D_{r,r}$. Then g is a proper map, or $f(Z) = \left(\begin{smallmatrix} Z & 0 \\ 0 & h(Z) \end{smallmatrix}\right)$ for some holomorphic function $h: \Omega_{r,r} \to \Delta$.*

3. Proper Monomial Rational Map from $D_{r,s}$ to $D_{r',s'}$

Let $g : D_{r,s} \to D_{r',s'}$ be a proper monomial rational map, and P, Q_P be homogeneous polynomials defined by (1.1) and (1.2). Then Q_P has the following properties:

(1) $Q_P(x)$ is a homogeneous polynomial, which is not identically zero on

$$\left\{ x = (x_1, \ldots, x_{r+s}) \in \mathbb{R}^{r+s} : \sum_{j=1}^{r} x_j = \sum_{j=r+1}^{r+s} x_j \right\}.$$

(2) $Q_P(x) > 0$ whenever $x_i > 0$ for all i, and $\sum_{j=1}^{r} x_j > \sum_{j=r+1}^{r+s} x_j$.

3.1. Classifying Proper Monomial Rational Map from $D_{r,r}$ to $D_{r+1,r+1}$

A situation of classifying proper rational maps between generalized balls is different from that of classifying proper holomorphic maps between unit balls in [2] since there are an infinite number of proper rational maps that are same in an open dense subset. For example, $g : D_{2,2} \to D_{3,3}$, $[z_1, \ldots, z_4] \mapsto [z_1 h, z_2 h, 0, z_3 h, z_4 h, 0]$ for any holomorphic function h of $\mathbb{C}^4$ that is not identically zero on $D_{2,2}$ are same in an open dense subset depending on the zero set of h. On the other hand, proper rational maps which are same in an open dense subset induce the same proper holomorphic map between corresponding bounded symmetric domains of type I. Hence, we consider an equivalence relation on proper monomial rational maps to incorporate these infinite number of rational maps.

DEFINITION 3.1. Let $g_1, g_2 : \mathbb{P}^{2r-1} \to \mathbb{P}^{2r+1}$ be rational maps. We say that g_1 and g_2 are rationally equivalent if there is a rational map $g : \mathbb{P}^{2r-1} \to \mathbb{P}^{2r+1}$ such that g is a common extension of g_1 and g_2.

We may assume that all components of $g : D_{r,s} \to D_{r',s'}$ have no common factor.

In the rest of this section, we characterize the induced polynomial $P(x)$ and the proper monomial rational maps from $D_{r,r}$ to $D_{r+1,r+1}$ to prove Theorem 1.2. For this aim, we will count the number of monomials of P for suitable Q_P. For a polynomial A, denote by $n_i(A)$ the number of monomials with maximal degree in x_i of A and by $n(A)$ the number of monomials in A.

LEMMA 3.2. *For a polynomial* $A = (b_1 x_1 + \cdots + b_k x_k)^m \tilde{A}$ *with nonzero polynomial* $\tilde{A}$, *positive integer* m, *and nonzero* b_i *for all* i, $1 \le i \le k$, *we have* $n(A) \ge \sum_{i=1}^{k} n_i(\tilde{A})$.

Proof. The term $(b_i x_i)^m$ times the monomial with the maximal degree of x_i in $\tilde{A}$ cannot be canceled. □

LEMMA 3.3. *Let* $P(x)$ *be a homogeneous polynomial on* $\mathbb{R}^k$ *of the form*

$$(b_1 x_1 + \cdots + b_k x_k)^m Q_P(x)$$

for some positive integer m, nonzero b_i for all i, $1 \le i \le k$, and homogeneous polynomial $Q_P(x)$ with nonnegative coefficients. Then if $m \ge 2$, then $n(P) \ge 2k - 1$.

Proof. Without loss of generality, we may assume that $Q_P(x)$ contains the x_1 variable with $b_1 > 0$ and $n(Q_P) \ge 2$. Let $Q_P(x) = A_0 + A_1 x_1 + A_2 x_1^2 + \cdots + A_\alpha x_1^\alpha$ be the expansion of $Q_P(x)$ with respect to the degree of the x_1 variable where α is the maximal degree of x_1 in $Q_P(x)$, A_l is a homogeneous polynomial without the x_1 variable having nonnegative coefficients, and A_0 and A_α are nonzero. Denote $B = b_2 x_2 + \cdots + b_k x_k$. Then

$$P(x) = A_0 B^m + x_1 B^{m-1}(m b_1 A_0 + A_1 B) + \cdots + x_1^{\alpha+m} A_\alpha.$$

Note that there are at least $k - 1$ monomials in $A_0 B^m$ and one monomial in $x_1^{\alpha+m} A_\alpha$. Notice that the second term $x_1 B^{m-1}(m b_1 A_0 + A_1 B)$ does not vanish and has at least $k - 1$ monomials. Hence, summing up, there are at least $2k - 1$ monomials in P when $m \ge 2$. $\qquad\square$

LEMMA 3.4. *Let $P(x)$ be a homogeneous polynomial on $\mathbb{R}^{2r}$ of the form*

$$(x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{2r}) Q_P(x)$$

for some homogeneous polynomial $Q_P(x)$ with nonnegative coefficients and $n(Q_P) \ge 2$. Then

(1) $n(P) \ge 3r - 1$ *if $r \ge 2$;*
(2) $n(P) \ge 9$ *if $r = 3$.*

Proof. As in the proof of Lemma 3.3, consider

$$P(x) = A_0 B + x_1(A_0 + A_1 B) + x_1^2(A_1 + A_2 B) + \cdots + A_\alpha x_1^{\alpha+1}.$$

Suppose that $A_i = 0$ but $A_{i+1} \ne 0$ for some i, $1 \le i \le \alpha - 1$. Then the coefficient of x_1^{i+1} is $A_{i+1} B$, and then there exist at least $2r - 1$ monomials that cannot be canceled. This implies that in this case, $n(P) \ge 4r - 1$. Hence, it is enough to consider the case where $A_i \ne 0$ for any i, $0 \le i \le \alpha$. In this case, there are at least $2r - 1$ monomials in $A_0 B$, $r - 1$ monomials in $x_1(A_0 + A_1 B)$, $r - 1$ monomials in $x_1^2(A_1 + A_2 B)$, and one monomial in $A_\alpha x_1^{\alpha+1}$. Hence, $n(P) \ge 3r - 1$.

Consider $r = 3$. We may assume that $A_i \ne 0$ for all i. Since $n(A_i + A_{i+1}) \ge 2$ for all i, it is enough to consider the case $\alpha = 1$. Then $P(x) = A_0 B + x_1(A_0 + A_1 B) + A_1 x_1^2$. If $A_0 = A_1(x_4 + x_5 + x_6)$, then $n(A_0 B) \ge 9$, and if $A_0 \ne A_1(x_4 + x_5 + x_6)$, then $n(x_1(A_0 + A_1 B)) \ge 3$. Hence, $n(P) \ge 9$. $\qquad\square$

LEMMA 3.5. *Let $P(x)$ be a nonzero homogeneous polynomial on $\mathbb{R}^k$ ($k \ge 1$) of the form*

$$(b_1 x_1 + \cdots + b_k x_k)^m (a_1 x_1 + \cdots + a_k x_k)$$

for some positive integer m, $a_i \in \mathbb{R}$ for i, $1 \le i \le k$, and nonzero b_i for all i, $1 \le i \le k$. Then

(1) *if $m \ge 2$, then $n(P) \ge 2k - 1$;*
(2) *if $m = 1$ and $n(a_1 x_1 + \cdots + a_k x_k) \ge 2$, then $n(P) \ge 2k - 2$.*

Proof. We will prove (1). The proof of (2) is similar.

If $n(a_1x_1 + \cdots + a_kx_k) = 1$, then there are $\binom{k+m-1}{m} \geq 2k-1$ monomials in P.

Suppose that $n(a_1x_1 + \cdots + a_kx_k) \geq 2$. We may assume that $a_1 \neq 0$. Put $A = a_2x_2 + \cdots + a_kx_k$ and $B = b_2x_2 + \cdots + b_kx_k$. Then

$$P(x) = B^m A + x_1 B^{m-1}(mb_1 A + a_1 B) + \cdots + a_1 x_1^{m+1}.$$

Consider the case $mb_1 A + a_1 B \neq 0$. Let x be the number of a_i that are zero, and y be the number of a_i that are nonzero. Then $n(B^m A) \geq y - 1 + x(y-1) = -y^2 + (k+2)y - 1 - k$ for y, $2 \leq y \leq k$. At $y = 2$, the minimum $k - 1$ appears. Hence, $n(P) \geq n(B^m A) + n(B^{m-1}(mb_1 A + a_1 B)) + n(a_1 x_1^{m+1}) \geq 2k - 1$.

If $mb_1 A + a_1 B = 0$, then $n(B^m A) = n(B^{m+1}) = \binom{k+m}{m} \geq 2k - 1$.   □

LEMMA 3.6. *Let* $P(x) = (x_1 + x_2 - x_3 - x_4)Q_P(x)$ *for* $Q_P(x) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$. *Suppose that* $n(P) \leq 6$ *and*

$$Q_P(x) > 0 \text{ whenever } x_1 + x_2 > x_3 + x_4 \text{ and } x_i > 0 \text{ for all } i, 1 \leq i \leq 4. \quad (3.1)$$

Then the $Q_P(x)$ *is one of the following up to multiplication of constants*:

$$x_1, x_2, x_3, x_4, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_1 + x_2 + x_3 + x_4.$$

Proof. We prove the lemma case by case. Write

$$P(x) = a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 + (a_2 + a_1)x_1x_2 + (a_3 - a_1)x_1x_3$$
$$+ (a_3 - a_2)x_2x_3 + (a_4 - a_1)x_1x_4 + (a_4 - a_2)x_2x_4 - (a_3 + a_4)x_3x_4.$$
$$(3.2)$$

(1) If only one a_i is zero and the others are nonzero, then Q_P is x_i for $1 \leq i \leq 4$.

(2) If $a_1 = 0$ and $a_i \neq 0$ where $2 \leq i \leq 4$, then there are monomials x_1x_i and x_i^2 for $2 \leq i \leq 4$ that cannot be canceled. Hence, $a_2 = a_3$, $a_2 = a_4$, $a_4 + a_3 = 0$, and this is a contradiction. If $a_j = 0$ and $a_k \neq 0$ for $k \neq j$, by the same way, this cannot happen.

(3) If $a_1 = a_2 = 0$, $a_3 \neq 0$, $a_4 \neq 0$, then $a_3 + a_4 = 0$. This contradicts condition (3.1). Similarly, there is no Q_P for $a_3 = a_4 = 0$, $a_1 \neq 0$, $a_2 \neq 0$.

(4) If $a_2 = a_4 = 0$, $a_1 \neq 0$, $a_3 \neq 0$, then $a_1 = a_3$ and $a_1 > 0$. This case corresponds to $Q_P(x) = x_1 + x_3$. Similarly, the cases $\{a_1 = a_3 = 0, a_2 \neq 0, a_4 \neq 0\}$, $\{a_1 = a_4 = 0, a_3 \neq 0, a_2 \neq 0\}$, $\{a_3 = a_2 = 0, a_1 \neq 0, a_4 \neq 0\}$ correspond to $x_2 + x_4$, $x_3 + x_2$, $x_1 + x_4$, respectively.

(5) If all a_i are nonzero, then, by (3.1), $a_1 > 0$, $a_2 > 0$. Hence, at least three monomials among $(a_3 - a_1)x_1x_3$, $(a_3 - a_2)x_2x_3$, $(a_4 - a_1)x_1x_4$, $(a_4 - a_2)x_2x_4$ should be zero. This implies that $a_1 = a_2 = a_3 = a_4$.   □

Proof of Theorem 1.2. Let

$$(x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{2r})^m Q_P(x)$$

be the homogeneous polynomial induced by g for some positive integer m and homogeneous polynomial $Q_P(x)$. Then P satisfies $n(P) \leq 2r + 2$. If $n(Q_P) = 1$, then g is rationally equivalent to (1). Hence, we only need to consider the case $n(Q_P) \geq 2$.

Suppose $m \geq 2$. Then, by Lemmas 3.3 and 3.5, $n(P) \geq 4r - 1 > 2r + 2$. Hence, $m = 1$. On the other hand, by Lemmas 3.5 and 3.4, $n(P) \geq 2r + 2$ for all $r \geq 3$.

For $m = 1$ and $r = 2$, by Lemma 3.6,

$$x_1 + x_2 - x_3 - x_4, x_1^2 + x_1 x_2 + x_2 x_3 - x_3^2 - x_1 x_4 - x_3 x_4,$$

$$x_2^2 + x_1 x_2 + x_1 x_4 - x_4^2 - x_2 x_3 - x_3 x_4, x_1^2 + x_1 x_2 + x_2 x_4 - x_4^2 - x_1 x_3 - x_3 x_4,$$

$$x_2^2 + x_1 x_2 + x_1 x_3 - x_3^2 - x_2 x_4 - x_3 x_4, x_1^2 + 2 x_1 x_2 + x_2^2 - x_3^2 - 2 x_3 x_4 - x_4^2.$$

Then the first one induces (1), and the last one induces the map (2b). The remaining induce the map equivalent to (2a). □

4. Proper Holomorphic Maps between Bounded Symmetric Domains

4.1. Construction of Proper Holomorphic Maps from $\Omega_{r,s}$ to $\Omega_{r',s'}$

In this section, using the relations between $(r - 1, s)$-subspaces in $\Omega_{r,s}$ and projective subspaces ($\cong \mathbb{P}^{r-1}$) in $D_{r,s}$ given in [4], we describe the construction of proper holomorphic mapping between bounded symmetric spaces of type I. To consider the boundary behavior of g, extend ϕ and ψ to

$$\tilde{\phi} : \mathbb{P}^{r-1} \times \overline{\Omega}_{r,s} \to \overline{\Omega}_{r,s}, \qquad ([X], Z) \mapsto Z,$$

$$\tilde{\psi} : \mathbb{P}^{r-1} \times \overline{\Omega}_{r,s} \to \overline{D}_{r,s}, \qquad ([X], Z) \mapsto [X, XZ].$$

For the boundary points, consider the fibral image with respect to this extended map. Let $z \in \partial \Omega_{r,s}$. This implies that z satisfies $I_{r,r} - z \overline{z}^t \geq 0$ and there is $a \in \mathbb{C}^r$ such that $a(I_{r,r} - z\overline{z}^t)\overline{a}^t = 0$. Hence, $z^\#$ may not be contained in $\partial D_{r,s}$, and

$$z^\# \cap \partial D_{r,s} = \{[a, az] \in \overline{D}_{r,s} : [a] \in \mathbb{P}^{r-1}, a(I_{r,r} - z\overline{z}^t)\overline{a}^t = 0\}. \qquad (4.1)$$

On the other hand, for $[a, b] \in \partial D_{r,s}$ where $a \in M(1, r, \mathbb{C})$ and $b \in M(1, s, \mathbb{C})$, if $z \in [a, b]^\#$, then $a\overline{a}^t = b\overline{b}^t = az\overline{(az)}^t = az\overline{z}^t\overline{a}^t$. Hence, for $[a, b] \in \partial D_{r,s}$, $[a, b]^\# \subset \partial \Omega_{r,s}$.

DEFINITION 4.1. For a rational map $g : D_{r,s} \to D_{r',s'}$, we say that a rational map g is *proper* if for any point $x \in \partial D_{r,s}$ and open neighborhood U of x that does not intersect the indeterminacy of g, g is proper on $U \cap D_{r,s}$.

PROPOSITION 4.2. *Let* $f : \Omega_{r,s} \to \Omega_{r',s'}$ *be a holomorphic map. Suppose that there is a proper rational map* $g : D_{r,s} \to D_{r',s'}$ *satisfying*

$$f(X^\#) \subset g(X)^\# \quad \text{for generic point } X \in D_{r,s}. \qquad (4.2)$$

Then f is proper.

Proof. Let $\{Z_j\}$ be a sequence in $\Omega_{r,s}$ such that $Z_j \to z \in \partial \Omega_{r,s}$. Choose points $X_j \in Z_j^\#$ and $x \in \partial D_{r,s} \cap z^\#$ such that $X_j \to x$. Then since $g(X_j) \to g(x)$, $f(Z_j) \in f(X_j^\#) \subset g(X_j)^\# \to g(x)^\# \subset \partial \Omega_{r',s'}$. Hence, f is proper. □

Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic map that is provided from a proper rational map $g : D_{r,s} \to D_{r',s'}$ satisfying the condition in Proposition 4.2. Let

$g = [g_1, g_2]$ where g_1 has r'-components and g_2 has s'-components. Let $X = [A, B] \in D_{r,s}$ and $Z \in X^{\#}$, that is, $B = AZ$. Then $f([A, AZ]^{\#}) \subset g([A, AZ])^{\#}$, and this implies that

$$g_1([A, AZ]) f(Z) = g_2([A, AZ]) \quad \text{for all } A \in \mathbb{P}^{r-1}. \tag{4.3}$$

REMARK 4.3. For a meromorphic map $g : D_{r,s} \to D_{r',s'}$ and a holomorphic map $f : \Omega_{r,s} \to \Omega_{r',s'}$ satisfying (4.2), put g' a meromorphic map $h \circ g_2 \circ h'$ for some $h' \in \text{Aut}(D_{r,s})$ and $h \in \text{Aut}(D_{r',s'})$. Then there are $H \in \text{Aut}(\Omega_{r,s})$ and $H' \in \text{Aut}(\Omega_{r',s'})$ such that g' and $f' := H' \circ f \circ H$ satisfy (4.2). This is due to the construction of (2.2), and for more detail, see [4].

4.2. Proof of Theorem 1.3

Note that two rationally equivalent proper monomial rational maps from $D_{r,r}$ to $D_{r+1,r+1}$ induce the same proper holomorphic map from $\Omega_{r,r}$ to $\Omega_{r+1,r+1}$. By Theorem 1.2 there are three possibilities to be g. Moreover there exists a holomorphic map g satisfying (4.2) for any proper holomorphic map $f : \Omega_{r,r} \to \Omega_{r+1,r+1}$ ($r \geq 2$). We will only induce the proper map (2a) since calculation of map (2b) is similar. A proper rational map is given by $g([x_1, x_2, x_3, x_4]) = [x_1^2, x_1 x_2, x_2 x_3, x_3^2, x_1 x_4, x_3 x_4]$. Let $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right) \in \Omega_{2,2}$. Then

$$Z^{\#} = \{[x_1, x_2, x_1 z_1 + x_2 z_3, x_1 z_2 + x_2 z_4] \in D_{2,2} : [x_1, x_2] \in \mathbb{P}^1\},$$
$$g([x_1, x_2, x_1 z_1 + x_2 z_3, x_1 z_2 + x_2 z_4]) = [A, B], \quad \text{where}$$
$$A = (x_1^2, x_1 x_2, x_2(x_1 z_1 + x_2 z_3)),$$
$$B = ((x_1 z_1 + x_2 z_3)^2, x_1(x_1 z_2 + x_2 z_4), (x_1 z_1 + x_2 z_3)(x_1 z_2 + x_2 z_4)).$$

Denote

$$f(Z) = \begin{pmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{pmatrix}.$$

Then

$$x_1^2 L_1 + x_1 x_2 L_2 + x_2(x_1 z_1 + x_2 z_3) L_3 = (x_1 z_1 + x_2 z_3)^2,$$
$$x_1^2 M_1 + x_1 x_2 M_2 + x_2(x_1 z_1 + x_2 z_3) M_3 = x_1(x_1 z_2 + x_2 z_4), \quad \text{and}$$
$$x_1^2 N_1 + x_1 x_2 N_2 + x_2(x_1 z_1 + x_2 z_3) N_3 = (x_1 z_1 + x_2 z_3)(x_1 z_2 + x_2 z_4)$$

for all $[x_1, x_2] \in \mathbb{P}^1$. Hence, we obtain (2).

Consider case (1) in Theorem 1.2. Suppose for simplicity that $g : D_{2,2} \to D_{3,3}$ is $g(x) = [x_1, x_2, x_1, x_3, x_4, x_1]$. This method can be applied to general r and homogeneous monomial linear map g. The induced map $f : \Omega_{2,2} \to \Omega_{3,3}$ has the form

$$f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1 - L & z_2 - M & 1 - N \\ z_3 & z_4 & 0 \\ L & M & N \end{pmatrix}$$

for some holomorphic functions L, M, N on $\Omega_{2,2}$. Notice that f is equivalent to

$$\tilde{f}\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} \frac{z_1}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ z_3 & z_4 & 0 \\ \tilde{L} & \tilde{M} & \tilde{N} \end{pmatrix}$$

for some suitable holomorphic functions $\tilde{L}$, $\tilde{M}$ and $\tilde{N}$ on $\Omega_{2,2}$. Since the map $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} \frac{z_1}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ z_3 & z_4 & 0 \end{pmatrix}$ is a proper holomorphic map from $\Omega_{2,2}$ to $\Omega_{2,3}$, it is equivalent to the embedding $Z \mapsto (Z \quad 0)$ and hence, $\tilde{f}$ is equivalent to

$$Z \mapsto \begin{pmatrix} Z & 0 \\ k(Z) & h(Z) \end{pmatrix}$$

for some holomorphic functions k_1, k_2, h on $\Omega_{2,2}$ where $k = (k_1, k_2)$. Then by the maximum principle and homogeneity of the domains, k should be zero. Hence, f should be of the form (1) in Theorem 1.3.

REMARK 4.4. Note that, in general, for one g, there could be several f. However, in case of $D_{2,2}$, $D_{3,3}$ and $\Omega_{2,2}$, $\Omega_{3,3}$, there is a unique f for each g since the number of equations and the number of unknowns are same.

4.3. More Examples

EXAMPLE 4.5. If the difference of dimension gets greater, then there are an infinite number of proper holomorphic maps that are not rationally equivalent up to the automorphisms. Consider the proper holomorphic maps from $D_{2,2}$ to $D_{4,4}$. By the same method, let $P_t(x) = (x_1 + x_2 - x_3 - x_4)Q_P(x)$ where $Q_{P_t}(x) = x_1 + x_2 + x_3 + x_4 - t(x_2 + x_4)$, $0 \le t \le 1$. Then

$$P_t(x) = x_1^2 + (2 - t)x_1x_2 + (1 - t)x_2^2 + tx_2x_3 - x_3^2$$
$$- (2 - t)x_3x_4 - (1 - t)x_4^2 - tx_1x_4,$$

and the induced proper holomorphic maps are

$$g_t([z_1, z_2, z_3, z_4])$$
$$= [z_1^2, \sqrt{2 - t}z_1z_2, \sqrt{1 - t}z_2^2, \sqrt{t}z_2z_3, z_3^2, \sqrt{2 - t}z_3z_4, \sqrt{1 - t}z_4^2, \sqrt{t}z_1z_4].$$

Then g_t induces an infinite number of proper holomorphic maps from $f_t : \Omega_{2,2} \to \Omega_{4,4}$, which are defined by

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} z_1^2 & \sqrt{2 - t}z_1z_2 & \sqrt{1 - t}z_2^2 & \sqrt{t}z_2 \\ \sqrt{2 - t}z_1z_3 & \frac{2(1-t)}{2-t}z_1z_4 + z_2z_3 & 2\sqrt{\frac{1-t}{2-t}}z_2z_4 & \sqrt{\frac{t}{2-t}}z_4 \\ \sqrt{1 - t}z_3^2 & 2\sqrt{\frac{1-t}{2-t}}z_3z_4 & z_4^2 & 0 \\ \sqrt{t}z_3 & \sqrt{\frac{t}{2-t}}z_4 & 0 & 0 \end{pmatrix}. \quad (4.4)$$

REMARK 4.6. (2) and (3) are homotopic to each other by (4.4).

EXAMPLE 4.7. There is a proper holomorphic map $f : \Omega_{2,2} \to \Omega_{4,4}$ that has a degree 3 polynomial in components. Let $Q_P(x) = x_1^2 + x_1 x_3 + x_3^2$. Then $P(x) = x_1^3 + x_1^2 x_2 + x_1 x_2 x_3 + x_2 x_3^2 - x_3^3 - x_1^2 x_4 - x_1 x_3 x_4 - x_3^2 x_4$ and hence

$$g([x_1, x_2, x_3, x_4]) = [x_1^3, x_1^2 x_2, x_1 x_2 x_3, x_2 x_3^2, x_3^3, x_1^2 x_4, x_1 x_3 x_4, x_3^2 x_4].$$

The corresponding proper holomorphic map $f : \Omega_{2,2} \to \Omega_{4,4}$ is

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} z_1^3 & z_2 & z_1 z_2 & z_1^2 z_2 \\ z_1^2 z_3 & z_4 & z_2 z_3 & z_1 z_2 z_3 \\ z_1 z_3 & 0 & z_4 & z_2 z_3 \\ z_3 & 0 & 0 & z_4 \end{pmatrix}.$$

EXAMPLE 4.8 (Generalized Whitney map). Consider

$$P(z) = (x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{r+s})(x_1 + x_{r+1}).$$

This polynomial induces the proper meromorphic map $g : D_{r,s} \to D_{2r-1,2s-1}$ defined by

$$g([z_1, \ldots, z_r, w_1, \ldots, w_s]) = [z_1^2, z_1 z_2, \ldots, z_1 z_r, w_1 z_2, \ldots, w_1 z_r,$$
$$w_1^2, w_1 w_2, \ldots, w_1 w_s, z_1 w_2, \ldots z_1 w_s].$$

The map g induces the proper holomorphic map $f^w : \Omega_{r,s} \to \Omega_{2r-1,2s-1}$ defined by

$$\begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \mapsto \begin{pmatrix} z_{11}^2 & z_{11} z_{12} & \cdots & z_{11} z_{1s} & z_{12} & \cdots & z_{1s} \\ z_{21} z_{11} & z_{21} z_{12} & \cdots & z_{21} z_{1r} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} z_{11} & z_{r1} z_{12} & \cdots & z_{r1} z_{1s} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & \cdots & 0 \end{pmatrix}. \quad (4.5)$$

This is a generalized proper holomorphic map of (2): if $r = s = 2$, then f^w is same with (2) in Theorem 1.3.

EXAMPLE 4.9. Consider the proper holomorphic maps from $D_{2,2}$ to $D_{3,4}$. Let $P_t(x) = (x_1 + x_2 - x_3 - x_4) Q_{P_t}(x)$ where $Q_{P_t}(x) = x_1 + t x_3$, $0 \le t \le 1$. Then the proper rational map $g_t : D_{2,2} \to D_{3,4}$ is given by

$$g_t([x_1, x_2, x_3, x_4]) = \left[x_1^2, x_1 x_2, \sqrt{t} x_2 x_3, \sqrt{t} x_3^2, \sqrt{t} x_3 x_4, \sqrt{1 - t} x_1 x_3, x_1 x_4 \right].$$

The induced proper holomorphic maps $f_t : \Omega_{2,2} \to \Omega_{3,4}$ is given by

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{t} z_1^2 & \sqrt{t} z_1 z_2 & \sqrt{1 - t} z_1 & z_2 \\ \sqrt{t} z_1 z_3 & \sqrt{t} z_2 z_3 & \sqrt{1 - t} z_3 & z_4 \\ z_3 & z_4 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

Furthermore, we can generalize the proper holomorphic map (4.6) to $F_t : \Omega_{r,s} \to \Omega_{2r-1,2s}$ given, for $Z = (z_{ij})_{1 \le i \le r, 1 \le j \le s}$, by

$$
Z \mapsto
\begin{pmatrix}
\sqrt{t}z_{11}^2 & \sqrt{t}z_{11}z_{12} & \cdots & \sqrt{t}z_{11}z_{1s} & \sqrt{1-t}z_{11} & z_{12} & \cdots & z_{1s} \\
\sqrt{t}z_{11}z_{21} & \sqrt{t}z_{21}z_{12} & \cdots & \sqrt{t}z_{21}z_{1r} & \sqrt{1-t}z_{21} & z_{22} & \cdots & z_{2s} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sqrt{t}z_{11}z_{r1} & \sqrt{t}z_{r1}z_{12} & \cdots & \sqrt{t}z_{r1}z_{1s} & \sqrt{1-t}z_{r1} & z_{r2} & \cdots & z_{rs} \\
z_{21} & z_{22} & \cdots & z_{2s} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & 0 & \cdots & 0
\end{pmatrix}.
$$

(4.7)

EXAMPLE 4.10. Consider

$$
P(x) = (x_1 + \cdots + x_r - y_1 - \cdots - y_s)(x_1 + \cdots + x_r + y_1 + \cdots + y_s)
$$

and the induced rational map $g : D_{r,s} \to D_{r',s'}$, where $r' = \frac{1}{2}r(r+1)$ and $s' = \frac{1}{2}s(s+1)$, defined by

$$
\begin{aligned}
g([x_1, \ldots, x_r, y_1, \ldots, y_s]) = \big[&x_1^2, \ldots, x_r^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_ix_j, \ldots, \sqrt{2}x_{r-1}x_r, \\
&y_1^2, \ldots, y_s^2, \sqrt{2}y_1y_2, \ldots, \sqrt{2}y_ky_l, \ldots, \sqrt{2}y_{s-1}y_s \big],
\end{aligned}
$$

where i, j, k, and l trace over $1 \le i < j \le r$ and $1 \le k < l \le s$. Then the induced proper holomorphic map $f : \Omega_{r,s} \to \Omega_{r',s'}$ is

$$
f\left(\begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \right) = (M, N), \quad \text{where}
$$

$$
M =
\begin{pmatrix}
z_{11}^2 & \cdots & z_{1s}^2 \\
\vdots & & \vdots \\
z_{r1}^2 & \cdots & z_{rs}^2 \\
\sqrt{2}z_{11}z_{21} & \cdots & \sqrt{2}z_{1s}z_{2s} \\
\vdots & & \vdots \\
\sqrt{2}z_{i1}z_{j1} & \cdots & \sqrt{2}z_{is}z_{js} \\
\vdots & & \vdots \\
\sqrt{2}z_{r-11}z_{r1} & \cdots & \sqrt{2}z_{r-1s}z_{rs}
\end{pmatrix},
$$

$$N = \begin{pmatrix} \sqrt{2}z_{11}z_{12} & \cdots & \sqrt{2}z_{1k}z_{1l} \\ \vdots & & \vdots \\ \sqrt{2}z_{r1}z_{r2} & \cdots & \sqrt{2}z_{rk}z_{rl} \\ z_{11}z_{22}+z_{12}z_{21} & \cdots & z_{1k}z_{2l}+z_{2k}z_{1l} \\ \vdots & & \vdots \\ z_{i1}z_{j2}+z_{j1}z_{i2} & \cdots & z_{ik}z_{jl}+z_{jk}z_{il} \\ \vdots & & \vdots \\ z_{r-11}z_{r2}+z_{r1}z_{r-12} & \cdots & z_{r-1k}z_{rl}+z_{rk}z_{r-1l} \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \sqrt{2}z_{1s-1}z_{1s} \\ & \vdots \\ \cdots & \sqrt{2}z_{rs-1}z_{rs} \\ \cdots & z_{1s-1}z_{2s}+z_{2s-1}z_{1s} \\ & \vdots \\ \cdots & z_{is-1}z_{js}+z_{js-1}z_{is} \\ & \vdots \\ \cdots & z_{r-1s-1}z_{rs}+z_{rs-1}z_{r-1s} \end{pmatrix}.$$

Here i, j, k, l trace over $1 \le i < j \le r$ and $1 \le k < l \le r$.

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