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Higher Derivatives of Length Functions along Earthquake Deformations

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1. Introduction

Let S be a closed surface of genus $g \geq 2$, and $T(S)$ the associated Teichmüller space of hyperbolic structures on S. Given $\gamma \in \pi_1(S)$, let $L_\gamma : T(S) \to \mathbb{R}$ be the associated length function, and $T_\gamma : T(S) \to \mathbb{R}$ the associated trace function. The functions L_γ, T_γ have a simple relation given by

$$T_\gamma = 2\cosh(L_\gamma/2). \tag{1}$$

Let β be the homotopy class of a simple multicurve (i.e., a union of disjoint simple nontrivial closed curves in S), and t_β the vector field on $T(S)$ associated with left twist along the geodesic representative of β (see [4]). In this paper, we describe a formula to calculate the higher-order derivatives of the functions L_γ, T_γ along t_β. In particular, we will find a formula for

$$t_\beta^k L_\gamma = t_\beta t_\beta \dots t_\beta L_\gamma.$$

The formulae we derive generalize formulae for the first two derivatives derived by Kerchoff [4] (first derivative) and Wolpert [5; 6] (first and second derivatives).

Kerckhoff and Wolpert both showed that the first derivative is given by

$$t_\beta L_\gamma = \sum_{p \in \beta' \cap \gamma'} \cos\theta_p, \tag{2}$$

where β', γ' are the geodesic representatives of β, γ, respectively, and θ_p is the angle of intersection at $p \in \beta' \cap \gamma'$. Kerckhoff [4] further generalized the formula for the case where β, γ are measured laminations.

Wolpert [6] derived the following formula for the second derivative:

$$\begin{aligned} t_\alpha t_\beta L_\gamma = &\sum_{(p,q) \in \beta'\cap\gamma' \times \alpha'\cap\gamma'} \frac{e^{l_{pq}} + e^{l_{qp}}}{2(e^{L_\gamma} - 1)} \sin\theta_p \sin\theta_q \\ &+ \sum_{(r,s) \in \beta'\cap\gamma' \times \beta'\cap\alpha'} \frac{e^{m_{rs}} + e^{m_{sr}}}{2(e^{L_\beta} - 1)} \sin\theta_r \sin\theta_s, \end{aligned}$$

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where l_{xy} is the length along γ between x, y, and, similarly, m_{xy} is the length along β.

It follows from Wolpert's formula that

$$t_\beta^2 L_\gamma = t_\beta t_\beta L_\gamma = \sum_{p,q\in\beta'\cap\gamma'} \frac{e^{l_{pq}} + e^{l_{qp}}}{2(e^{L_\gamma} - 1)} \sin\theta_p \sin\theta_q. \tag{3}$$

Our formula generalizes equations (2) and (3) to higher derivatives. Our approach is to derive a formula for the higher derivatives of T_γ and then use the functional relation in equation (1) to derive the formula for L_γ.

2. Higher-Derivative Formula

We take the geodesic representatives of β and γ. We let the geometric intersection number satisfy $i(\beta,\gamma) = n$, and we order the points of intersection $x_1,\ldots,x_n$ by choosing a base point on γ. We let θ_i be the angle of intersection of β, γ at x_i and l_i be the length along γ from x_1 to x_i. This gives us n-tuples $(l_1,\ldots,l_n)$ and $(\theta_1,\ldots,\theta_n)$.

In order to describe the formula for the higher derivatives, we first introduce some more notation.

Given r, we let $P(r)$ be the set of subsets of the set $\{1,\ldots,r\}$. Then $I \in P(r)$ will be denoted by $I = (i_1,\ldots,i_k)$ where $1 \le i_1 < i_2 < \cdots < i_k \le r$. We then define $\hat{I}$ to be the complementary subset. We also let $|I|$ be the cardinality of I.

We define the alternating length L_I for $I = (i_i,\ldots,i_k)$ by

$$L_I = \sum_{j=1}^{k} (-1)^j l_{i_j} = -l_{i_1} + l_{i_2} - l_{i_3} - \cdots + (-1)^k l_{i_k}.$$

We further define a signature for $I \in P(r)$. For $I = (i_1,\ldots,i_k)$, we can consider the integers in $\{1,\ldots,r\}$ in the ordered blocks $[1,i_1], [i_1,i_2],\ldots,[i_k,r]$. We take the sum of the cardinality of the even ordered blocks. Then

$$s(I) = (i_2 - i_1 + 1) + (i_4 - i_3 + 1) + \cdots + (i_k - i_{k-1} + 1), \quad k \text{ even},$$
$$s(I) = (i_2 - i_1 + 1) + (i_4 - i_3 + 1) + \cdots + (r - i_k + 1), \quad k \text{ odd}.$$

For $(\theta_1,\ldots,\theta_n)$, we also define

$$\cos(\theta_I) = \prod_{j=1}^{k} \cos(\theta_{i_j}) = \cos(\theta_{i_1})\cos(\theta_{i_2})\ldots\cos(\theta_{i_k})$$

and similarly define $f(\theta_I)$ for a trigonometric function f.

We let $u_j = l_j + i\theta_j$. The function F_r is given by

$$F_r(u_1,\ldots,u_r,L) = \sum_{I\in P(r),|I| \text{ even}} (-1)^{s(I)} \sin(\theta_I)\cos(\theta_{\hat{I}})(e^{L/2-L_I} + (-1)^r e^{L_I - L/2})$$

or, equivalently,

$$F_r(u_1, \dots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_{\hat{I}}) \cosh(L/2 - L_I)$$

for r even and

$$F_r(u_1, \dots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_{\hat{I}}) \sinh(L/2 - L_I)$$

for r odd.

We let $C(n, r)$ be the set of subsets of size r of the set $\{1, 2, \dots, n\}$. It is given by

$$C(n, r) = \{I = (i_1, i_2, \dots, i_r) \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\}.$$

Given $m \in \mathbb{N}$, we let $[m]$ be the parity of m, that is, $[m] = 0$ if m is even and $[m] = 1$ if m is odd.

THEOREM 1. *Let β be a homotopy class of a simple closed multicurve, and γ a homotopy class of nontrivial closed curve. Let the geometric intersection number $i(\beta, \gamma) = n$. Then*

$$t_\beta^k T_\gamma = \frac{1}{2^k} \sum_{\substack{r=0 \\ [r]=[k]}}^{k} B_{n,k,r} \sum_{I \in C(n,r)} F_r(u_{i_1}, \dots, u_{i_r}, L_\gamma),$$

where $B_{n,k,r}$ are constants described below.

The first two equations ($k = 1, 2$) correspond to formulae (2) and (3) for the derivatives of length. Taking $k = 3$, we derive the next case as an example.

THIRD DERIVATIVE. We use the formula of Theorem 1 to calculate the formula for the third derivative:

$$\begin{aligned} t_\beta^3 T_\gamma = \frac{1}{8}\Bigg(&(6n-4)\sinh(L_\gamma/2) \sum_{i=1}^{n} \cos(\theta_i) \\ &+ 12\Bigg(\sum_{i<j<k} \sinh(L_\gamma/2)\cos(\theta_i)\cos(\theta_j)\cos(\theta_k) \\ &+ \sinh(L_\gamma/2 - l_{ij})\sin(\theta_i)\sin(\theta_j)\cos(\theta_k) \\ &- \sinh(L_\gamma/2 - l_{ik})\sin(\theta_i)\cos(\theta_j)\sin(\theta_k) \\ &+ \sinh(L_\gamma/2 - l_{jk})\cos(\theta_i)\sin(\theta_j)\sin(\theta_k)\Bigg)\Bigg). \end{aligned}$$

2.1. *Constants $B_{n,k,r}$*

We denote by $P(k,n)$ the collection of partitions of k into n ordered nonnegative integers, that is,

$$P(k,n)=\left\{p=(p_1,p_2,\dots,p_n)\in\mathbb{N}_0^n \;\middle|\; \sum_{i=1}^{n}p_i=k\right\}.$$

For $p\in P(k,n)$, we define $[p]=([p_1],\dots,[p_n])$ where $[n]$ is the parity of n. We let $|p|=[p_1]+\cdots+[p_n]$. Then $[p]$ is an n-tuple of 0s and 1s with exactly $|p|$ 1s.

Given $p\in P(k,n)$, we define $B(p)$ as the sum of multinomials given by

$$B(p)=\sum_{q\in P(k,n),[q]=[p]}\binom{k}{q}.$$

It is easy to see that $B(p)$ only depends on n, k, and $r=|p|$. We therefore define

$$B_{n,k,r}=B(p)\quad\text{for some } p \text{ with } |p|=r.$$

In particular, if we let $p_r=(1,1,\dots,1,0,\dots,0)\in P(k,n)$, of r 1s followed by $(n-r)$ 0s, then we have

$$B_{n,k,r}=\sum_{p\in P(k,n),[p]=[p_r]}\binom{k}{p}.$$

A simple calculation gives

$$B_{n,k,k}=\binom{k}{p_k}=\binom{k}{1,1,1,\dots,0,0,\dots,0}=k!\,.$$

3. Twist Deformation

We consider $T(S)$ as the Fuchsian locus of the associated quasi-Fuchsian space $\mathrm{QF}(S)$. Let $X\in T(S)$ and $X=\mathbb{H}^2/\Gamma$, where Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ acting on upper half-space $\mathbb{H}^3=\{(u,v,w)\in\mathbb{R}^3 \mid w>0\}$ fixing the hyperbolic plane $\mathbb{H}^2=\{(u,0,w)\mid w>0\}$. Let Γ_z be the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ obtained by complex shear-bend along β by amount $z=s+it$, that is, left shear by amount s followed by bend of t. Then, for small z, $X_z=\mathbb{H}^3/\Gamma_z\in\mathrm{QF}(S)$. In the terminology of Epstin–Marden this is a quake-bend deformation. See Section II.3 of [3] for details on quake-bend deformations and Section II.3.9 for a detailed discussion of derivatives of length along quake-bend deformations.

Let $\gamma\in\Gamma$ be a hyperbolic element, let $\gamma(z)\in\Gamma_z$ be the element of the deformed group corresponding to γ, and let $L(z)$ the complex translation length of $\gamma(z)$. To see how γ is deformed, by conjugating, we assume that γ has as an axis the geodesic g with endpoints $0,\infty\in\hat{\mathbb{C}}$ and is given by

$$\gamma=\begin{pmatrix}\lambda & 0\\ 0 & 1/\lambda\end{pmatrix}\quad\text{with } \lambda=e^{L/2} \text{ where } L>0 \text{ is the translation length of } \gamma.$$

We consider the lifts of β that intersect the axis g of γ and normalize to have a lift of β labeled β_1 that intersects the axis g at height 1. We enumerate all other lifts by the order of the height of their intersection point with g starting with the intersection point of β_1. Let n be such that $\gamma\beta_1 = \beta_{n+1}$. Let $R_i(z)$ be the Möbius transformation corresponding to a complex bend about β_i of z. Then, under the complex bend about β, $\gamma(z)$ given by

$$\gamma(z) = R_1(z)R_2(z)\dots R_n(z)\gamma.$$

A similar description of the deformation of an element in the punctured surface case can be given in terms of shearing coordinates (see [2] for details).

Taking traces, we have

$$T(z) = \mathrm{Tr}(R_1(z)R_2(z)\dots R_n(z)\gamma) = 2\cosh(L(z)/2).$$

We can find the derivatives of $L(z)$ by differentiating this formula repeatedly. The final formula is obtained by applying symmetry relations on the derivatives and some elementary combinatorics.

We note that both $T(z)$ and $L(z)$ are holomorphic in z. Differentiating in the real direction, we have

$$t_\beta^k L_\gamma = \frac{d^k L}{dz^k}(0) = L^{(k)}(0).$$

Also, if we let b_β be the vector field on $T(S)$ given by pure bending along β, then by the analyticity of $L(z)$ we have

$$b_\beta^k L_\gamma = i^k L^{(k)}(0) = (i t_\beta)^k L_\gamma.$$

This corresponds to the observation that $b_\beta = J t_\beta$, where J is the complex structure on QF(S) (see [1]).

3.1. Derivation of First Two Derivatives

We now calculate the first two derivatives and recover Wolpert's formulae. By the product rule we have

$$\begin{aligned} T'(0) &= \sum_{i=1}^{n} \mathrm{Tr}(R_i'(0)\gamma), \\ T''(0) &= \sum_{i=1}^{n} \mathrm{Tr}(R_i''(0)\gamma) + 2\sum_{\substack{i,j=1\\ i<j}}^{n} \mathrm{Tr}(R_i'(0)R_j'(0)\gamma). \end{aligned} \tag{4}$$

We now describe $R_i(z)$. Let β_i have endpoints $a_i, b_i \in \mathbf{R}$ where $a_i > 0$ and $b_i < 0$. We let λ_i be the height at which β_i intersects g. We orient β_i from a_i to b_i and g from 0 to ∞ and let θ_i be the angle β_i makes with side g with respect to these orientations (see Figure 1).

Then

$$\lambda_i = \sqrt{-a_i b_i}, \qquad \cos\theta_i = -\left(\frac{a_i + b_i}{a_i - b_i}\right), \qquad \sin\theta_i = \frac{2\sqrt{-a_i b_i}}{a_i - b_i}.$$

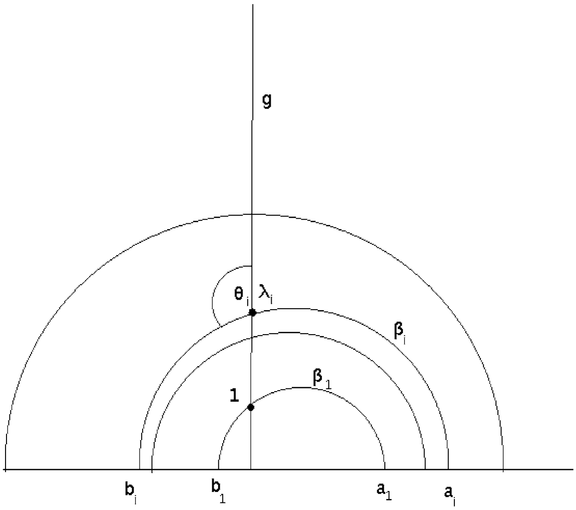

Figure 1 Lift of γ

Since β_1 intersects at height 1, the distance l_i between the intersection points of β_1 and β_i is given by $e^{l_i} = \lambda_i$. Then let $f_i \in \mathrm{SL}(2,\mathbb{R})$ act on the upper-half space by $f_i(z) = (z-a_i)/(z-b_i)$, and let $S(z) = f_i R_i(z) f_i^{-1}$. Then $S_i(z)$ is the complex translation given by

$$S(z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.$$

Thus, $R_i(z) = f_i^{-1} S(z) f_i$. Taking derivatives, we have $R_i'(0) = f_i^{-1} S'(0) f_i$ and

$$\begin{aligned} R_i'(0) &= \frac{1}{a_i - b_i} \begin{pmatrix} -b_i & a_i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -a_i \\ 1 & -b_i \end{pmatrix} \\ &= \frac{1}{2(a_i - b_i)} \begin{pmatrix} -(a_i+b_i) & 2a_i b_i \\ -2 & a_i + b_i \end{pmatrix}. \end{aligned}$$

Therefore,

$$R_i'(0) = \frac{1}{2} \begin{pmatrix} \cos\theta_i & -e^{l_i}\sin\theta_i \\ -e^{-l_i}\sin\theta_i & -\cos\theta_i \end{pmatrix}.$$

Also, since $S''(0) = \frac{1}{4}I$, we have $R_i''(0) = \frac{1}{4}I$. Using this, we have that

$$\begin{aligned} \mathrm{Tr}(R_i'(0)\gamma) &= \mathrm{Tr}\left(\frac{1}{2} \begin{pmatrix} \cos\theta_i & -e^{l_i}\sin\theta_i \\ -e^{-l_i}\sin\theta_i & -\cos\theta_i \end{pmatrix} \begin{pmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{pmatrix} \right) \\ &= \sinh(L/2)\cos\theta_i, \\ \mathrm{Tr}(R_i'(0)R_j'(0)\gamma) &= \mathrm{Tr}\left(\frac{1}{4} \begin{pmatrix} \cos\theta_i & -e^{l_i}\sin\theta_i \\ -e^{-l_i}\sin\theta_i & -\cos\theta_i \end{pmatrix} \right. \\ &\qquad \times \left. \begin{pmatrix} \cos\theta_j & -e^{l_j}\sin\theta_j \\ -e^{-l_j}\sin\theta_j & -\cos\theta_j \end{pmatrix} \begin{pmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{pmatrix} \right) \end{aligned} \tag{5}$$

$$
\begin{aligned}
&= \frac{1}{4}(\cos\theta_i \cos\theta_j (e^{L/2} + e^{-L/2}) \\
&\quad + \sin\theta_i \sin\theta_j (e^{L/2+l_i-l_j} + e^{-(L/2+l_i-l_j)})).
\end{aligned}
$$

Let l_{ij} be the distance along γ from β_i to β_j with respect to the orientation of γ. Then, for $i < j$, we have $l_{ij} = l_j - l_i$, and $l_{ji} = L - l_{ij}$ for $i > j$, so that

$$
\begin{aligned}
&\operatorname{Tr}(R_i'(0)R_j'(0)\gamma) \\
&\quad = \frac{1}{2}(\cos\theta_i \cos\theta_j \cosh(L/2) + \sin\theta_i \sin\theta_j \cosh(L/2 - l_{ij})). \qquad (6)
\end{aligned}
$$

Combining these, we obtain the first two derivatives of T_γ:

$$
\begin{aligned}
T'(0) &= \sinh(L/2) \sum_{i=1}^{n} \cos\theta_i, \\
T''(0) &= \sum_{\substack{i,j=1 \\ i<j}}^{n} (\cos\theta_i \cos\theta_j \cosh(L/2) + \sin\theta_i \sin\theta_j \cosh(L/2 - l_{ij})) \\
&\quad + \frac{n\cosh(L/2)}{2}.
\end{aligned}
$$

Since $T(z) = 2\cosh(L(z)/2)$, we have $T'(0) = \sinh(L/2)L'(0)$, which gives

$$
L'(0) = \sum_{i=1}^{n} \cos\theta_i .
$$

Also, $T''(0) = \frac{1}{2}\cosh(L/2)(L'(0))^2 + \sinh(L/2)L''(0)$. Therefore,

$$
\begin{aligned}
T''(0) &= \frac{\cosh(L/2)}{2}\left(n + 2\sum_{\substack{i,j=1 \\ i<j}}^{n} \cos\theta_i \cos\theta_j\right) \\
&\quad + \sum_{\substack{i,j=1 \\ i<j}} \sin\theta_i \sin\theta_j \cosh(L/2 - l_{ij}).
\end{aligned}
$$

We have

$$
n + 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \cos\theta_i \cos\theta_j = \left(\sum_{i=1}^{n} \cos\theta_i\right)^2 + \sum_{i=1}^{n} \sin^2\theta_i
$$

and

$$
\begin{aligned}
T''(0) &= \frac{\cosh(L/2)((\sum_{i=1}^{n} \cos\theta_i)^2 + \sum_{i=1}^{n} \sin^2\theta_i)}{2} \\
&\quad + \sum_{\substack{i,j=1 \\ i<j}} \sin\theta_i \sin\theta_j \cosh(L/2 - l_{ij}). \qquad (7)
\end{aligned}
$$

Solving for $L''(0)$, we obtain

$$L''(0) = \sum_{i=1}^{n} \frac{\sin^2\theta_i}{2\tanh(L/2)} + \sum_{\substack{i,j=1\\ i<j}}^{n} \frac{\sin\theta_i \sin\theta_j \cosh(L/2 - l_{ij})}{\sinh(L/2)}.$$

Since $l_{ii} = 0$, we can write

$$L''(0) = \sum_{i,j=1}^{n} \frac{e^{l_{ij}-L/2} + e^{L/2-l_{ij}}}{2(e^{L/2} - e^{-L/2})} \sin\theta_i \sin\theta_j = \sum_{i,j=1}^{n} \frac{e^{l_{ij}} + e^{l_{ji}}}{2(e^L - 1)} \sin\theta_i \sin\theta_j.$$

The formulae obtained give formulae (2) and (3), as described.

4. Higher Derivatives

We now derive the formula for higher derivatives. We have the formula

$$T(z) = \mathrm{Tr}(R_1(z)R_2(z)\dots R_n(z)\gamma).$$

Let $P(k,n)$ be the collection of partitions of k into n ordered nonnegative integers, that is,

$$P(k,n) = \left\{ p = (p_1, p_2, \dots, p_n) \in \mathbb{N}_0^n \;\middle|\; \sum_{i=1}^{n} p_i = k \right\}.$$

Then by the product rule the kth derivative of T at zero is

$$T^{(k)}(0) = \sum_{p \in P(k,n)} \binom{k}{p} \mathrm{Tr}(R_1^{(p_1)}(0)\dots R_n^{(p_n)}(0)\gamma).$$

Thus, we have $R_i(z) = f_i^{-1} S(z) f_i$, where

$$S(z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.$$

Since $S^{(2)}(z) = \frac{1}{4} S(z)$, for m even, we have

$$R_i^{(m)}(0) = \frac{1}{2^m} I,$$

and for m odd, we have

$$R_i^{(m)}(0) = \frac{1}{2^{m-1}} R_i'(0) = \frac{1}{2^m} \begin{pmatrix} \cos\theta_i & -e^{l_i}\sin\theta_i \\ -e^{-l_i}\sin\theta_i & -\cos\theta_i \end{pmatrix}.$$

Let $z = x + iy$ and define

$$A(z) = \begin{pmatrix} \cos y & -e^{x}\sin y \\ -e^{-x}\sin y & -\cos y \end{pmatrix}.$$

We let $u_j = l_j + i\theta_j$. Then

$$R_j^{(p)}(0) = \begin{cases} \frac{1}{2^p} A(u_j), & p \text{ odd}, \\ \frac{1}{2^p} I, & p \text{ even}. \end{cases}$$

Therefore,

$$T^{(k)}(0) = \frac{1}{2^k} \sum_{p \in P(k,n)} \binom{k}{p} \mathrm{Tr}(A(u_1)^{[p_1]} \dots A(u_n)^{[p_n]}\gamma),$$

where $[m]$ is the parity of m. We define

$$F_r(z_1, \dots, z_r, L) = \mathrm{Tr}(A(z_1) \dots A(z_r)\gamma).$$

Therefore, gathering terms, we have

$$T^{(k)}(0) = \frac{1}{2^k} \sum_{r=0}^{k} B_{n,k,r} \sum_{1 \le i_1 < \dots < i_r \le n} F_r(u_{i_1}, \dots, u_{i_r}, L),$$

where $B_{n,k,r}$ are the coefficients described before. We note that we only get nonzero terms for $[r] = [k]$, so we have $B_{n,k,r} = 0$ for $[k] \neq [r]$.

We define the function

$$G_r(u_1, \dots, u_n, L) = \sum_{I \in C(n,r)} F_r(u_{i_1}, \dots, u_{i_r}, L).$$

Then G_r is symmetric in $(u_1, \dots, u_n)$, and we have

$$t_\beta^k T_\gamma = \frac{1}{2^k} \sum_{\substack{r=0 \\ [r]=[k]}}^{k} B_{n,k,r} G_r(u_1, \dots, u_n, L_\gamma).$$

4.1. Function F_r

We now calculate the formula for F_r.

LEMMA 1. *The function F_r is given by*

$$\begin{aligned} &F_r(u_1, \dots, u_r, L) \\ &\quad = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} \sin(\theta_I) \cos(\theta_{\hat{I}})(e^{L/2 - L_I} + (-1)^r e^{L_I - L/2}) \end{aligned}$$

or, equivalently,

$$F_r(u_1, \dots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_{\hat{I}}) \cosh(L/2 - L_I)$$

for r even and

$$F_r(u_1, \dots, u_r, L) = \sum_{I \in P(r), |I| \text{ even}} (-1)^{s(I)} 2 \sin(\theta_I) \cos(\theta_{\hat{I}}) \sinh(L/2 - L_I)$$

for r odd.

Proof. We have

$$A(u) = \begin{pmatrix} \cos\theta & -e^l \sin\theta \\ -e^{-l} \sin\theta & -\cos\theta \end{pmatrix}.$$

Therefore, $F_r(u_1,\dots,u_r,L)=\mathrm{Tr}(A(u_1)\dots A(u_r)\gamma)$ has the form

$$F_r(u_1,\dots,u_r,L)=\sum_{I\in P(r)} a_I\sin(\theta_I)\cos(\theta_{\hat{I}})$$

for some coefficients a_I. Expanding the latter, we have

$$\begin{aligned}F_r(u_1,\dots,u_r,L)&=(A(u_1)\dots A(u_r)\gamma)^1_1+(A(u_1)\dots A(u_r)\gamma)^2_2\\&=e^{L/2}(A(u_1)\dots A(u_r))^1_1+e^{-L/2}(A(u_1)\dots A(u_r))^2_2.\end{aligned}$$

Similarly, we have

$$(A(u_1)\dots A(u_r))^i_j=\sum_{I\in P(r)} a^i_j(I)\sin(\theta_I)\cos(\theta_{\hat{I}})$$

and define

$$(A(u_1)\dots A(u_r))^i_j(I)=a^i_j(I)\sin(\theta_I)\cos(\theta_{\hat{I}}).$$

We prove the lemma by induction. Given $I=(i_1,\dots,i_k)\in P(r)$, we have $I_j=(i_1,i_2,\dots,i_{j-1})\in P(i_j)$.

The matrix $A(u)$ has cos terms on the diagonal and sin off the diagonal. Since $\sin(\theta_{i_k})$ is the last sin term in $(A(u_1),\dots,A(u_r))^1_1(I)$, we have

$$\begin{aligned}&(A(u_1)\dots A(u_r))^1_1(I)\\&\quad=(A(u_1)\dots A(u_{i_{k-1}}))^1_2(I_k)(A(u_{i_k})^2_1A(u_{i_k+1})^1_1\dots A(u_r)^1_1)\\&\quad=\cos(\theta_{i_k+1})\dots\cos(\theta_r)(-e^{-l_{i_k}}\sin(\theta_{i_k}))(A(u_1)\dots A(u_{i_{k-1}}))^1_2(I_k).\end{aligned}$$

Now, since the next sin is $\sin(\theta_{i_{k-1}})$, by iterating we have

$$\begin{aligned}&(A(u_1)\dots A(u_{i_{k-1}}))^1_2(I_k)\\&\quad=(A(u_1)\dots A(u_{i_{k-2}}))^1_1(I_{k-1})\\&\qquad\times A^1_2(u_{i_{k-1}})A^2_2(u_{i_{k-1}+1})A^2_2(u_{i_{k-1}+2})\dots A^2_2(u_{i_{k-1}})\\&\quad=(A(u_1)\dots A(u_{i_{k-2}}))^1_1(I_{k-1})(-e^{l_{i_{k-1}}}\sin(\theta_{i_{k-1}}))\\&\qquad\times(-\cos(\theta_{i_{k-1}+1}))(-\cos(\theta_{i_{k-1}+2}))\dots(-\cos(\theta_{i_k-1})).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{(A(u_1)\dots A(u_r))^1_1(I)}{(A(u_1)\dots A(u_{i_{k-2}}))^1_1(I_{k-1})}\\&\quad=(-1)^{i_k-i_{k-1}+1}e^{l_{i_{k-1}}-l_{i_k}}\sin(\theta_{i_{k-1}})\cos(\theta_{i_{k-1}+1})\dots\\&\qquad\times\cos(\theta_{i_k-1})\sin(\theta_{i_k})\cos(\theta_{i_k+1})\cos(\theta_{i_k+2})\dots\cos(\theta_r).\end{aligned}$$

Since each off-diagonal term switches the index, there must be an even number of off-diagonal terms in the trace, and therefore $|I|$ is even. Then by induction

$$(A(u_1)\dots A(u_r)\gamma)^1_1=(-1)^{s(I)}\sin(\theta_I)\cos(\theta_{\hat{I}})e^{L/2-L_I},$$

where

$$s(I)=(i_2-i_1+1)+(i_4-i_3+1)+\dots+(i_k-i_{k-1}+1)$$

and

$$L_I = \sum_{j=1}^{k} (-1)^j l_{i_j} = -l_{i_1} + l_{i_2} - l_{i_3} - \cdots + (-1)^k l_{i_k}.$$

Similarly,

$$\begin{aligned}
&\frac{(A(u_1)\dots A(u_r))_2^2(I)}{(A(u_1)\dots A(u_{i_{k-2}}))_2^2(I_{k-1})} \\
&\quad = (-e^{-l_{i_{k-1}}} \sin(\theta_{i_{k-1}}))(\cos(\theta_{i_{k-1}+1}))(\cos(\theta_{i_{k-1}+2}))\dots(\cos(\theta_{i_k-1})) \\
&\qquad \times (-e^{l_{i_k}} \sin(\theta_{i_k}))(-\cos(\theta_{i_k+1}))\dots(-\cos(\theta_r)) \\
&\quad = (-1)^{r-i_k+2} e^{-l_{i_{k-1}}+l_{i_k}} \sin\theta_{i_{k-1}} \cos(\theta_{i_{k-1}+1})\dots\cos(\theta_{i_k-1}) \\
&\qquad \times \sin(\theta_{i_k})\cos(\theta_{i_k+1})\cos(\theta_{i_k+2})\dots\cos(\theta_r).
\end{aligned}$$

Counting negative signs, we have $r - s(I) + |I|$ negative signs.

$$(A(u_1)\dots A(u_r)\gamma)_2^2 = (-1)^{r-s(I)+|I|} \sin(\theta_I)\cos(\theta_{\hat{I}}) e^{L_I - L/2}.$$

Since $|I|$ is even, we get

$$(A(u_1)\dots A(u_r)\gamma)_2^2 = (-1)^{r+s(I)} \sin(\theta_I)\cos(\theta_{\hat{I}}) e^{L_I - L/2},$$

giving the result. □

5. Some Examples

We have from the calculations in the last section that

$$F_0(L) = 2\cosh(L/2), \qquad F_1(u, L) = 2\sinh(L/2)\cos\theta,$$
$$F_2(u_1, u_2, L) = 2(\cos\theta_1\cos\theta_2\cosh(L/2) + \sin\theta_1\sin\theta_2\cosh(L/2 - l_{12})).$$

Calculating F_3, we have

$$\begin{aligned}
F_3(u_1, u_2, u_3, L) = {}& 2\sinh(L/2)\cos(\theta_1)\cos(\theta_2)\cos(\theta_3) \\
&+ 2\sinh(L/2 - l_{12})\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\
&- 2\sinh(L/2 - l_{13})\sin(\theta_1)\cos(\theta_2)\sin(\theta_3) \\
&+ 2\sinh(L/2 - l_{23})\cos(\theta_1)\sin(\theta_2)\sin(\theta_3).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
G_0(L) &= 2\cosh(L/2), \\
G_1(u_1, \dots, u_n, L) &= 2\sinh(L/2)\sum_{i=1}^{n}\cos\theta_i, \\
G_2(u_1, \dots, u_n, L) &= 2\sum_{\substack{i,j=1 \\ i<j}}^{n} (\cos\theta_i\cos\theta_j\cosh(L/2) \\
&\qquad + \sin\theta_i\sin\theta_j\cosh(L/2 - l_{ij})),
\end{aligned}$$

$$
\begin{aligned}
G_3(u_1,\dots,u_n,L) = 2\sum_{i<j<k} &(\sinh(L/2)\cos(\theta_i)\cos(\theta_j)\cos(\theta_k)\\
&+\sinh(L/2-l_{ij})\sin(\theta_i)\sin(\theta_j)\cos(\theta_k)\\
&-\sinh(L/2-l_{ik})\sin(\theta_i)\cos(\theta_j)\sin(\theta_k)\\
&+\sinh(L/2-l_{jk})\cos(\theta_i)\sin(\theta_j)\sin(\theta_k)).
\end{aligned}
$$

Since the functions G_r do not depend on k, once we have calculated all derivatives of orders less than k, we only need calculate G_k to find the kth derivative.

For $k=3$, we have

$$
t_\beta^3 T_\gamma = \frac{1}{8}(B_{n,3,1}G_1(u_1,\dots,u_n,L_\gamma) + B_{n,3,3}G_3(u_1,\dots,u_n,L_\gamma)),
$$

$$
B_{n,3,3} = 3! = 6, \qquad B_{n,3,1} = (n-1)\binom{3}{1,2} + \binom{3}{3} = 3(n-1)+1 = 3n-2,
$$

$$
\begin{aligned}
t_\beta^3 T_\gamma = \frac{1}{8}\Bigg(&(6n-4)\sinh(L_\gamma/2)\sum_{i=1}^{n}\cos(\theta_i)\\
&+12\Bigg(\sum_{i<j<k}\sinh(L_\gamma/2)\cos(\theta_i)\cos(\theta_j)\cos(\theta_k)\\
&+\sinh(L_\gamma/2-l_{ij})\sin(\theta_i)\sin(\theta_j)\cos(\theta_k)\\
&-\sinh(L_\gamma/2-l_{ik})\sin(\theta_i)\cos(\theta_j)\sin(\theta_k)\\
&+\sinh(L_\gamma/2-l_{jk})\cos(\theta_i)\sin(\theta_j)\sin(\theta_k)\Bigg)\Bigg).
\end{aligned}
$$

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