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Zeta Functions of Curves with No Rational Points

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Abstract. We show that the motivic zeta functions of smooth, geometrically connected curves with no rational points are rational functions. This was previously known only for curves whose smooth projective models have a rational point on each connected component. In the course of the proof we study the class of a Severi–Brauer scheme over a general base in the Grothendieck ring of varieties.

1. Introduction

Let k be a field, and $K_0(\mathrm{Var}_k)$ the Grothendieck ring of varieties over k. This is the free Abelian group on isomorphism classes $[X]$ of finite-type k-schemes, subject to the following relation:

$$[X] = [Y] + [X \setminus Y] \quad \text{for } Y \hookrightarrow X \text{ a closed embedding.}$$

Multiplication is given by

$$[X] \cdot [Y] = [X \times Y]$$

on classes of finite-type k-schemes and extended bilinearly. This ring was introduced by Grothendieck [12, Letter of 16 August 1964] in a letter to Serre. The Grothendieck ring of varieties is the universal ring through which all "motivic" invariants factor (e.g., Euler characteristic with compact support, Hodge–Deligne polynomial, virtual Chow motive, etc.).

Let $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_k)$ be the class of the affine line.

Example 1. Using the fact that $\mathbb{P}^n = \mathrm{pt} \cup \mathbb{A}^1 \cup \mathbb{A}^2 \cup \cdots \cup \mathbb{A}^n$, we have

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n.$$

In [6, 1.3], Kapranov introduces for each quasi-projective X/k a motivic zeta function $Z_X(t)$.

Definition 2 (Kapranov motivic zeta function). Let X be a quasi-projective k-scheme. Then the motivic zeta function $Z_X(t) \in K_0(\mathrm{Var}_k)[[t]]$ is

$$Z_X(t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)] t^n,$$

where $\mathrm{Sym}^n(X)$ is the quotient of X^n by the symmetric group Σ_n, and Σ_n acts on X^n by permuting the factors. The quotient exists since X is quasi-projective.

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If $k = \mathbb{F}_q$ is a finite field, then $Z_X(t)$ is an analogue of the Weil zeta function

$$\zeta_X(t) := \exp\left(\sum_{k=1}^{\infty} \frac{\#|X(\mathbb{F}_{q^k})|}{k} t^k\right) = \sum_{n=0}^{\infty} \#|\operatorname{Sym}^n(X)(\mathbb{F}_q)| t^n.$$

Indeed, in this case there is a natural homomorphism

$$\begin{aligned} \# :\ K_0(\mathrm{Var}_k) &\to \mathbb{Z}, \\ [X] &\mapsto \#X(\mathbb{F}_q), \end{aligned}$$

and $\#(Z_X(t)) = \zeta_X(t)$.

REMARK 3. The motivic zeta function is a homomorphism

$$K_0(\mathrm{Var}_k) \to 1 + tK_0(\mathrm{Var}_k)[[t]],$$

that is, if $[X] = [Y] + [W]$, then $Z_X(t) = Z_Y(t) \cdot Z_W(t)$. Thus, we may define the zeta function of any class in $K_0(\mathrm{Var}_k)$ by extending linearly from classes of quasi-projective k-schemes, which generate $K_0(\mathrm{Var}_k)$ additively.

EXAMPLE 4 ([8, Corollary 3.6]).

$$Z_{\mathbb{P}^n}(t) = \frac{1}{(1-t)(1-\mathbb{L}t)\cdots(1-\mathbb{L}^n t)}.$$

Kapranov shows the following:

PROPOSITION 5 ([6, 1.1.9], [10, Theorem 7.33]). *Let k be a field, and C/k a smooth, geometrically connected, projective curve of genus g with $C(k) \neq \emptyset$. Then*

$$Z_C(t)(1-t)(1-\mathbb{L}t) \in K_0(\mathrm{Var}_k)[[t]]$$

is a polynomial of degree $2g$.

This result is analogous to (and implies by applying $\#(-)$) the rationality of the Weil zeta function of C if k is a finite field.

REMARK 6. Kapranov speculates that $Z_X(t)$ may be a rational function for arbitrary k-varieties X [6, Remark 1.3.5(b)]. If k is finite, such a result would give a geometric explanation for the rationality of the Weil zeta function ζ_X. However, Larsen and Lunts show that for $k = \mathbb{C}$, when X is a surface with Kodaira dimension different from $-\infty$, $Z_X(t)$ is not rational [8, Theorem 1.1]. The problem of finding a natural quotient of $K_0(\mathrm{Var}_k)$ (through which "motivic" invariants still factor) over which $Z_X(t)$ becomes rational is of some importance.

Kapranov remarks that $Z_C(t)$ is still a rational function if $C(k) = \emptyset$; however, a correct proof of this fact has not yet appeared in the literature. The reason for writing the present paper was to rectify this lack since the proof is not trivial.

The main technical theorem of this paper is a description of the class in $K_0(\mathrm{Var}_k)$ of a Sever–Brauer variety over a general finite-type k-scheme.

THEOREM 7. *Let S be a finite-type k-scheme, and $\alpha \in \mathrm{Br}(S)$ a Brauer class. Then there are an element $P = P(\alpha, S) \in K_0(\mathrm{Var}_k)$ and an integer $r = r(\alpha, S)$ determined only by α and S, such that for any Severi–Brauer S-scheme V with Brauer class α,*

$$[V] = P(1 + \mathbb{L}^r + \mathbb{L}^{2r} + \cdots + \mathbb{L}^{nr})$$

for some n.

See Proposition 28 for a refined version of this result.

After giving this description of the class of a Severi–Brauer scheme, we will prove the main result of the paper.

THEOREM 8. *Let C be a smooth, projective, geometrically connected curve over a field k. Then there exists a polynomial*

$$p(t) \in 1 + tK_0(\mathrm{Var}_k)[t]$$

such that $p(t)Z_C(t) \in K_0(\mathrm{Var}_k)[[t]]$ is a polynomial with constant term 1.

REMARK 9. Informally, we say that C has "rational motivic zeta function." The fact that $p(t)$ and $p(t)Z_C(t)$ have constant term 1 is important: it implies that the numerator and denominator of this rational function are invertible in $K_0(\mathrm{Var}_k)[[t]]$.

REMARK 10. Much previous work [13; 7] studies the Chow motive of a Severi–Brauer variety. The methods of this paper may be used to recover many of the results of these works; we believe that our methods bear some similarity to those of [7].

2. Discussion of the Proof of Proposition 5

Let us briefly review the proof of Proposition 5 and then discuss how it fails if $C(k) = \emptyset$. Here $\mathrm{Pic}^n(C)$ denotes the moduli space of degree n line bundles on C, defined as in [3, 9.2].

Proof of Proposition 5. Observe that if $C(k) \neq \emptyset$, then the Abel–Jacobi map $\mathrm{Sym}^n(C) \to \mathrm{Pic}^n(C)$ is a (Zariski) $\mathbb{P}^{n-g}$-bundle for $n > 2g - 2$ [11, Theorem 4]; thus, for $n > 2g - 2$,

$$[\mathrm{Sym}^n(C)] = [\mathbb{P}^{n-g}][\mathrm{Pic}^n(C)] = \frac{1 - \mathbb{L}^{n-g+1}}{1 - \mathbb{L}}[\mathrm{Pic}^n(C)].$$

Furthermore, $\mathrm{Pic}^n(C) \simeq \mathrm{Pic}^0(C)$ for all n (again using the existence of a rational point on C). In particular,

$$Z_C(t) = \sum_{n=0}^{2g-2}[\mathrm{Sym}^n(C)]t^n + [\mathrm{Pic}^0(C)]\sum_{n=2g-1}^{\infty}\frac{1 - \mathbb{L}^{n-g+1}}{1 - \mathbb{L}}t^n,$$

and thus

$$(1-t)(1-\mathbb{L}t)Z_C(t)$$

is a polynomial of degree $2g$. $\square$

EXAMPLE 11. Unfortunately, the first step of this proof breaks if $C(k)=\emptyset$. For example, consider the curve X in $\mathbb{P}^2_{\mathbb{R}}$ defined in homogeneous coordinates by

$$x^2+y^2+z^2=0.$$

It is easy to see that $\mathrm{Pic}^n(X)=\mathrm{Spec}(\mathbb{R})$ for all n (see, e.g., [3, 9.2.4]), but $\mathrm{Sym}^n(X)$ has no rational points if n is odd. Thus the Abel–Jacobi morphism $\mathrm{Sym}^n(X)\to\mathrm{Pic}^n(X)$ is not a Zariski $\mathbb{P}^n$-bundle for odd n.

REMARK 12. Theorem 8 of this paper implies that in the example,

$$(1-\mathbb{L}^2t^2)(1-t^2)Z_X(t)$$

is a polynomial. Let us compare this with Remark 1.3.5(a) of [6]. The remark states that $(1-\mathbb{L}^n t^n)(1-t^n)Z_X(t)$ is a polynomial, where $n>0$ is minimal such that $\mathrm{Pic}^n(X)(k)\neq\emptyset$; in the example, $\mathrm{Pic}^1(X)=\mathrm{Spec}(\mathbb{R})$, so the remark suggests that $(1-\mathbb{L}t)(1-t)Z_X(t)$ is rational. We do not know a proof of this fact and do not believe it to be true (though we have no proof that it is false).

There is some ambiguity in Remark 1.3.5(a) of [6], which may allow preserving its correctness. In the case where X has no rational points, the scheme $\mathrm{Pic}(X)$ represents the fppf sheafification of the functor sending T to the set of isomorphism classes of line bundles on $X\times T$ modulo line bundles pulled back from T. If we take the comment to refer to the Zariski sheafification of this functor instead, the remark has some chance of being true—though again, we do not know a proof.

The issue identified in Example 11 is that $\mathrm{Sym}^n(C)\to\mathrm{Pic}^n(C)$ may not be a Zariski fiber bundle. Of course (if C is geometrically connected), after a finite extension of the base field, we recover the usual situation of a projective space bundle over the $\mathrm{Pic}^n(C)$, so in general $\mathrm{Sym}^n(C)\to\mathrm{Pic}^n(C)$ is a Severi–Brauer scheme over $\mathrm{Pic}^n(C)$. Thus, we will proceed by studying the class $[V]$ of a Severi–Brauer S-scheme V/S in $K_0(\mathrm{Var}_k)$. The main result of this study is a description of the class $[V]\in K_0(\mathrm{Var}_k)$ (Proposition 28). Before proceeding with this description, we require some preliminaries on the Brauer group and twisted sheaves.

3. Twisted Sheaves

Traditionally, the Brauer group of a scheme is studied by means of Azumaya algebras [4; 5] or Severi–Brauer varieties [1]; instead, we will find it convenient to use the notion of twisted sheaves.

We begin with a brief, largely self-contained, review of the facts about twisted sheaves that we will need; useful references establishing many of the basics of the theory are Căldăraru [2] or Lieblich [9]. For an elaboration of the following definition and a discussion of many of its basic properties, see [2, Definition 1.2.1].

DEFINITION 13 (The category of α-twisted sheaves, $\mathrm{QCoh}(X,\alpha)$). Let X be a scheme, and $\alpha \in H^2(X_{\text{ét}}, \mathbb{G}_m)$ a cohomology class represented by a Čech 2-cocycle $\lambda \in \Gamma(U \times_X U \times_X U, \mathbb{G}_m)$ for some étale cover $U \to X$. The objects of the category $\mathrm{QCoh}(X,\alpha)$ of α-twisted sheaves are "descent data for quasi-coherent sheaves" twisted by α. Namely, let $\pi_1, \pi_2 : U \times_X U \to U$ be the two projections, and similarly with $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$. An α-twisted sheaf is the data of a quasi-coherent sheaf $\mathcal{E}$ on U and an isomorphism $\phi : \pi_1^*\mathcal{E} \xrightarrow{\sim} \pi_2^*\mathcal{E}$, so that $\pi_{23}^*\phi \circ \pi_{12}^*\phi = \lambda \cdot \pi_{13}^*\phi$. Observe that if $\mathcal{E}$ is a vector bundle, then we may (after refining U to trivialize $\mathcal{E}$) view this descent data as the data of a section $g' \in \Gamma(U \times_X U, \mathrm{GL}_n)$; we call $(\mathcal{E}, \phi)$ an α-twisted vector bundle if $\mathcal{E}$ is a vector bundle.

A morphism $(\mathcal{E}, \phi) \to (\mathcal{E}', \phi')$ is defined as a morphism $f : \mathcal{E} \to \mathcal{E}'$ such that the diagram

$$\begin{tikzcd} \pi_1^*\mathcal{E} \arrow[r, "\phi"] \arrow[d, "\pi_1^* f"] & \pi_2^*\mathcal{E} \arrow[d, "\pi_2^* f"] \\ \pi_1^*\mathcal{E}' \arrow[r, "\phi'"] & \pi_2^*\mathcal{E}' \end{tikzcd}$$

commutes.

REMARK 14. A priori, the definition of $\mathrm{QCoh}(X,\alpha)$ depends on the choice of cocycle λ representing $\alpha \in H^2(X, \mathbb{G}_m)$. However, if λ and λ' are two cocycles representing α, then the categories of twisted sheaves they define are (noncanonically) equivalent [2, Lemma 1.2.8]. Namely, refine the covers on which λ and λ' are defined and choose a 1-cocycle β with $d\beta = \lambda^{-1}\lambda'$. Then the functor

$$(\mathcal{E}, \phi) \mapsto (\mathcal{E}, \beta\phi)$$

is an equivalence of categories

$$\mathrm{QCoh}(X, [\lambda]) \to \mathrm{QCoh}(X, [\lambda']).$$

This equivalence *does* depend on the choice of β; these equivalences (up to natural isomorphism) are a torsor under $H^1(X, \mathbb{G}_m)$ (which corresponds to the fact that there are autoequivalences of $\mathrm{QCoh}(X,\alpha)$ coming from the functors

$$(\mathcal{E}, \phi) \mapsto (\mathcal{E} \otimes \mathcal{L}, \phi \otimes \mathrm{id}),$$

where $\mathcal{L}$ is a line bundle on X).

PROPOSITION 15. *Let X be a scheme, and $\alpha, \alpha' \in H^2(X_{\text{ét}}, \mathbb{G}_m)$ be cohomology classes.*

(1) *α is a Brauer class if and only if there exists an α-twisted vector bundle.*
(2) $\mathrm{QCoh}(X,\alpha)$ *is an Abelian category with enough injectives.*

(3) *There are natural functors*

$$- \otimes - : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha + \alpha')$$

and

$$\mathrm{Hom}(-, -) : \mathrm{QCoh}(X, \alpha)^{\mathrm{op}} \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha' - \alpha)$$

given by $\otimes$ *and* Hom *on twisted descent data.*

(4) *Similarly,* $\bigwedge^n$ *and* Sym^n *extend to functors* $\mathrm{QCoh}(X, \alpha) \to \mathrm{QCoh}(X, n\alpha)$.

(5) *If* $f : X \to Y$ *is a morphism, then there is a natural functor*

$$f^* : \mathrm{QCoh}(Y, \alpha) \to \mathrm{QCoh}(X, f^*\alpha)$$

given by applying f^* *to twisted descent data.*

(6) $\mathrm{QCoh}(X, 0)$ *is the usual category of quasi-coherent sheaves on* X.

Proof. All the statements aside from (1) and (2) are immediate from the definitions. For a sketch proof of (2), see [9, Lemma 2.2.3.2] or [2, Lemma 2.1.1]. For (1), see [2, Theorem 1.3.5 and the subsequent remarks]. □

PROPOSITION 16. *Let* X *be a scheme. There is an* α*-twisted line bundle on* X *if and only if* $\alpha = 0$.

Proof. If $\alpha = 0$, then $\mathcal{O}_X$ is an α-twisted line bundle.

On the other hand, let $(\mathcal{L}, \phi)$ be an α-twisted line bundle given by twisted descent data on some étale cover $U \to X$. We may choose a cover $r : U' \to U$ so that $r^*\mathcal{L}$ is trivial; after choosing a trivialization, we may view $r^*\phi$ as an element of $\Gamma(U' \times_X U', \mathbb{G}_m)$, that is, a 1-cochain for $\mathbb{G}_m$. But then $[d(r^*\phi)] = \alpha$, so α is a coboundary. Thus, $\alpha = 0$. □

COROLLARY 17. *Suppose that* $\mathcal{E}$ *is an* α*-twisted vector bundle of rank* n*. Then* α *is* n*-torsion in* $\mathrm{Br}(X)$.

Proof. By Proposition 15(4), $\bigwedge^n \mathcal{E}$ is an $n\alpha$-twisted line bundle, and thus $n\alpha = 0$ in $\mathrm{Br}(X)$ by Proposition 16. □

Twisted vector bundles have many of the same properties of vector bundles.

PROPOSITION 18. *Suppose that* X *is an affine scheme and* α *a Brauer class on* X*. Then all short exact sequences of* α*-twisted vector bundles on* X *split.*

Proof. Suppose that

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is a short exact sequence of α-twisted vector bundles. We wish to show that $\mathrm{Ext}^1_{\mathrm{QCoh}(X,\alpha)}(\mathcal{E}_3, \mathcal{E}_1) = 0$. But we have

$$\mathrm{Ext}^1_{\mathrm{QCoh}(X,\alpha)}(\mathcal{E}_3, \mathcal{E}_1) = H^1(X_{\text{ét}}, \mathcal{E}_3^\vee \otimes \mathcal{E}_1) = 0,$$

where we use Serre vanishing and that étale cohomology of quasi-coherent sheaves is the same as Zariski cohomology (by étale descent for quasi-coherent sheaves). □

COROLLARY 19. *Let $\mathcal{E}$ be an α-twisted vector bundle over the spectrum of a field. Then $\mathcal{E}$ is simple if and only if* $\operatorname{End}(\mathcal{E})$ *is a division algebra.*

Proof. Suppose that $\mathcal{E}$ is simple. Then any nonzero endomorphism of $\mathcal{E}$ must have no kernel since the kernel would be an α-twisted subbundle of $\mathcal{E}$. But we are working over a field, so (working étale-locally) we see that an endomorphism with no kernel is an isomorphism.

On the other hand, if $\mathcal{E}$ is not simple, then Proposition 18 gives that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ for some nonzero $\mathcal{F}$ and $\mathcal{G}$; then projection to either factor is a noninvertible endomorphism. □

COROLLARY 20. *Let X be the spectrum of a field. Then there is a unique isomorphism class of nonzero simple α-twisted vector bundles over X.*

Proof. Suppose that D and D' are nonzero simple α-twisted vector bundles. Then

$$\operatorname{Hom}_{\mathrm{QCoh}(X,\alpha)}(D, D') \simeq H^0_{\mathrm{QCoh}(X)}(D^\vee \otimes D') \neq 0.$$

But since D and D' are simple, any nonzero morphism between them is an isomorphism. □

COROLLARY 21 (Artin–Wedderburn). *Let X be the spectrum of a field, and D the unique nonzero simple α-twisted vector bundle over X. Then any α-twisted vector bundle E is isomorphic to $D^{\oplus n}$ for some n.*

Proof. Let E be a nonzero α-twisted vector bundle. If E is simple, then it is isomorphic to D by Corollary 20. Otherwise, let E' be a nonzero proper subbundle; by induction on the rank, $E' \simeq D^{\oplus k}$ and $E/E' \simeq D^{\oplus k'}$. So there is a short exact sequence

$$0 \to D^{\oplus k} \to \mathcal{E} \to D^{\oplus k'} \to 0,$$

and we may conclude the corollary by Proposition 18. □

COROLLARY 22. *Let X be an integral Noetherian scheme, and $\mathcal{E}_1$, $\mathcal{E}_2$ be two α-twisted vector bundles on X with ranks $r_1 \leq r_2$. Then there exist a nonempty open set $U \subset X$ and a monomorphism $\iota : \mathcal{E}_1|_U \hookrightarrow \mathcal{E}_2|_U$ such that* $\operatorname{coker}(\iota)$ *is an α-twisted vector bundle.*

Proof. We apply Corollary 21 at the generic point η of X to obtain a monomorphism $\mathcal{E}_1|_\eta \hookrightarrow \mathcal{E}_2|_\eta$. Spreading out gives the claim. □

If $\mathcal{E}$ is a vector bundle, we may consider $\mathbb{P}(\mathcal{E})$, the scheme of hyperplanes in $\mathcal{E}$ (Grothendieck's convention). Similarly, given an α-twisted sheaf $\mathcal{E}$ over a scheme X, we may obtain a Severi–Brauer variety with Brauer class α by considering $\mathbb{P}(\mathcal{E})$, which is étale descent data for a scheme over X. Since $\mathbb{P}(\mathcal{E})$ is anticanonically polarized over X, these descent data are effective, and we obtain a Severi–Brauer variety over X. To obtain an Azumaya algebra with Brauer class α, simply consider $\operatorname{End}(\mathcal{E})$. It is not hard to see that every Severi–Brauer variety or Azumaya algebra is obtained in this fashion; indeed, take the PGL_n-cocycle defining the

Severi–Brauer variety or Azumaya algebra and lift it to an arbitrary cocycle for GL_n. (To do so, we may have to refine the cover on which the cocycle is defined.)

We will require the following well-known fact about Severi–Brauer schemes; we sketch a proof using twisted sheaves.

COROLLARY 23 ([1, Lemma 3.3]). *Let $\pi : \mathbb{P} \to S$ be a Severi–Brauer scheme over S. If π admits a section, then $\mathbb{P} = \mathbb{P}(\mathcal{E})$ for a vector bundle $\mathcal{E}$ over S.*

Proof. Let $\mathcal{E}$ be an α-twisted vector bundle such that $\mathbb{P} = \mathbb{P}(\mathcal{E})$; we wish to show that $\alpha = 0 \in H^2(S, \mathbb{G}_m)$. But the section to π corresponds to an α-twisted line bundle that is a quotient of $\mathcal{E}$; hence, by Proposition 16, α is trivial. □

4. The Class of a Severi–Brauer Variety

Suppose that S is a finite-type k-scheme and

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is a short exact sequence of α-twisted vector bundles on S. We wish to relate the classes of the Severi–Brauer schemes

$$\mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2), \mathbb{P}(\mathcal{E}_3)$$

in $K_0(\mathrm{Var}_k)$. The main result of this section is such a relationship.

THEOREM 24. *Suppose that $\mathcal{E}_1$, $\mathcal{E}_3$ have ranks r_1, r_3, respectively, so that $\mathcal{E}_2$ has rank $r_2 := r_1 + r_3$. Then*

$$[\mathbb{P}(\mathcal{E}_2)] = [\mathbb{P}(\mathcal{E}_1)] + \mathbb{L}^{r_1}[\mathbb{P}(\mathcal{E}_3)] = [\mathbb{P}(\mathcal{E}_3)] + \mathbb{L}^{r_3}[\mathbb{P}(\mathcal{E}_1)] \in K_0(\mathrm{Var}_k).$$

Before giving the proof, we need a lemma.

LEMMA 25. *Let S be a scheme, and*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$$

a split α-twisted vector bundle on S. Then

$$\mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{E}_2) \simeq \mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_1)/\mathbb{P}(\mathcal{E})})$$

over $\mathbb{P}(\mathcal{E}_1)$, where $\mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_1)/\mathbb{P}(\mathcal{E})})$ is the total space of the normal bundle of $\mathbb{P}(\mathcal{E}_1)$ in $\mathbb{P}(\mathcal{E})$.

Proof. The idea of this statement is that projection away from $\mathbb{P}(\mathcal{E}_2)$ induces the desired isomorphism. This is well known in the case that $\alpha \in H^2(S, \mathbb{G}_m)$ is trivial; that is, in the case where the $\mathcal{E}_i$ are ordinary (untwisted) vector bundles. We reduce to that case.

Observe that the projection maps $\mathbb{P}(\mathcal{E}_1) \times \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_1)$ admit a section (the diagonal map); thus, by Corollary 23, if $\pi_1 : \mathbb{P}(\mathcal{E}_1) \to S$ is the structure map, then

$$\pi_1^* \alpha = 0 \in H^2(\mathbb{P}(\mathcal{E}_1), \mathbb{G}_m).$$

Thus, in particular, $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_i) \times \mathbb{P}(\mathcal{E}_1)$ are trivial Severi–Brauer varieties over $\mathbb{P}(\mathcal{E}_1)$, so by the split case we have that there is a natural isomorphism

$$\mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_1)\times\mathbb{P}(\mathcal{E}_1)/\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E}_1)}) \simeq \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}_1) \setminus \mathbb{P}(\mathcal{E}_2) \times \mathbb{P}(\mathcal{E}_1)$$

over $\mathbb{P}(\mathcal{E}_1) \times \mathbb{P}(\mathcal{E}_1)$. Pulling back along the diagonal map $\Delta : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_1) \times \mathbb{P}(\mathcal{E}_1)$ gives the desired claim. □

Proof of Theorem 24. Without loss of generality, S is integral and affine, and the short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

splits (by Proposition 18), so it suffices to prove the first equality, and we may view $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_3)$ as (linear) Severi–Brauer subvarieties of $\mathbb{P}(\mathcal{E}_2)$.

The morphism $\mathcal{E}_1 \to \mathcal{E}_2$ induces a closed embedding $\mathbb{P}(\mathcal{E}_1) \hookrightarrow \mathbb{P}(\mathcal{E}_2)$, so

$$[\mathbb{P}(\mathcal{E}_2)] = [\mathbb{P}(\mathcal{E}_1)] + [U],$$

where $U := \mathbb{P}(\mathcal{E}_2) \setminus \mathbb{P}(\mathcal{E}_1)$. We wish to identify U with the total space of a vector bundle over $\mathbb{P}(\mathcal{E}_3)$. But projection away from $\mathbb{P}(\mathcal{E}_1)$ identifies U with the total space $\mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_3)/\mathbb{P}(\mathcal{E}_2)})$ of $\mathcal{N}_{\mathbb{P}(\mathcal{E}_3)/\mathbb{P}(\mathcal{E}_2)}$ by Lemma 25. $\mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_3)/\mathbb{P}(\mathcal{E}_2)})$ is a Zariski-locally trivial $\mathbb{A}^{r_1}$ fiber bundle over $\mathbb{P}(\mathcal{E}_3)$, so

$$[U] = [\mathrm{Tot}(\mathcal{N}_{\mathbb{P}(\mathcal{E}_3)/\mathbb{P}(\mathcal{E}_2)})] = \mathbb{L}^{r_1}[\mathbb{P}(\mathcal{E}_3)] \in K_0(\mathrm{Var}_k),$$

as desired. □

COROLLARY 26. *Suppose that $\mathcal{E}$ is an α-twisted vector bundle with $\mathcal{E} = \mathcal{F}^{\oplus n}$ for some α-twisted vector bundle $\mathcal{F}$ of rank r. Then*

$$[\mathbb{P}(\mathcal{E})] = [\mathbb{P}(\mathcal{F})](1 + \mathbb{L}^r + \cdots + \mathbb{L}^{r(n-1)}).$$

Proof. This is immediate from Theorem 24 and induction on n. □

PROPOSITION 27. *Let S be a finite-type k-scheme, and P_1, P_2 two Severi–Brauer varieties over S of the same dimension and with the same Brauer class α. Then*

$$[P_1] = [P_2] \in K_0(\mathrm{Var}_k).$$

Proof. We may immediately replace S with S_{red}. Suppose that $\mathcal{E}_1$, $\mathcal{E}_2$ are α-twisted sheaves with $P_i = \mathbb{P}(\mathcal{E}_i)$. Then by Corollary 22 (replacing S with an integral affine open subscheme) there is an open set $U \subset S$ such that $\mathcal{E}_1|_U \simeq \mathcal{E}_2|_U$. Thus, $P_1|_U \simeq P_2|_U$, and so $[P_1|_U] = [P_2|_U]$. Proceed by Noetherian induction. □

PROPOSITION 28 (Theorem 7 refined). *Let S be a finite-type k-scheme, and $\alpha \in$* $\mathrm{Br}(S)$ *a Brauer class. Let* $r = \gcd(\mathrm{rk}(\mathcal{E}))$, *where $\mathcal{E}$ runs over all α-twisted vector bundles. Then there exists a class* $P \in K_0(\mathrm{Var}_k)$ *such that for any Severi–Brauer S-scheme $\mathbb{P}(\mathcal{E})$ with Brauer class α and* $\mathrm{rk}(\mathcal{E}) = d$,

$$[\mathbb{P}(\mathcal{E})] = P(1 + \mathbb{L}^r + \mathbb{L}^{2r} + \cdots + \mathbb{L}^{d-r}).$$

Proof. We first show that given $\mathcal{E}$, there exists a desired P; then we show that the class of P does not depend on $\mathcal{E}$.

By Corollary 26 it suffices to find a stratification $\{S_i\}$ of S such that on each stratum $(S_i)_{\mathrm{red}}$, $\mathcal{E}|_{S_i} = \mathcal{F}_i^{\oplus k}$ for some α-twisted vector bundle $\mathcal{F}_i$ of rank r on $(S_i)_{\mathrm{red}}$ since then we may write $P = \sum_i [\mathbb{P}(\mathcal{F}_i)]$, and the result follows for $\mathcal{E}$.

Now let S_1 be any integral open affine; then at the generic point $\iota : \eta \hookrightarrow (S_1)_{\mathrm{red}}$, $\mathcal{E}|_\eta = D^{\oplus k}$ for the unique simple $\iota^*\alpha$-twisted vector bundle D. But $\mathrm{rk}(D)$ divides r since the generic fiber of any α-twisted vector bundle admits a similar decomposition, so after shrinking S_1, we may take $\mathcal{F}_1 = \mathcal{D}^{\oplus k'}$ for some k' and some $\mathcal{D}$ extending D. We now proceed by Noetherian induction.

To see that our choice of P is independent of $\mathcal{E}$, let $\mathcal{E}'$ be another α-twisted vector bundle, with associated stratification $\{S'_j\}$ and twisted vector bundles $\mathcal{F}'_j$ on $(S'_j)_{\mathrm{red}}$, and $P' = \sum_j [\mathbb{P}(\mathcal{F}'_j)]$. Then on each irreducible component of

$$U_{ij} = (S_i \cap S'_j)_{\mathrm{red}},$$

$\mathbb{P}(\mathcal{F}_i|_{U_{ij}})$ and $\mathbb{P}(\mathcal{F}'_j|_{U_{ij}})$ satisfy the hypothesis of Proposition 27. Thus, $P = P'$, as desired. $\square$

REMARK 29. This result is an analogue of the main result of [7] with the features that (1) equality holds in the Grothendieck ring of varieties and (2) the result is proven in the relative setting. The methods here may be used to obtain relative versions of the many of the results of [13; 7]; for example, the main theorem [7, Theorem 1.3.1], which gives a decomposition of the motive of a Severi–Brauer variety over a field, may be extended to Severi–Brauer schemes over arbitrary k-varieties.

5. The Abel–Jacobi Morphism

Let C be a smooth, projective, geometrically connected curve over a field k with genus g. We now consider the Abel–Jacobi morphism

$$AJ^n : \mathrm{Sym}^n(C) \to \mathrm{Pic}^n(C),$$

where $n > 2g - 2$, sending a divisor to the associated line bundle. If C has a rational point, then this is a Zariski $\mathbb{P}^{n-g}$-bundle; so, in general, AJ^n exhibits $\mathrm{Sym}^n(C)$ as a Severi–Brauer variety over $\mathrm{Pic}^n(C)$.

Let K/k be a finite separable extension over which C obtains a rational point, so that there is a universal line bundle $\mathcal{L}_n$ over $C_K \times \mathrm{Pic}^n(C)_K$, and let $p : C \times \mathrm{Pic}^n(C) \to \mathrm{Pic}^n(C)$ and $q : C \times \mathrm{Pic}^n(C) \to C$ be the natural projections; we let p_K, q_K be the maps obtained by extending scalars to K. Then, by [11, Theorem 4],

$$\mathrm{Sym}^n(C)_K \simeq \mathbb{P}_{\mathrm{Pic}^n(C)_K}(p_{K*}\mathcal{L}_n)$$

for $n > 2g - 2$. Viewing $\mathrm{Sym}^n(C)$ as a descent of $\mathbb{P}_{\mathrm{Pic}^n(C)_K}(p_{K*}\mathcal{L}_n)$ induces descent data on $\mathbb{P}_{\mathrm{Pic}^n(C)_K}(p_{K*}\mathcal{L}_n)$, which we may view as a 1-cocycle valued in $\mathrm{PGL}(p_{K*}\mathcal{L}_n)$. Choosing an arbitrary lift of this 1-cocycle to a 1-cocycle valued in $\mathrm{GL}(p_{K*}\mathcal{L}_n)$ (to do so, we may have to refine the cover $\mathrm{Pic}^n(C)_K \to$

$\mathrm{Pic}^n(C)$), we may view $p_{K*}\mathcal{L}_n$ as an α-twisted sheaf $\mathcal{F}_n$ on $\mathrm{Pic}^n(C)$ for some $\alpha \in H^2(\mathrm{Pic}^n(C), \mathbb{G}_m)$, and $\mathrm{Sym}^n(C) = \mathbb{P}_{\mathrm{Pic}^n(C)}(\mathcal{F}_n)$.

Proof of Theorem 8. Let g be the genus of C.

Let D be a k-rational effective 0-cycle on C of degree n, that is, a rational point of $\mathrm{Sym}^n(C)$ for some n. Let $f \in \Gamma(C, \mathcal{O}_C(D))$ be such that

$$0 \to \mathcal{O}_C(-D) \xrightarrow{\cdot f} \mathcal{O}_C \to \mathcal{O}_D \to 0$$

is exact. Let

$$a_D^m : \mathrm{Pic}^m(C) \xrightarrow{\sim} \mathrm{Pic}^{m+n}(C)$$

be the map induced by multiplication by $AJ^n(D)$. Note that after changing base to K, there is an isomorphism $a_D^{m*}\mathcal{L}_{m+n} \simeq \mathcal{L}_m \otimes q_K^*\mathcal{O}_C(D)$; since f is defined over k, multiplication by f induces a morphism $b_D^m : \mathcal{F}_m \to a_D^{m*}\mathcal{F}_{m+n}$. We may check that b_D^m is a monomorphism by changing base to K. For $m > 2g-2$, the induced map

$$\mathbb{P}(b_D^m) : \mathrm{Sym}^m(C) \simeq \mathbb{P}(\mathcal{F}_m) \to \mathbb{P}(a_D^{m*}\mathcal{F}_{m+n}) \simeq \mathrm{Sym}^{m+n}(C)$$

agrees with the morphism $\mathrm{Sym}^m(C) \to \mathrm{Sym}^{n+m}(C)$ sending a effective degree m 0-cycle R to $R+D$. Furthermore, the existence of the morphism b_D^m implies that $\mathcal{F}_m$, $a_D^{m*}\mathcal{F}_{n+m}$ are vector bundles twisted by the same class $[\alpha] \in H^2(\mathrm{Pic}^m(C), \mathbb{G}_m)$.

Let $R_m = \mathrm{coker}(b_D^m)$; by extending scalars to K, we see that R_m is an α-twisted vector bundle of rank n. Thus,

$$\begin{aligned}[\mathrm{Sym}^{n+m}(C)] &= [\mathbb{P}(a_D^{m*}\mathcal{F}_{n+m})] = [\mathbb{P}(\mathcal{F}_m)] + \mathbb{L}^{m-g+1}[\mathbb{P}(R_m)] \\ &= \mathbb{L}^n[\mathrm{Sym}^m(C)] + [\mathbb{P}(R_m)]\end{aligned}$$

by Theorem 24. Observe that R_m and $a_D^{m*}R_{m+n}$ are α-twisted vector bundles of the same rank; thus, by Proposition 27,

$$[\mathbb{P}(R_m)] = [\mathbb{P}(a_D^{m*}R_{m+n})] = [\mathbb{P}(R_{m+n})].$$

Let $[P_m] \in K_0(\mathrm{Var}_k)$ be this class. Then by induction we have that

$$[\mathrm{Sym}^{m'+n}(C)] = [P_m] + \mathbb{L}^n[\mathrm{Sym}^{m'}(C)]$$

for all

$$m' \equiv m \bmod n, \quad m' > 2g-2.$$

Thus, there exists a polynomial $p(t) \in K_0(\mathrm{Var}_k)[t]$ such that

$$Z_C(t) = p(t) + \mathbb{L}^n t^n Z_C(t) + \sum_{m=2g-1}^{2g+n-2} \frac{[P_m]t^m}{1-t^n}.$$

In particular,

$$(1-\mathbb{L}^n t^n)(1-t^n)Z_C(t)$$

is a polynomial. Since $Z_C(t)$ has constant term 1, so does $(1-\mathbb{L}^n t^n)(1-t^n)Z_C(t)$. $\square$

COROLLARY 30. *Let C be a curve over k such that each irreducible component of $\widetilde{C_{\mathrm{red}}}$ (the normalization of the underlying reduced curve C_{red}) is geometrically irreducible. Then there exists a polynomial $p(t) \in 1 + tK_0(\mathrm{Var}_k)[t]$ such that $p(t)Z_C(t)$ is a polynomial with constant term* 1.

Proof. We reduce to the case where C is smooth and projective. Indeed, we may assume that C is reduced as $[C] = [C_{\mathrm{red}}]$; let $\tilde{C}$ be the smooth projective model of C. Then $[C] = [\tilde{C}] + [X] - [Y]$, where X and Y are zero-dimensional schemes. In particular,

$$Z_C(t)Z_Y(t) = Z_{\tilde{C}}(t)Z_X(t)$$

by Remark 3. We leave to the reader to show that there exist polynomials $p_X(t), p_Y(t) \in 1 + tK_0(\mathrm{Var}_k)[t]$ such that

$$p_X(t)Z_X(t),\ p_Y(t)Z_Y(t)$$

are polynomials with constant term one; thus, to prove the theorem for C, it suffices to prove it for $\tilde{C}$. But $\tilde{C}$ is a disjoint union of components C_i satisfying the conditions of Theorem 8, and

$$Z_C(t) = \prod_i Z_{C_i}(t),$$

so we are done. □

REMARK 31. It is natural to guess that the motivic zeta function of *any* curve is rational, that is, we may drop the condition of geometric connectedness in Theorem 8 and the rather artificial hypothesis of Corollary 30.

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