General Hilbert Stacks and Quot Schemes

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In memory of Dan Laksov

ABSTRACT. We prove the algebraicity of the Hilbert functor, the Hilbert stack, the Quot functor, and the stack of coherent sheaves on an algebraic stack X with (quasi-)finite diagonal without any finiteness assumptions on X. We also give similar results for Hom stacks and Weil restrictions.

Introduction

Let S be a scheme, and let $f : X \to S$ be a morphism between algebraic stacks that is locally of finite presentation. If f is separated, then it is well known that the Hilbert functor $\mathcal{H}ilb_{X/S}$ is an algebraic space, locally of finite presentation over S [Art69; OS03; Ols05]. If f is not separated but has a quasi-compact and separated diagonal with affine stabilizers, then one can instead prove that the Hilbert stack $\mathcal{H}^{\mathrm{qfin}}_{X/S}$—parameterizing proper flat families with a quasi-finite morphism to X—is an algebraic stack, locally of finite presentation over S [HR14; Ryd11]. The first main result of this paper is a partial generalization of these two results to stacks that are not locally of finite presentation.

THEOREM A. *Let S be a scheme, and let X be an algebraic stack over S.*

  (i) *If $X \to S$ has a finite diagonal, then $\mathcal{H}ilb_{X/S}$ is a separated algebraic space, and $\mathcal{H}^{\mathrm{qfin}}_{X/S}$ is an algebraic stack with affine diagonal.*

 (ii) *If $X \to S$ has a quasi-compact and separated diagonal with affine stabilizers, then $\mathcal{H}^{\mathrm{qfin}}_{X/S}$ is an algebraic stack with quasi-affine diagonal.*

In particular, if X is any separated scheme, algebraic space, or Deligne–Mumford stack, then $\mathcal{H}ilb_{X/S}$ is an algebraic space.

Our second result is about stacks of sheaves. Let us again first recall the classical situation. So, let $f : X \to S$ be a separated morphism between algebraic stacks that is locally of finite presentation. Then $\mathcal{C}oh(X/S)$—the stack of finitely presented sheaves on X that are flat and proper over S—is an algebraic stack, locally of finite presentation over S with affine diagonal [Lie06, Thm. 2.1], [Hal14b, Thm. 8.1]. If we are also given a quasi-coherent sheaf $\mathcal{F}$ on X, then $\mathrm{Quot}(X/S, \mathcal{F})$ is a separated algebraic space [Hal14b, Cor. 8.2]. Usually, it is

also assumed that $\mathcal{F}$ is finitely presented. Then $\text{Quot}(X/S, \mathcal{F})$ is locally of finite presentation over S, and the result goes back to [Art69; OS03; Ols05]. Again, we are able to remove the hypothesis that $X \to S$ is locally of finite presentation.

THEOREM B. *Let X be an algebraic stack with finite diagonal over S. Then the stack $\mathscr{C}oh(X/S)$ is algebraic with affine diagonal. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\text{Quot}(X/S, \mathcal{F})$ is a separated algebraic space over S.*

When $X \to S$ is not locally of finite presentation, the definitions of $\mathscr{C}oh(X/S)$ and $\text{Quot}(X/S, \mathcal{F})$ are somewhat subtle. The objects are quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ that are flat, *intrinsically of finite presentation* and *intrinsically proper* over S (together with a surjective homomorphism $\mathcal{F} \to \mathcal{G}$ for Quot). Modules intrinsically of finite presentation over S are of finite type as $\mathcal{O}_X$-modules. They are, however, not necessarily of finite presentation as $\mathcal{O}_X$-modules, and not every finitely presented $\mathcal{O}_X$-module is intrinsically of finite presentation.

There are two key ingredients in the proofs. The first is the approximation result [Ryd15, Thm. D]: every algebraic stack with quasi-finite diagonal can be approximated by algebraic stacks of finite presentation. The second is the representability result [Hal14a, Thm. D]: if $X \to S$ is separated and locally of finite presentation, and given $\mathcal{F}, \mathcal{G} \in \mathbf{QCoh}(X)$ such that $\mathcal{G}$ is of finite presentation, flat over S and with support proper over S, then $\text{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is *affine* over S.

The first result shows that the morphisms $f : X \to S$ appearing in the main theorems can be factored as $X \to X_0 \to S$, where $X \to X_0$ is affine and $X_0 \to S$ is of finite presentation. If $P_{X/S}$ is one of the stacks figuring in the main theorems, then we will describe natural morphisms $P_{X/S} \to P_{X_0/S}$. The second result will show that these morphisms are affine.

Independently, Di Brino used similar methods to prove that $\text{Quot}(\mathcal{F})$ is a scheme when $\mathcal{F}$ is a quasi-coherent sheaf on a projective scheme [DB12]. Approximation for the Quot and Hilbert functors is somewhat complicated since a homomorphism $\mathcal{F}_\lambda \to \mathcal{F}$ only gives rise to a rational map $\text{Quot}(\mathcal{F}) \dashrightarrow \text{Quot}(\mathcal{F}_\lambda)$. We apply the approximation step to $\text{Hom}(\mathcal{F}, \mathcal{G})$, $\mathscr{C}oh(X/S)$ and the Hilbert stack $\mathscr{H}_{X/S}$ where this inconvenience is absent. The algebraicity of $\text{Quot}(\mathcal{F})$ and $\mathcal{H}ilb_{X/S}$ then follows from the algebraicity of $\text{Hom}(\mathcal{F}, \mathcal{G})$, $\mathscr{C}oh(X/S)$ and $\mathscr{H}_{X/S}$. For zero-dimensional families, our results have appeared in [Ryd11] and [GLS07a; GLS07b; Skj11] using étale localization and explicit equations in the affine case.

The last two decades have witnessed an increased interest in the usage of objects that are not of finite type—particularly in non-Archimedean and arithmetic geometry. That being said, this paper was not written with a particular application in mind. Rather, it was the startling realization that recent techniques implied the existence of parameter spaces in such a great generality—in contrast to the preconceptions of the authors—that led to this paper.

1. Approximation

Let S be an affine scheme (or more generally a pseudo-Noetherian stack). Recall that an algebraic stack $X \to S$ has an *approximation* if there exists a factorization $X \to X_0 \to S$ where $X \to X_0$ is affine and $X_0 \to S$ is of finite presentation [Ryd15, Def. 7.1]. Equivalently, there is an inverse system $\{X_\lambda\}$ of algebraic stacks of finite presentation over S with affine bonding maps and inverse limit X [Ryd15, Prop. 7.3].

We say that a morphism $X \to S$ is *locally of approximation type* if there exist a faithfully flat morphism $S' \to S$ that is locally of finite presentation and an étale representable surjective morphism $X' \to X \times_S S'$ such that $X' \to S'$ is a composition of a finite number of morphisms that are either affine or locally of finite presentation and quasi-separated.

The condition of being locally of approximation type is clearly (i) stable under base change, (ii) stable under precomposition with morphisms that are either affine or locally of finite presentation and quasi-separated, (iii) fppf-local on the base, and (iv) étale-local on the source.

LEMMA 1.1. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The following are equivalent*:

(i) *f is locally of approximation type.*
(ii) *There exists an fppf-covering $\{S_i \to S\}$ such that S_i is affine and $X_i = X \times_S S_i \to S_i$ has an approximation.*

Proof. Clearly (ii) implies (i). For the converse, we may assume that S is affine and that there exists an étale representable surjective morphism $X' \to X$ such that $X' \to S$ is a composition of morphisms that are either affine or locally of finite presentation and quasi-separated. Since X is quasi-compact, we may further assume that these morphisms are quasi-compact. Then $X' \to X$ is of finite presentation, and $X \to S$ is of approximation type [Ryd15, Def. 2.9]. It then has an approximation by [Ryd15, Thm. 7.10]. □
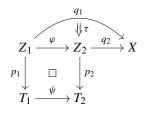
2. Stacks of Spaces

In this section we prove Theorem A and some related algebraicity results for Hom-stacks and Weil restrictions.

DEFINITION 2.1. Let $f : X \to S$ be a morphism of algebraic stacks. The Hilbert stack $\mathscr{H}_{X/S}$ is the category where:

- objects are pairs of morphisms $(p : Z \to T, q : Z \to X)$, where T is an S-scheme, such that p is flat, proper, and of finite presentation and the induced morphism $(q, p) : Z \to X \times_S T$ is representable;

• morphisms are triples (φ, ψ, τ) fitting into a 2-commutative diagram

$$
\begin{array}{ccc}
& \overset{q_1}{\overbrace{}} & \\
& \Downarrow \tau & \\
Z_1 \overset{\varphi}{\longrightarrow} & Z_2 \overset{q_2}{\longrightarrow} & X \\
\downarrow p_1 \quad \square & \downarrow p_2 & \\
T_1 \underset{\psi}{\longrightarrow} & T_2 &
\end{array}
$$

such that the square is Cartesian.

The category $\mathscr{H}_{X/S}$ is fibered in groupoids over $\mathbf{Sch}_{/S}$, and by étale descent it follows that $\mathscr{H}_{X/S}$ is a stack. We call $\mathscr{H}_{X/S}$ the *Hilbert stack* of X.

The substack of objects such that $(q, p) : Z \to X \times_S T$ is locally quasi-finite (resp. unramified, resp. a closed immersion) is denoted $\mathscr{H}_X^{\mathrm{qfin}}$ (resp. $\mathscr{H}_X^{\mathrm{unram}}$, resp. $\mathcal{H}ilb_{X/S}$). The first two substacks are always open substacks, and the third substack—the Hilbert functor—is an open substack if $X \to S$ is separated [Ryd11, Prop. 1.9]. When $X \to S$ has a quasi-compact and separated diagonal, then (q, p) is quasi-finite and separated for every object of $\mathscr{H}_{X/S}^{\mathrm{qfin}}$. Thus, the stack $\mathscr{H}_{X/S}^{\mathrm{qfin}}$ coincides with the Hilbert stack figuring in [HR14].

THEOREM 2.2. *Let $f : X \to S$ be a morphism of algebraic stacks that is quasi-separated and locally of approximation type. If f is separated (resp. has quasi-finite and separated diagonal), then $\mathscr{H}_{X/S}^{\mathrm{qfin}}$ is an algebraic stack with affine (resp. quasi-affine) diagonal.*

Theorem A is a consequence of Theorem 2.2 and the following two facts:
(i) an algebraic stack with quasi-finite and separated diagonal is locally of approximation type [Ryd15, Thm. D]; and
(ii) if X has affine stabilizers, then $\mathscr{H}_{X/S}^{\mathrm{qfin}} = \mathscr{H}_{X^{\mathrm{qfin}}/S}^{\mathrm{qfin}}$ where $X^{\mathrm{qfin}} \subseteq X$ denotes the open locus where X has finite stabilizers [HR14, Pf. of Thm. 4.3].

Before we prove Theorem 2.2, we will state a result on the algebraicity of Weil restrictions.

THEOREM 2.3. *Let $f : Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks. Let $g : W \to Z$ be a morphism of algebraic stacks, and let $f_* W = \mathbf{R}_{Z/S}(W) = \mathbf{Sec}_{Z/S}(W/Z)$ be the Weil restriction of W along f.*
(i) *If $g : W \to Z$ is affine, then $f_* W \to S$ is affine.*
(ii) *If $g : W \to Z$ is quasi-affine, then $f_* W \to S$ is quasi-affine.*
(iii) *If $g : W \to Z$ is a quasi-compact open immersion, then so is $f_* W \to S$.*
(iv) *If $g : W \to Z$ is a closed immersion, then so is $f_* W \to S$.*
(v) *If $f : Z \to S$ has a finite diagonal and $g : W \to Z$ has a finite diagonal, then $f_* W \to S$ is algebraic with affine diagonal.*
(vi) *If $f : Z \to S$ has a finite diagonal and $g : W \to Z$ has a quasi-finite and separated diagonal, then $f_* W \to S$ is algebraic with quasi-affine diagonal.*

Proof of Theorems 2.2 *and* 2.3. We begin with Theorem 2.3 (i). Let $g : W \to Z$ be affine. Recall that the functor $\mathrm{Hom}_{\mathcal{O}_Z/S}(g_*\mathcal{O}_W, \mathcal{O}_Z)$ is affine over S [Hal14a, Thm. D]. There is a functor $f_*W \to \mathrm{Hom}_{\mathcal{O}_Z/S}(g_*\mathcal{O}_W, \mathcal{O}_Z)$ taking a section $s : Z \to W$ of g to the corresponding $\mathcal{O}_Z$-module homomorphism. This functor is represented by closed immersions. To see this, let $\varphi : g_*\mathcal{O}_W \to \mathcal{O}_Z$ be an $\mathcal{O}_Z$-module homomorphism. This gives a section of $g : W \to Z$ if and only if the following maps vanish:

$$\mathrm{id}_{\mathcal{O}_Z} - \varphi \circ \eta : \mathcal{O}_Z \to \mathcal{O}_Z,$$
$$\varphi \circ \mu - \varphi \otimes \varphi : g_*\mathcal{O}_W \otimes_{\mathcal{O}_Z} g_*\mathcal{O}_W \to \mathcal{O}_Z,$$

where $\eta : \mathcal{O}_Z \to g_*\mathcal{O}_W$ is the unit homomorphism, and μ defines the multiplication on $g_*\mathcal{O}_W$. These conditions are closed since $\mathrm{Hom}_{\mathcal{O}_Z/S}(\mathcal{F}, \mathcal{O}_Z)$ is affine, and hence separated, for all quasi-coherent $\mathcal{O}_Z$-modules $\mathcal{F}$ [Hal14a, Thm. D].

For Theorem 2.3 (ii), the question easily reduces to (iii): if $W \to Z$ is a quasi-compact open immersion, then so is $f_*W \to S$. Since $f_*W = S \setminus f(Z \setminus W)$ is open and constructible, it is quasi-compact and open.

For Theorem 2.3 (iv), we first assume that $g : W \to Z$ is a closed immersion of finite presentation. Then $f_*W \to S$ is affine, of finite presentation [Hal14a, Thm. D], and a monomorphism. To show that $f_*W \to S$ is a closed immersion, it is thus enough to verify the valuative criterion for properness. This is readily verified since if S is the spectrum of a valuation ring with generic point ξ, then Z_ξ is schematically dense in Z by flatness of $Z \to S$.

Now suppose that $W \to Z$ is a closed immersion merely of finite type. Working locally on S, we may assume that S is affine. Then $W \to Z$ can be written as an inverse limit $W = \varprojlim W_\lambda$ of finitely presented closed immersions $W_\lambda \to Z$. It follows that $f_*W = \varprojlim f_*W_\lambda$ is a closed immersion.

Now, we prove Theorem 2.2. The question is fppf-local on S, so we may assume that S is affine. Given a representable morphism $g : X \to Y$ of algebraic stacks over S, there is a natural functor $g_* : \mathscr{H}_{X/S} \to \mathscr{H}_{Y/S}$ taking $Z \to X$ to $Z \to X \to Y$. If $Z \to X \to Y$ is a representable morphism, then so is $Z \to X$.

Now assume that g is quasi-affine. If $Y \to S$ is separated or has a quasi-finite and separated diagonal, then we obtain an induced morphism $g_* : \mathscr{H}_{X/S}^{\mathrm{qfin}} \to \mathscr{H}_{Y/S}^{\mathrm{qfin}}$. Indeed, let $Z \to S$ be a proper morphism together with a quasi-finite S-morphism $Z \to X$. If $Y \to S$ is separated, then $Z \to X$ is finite, so that $Z \to X \to Y$ is proper and quasi-affine, hence finite. If instead $Y \to S$ has a quasi-finite and separated diagonal, then $Z \to X \to Y$ is quasi-affine, of finite type, and has proper fibers. The last fact follows from the observation that the residual gerbe $\mathscr{G}_y$ is separated for every $y \in |Y|$. It follows that $Z \to Y$ is quasi-finite, so g_* is well defined.

Also, if g is quasi-affine (resp. affine, resp. a quasi-compact open immersion), we note that $g_* : \mathscr{H}_{X/S} \to \mathscr{H}_{Y/S}$ and $g_* : \mathscr{H}_{X/S}^{\mathrm{qfin}} \to \mathscr{H}_{Y/S}^{\mathrm{qfin}}$ are quasi-affine (resp. affine, resp. quasi-compact open immersions). Indeed, given a morphism $T \to \mathscr{H}_{Y/S}^{\mathrm{qfin}}$ corresponding to maps $Z \to T$ and $Z \to Y$, then the pull-back of g_* to

T is $\mathbf{R}_{Z/T}(X \times_Y Z/Z)$, which is quasi-affine (resp. affine, resp. a quasi-compact open immersion), by Theorem 2.3 (i)–(iii).

It is now readily deduced that $\mathscr{H}^{\mathrm{qfin}}_{X/S} = \bigcup_U \mathscr{H}^{\mathrm{qfin}}_{U/S}$, where the union is over all open quasi-compact substacks $U \subseteq X$. We can thus assume that $X \to S$ is quasi-compact. As the question of algebraicity is fppf-local on S, we can also assume that $X \to S$ has an approximation $X \to X_0 \to S$. If $X \to S$ is separated (resp. has a quasi-finite and separated diagonal), then it can be arranged so that $X_0 \to S$ is also separated (resp. has a quasi-finite and separated diagonal) [Ryd15, Thm. C]. The stack $\mathscr{H}^{\mathrm{qfin}}_{X_0/S}$ is thus algebraic and has an affine (resp. quasi-affine) diagonal [Hal14b, Thm. 9.1] and [HR14, Thm. 2]. As we have seen before, the morphism $\mathscr{H}^{\mathrm{qfin}}_{X/S} \to \mathscr{H}^{\mathrm{qfin}}_{X_0/S}$ is affine. This proves Theorem 2.2.

Parts (v)–(vi) of Theorem 2.3 follow from Theorem 2.2 since the morphism $\mathbf{R}_{Z/S}(W/Z) \to \mathscr{H}^{\mathrm{qfin}}_{W/S}$, taking a section to its graph, is an open immersion.   □

COROLLARY 2.4. *Let $f : Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks, and let $g : X \to S$ be a morphism of algebraic stacks.*

(i) *If $Z \to S$ has a finite diagonal and $X \to S$ has a quasi-finite and separated diagonal, then $\mathrm{Hom}_S(Z, X)$ is an algebraic stack with quasi-affine diagonal.*

(ii) *If $X \to S$ has a finite diagonal, then $\mathrm{Hom}_S(Z, X)$ is an algebraic stack with affine diagonal.*

Proof. Note that there is an isomorphism $\mathrm{Hom}_S(Z, X) \to \mathbf{R}_{Z/S}(X \times_S Z/Z)$ taking a morphism $h : Z \to X$ to the section $(h, \mathrm{id}_Z) : Z \to X \times_S Z$. Thus, in the first case, the corollary follows immediately from Theorem 2.3. In the second case, we reduce as before to the case where S is affine and $X \to S$ is quasi-compact, so there is an approximation $X \to X_0 \to S$. As before, $\mathrm{Hom}_S(Z, X) \to \mathrm{Hom}_S(Z, X_0)$ is affine since for any $T \to \mathrm{Hom}_S(Z, X_0)$, the pull-back is the Weil restriction $\mathbf{R}_{Z \times_S T/T}(X \times_{X_0} Z \times_S T)$. Finally, $\mathrm{Hom}_S(Z, X_0)$ is algebraic with affine diagonal by [Hal14b, Cor. 9.2].   □

REMARK 2.5. Let $Z \to S$ be as in Theorem 2.3, and let $h : W_1 \to W_2$ be a morphism between stacks over Z such that either $W_i \to Z$ are quasi-affine or $Z \to S$ and $W_i \to Z$ have quasi-finite and separated diagonals. Then $f_* W_1$ and $f_* W_2$ are algebraic by Theorem 2.3. If h has one of the properties: affine, quasi-affine, closed immersion, open immersion, quasi-compact open immersion, monomorphism, Deligne–Mumford, representable, representable and separated, locally of finite presentation, locally of finite type, unramified, étale; then so has $f_* W_1 \to f_* W_2$. These are routine verifications. Indeed, first reduce to the case $W_2 = Z$ and then apply Theorem 2.3 or argue by functorial characterizations and diagonals; cf. [Ryd11, Props. 3.5 and 3.8]. The only exception is "locally of finite type". In this case, one first easily reduces to the situation where $W_1 = W \to W_2 = Z$ is of finite type. Then $W \to Z$ has an approximation by [Ryd15, Prop. 7.6 or Thm. D]. This means that we can write $W \hookrightarrow W_0 \to Z$ with

$W \hookrightarrow W_0$ a closed immersion and $W_0 \to Z$ of finite presentation, and the result follows.

Similarly, if $Z \to S$ and $X \to S$ are as in Corollary 2.4 and in addition $X \to S$ has one of the properties: affine, quasi-affine, Deligne–Mumford, representable, representable and separated, locally of finite presentation, locally of finite type, unramified, étale; then so has $\mathrm{Hom}_S(Z, X)$.

REMARK 2.6 (Smoothness). If $W \to Z$ is smooth, then this does not imply that $f_*W \to S$ is smooth unless $Z \to S$ is finite. The proof of [Ryd11, Prop. 3.5 (iv)] does not apply since formal smoothness only implies that the infinitesimal lifting property holds for thickenings of *affine* schemes. For a counterexample, let $Z \to S$ be a one-parameter family of twisted cubics degenerating to a nodal plane curve with an embedded component. Then $\mathrm{Hom}_S(Z, \mathbb{A}_S^1) = \mathbf{R}_{Z/S}(\mathbb{A}_Z^1)$ is an affine scheme over S with generic fiber $\mathbb{A}^1$ and special fiber $\mathbb{A}^2$, hence not smooth. We thank the referee for making us aware of this fact.

REMARK 2.7 (Boundedness). If $W \to Z$ is quasi-compact, then it is nontrivial to show that $f_*W \to S$ is quasi-compact. Some results are available, however, such as [Ols07], [AOV11, App. C], and [Ryd11, Prop. 3.8]. These results imply corresponding boundedness results for Hom-stacks and have, for example, been used to deduce that the stack of twisted stable maps has quasi-compact components under mild hypotheses.

REMARK 2.8. The proof of Theorem 2.2 also shows that if $X = \varprojlim_\lambda X_\lambda$, where $\{X_\lambda\}_\lambda$ is an inverse system of algebraic stacks of finite presentation over S with affine bonding maps, then $\mathscr{H}_{X/S}^{\mathrm{qfin}} = \varprojlim_\lambda \mathscr{H}_{X_\lambda/S}^{\mathrm{qfin}}$, and this inverse system has affine bonding maps.

3. Intrinsic Finiteness for Sheaves

In this section, we introduce the relative finiteness notion—intrinsically of finite presentation—referred to in the introduction. This notion is needed in the definition of the stack $\mathscr{C}oh(X/S)$ and the sheaf $\mathrm{Quot}(X/S, \mathcal{F})$ when $X \to S$ is not locally of finite presentation. To motivate this definition, note that if $q : Z \to X$ is a finite morphism and $p : Z \to X \to S$ is of finite presentation, then $q_*\mathcal{O}_Z$ is of finite type but not necessarily of finite presentation. Conversely, if $q_*\mathcal{O}_Z$ is of finite presentation, then this does not imply that p is of finite presentation. The new finiteness notion fixes this: $q_*\mathcal{O}_Z$ is intrinsically of finite presentation over S exactly when p is of finite presentation. Moreover, this notion is also defined for sheaves of $\mathcal{O}_X$-modules. We begin with the affine case.

DEFINITION 3.1. Let A be a ring, let B be an A-algebra, and let M be a B-module. We say that M is *intrinsically of finite presentation over A* if there exist a polynomial ring $A[x_1, x_2, \ldots, x_n]$ and a homomorphism $A[x_1, x_2, \ldots, x_n] \to B$ such that M is of finite presentation as an $A[x_1, x_2, \ldots, x_n]$-module.

Although quite natural, we have not been able to find this definition in the literature except in the special case where B is of finite type [SP, 0659] under the name "finitely presented relative to A". The following lemma is of fundamental importance.

LEMMA 3.2. *Let A be a ring, let B be an A-algebra, and let M be a B-module.*

(i) *If M is finitely generated as an A-module, then M is finitely generated as a B-module, and $B/\operatorname{Ann}_B M$ is integral over A. In particular, the image of $\operatorname{Supp}_B M$ along $\operatorname{Spec} B \to \operatorname{Spec} A$ is $\operatorname{Supp}_A M$.*

(ii) *If B is finitely generated as an A-algebra and M is finitely presented as an A-module, then M is finitely presented as a B-module.*

Proof. If M is finitely generated as an A-module, then clearly M is finitely generated as a B-module. To see that $B/\operatorname{Ann}_B M$ is integral over A, we may replace B with $B/\operatorname{Ann}_B M$, so $B \to \operatorname{End}_A M$ becomes injective. Now Cayley–Hamilton's theorem [Eis95, Thm. 4.3] shows that every $b \in B$ satisfies an integral equation with coefficients in A.

To prove the second statement, choose a surjection $A^n \twoheadrightarrow M$ and note that the kernel K is a finitely generated A-module. The kernel K_B of $B^n \twoheadrightarrow M \otimes_A B$ is thus also finitely generated. Let L be the kernel of the surjective homomorphism $B^n \twoheadrightarrow M \otimes_A B \twoheadrightarrow M$, and let N be the kernel of the surjective homomorphism $M \otimes_A B \to M$. Then L is an extension of N by K_B, so it is enough to show that N is a finitely generated B-module. If $b_1, b_2, \ldots, b_n$ are generators of B as an A-algebra and $m_1, m_2, \ldots, m_r$ are generators of M as an A-module, then $m_j \otimes b_i - (b_i m_j) \otimes 1$ are generators of N as a B-module. $\qquad\square$

Let A be a ring, let B be an A-algebra, and let M be a B-module. Then from the previous lemma we obtain that

$$M \text{ is i.f.p. over } A \quad \Longrightarrow \quad M \text{ is f.g. as a } B\text{-module.}$$

If B is an A-algebra of finite type, then

$$M \text{ is i.f.p. over } A \quad \Longrightarrow \quad M \text{ is f.p. as a } B\text{-module,}$$

and the converse holds if B is an A-algebra of finite presentation. Finally, if C is a B-algebra, then

$$C \text{ is i.f.p. over } A \quad \Longleftrightarrow \quad C \text{ is f.g. as a } B\text{-module and f.p. as an } A\text{-algebra.}$$

LEMMA 3.3. *If $B = \varinjlim_\lambda B_\lambda$ is a direct limit of A-algebras of finite presentation and M is a B-module, then the following are equivalent*:

(i) *M is intrinsically of finite presentation over A,*

(ii) *M is a finitely presented B_λ-module for all sufficiently large λ,*

(iii) *M is a finitely presented B_λ-module for some λ.*

Proof. This follows from Lemma 3.2 and the observation that every homomorphism $A[x_1, x_2, \ldots, x_n] \to B$ factors through B_λ for all sufficiently large λ. $\qquad\square$

From the characterization in Lemma 3.3 we easily obtain that the property "intrinsically of finite presentation over the base" is stable under base change, fpqc-local on the base, stable on the source under pull-back by finitely presented morphisms, and fppf-local on the source:

- Let A' be an A-algebra. If M is i.f.p. over A, then $M \otimes_A A'$ is i.f.p. over A'. The converse holds if $A \hookrightarrow A'$ is faithfully flat.
- Let B' be a finitely presented B-algebra. If M is i.f.p. over A, then $M \otimes_B B'$ is i.f.p. over A. The converse holds if $B \hookrightarrow B'$ is faithfully flat.

In particular, the property is fppf-local on source and target, so we may extend the definition to algebraic stacks as follows.

DEFINITION 3.4. Let $f : X \to S$ be a morphism of algebraic stacks. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *intrinsically of finite presentation over S* if fppf-locally on X and S, it is intrinsically of finite presentation. If, in addition, $\operatorname{Supp} \mathcal{F}$ is universally closed, quasi-compact and separated over S, then we say that $\mathcal{F}$ is *intrinsically proper over S*.

PROPOSITION 3.5. *Let $f : X \to S$ be a morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(i) *If f is of finite presentation, then $\mathcal{F}$ is intrinsically of finite presentation over S (resp. intrinsically proper over S) if and only if $\mathcal{F}$ is of finite presentation (resp. has proper support over S).*

(ii) *Stability under base change: if $\mathcal{F}$ is intrinsically of finite presentation over S (resp. intrinsically proper over S) and $S' \to S$ is any morphism, then so is the base change $\mathcal{F}'$ over S'.*

(iii) *Suppose that $X = \varprojlim X_\lambda$ is the limit of an inverse system of finitely presented S-stacks $\{X_\lambda \to S\}$ with affine bonding maps $X_\lambda \to X_\mu$ and let $h_\lambda : X \to X_\lambda$ denote the canonical morphism. Then the following are equivalent:*
   (a) *$\mathcal{F}$ is intrinsically of finite presentation over S.*
   (b) *There exists an index α such that $(h_\alpha)_* \mathcal{F}$ is of finite presentation.*
   (c) *There exists an index α such that $(h_\lambda)_* \mathcal{F}$ is of finite presentation for all $\lambda \geq \alpha$.*

(iv) *Given an approximation $X \xrightarrow{h} X_0 \to S$, with $X_0 \to S$ separated, the following are equivalent:*
   (a) *$\mathcal{F}$ is intrinsically of finite presentation and intrinsically proper over S.*
   (b) *$h_* \mathcal{F}$ is of finite presentation with proper support over S.*

Proof. (i)–(iii) follow directly from the affine case, so it remains to prove (iv). If $h_* \mathcal{F}$ is of finite presentation, then $\mathcal{F}$ is intrinsically of finite presentation by (iii). Also, $\mathcal{F}$ is finitely generated, and $\operatorname{Supp} \mathcal{F} \hookrightarrow X \to X_0$ is integral with image $\operatorname{Supp} h_* \mathcal{F}$ (Lemma 3.2). It follows that $\mathcal{F}$ is intrinsically of finite presentation and intrinsically proper over S.

For the converse, we may work locally on S and assume that S is affine. Then X_0 is pseudo-Noetherian, so we may write $X = \varprojlim X_\lambda$ where $g_\lambda : X_\lambda \to X_0$ is affine of finite presentation. Let h_λ denote the induced map $X \to X_\lambda$. The

push-forward $(h_\lambda)_*\mathcal{F}$ is of finite presentation for sufficiently large λ by (iii). Let $Z_\lambda \hookrightarrow X_\lambda$ denote the closed substack defined by the zeroth Fitting ideal of $(h_\lambda)_*\mathcal{F}$. Since $(h_\lambda)_*\mathcal{F}$ is finitely presented, the Fitting ideal is finitely generated, so $j : Z_\lambda \hookrightarrow X_\lambda$ is of finite presentation. The Fitting ideal is contained in the annihilator of $(h_\lambda)_*\mathcal{F}$ and contains a power of the annihilator [Eis95, Prop. 20.7]. Thus, $(h_\lambda)_*\mathcal{F} = j_*j^*(h_\lambda)_*\mathcal{F}$ and $|Z_\lambda| = \mathrm{Supp}\,\mathcal{F}$. Moreover, since $X_0 \to S$ is separated, it follows that $Z_\lambda \to X_0$ is proper and affine, hence finite. We conclude that $h_*\mathcal{F} = (g_\lambda)_*(h_\lambda)_*\mathcal{F}$ is of finite presentation with proper support. □

4. Stacks of Sheaves

In this section, we prove Theorem B and related results on Hom-spaces of sheaves.

DEFINITION 4.1. Let $f : X \to S$ be a separated morphism of algebraic stacks.

- The stack of coherent sheaves $\mathscr{C}oh(X/S)$ is the category with objects $(T, \mathcal{G})$ where T is an S-scheme and $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_{X\times_S T}$-modules that is flat over T, intrinsically of finite presentation over T and intrinsically proper over T.
- The stack of coherent algebras $\mathscr{C}oh^{\mathrm{alg}}(X/S)$ is the analogous category with finite algebras instead of modules.
- Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The functor $\mathrm{Quot}(X/S, \mathcal{F})$ takes an S-scheme T to the set of quotients $\mathcal{F}_{X\times_S T} \twoheadrightarrow \mathcal{G}$ (up to isomorphism) such that $\mathcal{G}$ is flat over T, intrinsically of finite presentation over T, and intrinsically proper over T.

There are natural isomorphisms $\mathcal{H}ilb_{X/S} = \mathrm{Quot}(X/S, \mathcal{O}_X)$ and $\mathscr{H}_{X/S}^{\mathrm{qfin}} = \mathscr{C}oh^{\mathrm{alg}}(X/S)$ that take a family $(p : Z \to T, q : Z \to X)$ to the $\mathcal{O}_{X\times_S T}$-module $(q, p)_*\mathcal{O}_Z$, noting that (q, p) is finite since $X \to S$ is separated. Moreover, the natural forgetful morphism $\mathscr{C}oh^{\mathrm{alg}}(X/S) \to \mathscr{C}oh(X/S)$ is represented by affine morphisms. This follows as in [Lie06, Prop. 2.5] and [Ryd11, Lem. 4.2] using Theorem 4.2 below. Thus, in the separated case, Theorem A follows from Theorem B.

THEOREM 4.2. *Let $f : X \to S$ be a morphism of algebraic stacks that is separated and locally of approximation type. Let $\mathcal{F}, \mathcal{G} \in \mathbf{QCoh}(X)$ and assume that $\mathcal{G}$ is flat, intrinsically of finite presentation, and intrinsically proper over S. Then $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is affine. If, in addition, $\mathcal{F}$ is intrinsically of finite presentation, then $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is of finite type.*

Proof. The question is fppf-local on S, so we assume that S is affine. We may replace X with the closed substack defined by $\mathrm{Ann}_{\mathcal{O}_X}\mathcal{G}$ and assume that $X \to S$ is quasi-compact and universally closed. After replacing S with an fppf-covering, we may then assume that $X \to S$ has an approximation $X \xrightarrow{h} X_0 \to S$ with $X_0 \to S$ separated. Then $h_*\mathcal{G}$ is of finite presentation with proper support over S.

The natural morphism $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{X_0}/S}(h_*\mathcal{F}, h_*\mathcal{G})$ is a monomorphism since h is affine. As we will see, this monomorphism is represented by closed immersions. Since $\mathrm{Hom}_{\mathcal{O}_{X_0}/S}(h_*\mathcal{F}, h_*\mathcal{G})$ is affine [Hal14a, Thm. D], this will prove that $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is affine.

Let $\mathcal{K}$ be the kernel of the surjection $h^*h_*\mathcal{F} \to \mathcal{F}$. A homomorphism $h_*\mathcal{F} \to h_*\mathcal{G}$ induces a homomorphism $h^*h_*\mathcal{F} \to \mathcal{G}$, which factors uniquely through $\mathcal{F}$ if and only if the composition $\mathcal{K} \to \mathcal{G}$ is zero. This happens if and only if $h_*\mathcal{K} \to h_*\mathcal{G}$ is zero. This is a closed condition since $\mathrm{Hom}_{\mathcal{O}_{X_0}/S}(h_*\mathcal{K}, h_*\mathcal{G})$ is separated (even affine by [Hal14a, Thm. D]).

If $\mathcal{F}$ is also intrinsically of finite presentation, then $h_*\mathcal{F}$ is of finite presentation. It follows that $\mathrm{Hom}_{\mathcal{O}_{X_0}/S}(h_*\mathcal{F}, h_*\mathcal{G})$ is of finite presentation, and we conclude that $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is of finite type. □

COROLLARY 4.3. *Let $f : X \to S$ be a morphism of algebraic stacks that is separated and locally of approximation type. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Let $\mathcal{G} \in \mathbf{QCoh}(X)$ be flat, intrinsically of finite presentation, and intrinsically proper over S. Then the sheaf $\mathbf{R}_{\mathcal{G}/S}(\mathcal{A})$, which takes a morphism $h : T \to S$ to the set of $h^*\mathcal{A}$-module structures on $h^*\mathcal{G}$, is affine over S.*

Note that $\mathbf{R}_{\mathcal{G}/S}(\mathcal{A}) = \mathbf{R}_{\mathrm{Spec}\,\mathcal{G} \to S}(\mathrm{Spec}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{G})/\mathrm{Spec}\,\mathcal{G})$ when $\mathcal{G}$ is a quotient sheaf of $\mathcal{O}_X$, explaining the notation. Corollary 4.3 generalizes [Skj11, Thm. 3.5]: if $f : X \to S$ is *affine,* then $\mathcal{G} \in \mathbf{QCoh}(X)$ is intrinsically of finite presentation and intrinsically proper over S if and only if $f_*\mathcal{G}$ is of finite presentation (Proposition 3.5). Thus, $\mathbf{R}_{\mathcal{G}/S}(\mathcal{A})$ equals the module restriction functor $\mathscr{M}od^M_{B \to R}$, and we recover [Skj11, Thm. 3.5].

Proof of Corollary 4.3. The question is local on S, so we can assume that S is affine and $X \to S$ is quasi-compact and admits an approximation $X \to X_0 \to S$.

Consider $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{G})$, which is an affine S-scheme by Theorem 4.2. Let $\varphi : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{G}$ denote the universal homomorphism (after replacing S with the Hom-space). The Weil restriction $\mathbf{R}_{\mathcal{G}/S}(\mathcal{A})$ is then the subfunctor given by the conditions that the maps

$$\mathrm{id}_{\mathcal{G}} - \varphi \circ (\eta \otimes \mathrm{id}_{\mathcal{G}}) : \mathcal{G} \to \mathcal{G},$$
$$\varphi \circ (\mu \otimes \mathrm{id}_{\mathcal{G}}) - \varphi \circ (\mathrm{id}_{\mathcal{A}} \otimes \varphi) : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{G}$$

vanish, where $\eta : \mathcal{O}_X \to \mathcal{A}$ is the unit, and $\mu : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ is the multiplication. This is a closed subfunctor since $\mathrm{Hom}_{\mathcal{O}_X/S}(\mathcal{F}, \mathcal{G})$ is affine for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. □

THEOREM 4.4. *Let $f : X \to S$ be a morphism of algebraic stacks that is separated and locally of approximation type. The stack $\mathscr{C}oh(X/S)$ is algebraic with affine diagonal. If $\mathcal{F} \in \mathbf{QCoh}(X)$, then $\mathrm{Quot}(X/S, \mathcal{F})$ is a separated algebraic space.*

Proof. We argue almost exactly as in the proof of Theorem 2.2. First, we reduce to the case where S is affine and X quasi-compact. Next, we further reduce to the case where there is an approximation $X \xrightarrow{h} X_0 \to S$. Then there is a natural

morphism $h_* : \mathscr{C}oh(X/S) \to \mathscr{C}oh(X_0/S)$ that takes a sheaf $\mathcal{G}$ to $h_*\mathcal{G}$. The stack $\mathscr{C}oh(X_0/S)$ is algebraic, locally of finite presentation over S, and has affine diagonal [Hal14b, Thm. 8.1]. The morphism h_* is represented by affine morphisms: given a morphism $T \to \mathscr{C}oh(X_0/S)$, corresponding to a finitely presented sheaf $\mathcal{G}$ on $X_0 \times_S T$, the liftings to $\mathscr{C}oh(X/S)$ correspond to the $h_*\mathcal{O}_X$-module structures on $\mathcal{G}$. Thus, h_* is represented by $\mathbf{R}_{\mathcal{G}/T}(h_*\mathcal{O}_X)$, which is affine by Corollary 4.3. □

If f has a finite diagonal, then f is locally of approximation type. Thus, Theorem B is an immediate consequence of Theorem 4.4.

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