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Srinivas' Problem for Rational Double Points

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Abstract. For the completion B of a local geometric normal domain, V. Srinivas asked which subgroups of $\mathrm{Cl}\, B$ arise as the image of the map $\mathrm{Cl}\, A \to \mathrm{Cl}\, B$ on class groups as A varies among normal geometric domains with $B \cong \hat{A}$. For two-dimensional rational double point singularities, we show that all subgroups arise in this way by applying Noether–Lefschetz theory to linear systems with nonreduced base loci. By a similar technique we also show that in any dimension, every local ring of a normal hypersurface singularity has completion isomorphic to the completion of a geometric UFD.

1. Introduction

V. Srinivas posed several interesting problems about class groups of Noetherian local normal domains in his survey paper on geometric methods in commutative algebra [21, §3]. Recall that if A is such a ring with completion $\hat{A}$, then there is a well-defined injective map on divisor class groups $j: \mathrm{Cl}\, A \to \mathrm{Cl}\, \hat{A}$ [19, §1, Proposition 1] arising from valuation theory. For geometric local rings, that is, localizations of $\mathbb{C}$-algebras of finite type, Srinivas asks about the possible images of the map j [21, Questions 3.1 and 3.7].

Problem 1.1. Let B be the completion of a local geometric normal domain.

(a) What are the possible images of $\mathrm{Cl}\, A \hookrightarrow \mathrm{Cl}\, B$ as A ranges over all geometric local normal domains with $\hat{A} \cong B$?
(b) Is there a geometric normal local domain A with $\hat{A} \cong B$ and $\mathrm{Cl}\, A = \langle \omega_B \rangle \subset \mathrm{Cl}\, B$?

While we are mainly interested in (a), let us review the progress on Problem 1.1 (b). Since the dualizing module ω_B is necessarily in the image of $\mathrm{Cl}\, A \hookrightarrow \mathrm{Cl}\, B$ whenever A is a quotient of a regular local ring [15], part (b) asks whether the image that is a priori the *smallest* possible can be achieved. Moreover, if B is Gorenstein, then ω_B is trivial in $\mathrm{Cl}\, B$, and part (b) asks whether $\mathrm{Cl}\, A = 0$ is possible; in other words, whether B is the completion of a unique factorization domain (UFD). For arbitrary rings, Heitmann [9] proved that B is the completion of a UFD if and only if B is a field, a discrete valuation ring, or $\dim B \geq 2$, depth $B \geq 2$, and every integer is a unit in A, but for $\dim A \geq 2$, his constructions produce rings that are far from geometric.

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For Problem 1.1 (b) as stated, perhaps the most famous result is Grothendieck's solution to Samuel's conjecture [6, XI, Corollaire 3.14], which says that if B is a complete intersection that is factorial in codimension ≤ 3, then $\mathrm{Cl}\, B = 0$, so that B is already a UFD. In particular, any complete intersection ring B of dimension ≥ 4 with an isolated singularity is a UFD. Parameswaran and Srinivas [16] showed that such rings B of dimension $d = 2, 3$ are completions of UFDs, extending the earlier result of Srinivas for rational double points [20]. Hartshorne and Ogus [8] proved that any ring B with an isolated singularity and depth ≥ 3 having codimension ≥ 3 in a regular local ring is a UFD. Parameswaran and van Straten [17] answered Problem 1.1 (b) positively for arbitrary normal surface singularities, the only result that discusses nontrivial subgroups. We offer the following.

THEOREM 1.2. *Let B be a completed normal hypersurface singularity. Then there exist a hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ and a point $p \in X$ such that $A = \mathcal{O}_{X,p}$ is a UFD and $\hat{A} \cong B$.*

Unlike the results mentioned before for isolated singularities, our proof is very short (about a page) and uniformly handles all dimensions, though it only addresses rings B of the form $\mathbb{C}[[x_1, \ldots, x_n]]/(f)$ where f defines a variety V normal at the origin. We expect to extend our method to all normal local complete intersection singularities.

On the other hand, we have seen no work addressing the more difficult Problem 1.1 (a) since it appeared ten years ago. Noting an example of a ring B for which $\mathrm{Cl}\, B \cong \mathbb{C}$ but the images $\mathrm{Cl}\, A$ are finitely generated [21, Example 3.9], Srinivas seems pessimistic about the chances, saying that Problem 1.1 (b) is "probably the only reasonable general question in the direction of Problem 1.1 (a)", but his example at least suggests the following.

QUESTION 1.3. For B as before, is every finitely generated subgroup $H \subset \mathrm{Cl}\, B$ containing $\langle \omega_B \rangle$ the image of $\mathrm{Cl}\, A$ for a local geometric normal domain A with $\hat{A} \cong B$?

Our first result is a positive answer for the best understood surface singularities.

THEOREM 1.4. *Let B be the completion of the local ring of a rational double point on a surface. Then for any subgroup $H \subset \mathrm{Cl}\, B$, there is a local geometric normal ring A with $B \cong \hat{A}$ and $H = \mathrm{Cl}\, A \subset \mathrm{Cl}\, B$.*

There is a well-known classification of the surface rational double points, namely $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ [11]; in each case, we produce an algebraic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ and a rational double point $p \in S$ such that $\mathrm{Cl}\, \mathcal{O}_{S,p} \cong H$. This result comes as a surprise in view of Mohan Kumar's result [14]. He proved that for almost all $\mathbf{A}_n$- and $\mathbf{E}_n$-type singularities on a *rational* surface over $\mathbb{C}$, the analytic isomorphism class *determines* the algebraic isomorphism class; the exceptions are $\mathbf{A}_7$, $\mathbf{A}_8$, and $\mathbf{E}_8$, for which there are two possibilities each. In particular, the possibility for

$\mathrm{Cl}(A) \hookrightarrow \mathrm{Cl}(\hat{A})$ is unique, except for these cases (and also the $\mathbf{E}_8$ case since the complete local ring is a UFD).

Whereas the results cited are local algebraic, our method of proof uses global algebraic geometry. Our idea in each case is to exhibit a base locus $Y \subset \mathbb{P}^n$ and a point $p \in Y$ constructed so that the general hypersurface X containing Y has the desired singularity type at p, meaning that $B \cong \widehat{\mathcal{O}}_{X,p}$. Taking $A = \mathcal{O}_{X,p}$, an honest local ring from the variety X, we can read off the generators of $\mathrm{Cl}\, A$ by the following consequence of our extension of the Noether–Lefschetz theorem to linear systems with base locus [3].

THEOREM 1.5. *Let $Y \subset \mathbb{P}^n_{\mathbb{C}}$ be a closed subscheme that is properly contained in a normal hypersurface and suppose $p \in Y$. Then the very general hypersurface X of degree $d \gg 0$ containing Y is normal, and $\mathrm{Cl}\, \mathcal{O}_{X,p}$ is generated by supports of the components of Y having codimension two in $\mathbb{P}^n$. In particular,* $\mathrm{codim}(Y, \mathbb{P}^n) > 2 \Rightarrow \mathrm{Cl}\, \mathcal{O}_{X,p} = 0$.

REMARK 1.6. As is the case with many results of Noether–Lefschetz type, the conclusion of Theorem 1.5 holds for *Zariski* general $X \in |H^0(\mathcal{I}_Y(d))|$ when $n > 3$.

After applying Theorem 1.5, we use power series techniques to compute the images of the local class groups to obtain the results. For example, the proof of Theorem 1.2 combines Theorem 1.5 with a power series lemma due to Ruiz [18]. The constructions in Theorem 1.4 are more complicated and rather interesting. For $\mathbf{A}_n$ singularities, we used a Cohen–Macaulay multiplicity structure on a smooth curve for Y, but for $\mathbf{D}_n$ singularities, we found it necessary to use a line (or a double line, depending on the parity of n) with an embedded point at p. In the case of a $\mathbf{D}_n$ singularity given locally by $x^2 + y^2 z + z^{n-1} = 0$, the completed local ring has class group $\mathbb{Z}/4\mathbb{Z}$ when n is odd and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ when n is even. In the odd case there is only one subgroup of order 2, but in the even case there are *three* of them, two of which are indistinguishable up to automorphism of the complete local ring since their generators correspond to conjugate points in the associated Dynkin diagram [10]. In this case we prove the stronger statement that each of the two nonequivalent subgroups arises as the class group of a local ring on a surface. Finally, in Proposition 4.9 we prove that any divisor class of a rational double point surface singularity has a representative that is smooth at that point.

Regarding organization, we prove Theorem 1.5 and Theorem 1.2 in Sections 2 and 3, respectively. Section 4 is devoted to Theorem 1.4. We work throughout over the field $k = \mathbb{C}$ of complex numbers except as noted. Whereas the base locus Y should be projective to apply Theorem 1.5, we often give a local ideal for $Y \subset \mathbb{A}^n$ and apply the theorem to $\overline{Y} \subset \mathbb{P}^n$.

2. Geometric Generators of Local Class Groups

In this section we prove Theorem 1.5 from the Introduction, which identifies the generators of the local class group of a general member of a linear system of hypersurfaces at a point lying in the base locus. The proof is quite short, being a straightforward consequence of our Noether–Lefschetz theorem with base locus [3], and we present it without further ado.

Proof of Theorem 1.5. The condition that Y is properly contained in a normal hypersurface implies that Y has codimension ≥ 2 in $\mathbb{P}^n$ and its points of embedding dimension $= n$ have codimension ≥ 3 in $\mathbb{P}^n$, which is to say that Y is *superficial* in the language used in [3] (the converse is also easy to see). Thus, we may apply [3, Theorems 1.1 and 1.7] to see that X is normal and $\operatorname{Cl} X$ is (freely) generated by $\mathcal{O}_X(1)$ and the supports of the components of Y having codimension two in $\mathbb{P}^n$. On the other hand, the natural restriction map $\operatorname{Cl} X \to \operatorname{Cl} \mathcal{O}_{X,p}$ is surjective (because height one primes in $\mathcal{O}_{X,p}$ lift to global Weil divisors on X), so $\operatorname{Cl} \mathcal{O}_{X,p}$ is generated by these same classes. Since $\mathcal{O}_X(1)$ and the supports of the components of Y having codimension two in $\mathbb{P}^n$ not passing through p have trivial restriction in $\operatorname{Cl} \mathcal{O}_{X,p}$, it follows that $\operatorname{Cl} \mathcal{O}_{X,p}$ is generated by the remaining supports of components of Y having codimension two in $\mathbb{P}^n$ that pass through p. $\square$

One can use Theorem 1.5 together with the natural injection $\operatorname{Cl} \mathcal{O}_{X,p} \to \operatorname{Cl} \widehat{\mathcal{O}}_{X,p}$ to calculate class groups of local rings by carrying out calculations inside power series rings. We will use this idea to prove our main results; for further applications, see [5]. The following restates Theorem 1.5 in algebraic terms.

Corollary 2.1. *Let $I \subset R = \mathbb{C}[x_1, \ldots, x_n]$ be an ideal of height ≥ 2. In the primary decomposition $I = \bigcap q_i$ with q_i a p_i-primary ideal, assume that $q_i \not\subset p_i^2$ for each height two prime p_i. Then for the very general $f \in I$ and $A = \mathbb{C}[x_1, \ldots, x_n]/(f)$, the image of $\operatorname{Cl}(A) \hookrightarrow \operatorname{Cl}(\hat{A})$ is generated by the height two primes associated to I.*

3. Completions of Unique Factorization Domains

Given a complete local ring A, what is the nicest ring R having completion A? Heitmann [9] shows that A is the completion of a UFD if and only if A is a field, A is a DVR, or A has depth ≥ 2 and no integer is a zero-divisor of A; however, his constructions need not lead to excellent rings. Loepp [12] shows with minimal hypothesis that A is the completion of an *excellent* local ring, though her construction need not produce a UFD. Using geometric methods, Parameswaran and Srinivas [16] show that the completion of the local ring at an isolated local complete intersection singularity is the completion of a UFD that is the local ring for a variety. We prove the same for normal hypersurface singularities over $\mathbb{C}$ that need not be isolated. We first need a lemma, which can be found in Ruiz' book on power series [18, V, Lemma 2.2].

Lemma 3.1. *Let $\mathfrak{m} \subset \mathbb{C}[[x_1, \ldots, x_n]]$ denote the maximal ideal, fix $f \in \mathfrak{m}^2$, and define $J_f = (f_{x_1}, \ldots, f_{x_n})$ to be the ideal generated by the partial derivatives of f. Then for any $g \in \mathbb{C}[[x_1, \ldots, x_n]]$ such that $f - g \in \mathfrak{m} \cdot J_f^2$, $\mathbb{C}[[x_1, \ldots, x_n]]/(f) \cong \mathbb{C}[[x_1, \ldots, x_n]]/(g)$.*

REMARK 3.2. Ruiz actually proves Lemma 3.1 for the ring $\mathbb{C}\{x_1, \dots, x_n\}$ of convergent power series and notes that the same proof goes through in the formal case.

We proceed to prove Theorem 1.2.

Proof of Theorem 1.2. Let $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ be the equation of a hypersurface V that is singular and normal at the origin p, corresponding to the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, and let $B = \mathbb{C}[[x_1, x_2, \dots, x_n]]/(f)$ be the completion of $\mathcal{O}_{V,p}$. The singular locus D of V is given by the ideal $(f) + J_f$, where $J_f = (f_{x_1}, \dots, f_{x_n})$. Using primary decomposition in the ring $\mathbb{C}[x_1, \dots, x_n]$, we may write

$$J_f = \bigcap_{p_i \subset \mathfrak{m}} q_i \cap \bigcap_{p_i \not\subset \mathfrak{m}} q_i,$$

where q_i is p_i-primary, and we have sorted into components that meet the origin and those that do not. Denote by K the intersection on the left and J the intersection on the right; localizing at $\mathfrak{m}$, we find that $(J_f)_\mathfrak{m} = K_\mathfrak{m}$ because $J_\mathfrak{m} = (1)$.

Now if $K = (k_1, \dots, k_r)$ gives a polynomial generating set for K, the closed subscheme Y defined by the ideal $I_Y = (f, k_1^3, \dots, k_r^3)$ is supported on the components of the singular locus of V that contain the origin; hence, Y has codimension ≥ 3 in $\mathbb{P}^n$ by normality of V at the origin. The very general hypersurface X containing Y satisfies $\mathrm{Cl}\,\mathcal{O}_{X,p} = 0$ by Theorem 1.5, so $\mathcal{O}_{X,p}$ is a UFD [7, Prop. 6.2]. Moreover, X has the local equation

$$g = f + a_1 k_1^3 + \cdots + a_r k_r^3$$

for units a_i, and clearly $f - g \in K^3$. Since $K_\mathfrak{m} = (J_f)_\mathfrak{m}$, their completions are equal in $\mathbb{C}[[x_1, \dots, x_n]]$. Therefore, $f - g \in J_f^3 \subset \mathfrak{m} J_f^2$, and it follows from Lemma 3.1 that $\widehat{\mathcal{O}}_{X,p} = \mathbb{C}[[x_1, \dots, x_n]]/(g) \cong \mathbb{C}[[x_1, \dots, x_n]]/(f) = B$. $\square$

EXAMPLE 3.3. For an *isolated* singularity, we can give the following proof based on the Mather–Yau theorem [13]. The ideal $I = (f, f_{x_1}, \dots, f_{x_n})$ generated by f and its partial derivatives define a 0-dimensional scheme Y supported at the origin; hence, $(x_1, \dots, x_n)^N \subset I$ for some $N > 0$. The scheme Z defined by $(f, x_1^{N+2}, \dots, x_n^{N+2})$ is also supported at the origin p, so by Theorem 1.5 the very general surface S containing Z satisfies $\mathrm{Cl}\,\mathcal{O}_{S,p} = 0$. The local equation of S has the form $g = f + \sum a_i x_i^{N+2}$ for units a_i in the local ring $\mathcal{O}_{\mathbb{P}^n,p} \cong \mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. Observe that $J = (g, g_{x_1}, \dots, g_{x_n}) \subset I$ because $g - f$ and its partials lie in $(x_1, \dots, x_n)^{N+1} \subset I$. These ideals are equal because the induced map $J \to I/(x_1, \dots, x_n)I$ is obviously surjective, therefore so is the map $J \to I$ by Nakayama's lemma. It follows that $I = J$ in the ring $\mathbb{C}\{x_1, \dots, x_n\}$ of germs of holomorphic functions as well, so f and g define (complex-)analytically isomorphic singularities; this isomorphism lifts to a formal-analytic isomorphism. We have thus produced a UFD, namely $\mathcal{O}_{S,p}$, whose completion is isomorphic to A.

4. Local Class Groups of Rational Double Points

As noted in the Introduction, Mohan Kumar showed that for $\mathbf{A}_n$ and $\mathbf{E}_n$ double points on a *rational* surface, the analytic isomorphism class determines the algebraic isomorphism class of the local ring, with the three exceptions $\mathbf{A}_7$, $\mathbf{A}_8$, $\mathbf{E}_8$ for which there are two possibilities each. Regarding the question of Srinivas mentioned in the Introduction, this means that there is one (or sometimes two) possibilities for the inclusion $\mathrm{Cl}\, A \hookrightarrow \mathrm{Cl}\, \hat{A}$ for the corresponding local rings. Our main goal in this section is to prove Theorem 1.4 from the Introduction, which says that without the rationality hypothesis, *all* subgroups arise in this way. Note that the case of the trivial class group is Theorem 1.2, so we only need prove the result for nontrivial class groups.

We will prove this for each singularity type $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$ separately, noting their standard equations as we will use, the class group $\mathrm{Cl}\, \hat{A}$ of the completion of a local ring A having the given type, and the Dynkin diagram of exceptional (-2)-curves for the minimal resolution (these have been known for a long time; cf. [1; 11]). The class of any curve is determined by its intersection numbers with the exceptional curves E_j; denote by e_j the class-group element corresponding to a curve meeting e_j once and no other e_i.

Now we prove Theorem 1.4. For the trivial subgroup, we can apply Theorem 1.2 or Srinivas' calculation [20]. The approach in the nontrivial cases is to find a base locus that forces both the desired singularity type and a curve, guaranteed by Theorem 1.5 to generate the class group, whose strict transform has the desired intersection properties with the exceptional locus.

4.1. $\mathbf{A}_n$ *Singularities*

The $\mathbf{A}_n$ singularity for $n \geq 1$ is analytically isomorphic to that given by the equation $xy - z^{n+1}$ at the origin; its class group is $\mathbb{Z}/(n+1)\mathbb{Z}$, and the Dynkin diagram is

$$\underset{E_1}{\circ} \;\text{———}\; \underset{E_2}{\circ} \;\text{———}\; \cdots \;\text{———}\; \underset{E_n}{\circ}$$

Here e_1 is one generator, and $e_j = je_1$ in the local class group.

The following, which immediately implies Theorem 1.4 for $\mathbf{A}_n$ singularities, is Proposition 4.1 from [5]; there we used it to calculate the class groups of general surfaces containing base loci consisting of certain multiplicity structures on smooth curves.

PROPOSITION 4.1. *Let $Z \subset \mathbb{P}^3_{\mathbb{C}}$ be the subscheme with ideal $\mathcal{I}_Z = (x^2, xy, xz^q - y^{m-1}, y^m)$ for $m \geq 3$. Then the very general surface S containing Z has an $\mathbf{A}_{(m-1)q-1}$ singularity at $p = (0,0,0,1)$, and $\mathrm{Cl}(\mathcal{O}_{S,p}) \cong \mathbb{Z}/(m-1)\mathbb{Z}$ is generated by C.*

4.2. $\mathbf{D}_n$ *Singularities*

The $\mathbf{D}_n$ singularity for $n \geq 4$ is analytically isomorphic to that given by $x^2 + y^2 z + z^{n-1}$ at the origin. The class group of the completion is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if n is even and $\mathbb{Z}/4\mathbb{Z}$ if n odd, and the Dynkin diagram is

(1)

Here $e_j = je_1$ for $j < n-1$ as in the $\mathbf{A}_n$ case. If n is odd, then E_{n-1} and E_n are the two distinct generators of $\mathrm{Cl}\,\hat{A}$, and e_1 corresponds to the element 2; if n is even, then e_1, e_{n-1}, and e_n are the three distinct nonzero elements.

We will need the following lemmas.

LEMMA 4.2 [5, Lemma 2.2]. *Let $(R, \mathfrak{m})$ be a complete local domain, and let n be a positive integer that is a unit in R. If $a_0 \in R$ is a unit and $u \equiv a_0^n \bmod \mathfrak{m}^k$ for some fixed $k > 0$, then there exists $a \in R$ such that $a^n = u$ and $a \equiv a_0 \bmod \mathfrak{m}^k$.*

LEMMA 4.3. *Let $R = k[[y, z]]$ with maximal ideal $\mathfrak{m} \subset R$. For integers a, s, t with $s > a > 1$, $t > a + 1$, and $b \in R$ a unit, there is a change of coordinates Y, Z such that*

$$f = y^a z + z^s - by^t = Y^a Z + Z^s.$$

Furthermore X, Y may be chosen so that $y \equiv Y \bmod \mathfrak{m}^2$ and $z \equiv Z \bmod \mathfrak{m}^2$.

Proof. We produce coordinate changes y_i, z_i such that $y_{i+1} \equiv y_i \bmod \mathfrak{m}^{i+1}$, $z_{i+1} \equiv z_i \bmod \mathfrak{m}^i$, and $f = y_i^a z_i + z_i^s - b_i y_i^{k_i}$ with $k_i \geq i + a + 1$ and b_i a unit. By hypothesis, $y_1 = y$ and $z_1 = z$ give the base step $i = 1$.

For the induction step, let $z_{i+1} = z_i - b_i y^{k_i - a} \equiv z_i \bmod \mathfrak{m}^{i+1}$, so that

$$\begin{aligned} f &= y_i^a z_{i+1} + (z_{i+1}^s + s b_i z_{i+1}^{s-1} y_i^{k_i - a} + \cdots + s b_i^{s-1} z_{i+1} y_i^{(s-1)(k_i-a)} + y_i^{s(k_i-a)}) \\ &= y_i^a z_{i+1} \underbrace{[1 + s b_i z_{i+1}^{s-2} y_i^{k_i-2a} + \cdots + s b_i^{s-1} y_i^{(s-1)(k_i-2a)}]}_{v_i} + z_{i+1}^s + b_i^s y_i^{s(k_i-a)}, \end{aligned}$$

where v_i is a unit with lowest-degree term after the leading 1 is of degree $s - 2 + k_i - 2a \geq i$. By Lemma 4.2, v_i has an ath root w_i that is congruent to $1 \bmod \mathfrak{m}^i$. Then $y_{i+1} = w_i y_i \equiv y_i \bmod \mathfrak{m}^{i+1}$, so that $f = y_{i+1}^a z_{i+1} + z_{i+1}^s - b_{i+1} y_{i+1}^{k_{i+1}}$, where $b_{i+1} = -b_i^s w_i^{-s(k_i-a)}$ is a unit, and $k_{i+1} = s(k_i - a) \geq s(i+1) \geq (a+1)(i+1) \geq a + i + 2$, completing the induction. □

REMARK 4.4. For the convenience of the reader, we recall the well-known minimal resolution for $\mathbf{D}_n$ singularities for n even [11, §14 and §17].

(a) Blowing up a $\mathbf{D}_4$ results in a single rational exceptional curve E_2 containing three $\mathbf{A}_1$ singularities, which resolve by blowing up to obtain three rational exceptional curves E_1, E_3, E_4 that cannot be distinguished in the Dynkin diagram (1) and therefore by [10, Theorem 1.1] are equivalent up to automorphism of the completed local ring.

(b) For $n \geq 6$, we start with the standard form $x^2+y^2z+z^{n-1}$, blow up following our usual conventions, and look on the $Z=1$ patch, where the local equation becomes $X^2+Y^2z+z^{n-3}$. This has a $\mathbf{D}_{n-2}$ singularity at the origin and no other when $n \geq 6$. On the patch $Y=1$ we get $X^2+yZ+y^{n-3}Z^{n-1}$, which is an $\mathbf{A}_1$ singularity by Lemma 3.1 applied to the quadric cone X^2+yZ, and there are no further singularities. Resolving the $\mathbf{A}_1$ and the $\mathbf{D}_{n-2}$ and continuing by induction lead to the Dynkin diagram (1). Note that for $n \geq 5$, the exceptional curve in this blow-up becomes E_1 according to our conventions in the full resolution.

PROPOSITION 4.5. *Theorem 1.4 holds for* $\mathbf{D}_n$ *singularities.*

Proof. The class group of a complete $\mathbf{D}_n$ singularity is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; we first produce the subgroups of order two as class groups of local rings of surfaces. The scheme Z defined by $I_Z=(x^2, y^2z, z^{n-1}, xy^n)$ consists of the line $L\colon\ x=z=0$ and an embedded point at the origin p; hence, the local ring at p of the very general surface S containing Z has class group $\mathrm{Cl}(\mathcal{O}_{S,p})$ generated by L by Theorem 1.5; $L \neq 0$ in this group because L is not Cartier at p since L is smooth at p whereas S is not. The local equation of S has the form $ax^2+by^2z+z^{n-1}+cxy^n$ with units $a, b, c \in \mathcal{O}_{\mathbb{P}^3,p}$. After we use Lemma 4.2 to take square roots of a, b in $\widehat{\mathcal{O}}_{\mathbb{P}^3,p}$, the equation becomes

$$x^2+y^2z+z^{n-1}+cxy^n = \Big(\underbrace{x+\frac{c}{2}y^n}_{x_1}\Big)^2+y^2z+z^{n-1}-\frac{c^2}{4}y^{2n},$$

and applying the coordinate change of Lemma 4.3 exhibits the $\mathbf{D}_n$ singularity.

To determine the class of L, we blow up S at p. The local equation of $\tilde{S}$ on the patch $Y=1$ is $aX^2+byZ+y^{n-3}Z^{n-1}+cXy^{n-1}$, and the strict transform $\tilde{L}$ has ideal (X, Z); hence, $\tilde{L}$ meets the exceptional curve at the new origin of this patch, which is the $\mathbf{A}_1$ singularity whose blow-up will produce the exceptional divisor E_1. Resolving this $\mathbf{A}_1$ shows that $\tilde{L}$ meets E_1 but not E_2, so L gives the class u_1 defined previously. In particular, $2L=0$ and $\mathrm{Cl}(\mathcal{O}_{S,p}) \cong \mathbb{Z}/2\mathbb{Z}$ (see also [7, II, Ex. 6.5.2]).

When n is odd, $\langle u_1\rangle$ is the only subgroup of order 2 in $\mathrm{Cl}(\mathcal{O}_{S,p})$, and we are finished. For $n=4$, the exceptional curves E_1, E_3, E_4 are indistinguishable (see Remark 4.4), so there is essentially only one subgroup of order 2 to consider. When n is even and $n \geq 6$, there are three such subgroups $\langle u_1\rangle$, $\langle u_{n-1}\rangle$, and $\langle u_n\rangle$, the last two being distinguishable from the first, but not from one another, since they correspond to the two exceptional curves in the final blow-up. Since the

Table 1 Fundamental cycle for rational double points

Type	ξ_0
$\mathbf{A}_n$	$E_1 + E_2 + \cdots + E_n$
$\mathbf{D}_n$	$E_1 + 2E_2 + 2E_3 + \cdots + 2E_{n-2} + E_{n-1} + E_n$
$\mathbf{E}_6$	$E_1 + 2E_2 + 3E_3 + 2E_4 + 2E_5 + E_6$
$\mathbf{E}_7$	$E_1 + 2E_2 + 3E_3 + 4E_4 + 2E_5 + 3E_6 + 2E_7$
$\mathbf{E}_8$	$2E_1 + 3E_2 + 4E_3 + 5E_4 + 6E_5 + 3E_6 + 4E_7 + 2E_8$

previous paragraph achieves the subgroup generated by u_1, we therefore construct a $\mathbf{D}_n$ singularity on S at p, n even, such that $\mathrm{Cl}\,\mathcal{O}_{S,p}$ is generated by u_{n-1} or u_n.

Write $n = 2r$ and define Z by the ideal $I_Z = (x^2, y^2z - z^{2r-1}, y^5 - z^{5r-5})$. The last two generators show that $y \neq 0 \iff z \neq 0$ along Z, when I_Z is locally equal to $(x^2, y - z^{r-1})$; thus, Z consists of a double structure on the smooth curve C with ideal $(x, y - z^{r-1})$ and an embedded point at the origin p. Therefore, if S is a very general surface containing Z, then as before C generates $\mathrm{Cl}(\mathcal{O}_{S,p})$ and is nonzero. The local equation for S has the form $ax^2 + y^2z - z^{2r-1} + by^5 - bz^{5r-5}$ for units $a, b \in \mathcal{O}_{\mathbb{P}^3,p}$. Passing to the completion and adjusting the variables by appropriate roots of units, the equation of S becomes $x^2 + y^2z + z^{2r-1} + y^5$. Changing variables via Lemma 4.3, the equation becomes $x^2 + Y^2Z + Z^{2r-1}$, and we see the $\mathbf{D}_{2r}$-singularity.

To determine the class of C in $\mathrm{Cl}(\mathcal{O}_{S,p})$, we return to the original form of the equation for S and blow up p. Following our usual conventions, look on the patch $Z = 1$, where the blow-up $\tilde{S}$ has the equation $aX^2 + Y^2z - z^{2r-3} + bY^5z^3 - bz^{5r-7}$. Note that the strict transform $\tilde{C}$ of C has the ideal $(X, Y - z^{r-2})$ and so meets the exceptional curve $X = z = 0$ transversely. Rather than go through coordinate changes to identify the singularity of this new surface at the origin, we use Remark 4.4: There are two singular points in the blow-up, namely a $\mathbf{D}_{n-2}$ and an $\mathbf{A}_1$, and since at the origin the leading term is a square, this must be the $\mathbf{D}_{2r-2}$. In the minimal resolution of singularities the strict transform $\tilde{C} \subset \tilde{S}$ maps to the smooth curve C, which therefore has multiplicity one at $p \in S$, but by Remark 4.7 this multiplicity is given by $\xi_0.\tilde{C}$, where ξ_0 is the fundamental cycle for the $\mathbf{D}_n$ singularity (see Section 4.4 for generalities): glancing at the coefficients of the exceptional divisors E_i in ξ_0 in Table 1, we see that $\tilde{C}$ must meet either E_1, E_{n-1}, or E_n with multiplicity one. By the transversality noted before, however, it does not meet E_1 in the next blowup. We conclude that it meets E_{n-1} or E_n, and so we have produced the desired subgroup.

Finally, we need an example for which $\mathrm{Cl}(\mathcal{O}_{S,p}) = \mathrm{Cl}(\widehat{\mathcal{O}}_{S,p})$. To this end, consider the surface S defined by $x^2 + y^2z - z^{n-1}$, which has a $\mathbf{D}_n$ singularity at the origin (change coordinates $z \mapsto z' = \exp(\frac{\pi i}{n-1})z$, $y \mapsto y' = \exp(-\frac{\pi i}{2(n-1)}y)$). As before, the curve with ideal (x, z) corresponds to u_1, so it suffices in all cases to find a curve on this surface that gives the element u_n (or u_{n-1}).

For even values of n, the curve $(x, y - z^{(n-2)/2})$ gives the other generator by an argument analogous to, but much easier than, the one for n even in the previous case considered.

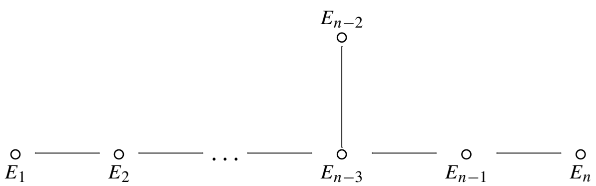
For odd values of n, write $n = 2r + 1$, use the form $x^2 + y^2z - z^{2r}$, and let C be the curve having the ideal $(x - z^r, y)$. Looking on the patch $Z = 1$ of the blow-up gives the surface $X^2 + Y^2z - 2z^{2r-2}$ and the curve $(X - z^{r-1}, Y)$, so by induction it suffices to prove that, for $n = 5$, that is, $r = 2$, the curve C having the ideal $(x - z^2, y)$ gives one of the classes u_4, u_5 in $\mathrm{Cl}(\mathcal{O}_{S,p})$, where S has the equation $x^2 + y^2z - 2z^4$, and p is the origin. On the patch $Z = 1$ of the blow-up $\tilde{S}$, the equation is $X^2 + Y^2z - Z^2$, which has the $\mathbf{A}_3$ at the origin, and $\tilde{C}$ has the ideal $(X - z, Y)$. A further blow-up, again on the patch $Z = 1$, gives the equation $X^2 + Y^2z - 1$ for $\tilde{S}$ and the ideal $(X - 1, Y)$ for $\tilde{C}$. The curve C meets the exceptional locus only at the smooth point with coordinates $(1, 0, 0)$ relative to this patch, so in the full resolution of singularities, $\tilde{C}$ meets one of the two exceptional curves arising from the blow-up of the $\mathbf{A}_3$, which correspond to E_4 and E_5 in the Dynkin diagram. □

4.3. $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ *Singularities*

The standard equations for the analytic isomorphism types and class groups for these singularities are as follows:

Type	Equation	$\mathrm{Cl}\,\hat{A}$
$\mathbf{E}_6$	$x^2 + y^3 + z^4$	$\mathbb{Z}/3\mathbb{Z}$
$\mathbf{E}_7$	$x^2 + y^3 + yz^3$	$\mathbb{Z}/2\mathbb{Z}$
$\mathbf{E}_8$	$x^2 + y^3 + z^5$	0

and the $\mathbf{E}_n$ Dynkin diagram looks like



For our purposes, as it happens, the way that the exceptional curves correspond to class group elements is not important.

PROPOSITION 4.6. *Theorem 1.4 holds for* $\mathbf{E}_6$, $\mathbf{E}_7$, *and* $\mathbf{E}_8$ *singularities.*

Proof. The statement holds automatically for $\mathbf{E}_8$ singularities because the class group is trivial. For the $\mathbf{E}_6$ singularity, consider the affine quartic surface S given by $x^2 + y^3 + x^4 = 0$ for which the $\mathbf{E}_6$ singularity at the origin p is clear. The smooth curve C with ideal $(x - iz^2, y)$ lies on S and passes through p. If C

restricts to 0 in $\mathrm{Cl}\,\mathcal{O}_{S,p}$, then C is Cartier on S at p, impossible because C is smooth at p whereas S is not: thus, C defines a nonzero element in $\mathrm{Cl}\,\mathcal{O}_{S,p}$, which must generate all of $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/3\mathbb{Z}$, the only nontrivial subgroup. For the $\mathbf{E}_7$ singularity, we use the affine quartic S given by $x^2 + y^3 + yz^3 = 0$ at the origin p when $\mathrm{Cl}\,\widehat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the smooth z-axis C. □

4.4. The Fundamental Cycle and Smooth Representatives for Local Class Groups

In this section we use the fundamental cycle to prove that at any rational double point, each element of the class group of the local ring can be lifted to a locally smooth curve.

If $p \in S$ is a rational double point with minimal desingularization $X \to S$, then the *fundamental cycle* ξ_0 is the unique minimal effective nonzero exceptionally supported divisor on X having nonpositive intersection with each exceptional curve [1, pp. 131–132]. With the conventions used in describing the minimal desingularizations, the fundamental cycles for each singularity type are given in Table 1.

REMARK 4.7. The fundamental cycle has the property that for a nonexceptional smooth curve D on X, $D.\xi_0$ is equal to the multiplicity of $\pi(D)$ on S [2, Prop. 5.3]. For a given rational singularity, the exceptional divisors E_i with coefficient 1 in the fundamental cycle ξ_0 are said to be *permissible* [10].

This lemma generalizes and makes a small correction to [2, Prop. 5.5].

LEMMA 4.8. *Let $X \to S$ be a minimal desingularization of a rational double point $p \in S$ with exceptional divisors $E_1, \dots, E_n$ and fix a divisor F on X such that $F.E_i \geq 0$ for all i. Then there exists an effectively supported divisor G on X such that $(F+G).E_i = 0$ for all but at most one i; if such i exists, then $(F+G).E_i = 1$, and E_i is permissible.*

Proof. Letting $s_i = F.E_i$, we are done if $\sum s_i \leq 1$. Assuming that $\sum s_i > 1$, we show how to add sums of the E_i to achieve the statement in each case.

For an $\mathbf{A}_n$ singularity, let j (resp. k) be the least (resp. greatest) index i with $s_i > 0$. If $j < k$, then adding $E_j + E_{j+1} + \cdots + E_k$ to F decreases s_j, s_k by 1, increases s_{j-1}, s_{k+1} by 1 (if these exist), and has no effect on the remaining s_i. Repeat until $j = 1$ or $k = n$ at which the point $\sum s_i$ decreases. If $1 < j = k < n$ and $s_j \geq 2$, then adding E_j reduces to the previous case. Eventually $\sum s_i = 1$, and we are done, and the final statement is clear because the coefficient of each E_i in ξ_0 is 1.

For a $\mathbf{D}_n$ singularity, note that adding E_{n-1} (resp. E_n) decreases s_{n-1} (resp. s_n) by 2 and increases E_{n-2} by 1, so we may assume $0 \leq s_{n-1}, s_n \leq 1$. We may assume $s_i = 0$ for $1 < i < n-1$. If this is not so, let k be the largest i in this range with $s_k > 0$. Adding $s_k(E_k + 2E_{k+1} + \cdots + 2E_{n-2} + E_{n-1} + E_n)$ increases s_{k-1} by s_k, decreases s_k to 0, and leaves the remaining s_i fixed; thus, k decreases, and we may continue until $k = 1$. Adding $2E_1 + \cdots + 2E_{n-2} + E_{n-1} + E_n$ decreases s_1 by 2 and fixes the remaining s_i, so we may assume that $0 \leq s_1, s_{n-1}, s_n \leq 1$ (the rest are zero). We are done if at most one of these is nonzero; otherwise, adding $s_1E_1 + s_2E_2 + E_3 + \cdots + E_{n-2} + s_{n-1}E_{n-1} + s_nE_n$ switches s_1, s_{n-1}, s_n from 0 to 1 or 1 to 0 and fixes the rest, finishing the proof.

For an $\mathbf{E}_6$, note that adding ξ_0 reduces s_4 while fixing the rest, so we may assume that $s_4 = 0$ as needed. Now applying the $\mathbf{A}_5$ strategy to the remaining chain reduces to the case that at most one of $\{s_i\}_{i\neq 4}$ is 1 while the rest of these are zero; if $s_1 = 1$ or $s_6 = 1$, we are done; if $s_2 = 1$, adding $E_1 + 2E_2 + 2E_3 + E_4 + E_5$ sets $s_2 = 0$ and $s_6 = 1$ (the case $s_5 = 1$ is similar); if $s_3 = 1$, adding $E_1 + 2E_2 + 3E_3 + E_4 + 2E_5 + E_6$ decreases s_3 by 1 and increases s_4 by 1.

For an $\mathbf{E}_7$ singularity, applying the $\mathbf{D}_6$ strategy to $E_1, \ldots, E_6$ reduces to the case $s_2 = s_3 = s_4 = 0$ and at most one of s_1, s_5, s_6 is 1; s_7 will likely be increased in the process, but adding ξ_0 decreases s_7 by one while fixing the rest, so we may assume that $s_7 = 0$. If $s_1 = 1$, we are done; if $s_5 = 1$, adding $E_2 + 2E_3 + 3E_4 + 2E_5 + 2E_6 + E_7$ sets s_5 to 0 while increasing s_1 to 1; if $s_6 = 1$, adding $2E_1 + 4E_2 + 6E_3 + 8E_4 + 4E_5 + 6E_3 + 3E_7$ sets $s_6 = 0$ while fixing the rest.

Finally, for an $\mathbf{E}_8$, the exceptional curves $E_2, \ldots, E_8$ form an $\mathbf{E}_7$ singularity; as before, we can add exceptional curves to reach a point where $s_2 = 0$ or 1, $s_3 = s_4 = \cdots = s_8 = 0$, and s_1 is still positive. Adding ξ_0 has the effect of decreasing s_1 by 1 and fixing the other s_j, so repeated addition of ξ_0 brings us to the situation where all the s_j are 0, in which case we are finished, or $s_2 = 1$ and all other s_j are 0. In this latter case add $3E_1 + 6E_2 + 8E_3 + 10E_4 + 12E_5 + 6E_6 + 8E_7 + 4E_8$, which has the effect of decreasing s_1 to 0 and leaving all the other s_j constant. □

PROPOSITION 4.9. *Let p be a rational double point on a projective surface S, and let $x \in \mathrm{Cl}(\mathcal{O}_{S,p})$. Then there is an effective divisor $D \in \mathrm{Cl}\, S$ smooth at p restricting to x.*

Proof. The class x lifts to a global Weil divisor $C \in \mathrm{Cl}\, S$ because any height one prime in the local ring lifts to a height one prime in the ring corresponding to an open affine. We may assume that C is effective after adding a high multiple of $\mathcal{O}_S(1)$ since $\mathcal{O}_S(1)$ has trivial restriction to $\mathrm{Cl}(\mathcal{O}_{S,p})$. Let $\pi : X \to S$ be the minimal resolution of singularities with irreducible exceptional curves E_i, and let $\tilde{C}$ be the strict transform of C on X. By Lemma 4.8 there exists an effective exceptionally supported divisor G on X such that $(\tilde{C} + G).E_i = 0$ for all but at most one of the E_i, and if such an E_i exists, then $(\tilde{C} + G).E_i = 1$, and E_i is admissible.

Setting $A = \tilde{C} + G$, we claim that no exceptional curve E_i is a fixed component of the linear system $|H^0(\mathcal{O}_X(A))|$. Indeed, the composite map $\mathcal{O}_X \to \mathcal{O}_X(A)|_{E_i}$ in the upper right corner of the diagram

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_X(-E_i) & \to & \mathcal{O}_X & \to & \mathcal{O}_{E_i} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_X(A - E_i) & \to & \mathcal{O}_X(A) & \to & \mathcal{O}_X(A)|_{E_i} & \to 0
\end{array}$$

is nonzero on global sections, the first map being surjective and the second injective. Therefore, the map $H^0(\mathcal{O}_X(A)) \to H^0(\mathcal{O}_X(A)|_{E_i})$ is also nonzero, which implies that $H^0(\mathcal{O}_X(A - E_i)) \to H^0(\mathcal{O}_X(A))$ is not surjective, verifying the claim.

Now consider the general member $\tilde{D} \in |H^0(\mathcal{O}_X(A))|$. If $A.E_i = 0$, then $\tilde{D}$ misses E_i because E_i is not a fixed component. Therefore, if $A.E_i = 0$ for all i, then $D = \pi(\tilde{D})$ misses p, and we are done. Otherwise, there is at most one E_i for which $A.E_i = 1$ and $|H^0(\mathcal{O}_X(A))|$ may have *one* fixed point $P_0 \in E_i$. By Bertini's theorem the only singular point for the general member of the linear system on the exceptional locus can be P_0, but even in this case $\tilde{D}$ is smooth at P_0 because $\tilde{D}.E_i = 1$. Therefore, $\tilde{D}$ is smooth along $E = \bigcup E_i$ and meets ξ_0 with multiplicity one, so $\operatorname{mult}_p(\pi(\tilde{D})) = \tilde{D}.\xi_0 = 1$ [2, Prop. 5.4], and $D = \pi(\tilde{D})$ is smooth at p. Moreover, D has the same class $x \in \operatorname{Cl}(\mathcal{O}_{S,p})$ as C because $\tilde{D}$ and $\tilde{C}$ differ by a sum of exceptional divisors. □

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