

On Undulation Invariants of Plane Curves

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ABSTRACT. A classical problem introduced by A. Cayley and G. Salmon in 1852 is to determine if a given plane curve of degree $r > 3$ has undulation points, the points where the tangent line meets the curve with multiplicity four. They proved that there exists an invariant of degree $6(r - 3)(3r - 2)$ that vanishes if and only if the curve has undulation points. In this paper we give explicit formulas for this invariant in the case of quartics ($r = 4$) and quintics ($r = 5$), expressing it as the determinant of a matrix with polynomial entries, of sizes 21×21 and 36×36 , respectively.

1. Introduction

This paper is devoted to a problem in classical invariant theory of plane curves, due to A. Cayley and G. Salmon (see [1], p. 362). Consider, on the projective plane \mathbb{CP}^2 with homogeneous coordinates $x_1 : x_2 : x_3$, a plane curve

$$P(x_1, x_2, x_3) = \sum_{i+j+k=r} C_{ijk} x_1^i x_2^j x_3^k = 0,$$

where P is a homogeneous irreducible degree r polynomial. By the Bezout theorem, any line in \mathbb{CP}^2 crosses this curve in exactly r points, if counted with multiplicities. The types of possible intersections thus can be put into correspondence with partitions $r = m_1 + m_2 + \dots$, where the parts m_i of the partition are the multiplicities of intersection points. An illustration of this for the case of quartics is given on Figure 1.

If a line is generic, then it intersects the curve in r distinct points with all multiplicities 1, that is, it corresponds to the partition $(1, 1, \dots, 1)$. The simplest nongeneric intersection occurs for the *tangent line* to a curve: then one of the intersection points has multiplicity 2 (the point of tangency), whereas all the other intersection points have multiplicity 1. This type of intersection corresponds to the partition $(2, 1, 1, \dots)$. The next-to-simplest types of intersection are, respectively, $(3, 1, \dots, 1)$ and $(2, 2, 1, \dots, 1)$; the former situation is called a *line of inflection*, whereas the latter is called a *bitangent* since in this case the line is simultaneously tangent to a curve in two distinct points. One can continue further by considering lines of type $(4, 1, \dots, 1)$, $(3, 2, 1, \dots, 1)$, and so on. These generally do not have given names, with one notable exception: a line of type $(4, 1, \dots, 1)$ is called a *line of undulation*, and the corresponding point of intersection is called an *undulation*

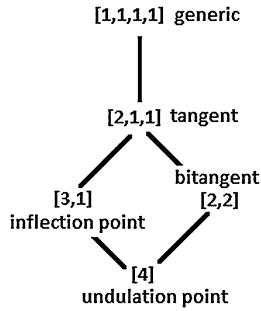


Figure 1 Types of intersection of a plane quartic and a line

point. Note that conventionally intersection with at least one part of multiplicity higher than 4 is also called so, motivating the following Definition 1.1.

DEFINITION 1.1. A point is called an undulation point of a plane curve $P(x_1, x_2, x_3) = 0$ if a tangent line at that point meets the curve with multiplicity four or higher.

It is classically known (and it is easy to estimate from degree counting) that a generic plane curve has no undulation points. For a plane curve to possess undulation points, it should be nongeneric, that is, P should satisfy some algebraic equation(s). In the classical book [1] Salmon, building on the work of Cayley, studied this question and proved the following existence theorem.

THEOREM 1.2 ([1], Chapter IX, p. 362). *There exists a unique up to rescaling function I , which is a homogeneous polynomial in $(r + 1)(r + 2)/2$ coefficients C of degree $6(r - 3)(3r - 2)$, such that*

$$I(C) = 0$$

is a necessary and sufficient condition for the curve $P(x_1, x_2, x_3) = 0$ to have undulation points.

Unfortunately, this theorem only justifies the existence of such an invariant. In practice, it is useful to have not only that but also an explicit formula. In this paper we address the problem of finding an explicit polynomial formula for the undulation invariant I . Such a formula was not given in the literature devoted to undulation [1; 2; 3; 4; 5], and the aim of this paper is to fill this gap. We will show that $I(C)$ is given by a determinant with polynomial entries, of size 21×21 for $r = 4$ and of size 36×36 for $r = 5$.

We believe that one of the reasons that the explicit formula that we present here was not found before is the complexity of the invariants: even in the simplest nontrivial case of quartic curves, for $r = 4$, the invariant I is a homogeneous polynomial of degree 60 in the 15 coefficients of the quartic and hence has a really impressive length (number of monomials). This is a typical phenomenon in invari-

ant theory (see, e.g., Appendix to the book version of [6] for another example). It is, however, not terrifying at all: these enormous invariant polynomials often possess a lot of nice properties and can be expressed by simple elegant formulae. In the previous example, as we find in this paper, the invariant of degree 60 turns out to be a determinant of a relatively simple 21×21 matrix. To find and explore such formulae, it is often beneficial to use modern computers and software (e.g., MAPLE; see Section 6 for more details on the technical tools we used).

2. The Undulation Ideal

The undulation problem is so far defined in the geometric terms of tangent lines and multiplicities. To proceed to solution of this problem, we will reformulate it in terms of properties of a certain polynomial ideal called the undulation ideal. Analyzing this ideal with a combination of relatively simple tools—linear algebra, representation theory of $SL(3)$, and computer algebra methods—we will be able to obtain the desired result, the determinantal formula for I .

DEFINITION 2.1. The *undulation ideal* \mathcal{I} is the set of all polynomials in the variables C, v_1, v_2, v_3 that vanish whenever $v_1x_1 + v_2x_2 + v_3x_3 = 0$ is the undulation line of the curve $P(x_1, x_2, x_3) = 0$:

$$\mathcal{I} = \{f \in \mathbb{C}[C, v_1, v_2, v_3] \mid f(C, v_1, v_2, v_3) = 0 \text{ if } v_1x_1 + v_2x_2 + v_3x_3 = 0 \text{ is an undulation line for the curve } P(x_1, x_2, x_3) = 0\}.$$

The motivation to consider such an ideal essentially comes from the Cayley–Salmon theorem, Theorem 1.2. In other words, this theorem can be stated as a fact that the simpler ideal

$\mathcal{I}' = \{f \in \mathbb{C}[C] \mid f(C) = 0 \text{ if } P(x_1, x_2, x_3) = 0 \text{ has at least one undulation line}\}$ is generated by a single element, the undulation invariant:

$$\mathcal{I}' = \langle I(C) \rangle.$$

Following the general wisdom “to understand something, deform/generalize it”, we propose to extend \mathcal{I}' to a bigger ideal \mathcal{I} , in a hope that this could reveal an additional structure and thus shed some light on the nature of the element $I(C)$. As we will see, this will happen to be the case.

The ideal \mathcal{I} admits three useful gradings. The first two are the obvious gradings w.r.t. the total degree in all the coefficients C , and the total degree in all the coefficients v :

$$\begin{aligned} \deg_C(v_i) &= 0, & \deg_C(C_{ijk}) &= 1; \\ \deg_v(v_i) &= 1, & \deg_v(C_{ijk}) &= 0. \end{aligned}$$

The last, third, grading is more refined and is determined by

$$\begin{aligned} \overline{\deg}(v_1) &= (1, 0, 0), & \overline{\deg}(v_2) &= (0, 1, 0), \\ \overline{\deg}(v_3) &= (0, 0, 1), & \overline{\deg}(C_{ijk}) &= (i, j, k). \end{aligned}$$

With respect to these gradings, the ideal \mathcal{I} is decomposed into a direct sum of graded components.

DEFINITION 2.2. Let $\mathcal{I}_{n,m}$ be the graded components of \mathcal{I} w.r.t. the first two gradings, and $\mathcal{I}_{n,m_1,m_2,m_3}$ be the graded components w.r.t. all the three gradings:

$$\begin{aligned} \mathcal{I}_{n,m} &= \{f \in \mathcal{I} \mid \deg_C(f) = n, \deg_v(f) = m\}, \\ \mathcal{I}_{n,m_1,m_2,m_3} &= \{f \in \mathcal{I} \mid \deg_C(f) = n, \overline{\deg}(f) = (m_1, m_2, m_3)\}. \end{aligned}$$

Note that

$$\mathcal{I}_{n,m} = \bigoplus_{m_1+m_2+m_3=rn+m} \mathcal{I}_{n,m_1,m_2,m_3}.$$

Note also, that the Cayley–Salmon ideal is nothing but $\mathcal{I}' = \bigoplus_n \mathcal{I}_{n,0}$.

3. The Structure of the Ideal

To understand the structure of the graded components $\mathcal{I}_{n,m}$, various methods can be used. It is interesting that for the purpose of this paper, it is enough to use the most basic and straightforward approach possible, direct calculation of the spaces $\mathcal{I}_{n,m}$ using only linear algebra.

For this, we will need the following simple lemma.

LEMMA 3.1. *A line*

$$v(x_1, x_2, x_3) = v_1x_1 + v_2x_2 + v_3x_3 = 0 \tag{1}$$

is an undulation line of a plane curve $P(x_1, x_2, x_3) = 0$ iff P can be decomposed in the form

$$P(x_1, x_2, x_3) = u(x_1, x_2, x_3)^4 h(x_1, x_2, x_3) + v(x_1, x_2, x_3)w(x_1, x_2, x_3), \tag{2}$$

where $u(x_1, x_2, x_3) = u_1x_1 + u_2x_2 + u_3x_3$ is some linear polynomial, and $h(x_1, x_2, x_3)$ and $w(x_1, x_2, x_3)$ are some homogeneous polynomials of degrees $r - 4$ and $r - 1$, respectively.

Proof. (\Leftarrow) This is true by the definition of multiplicity.

(\Rightarrow) Suppose that $v = 0$ is tangent to the curve at a point X with intersection multiplicity at least 4. Assuming without loss of generality that $(X_1 : X_2 : X_3) = (0 : 0 : 1)$ and $v(x_1, x_2, x_3) = x_1$, the intersection multiplicity equals multiplicity of the zero root of $P(0, z, 1)$. This implies $P(0, z, 1) = z^4 w(z)$ for some polynomial $w(z)$. In turn, this implies $P(0, x_2, x_3) = x_2^4 w(x_2, x_3)$ for some homogeneous polynomial $w(x_2, x_3)$ and, ultimately, $P(x_1, x_2, x_3) = x_2^4 w(x_2, x_3) + x_1 h(x_1, x_2, x_3)$ for some homogeneous polynomial $h(x_1, x_2, x_3)$. \square

Lemma 3.1 has an important corollary.

COROLLARY 3.2. *Each $\mathcal{I}_{n,m}$ can be computed as a solution to a finite linear system of equations.*

Proof. Denote

$$s_{ijk}(u, h, v, w) = \text{coefficient in front of } x_1^i x_2^j x_3^k \text{ in } (u^4 h + vw)$$

with u, h, v, w as in Lemma 3.1. Then, a homogeneous polynomial $f \in \mathbb{C}[C, v_1, v_2, v_3]$ of degrees $\deg_C(f) = n, \deg_v(f) = m$ belongs to $\mathcal{I}_{n,m}$ iff the following system of equations is satisfied:

$$f(s(u, h, v, w), v_1, v_2, v_3) = 0 \quad \forall u, v, h, w. \tag{3}$$

This is a system of finitely many linear equations, where the coefficients of the polynomial f are treated as indeterminates. For given pair of natural numbers n, m , there are only finitely many these coefficients. Therefore, for any given n, m , one can (at least in principle) write and explicitly solve the corresponding linear system, obtaining $\mathcal{I}_{n,m}$ as its solution space. \square

Despite the size and complexity of the above linear systems grows quite fast with n, m , we will see below that this straightforward approach suffices to investigate the simplest properties of the ideal \mathcal{I} . In particular, in the next section we will use this approach to find several lowest $\mathcal{I}_{n,m}$ for $r = 4$ and show that the elements of these linear spaces can be naturally put together to form a 21×21 matrix, the determinant of which is the Cayley–Salmon invariant. This is the main new result of this paper, which calls for further research in the nearby directions. Then, in the next section, we will do the same for $r = 5$ and obtain similar results, thus giving evidence that the $r = 4$ result is not an accident but rather the first step toward generalizations.

4. Determinantal Formula for $r = 4$

Using the approach explained in the previous section, we obtain the following.

THEOREM 4.1. *For plane quartics ($r = 4$), the dimensions $\dim \mathcal{I}_{n,m}$ of a few low-graded components of the undulation ideal are given by the following numbers:*

$n \backslash m$	0	1	2	3	4	5	6	7	...
0	0	0	0	0	0	0	0	0	...
1	0	0	0	0	0	0	0	0	...
2	0	0	0	0	1	3	21	45	...
3	0	0	0	0	15	63	325
...

Proof. Direct calculation via Corollary 3.2. \square

The spaces $\mathcal{I}_{2,5}$ and $\mathcal{I}_{3,5}$ are spanned, as linear spaces, by 3 and 63 polynomials, respectively:

$$\mathcal{I}_{2,5} = \left\{ \sum_{i=1}^3 c_i \alpha_i \mid c_1, \dots, c_3 \in \mathbb{C} \right\}, \quad \mathcal{I}_{3,5} = \left\{ \sum_{i=1}^{63} c_i \beta_i \mid c_1, \dots, c_{63} \in \mathbb{C} \right\}.$$

Since \mathcal{I} is an ideal, the product of any element of $\mathcal{I}_{2,5}$ and any element of \mathcal{C} belongs to $\mathcal{I}_{3,5}$. Computation shows that there are no relations between such products, that is, the subspace spanned by them has dimension 45. $\mathcal{I}_{3,5}$ can be then decomposed¹ as a sum of this 45-dimensional subspace and a complementary 18-dimensional subspace. Let $\beta_1, \dots, \beta_{18}$ be the basis elements of that 18-dimensional subspace. Together with the three basis elements of $\mathcal{I}_{2,5}$, they form a set of 21 linearly independent polynomials of degree 5 in v_1, v_2, v_3 . At the same time, the dimension of the space of homogeneous polynomials of degree 5 in three variables v_1, v_2, v_3 is exactly 21! This allows us to arrange these 3 + 18 polynomials into a 21×21 matrix with the following remarkable property.

THEOREM 4.2. *Let \mathcal{M} be the 21×21 matrix the rows of which are obtained by expanding the 21 polynomials $\alpha_1, \dots, \alpha_3; \beta_1, \dots, \beta_{18}$ in the 21 homogeneous monomials of degree 5 in v_1, v_2, v_3 . Then the determinant of this matrix is the Cayley–Salmon undulation invariant of plane quartics:*

$$I(C)_{r=4} = \det_{21 \times 21} \mathcal{M}. \quad (4)$$

Proof. By construction, if the curve $P(x_1, x_2, x_3) = 0$ possesses an undulation line $V_1 x_1 + V_2 x_2 + V_3 x_3 = 0$, then all the polynomials $\alpha_1, \dots, \alpha_3; \beta_1, \dots, \beta_{18}$ vanish at $v = V$. This implies that the 21-dimensional vector the components of which are the 21 monomials of degree 5 in V_1, V_2, V_3 belongs to the kernel of \mathcal{M} . Hence, \mathcal{M} is degenerate whenever the curve $P(x_1, x_2, x_3) = 0$ possesses an undulation line. Hence, its determinant is an element of the Cayley–Salmon undulation ideal:

$$\det_{21 \times 21} \mathcal{M} \in \mathcal{I}' = \langle I(C) \rangle.$$

By the Cayley–Salmon theorem this ideal is generated by a unique element, the Cayley–Salmon undulation invariant $I(C)$, and therefore $\det \mathcal{M}$ has to be proportional to $I(C)$:

$$\det_{21 \times 21} \mathcal{M} = i(C) \cdot I(C).$$

Here $i(C)$ is some polynomial in C . It is easy to calculate its degree:

$$\deg_C i(C) = \deg_C \det_{21 \times 21} \mathcal{M} - \deg_C I(C) = 3 \cdot 2 + 18 \cdot 3 - 60 = 0.$$

¹This decomposition is, of course, not unique, but the results of this section are valid for any choice of it. This choice affects only the shape and simplicity of the resulting determinantal formula. A natural choice of the decomposition, which also leads to the simplest shape of the determinantal formula, comes from $SL(3)$ representation theory, discussed in the next section.

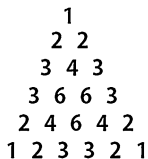


Figure 2 Structure of $\mathcal{I}_{3,5}$

So $i(C)$ does not depend on C , that is, is just a constant. The determinant of \mathcal{M} thus has the same degree as the Cayley–Salmon undulation invariant and coincides with it up to an overall constant. Computing this determinant for any curve without undulation lines, say, for $x_1^4 + x_2^4 + x_3^4 + (x_1 + x_2 + x_3)^4 = 0$, shows that this constant is not zero. Consequently, it can be always put to 1 (since $I(C)$ is itself defined up to rescaling). \square

5. Explicit Formulas

As usual, the symmetry group of the problem (in our case, $SL(3)$) acts on the space of solutions, decomposing it into irreducible representations. To find this decomposition, the easiest way is to consider, instead of the graded components $\mathcal{I}_{n,m}$, the more refined components $\mathcal{I}_{n,m_1,m_2,m_3}$. In complete analogy, their dimensions and spanning polynomials can be computed via Corollary 3.2. For $n = 3$ and $m = 5$, this gives the following triangle of integers, as shown in Fig. 2. Decomposing this triangle into the usual multiplicity diagrams of irreducible representations of $SL(3)$, we find

$$\begin{array}{c}
 1 \\
 2 \ 2 \\
 3 \ 4 \ 3 \\
 3 \ 6 \ 6 \ 3 \\
 2 \ 4 \ 6 \ 4 \ 2 \\
 1 \ 2 \ 3 \ 3 \ 2 \ 1 \\
 \hline
 63
 \end{array}
 =
 \begin{array}{c}
 1 \\
 1 \ 1 \\
 1 \ 1 \ 1 \\
 1 \ 1 \ 1 \ 1 \\
 1 \ 1 \ 1 \ 1 \ 1 \\
 \hline
 15 \cdot 3
 \end{array}
 \otimes
 \begin{array}{c}
 1 \\
 1 \ 1 \\
 \hline
 3
 \end{array}
 +
 \begin{array}{c}
 1 \ 1 \ 1 \\
 1 \ 2 \ 2 \ 1 \\
 1 \ 2 \ 1 \\
 1 \ 1 \\
 \hline
 18
 \end{array}
 +
 \begin{array}{c}
 1 \ 1 \\
 1 \\
 \hline
 3
 \end{array}$$

This indicates that the 63 basis polynomials of $\mathcal{I}_{3,5}$ consist of the uninteresting $15 \cdot 3 = 45$ polynomials obtained from $\mathcal{I}_{3,4}$, the 15 polynomials transforming in the irreducible representation $(3, 2)$, and the three polynomials transforming in the irreducible representation $(1, 1) = \bar{1}$. This completes the description of the decomposition of $\mathcal{I}_{3,5}$ into irreducibles. Finally, it is an easy exercise to check that $\mathcal{I}_{2,5}$ is itself an irreducible representation of $SL(3)$, namely, just the fundamental representation (1) . Together, these $21 = 15 + 3 + 3$ polynomials, transforming in representations $(3, 2) \oplus (1, 1) \oplus (1)$, form the matrix \mathcal{M} . To describe this matrix, it suffices to give explicit formulas for the highest weight vectors in the respective representations:

$$(1): \quad \alpha_1 = \mathcal{A}_3, \tag{5}$$

$$(1, 1): \beta_1 = 2\mathcal{B}_{32312} - 3\mathcal{B}_{12332} + \mathcal{B}_{31232} + \mathcal{B}_{22331} - \mathcal{B}_{22133}, \tag{6}$$

$$(3, 2): \beta_4 = 2\mathcal{B}_{23233} - \mathcal{B}_{23332} - \mathcal{B}_{32233}, \tag{7}$$

where, explicitly in components,

$$\mathcal{A}_i = \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} \epsilon^{d_1 d_2 d_3} P_{a_1 b_1 c_1 d_1} P_{a_2 b_2 c_2 d_2} v_{a_3} v_{b_3} v_{c_3} v_{d_3} v_i, \tag{8}$$

$$\begin{aligned} \mathcal{B}_{i_1 i_2 i_3 i_4 i_5} &= \epsilon^{a_1 a_2 a_3} \epsilon^{b_1 b_2 b_3} \epsilon^{c_1 c_2 c_3} \epsilon^{j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3} \\ &P_{a_1 b_1 i_2 i_3} P_{a_2 c_1 i_4 j_1} P_{b_2 c_2 i_5 k_1} v_{a_3} v_{b_3} v_{c_3} v_{j_3} v_{k_3}. \end{aligned} \tag{9}$$

Here $P_{ijkl} = \partial_{x_i} \partial_{x_j} \partial_{x_k} \partial_{x_l} P$ is the symmetric tensor of derivatives, ϵ^{ijk} stands for the completely antisymmetric tensor, and the summation over repeated indices is assumed. The polynomials $\alpha_1, \dots, \alpha_3, \beta_1, \dots, \beta_3$ and $\beta_4, \dots, \beta_{18}$ are obtained from these highest weight vectors by the action of $SL(3)$. One can straightforwardly check that they indeed vanish, provided that $P = u^4 h + v w$.

6. Determinantal Formula for $r = 5$

Having solved the undulation problem for $r = 4$, it is natural to go further and consider the case of quintics, $r = 5$. For plane quintics, the undulation invariant has degree $6(r - 3)(3r - 2) = 156$. Despite this is an impressively large degree, our calculations suggest that the same structure that we observed in the case of plane quartics exists for plane quintics as well.

PROPOSITION 6.1. *For plane quintics ($r = 5$), the dimensions $\dim \mathcal{I}_{n,m}$ of a few lowest graded components of the undulation ideal are given by the following numbers:*

$n \setminus m$	≤ 5	6	7	...
0	0	0	0	...
1	0	0	0	...
2	0	6	15	...
3	0	126	315	...
...
6	0	63,756	159,411	...

Proof. Direct calculation via Corollary 3.2. For $\dim \mathcal{I}_{2,7}$, this calculation is just as direct as for Theorem 5.1 and gives precisely 15. However, for $\dim \mathcal{I}_{6,7}$, it is significantly harder because sizes of linear systems (3), which define the generators of the ideal, become so large that solving them with MAPLE (and even finding their rank) is no longer possible.

We tackle this technical problem by utilizing the linear algebra package “linbox”. However, we are not using it directly, writing program to compute the ranks in pure C. We rather use SAGE, which is a great tool for mathematicians, written in Python and has bindings for “linbox”. To glue our MAPLE and SAGE codes

together (that is, to convert linear systems from polynomial form of notation, generated by the former to sparse-matrix form, understood by the latter), we use a simple Perl script. We believe, that such an approach of using several distinct computational and modeling tools, each of which is well suited for a particular task—rather than using all-in-one swiss-knives—and then gluing them together with the help of scripting languages (such as Perl, Python, and Lisp) will turn out to be very useful in attacking future problems of mathematics and physics that require computation. With this approach, we obtain the following dimensions of solution spaces for $\mathcal{I}_{6,m_1,m_2,m_3}$ (horizontal axis is m_1 , and vertical axis is m_2 , whereas $m_3 = 7 - m_2 - m_1$):

0	0	1	2	3	5	8	11	14	17	20	22	23	23	22	20	17	14	11	8	5	3	2	1
0	1	4	8	14	24	37	51	66	81	93	101	104	101	93	81	66	51	37	24	14	8	4	1
1	4	12	25	46	77	116	160	207	249	281	299	299	281	249	207	160	116	77	46	25	12	4	1
2	8	25	55	104	173	261	360	460	544	602	622	602	544	460	360	261	173	104	55	25	8	2	0
3	14	46	104	196	326	490	668	838	971	1044	1044	971	838	668	490	326	196	104	46	14	3	0	0
5	24	77	173	326	539	798	1068	1311	1476	1536	1476	1311	1068	798	539	326	173	77	24	5	0	0	0
8	37	116	261	490	798	1159	1518	1810	1972	1972	1810	1518	1159	798	490	261	116	37	8	0	0	0	0
11	51	160	360	668	1068	1518	1932	2229	2334	2229	1932	1518	1068	668	360	160	51	11	0	0	0	0	0
14	66	207	460	838	1311	1810	2229	2471	2471	2229	1810	1311	838	460	207	66	14	0	0	0	0	0	0
17	81	249	544	971	1476	1972	2334	2471	2334	1972	1476	971	544	249	81	17	0	0	0	0	0	0	0
20	93	281	602	1044	1536	1972	2229	2229	1972	1536	1044	602	281	93	20	0	0	0	0	0	0	0	0
22	101	299	622	1044	1476	1810	1932	1810	1476	1044	622	299	101	22	0	0	0	0	0	0	0	0	0
23	104	299	602	971	1311	1518	1518	1311	971	602	299	104	23	0	0	0	0	0	0	0	0	0	0
23	101	281	544	838	1068	1159	1068	838	544	281	101	23	0	0	0	0	0	0	0	0	0	0	0
22	93	249	460	668	798	798	668	460	249	93	22	0	0	0	0	0	0	0	0	0	0	0	0
20	81	207	360	490	539	490	360	207	81	20	0	0	0	0	0	0	0	0	0	0	0	0	0
17	66	160	261	326	326	261	160	66	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	51	116	173	196	173	116	51	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	37	77	104	104	77	37	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	24	46	55	46	24	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	14	25	25	14	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	8	12	8	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	4	4	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The sum of entries in this table is, indeed, equal to 159,411. □

By Proposition 6.1, the spaces $\mathcal{I}_{2,7}$ and $\mathcal{I}_{6,7}$ are spanned, as linear spaces, by 15 and 159,411 polynomials, respectively. As before, let us denote the basis polynomials in these spaces by $\alpha_1, \dots, \alpha_{15}$ and $\beta_1, \dots, \beta_{159,411}$. Since \mathcal{I} is an ideal, the product of any element of $\mathcal{I}_{2,7}$ and any polynomial of degree 4 of C belongs to $\mathcal{I}_{6,7}$. Computation shows that there are no relations between such products, that is, the subspace spanned by them has dimension $15 \dim(S^4 \text{span}(C))$. The dimension of the complementary subspace in decomposition of $\mathcal{I}_{6,7}$ is therefore

$$\begin{aligned}
 159,411 - 15 \cdot \dim(S^4 \text{span}(C)) &= 159,411 - 15 \cdot \frac{(21+4-1)!}{4!(21-1)!} \\
 &= 159,411 - 159,390 = 21.
 \end{aligned}$$

Let $\beta_1, \dots, \beta_{21}$ be the basis elements of that 21-dimensional subspace. Together with the 15 basis elements of $\mathcal{I}_{2,6}$, they form a set of 36 linearly independent polynomials of degree 7 in v_1, v_2, v_3 . At the same time, the dimension of the space of homogeneous polynomials of degree 7 in three variables v_1, v_2, v_3 is exactly 36! This allows us to arrange these 15 + 21 polynomials into a 36×36 matrix with the following remarkable property.

THEOREM 6.2. *Let \mathcal{M} be the 36×36 matrix, the rows of which are obtained by expanding the 36 polynomials $\alpha_1, \dots, \alpha_{15}; \beta_1, \dots, \beta_{21}$ in the 36 homogeneous monomials of degree 7 in v_1, v_2, v_3 . Then $\det \mathcal{M}$ is the Cayley–Salmon undulation invariant of plane quintics:*

$$I(C)_{r=5} = \det_{36 \times 36} \mathcal{M}. \tag{10}$$

Proof. Analogously to Theorem 4.2, the statement simply follows from the fact that $\det \mathcal{M}$ vanishes when the curve has undulation points, has the correct degree $15 \cdot 2 + 21 \cdot 6 = 156$, and is nonvanishing for one curve that has no undulation lines, say, for $x_1^5 + x_2^5 + x_3^5 + (x_1 + x_2 + x_3)^5 = 0$. \square

7. Conclusion

In this paper we have found an explicit polynomial formula for the Cayley–Salmon invariant of plane quartics and plane quintics. The formula is expressed as a determinant of a finite-size matrix with polynomial entries; it is therefore very convenient for practical calculations and allows one to determine in reasonable time and space whether the curve has undulation points or not.

The existence of such a formula rises several interesting questions:

- It would be interesting to find a generalization of these formulae to higher $r > 5$ or to make sure that such a generalization does not exist.
- It would be interesting to apply the method used in this paper to other types of invariants associated with various other types of decomposition of curves. The undulation condition is associated with the decomposition

$$P = a_1 a_1 a_1 a_1 + b_1 c_3,$$

where the letters denote different polynomials, and the indices show their degrees. Similarly, we can consider other different types of decompositions, in particular,

$$P = a_1 a_1 a_1 a_1 + b_2 c_2,$$

$$P = a_2 b_1 b_1 + c_2 d_1 d_1,$$

$$P = a_1 b_1 c_1 d_1 + a_1 b_1 c_1 e_1 + a_1 b_1 d_1 e_1 + a_1 c_1 d_1 e_1 + b_1 c_1 d_1 e_1,$$

$$P = a_1 a_1 a_1 a_1 + b_1 b_1 b_1 b_1 + c_1 c_1 c_1 c_1 + d_1 d_1 d_1 d_1 + e_1 e_1 e_1 e_1,$$

$$P = a_1 a_1 a_1 a_1 + b_2 b_2 + c_2 c_2.$$

It is easy to show that existence of each of these decompositions is equivalent to vanishing of a certain invariant polynomial in coefficients of P . Some of these

decompositions were discussed already for a long time: for example, the fourth decomposition corresponds to the Clebsch invariant, which has a well-known determinantal representation of degree six (see [7], p. 283). A more complicated case is the third decomposition, which defines the so-called Luroth quartics [8], and the corresponding invariant is called the Luroth invariant. In analogue with the Cayley–Salmon undulation invariant, it has a high degree (54). An explicit formula for this invariant in terms of the elementary system of invariants has been recently obtained in [9]; it would be interesting to see whether the method of this paper can produce a determinantal formula for it.

ACKNOWLEDGMENTS. We are indebted to B. Sturmfels for letting us know about the undulation problem. We are grateful to A. Morozov for stimulating discussions and to the anonymous referee for valuable comments and suggestions. Work is partly supported by RFBR grants 10-02-00509, 12-02-92108-YaF-a 11-01-92612-KO 11-02-90453-UKR (A.P), RFBR grant 12-01-00525 (Sh.Sh.), grant for support of scientific schools NSh-3349.2012.2, and government contract 8206.

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