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On the Geometry of Abel Maps for Nodal Curves

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Abstract. In this paper, we give local conditions to the existence of Abel maps for smoothings of nodal curves extending the Abel maps for the generic fiber. We use this result to construct Abel maps of any degree for nodal curves with two components.

1. Introduction

1.1. History

Let C be a smooth projective curve over an algebraically closed field and fix a point P in C. A degree-d Abel map is a map $\alpha^d_L : C^d \to J_C$ from the product of d copies of C to its Jacobian J_C, sending $(Q_1, \dots, Q_d)$ to the invertible sheaf $L(dP - Q_1 - \cdots - Q_d)$, where L is an invertible sheaf on C. It is classically known that this map encodes many geometric properties of the curve C. For instance, the Abel theorem states that the fibers of α^d_L are complete linear series on C, up to the action of the dth symmetric group. Thus, all possible embeddings of C in projective spaces are known once we know its Abel maps.

Often, to study linear series on smooth curves, we resort to degenerations to singular curves. Then, it is important to understand how linear series behave under such degenerations. It was through the study of these degenerations that Griffiths and Harris proved the celebrated Brill–Noether theorem in [14], and later Gieseker proved Petri's conjecture in [13]. This inspired the seminal work of Eisenbud and Harris [9], where they introduced the theory of limit linear series for curves of compact type. Nevertheless, a satisfactory general theory of limit linear series has not yet been obtained, although there are several works in this direction for curves with two components, for instance, Coppens and Gatto [8] and Esteves and Medeiros [11]. More recently, Osserman [15] gave a more refined notion of limit linear series for a curve of compact type with two components.

Since there is a relationship between linear series and Abel maps for smooth curves, an interplay between limit linear series and Abel maps for singular curves is expected. This interplay was explored by Esteves and Osserman [12] for curves of compact type with two components, for which natural Abel maps exist. However, Abel maps for singular curves have been constructed only in a few cases: for irreducible curves in [1], in degree one in [3] and [4], in degree two in [5], [6], [16], and [17], and for curves of compact type and in any degree, in [7].

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There are two main compactifications of the Jacobian employed as targets of these Abel maps, namely Caporaso–Pandharipande's compactified Jacobian constructed in [2] and [18] and Esteves' compactified Jacobian constructed in [10] based on the previous work of Altman and Kleiman [1]. The principal goal of this paper is to construct Abel maps of any degree for curves with two components with Esteves' compactified Jacobian as target.

In a different paper, we plan to describe the fibers of this map and unreveal their relationship with degenerations of linear series on smooth curves in the spirit of paper [12] and possibly compare the results with the work of Esteves and Medeiros [11].

1.2. *Main Results*

Let us explain in details our main results. Let C be a nodal curve over an algebraically closed field K. Let $\pi : \mathcal{C} \to B$ be a family of curves over $B := \operatorname{Spec}(K[[t]])$ with smooth total space $\mathcal{C}$ and C as a special fiber. Let $\sigma : B \to \mathcal{C}$ be a section of π through its smooth locus, and $\mathcal{L}$ be an invertible sheaf on $\mathcal{C}$ of relative degree e. Since the generic fiber of π is a pointed smooth curve, there exists a rational Abel map $\alpha^d_{\mathcal{L}} : \mathcal{C}^d \dashrightarrow \overline{\mathcal{J}_{\underline{e}}}$ from the product of d copies of $\mathcal{C}$ over B to the compactified Jacobian of π. Here, $\overline{\mathcal{J}_{\underline{e}}}$ is the fine moduli scheme, introduced by Esteves in [10], parameterizing rank-1 torsion-free sheaves of degree e that are σ-quasi-stable with respect to a polarization $\underline{e}$ of degree e (see Section 2 for more details).

We will resolve the map $\alpha^d_{\mathcal{L}}$ in the case where C has two smooth components C_1 and C_2. To do that, we construct a desingularization $\widetilde{\mathcal{C}}^d$ of $\mathcal{C}^d$ recursively on d. More precisely, we will perform a sequence of blowups along Weil divisors as follows. Set $\widetilde{\mathcal{C}}^1 := \mathcal{C}^1$. Assume that $\widetilde{\mathcal{C}}^d$ is constructed and let $\widetilde{\mathcal{C}}^{d+1} \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ be the sequence of blowups along the strict transforms of the following Weil divisors in the stated order:

$$\Delta_{d,d+1}, \Delta_{d-1,d+1}, \dots, \Delta_{1,d+1},$$

and then

$$C_1^{d+1}, C_1^d \times C_2, C_1^{d-1} \times C_2 \times C_1, C_1^{d-1} \times C_2^2,$$
$$\dots, C_2^{d-1} \times C_1 \times C_2, C_2^d \times C_1, C_2^{d+1},$$

where $\Delta_{i,d+1}$ is the "ith diagonal," that is, the image of the section $\widetilde{\mathcal{C}}^d \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ induced by the composition $\delta_i : \widetilde{\mathcal{C}}^d \to \mathcal{C}^d \to \mathcal{C}$ of the desingularization map with the projection onto the ith factor.

THEOREM. *There exists a modular map $\overline{\alpha}^d_{\mathcal{L}} : \widetilde{\mathcal{C}}^d \to \overline{\mathcal{J}_{\underline{e}}}$ extending the map $\alpha^d_{\mathcal{L}}$.*

We note that the order in which these Weil divisors are blown up is important to the resolution of the map. Indeed, it is not difficult to find examples in which a different sequence of blowups does not give rise to a resolution. Moreover, the desingularization $\widetilde{\mathcal{C}}^d$ is independent of the polarization $\underline{e}$ and the sheaf $\mathcal{L}$. We refer to [6, Section 7] for examples of resolutions for more general curves in degree 2.

In order to prove the result, we consider a local formulation of our problem. Indeed, we note that the completion of the local ring of $\widetilde{\mathcal{C}}^d$ at a point is given by $K[[u_1, \ldots, u_{d+1}]]$, and, in the relevant cases, the map $\widetilde{\mathcal{C}}^d \to B$ is given by $t = u_1 \cdot \ldots \cdot u_{d+1}$. For this reason, we consider $S := \operatorname{Spec}(K[[u_1, \ldots, u_{d+1}]])$ and the map $S \to B$ given by $t = u_1 \cdot \ldots \cdot u_{d+1}$. Let $\mathcal{C}_S := \mathcal{C} \times_B S$ and $\delta_1, \ldots, \delta_m$ be sections of $\pi_S : \mathcal{C}_S \to S$. Since the generic fiber $\mathcal{C}_\eta$ over the generic point η of B is smooth, we have a rational map $\alpha_{\mathcal{L}} : S \dashrightarrow \overline{\mathcal{J}_{\underline{e}}}$ sending η_S, the generic point of S, to the invertible sheaf

$$\mathcal{L}|_{\mathcal{C}_\eta}(m\sigma(\eta) - \delta_1(\eta_S) - \cdots - \delta_m(\eta_S)),$$

where the sections δ_i are identified with their composition with the projection $\mathcal{C}_S \to \mathcal{C}$.

In Theorem 4.2, we give numerical conditions to the existence of a map $\overline{\alpha}_{\mathcal{L}} : S \to \overline{\mathcal{J}_{\underline{e}}}$ extending $\alpha_{\mathcal{L}}$. In fact, this result holds for curves with any number of components.

To check that these conditions hold for a desingularization of $\mathcal{C}^d$, we need to understand its local geometry, that is, to understand how $\mathcal{C}^d$ behaves under the sequence of blowups performed. To do that, in Section 3, we give a local description of blowups along certain Weil divisors. Since this approach is local, it can be applied to curves with any number of components.

Altough we only obtained a sequence of blowups for curves with two components, our techniques might be applied more generally to determine algorithmically whether or not a given sequence of blowups resolves the map $\alpha^d_{\mathcal{L}}$ for any nodal curve. This approach is similar to that in [6], where a script to determine the existence of the degree-2 Abel map was produced.

1.3. Notation and Terminology

Throughout the paper, we will use the following notation.

We work over an algebraically closed field K. A *curve* is a connected, projective, and reduced scheme of dimension 1 over K. We will always consider curves with nodal singularities. A *pointed curve* is a curve C with a marked point P in the smooth locus of C, usually denoted by (C, P).

Let C be a curve. We denote the irreducible components of C by $C_1, \ldots, C_p$ and by C^{sing} the set of its nodes. A *subcurve* of C is a union of irreducible components of C. If Y is a proper subcurve of C, we let $Y^c := \overline{C \setminus Y}$ and call it the *complement* of Y. We denote $\Sigma_Y := Y \cap Y^c$ and $k_Y := \#\Sigma_Y$; a node in Σ_Y is called an *extremal* node of Y. We always consider curves with smooth irreducible components.

Given a map of curves $\phi : C' \to C$, we say that an irreducible component of C' is *ϕ-exceptional* if it is a smooth rational curve and is contracted by the map. A *chain of rational curves of length d* is a curve that is the union of smooth rational curves $E_1, \ldots, E_d$ such that $E_i \cap E_j$ is empty if $|i - j| > 1$

and $\#(E_i \cap E_{i+1}) = 1$. A *chain of ϕ-exceptional components* is a chain of ϕ-exceptional curves. We define the curve $C(d)$ as the curve

$$C(d) := C^\nu \cup \coprod_N E_N,$$

where C^ν is the normalization of C, N runs through all the nodes of C, and E_N is a chain of rational curves of length d such that each E_N intersects transversally C^ν at the two branches over the node N and each irreducible component $E_{N,i}$ of E_N intersects $\overline{C(d)\setminus E_{N,i}}$ in exactly two points. Note that $C(d)$ is endowed with a map $\phi : C(d) \to C$ such that ϕ is an isomorphism over the smooth locus of C and the preimage of each node of C consists of a chain of ϕ-exceptional components of length d. If (C, P) is a pointed curve, we abuse notation denoting by P its preimage in $C(d)$, so that $(C(d), P)$ is also a pointed curve.

A *family of curves* is a proper and flat morphism $\pi : \mathcal{C} \to B$ whose fibers are curves. If $b \in B$, then we denote $\mathcal{C}_b := \pi^{-1}(b)$ its fiber. The family $\pi : \mathcal{C} \to B$ is called *local* if $B = \operatorname{Spec}(K[[t]])$, *regular* if $\mathcal{C}$ is regular, and *pointed* if it is endowed with a section $\sigma : B \to \mathcal{C}$ through the smooth locus of π. A *smoothing* of a curve C is a regular local family $\pi : \mathcal{C} \to B$ with special fiber C. Given a pointed smoothing $\pi : \mathcal{C} \to B$ of a curve C with section $\sigma : B \to \mathcal{C}$, we define $P := \sigma(0)$. If $f : \mathcal{C} \to B$ is a family of curves, we denote by $\mathcal{C}^d$ the product of d copies of $\mathcal{C}$ over B.

Let I be a coherent sheaf on a curve C. We say that I is *torsion-free* if its associated points are generic points of C. We say that I is of *rank 1* if I is invertible on a dense open subset of C. Each invertible sheaf on C is a rank-1 torsion-free sheaf. If I is a rank-1 torsion-free sheaf, we call $\deg(I) := \chi(I) - \chi(\mathcal{O}_C)$ the *degree* of I. An invertible sheaf I over $\phi : C(d) \to C$ is *ϕ-admissible* if $\deg(I|_E) \in \{-1, 0, 1\}$ for every chain of ϕ-exceptional components E.

We fix $B := \operatorname{Spec}(K[[t]])$, $S := \operatorname{Spec}(K[[u_1, \dots, u_{d+1}]])$, and the map $S \to B$ given by $t = u_1 \cdot u_2 \cdot \dots \cdot u_{d+1}$. We will call the closed point of both B and S by 0 when no confusion may arise. Moreover, we denote by Q_i the generic point of $V(u_i)$ in S. Given a smoothing $\pi : \mathcal{C} \to B$ of a curve C, define $\mathcal{C}_S := \mathcal{C} \times_B S$ and let $\pi_S : \mathcal{C}_S \to S$ be the induced map.

2. Jacobians and Abel Maps

Let $\pi : \mathcal{C} \to B$ be a pointed regular local family of nodal curves with section $\sigma : B \to \mathcal{C}$. The *degree-$e$ Jacobian* of π is the scheme parameterizing the equivalence classes of degree-e invertible sheaves on the fibers. In general, this scheme is neither proper nor of finite type. To solve these issues, we resort to rank-1 torsion-free sheaves and to stability conditions.

Let C be a nodal curve with p irreducible components $C_1, \dots, C_p$, and P be a smooth point of C. A *polarization of degree e on C* is any p-tuple of rational numbers $\underline{e} = (e_1, \dots, e_p)$ summing up to e. Let Y be a proper subcurve of C. We set

$$e_Y := \sum_{C_i \subset Y} e_i.$$

Let I be a rank-1 degree-e torsion-free sheaf on C. We define the sheaf I_Y as the sheaf $I|_Y$ modulo torsion. We say that I is *P-quasi-stable over Y* (*with respect to* $\underline{e}$) if the following condition holds:

$$\frac{-k_Y}{2} < \deg(I_Y) - e_Y \leq \frac{k_Y}{2} \quad \text{if } P \in Y,$$
$$\frac{-k_Y}{2} \leq \deg(I_Y) - e_Y < \frac{k_Y}{2} \quad \text{if } P \notin Y.$$

Equivalently, I is P-quasi-stable over Y if the following conditions hold:

$$\frac{-k_Y}{2} < \deg(I_Y) - e_Y \quad \text{and} \quad \frac{-k_Y}{2} \leq \deg(I_{Y^c}) - e_{Y^c} \quad \text{if } P \in Y,$$
$$\frac{-k_Y}{2} \leq \deg(I_Y) - e_Y \quad \text{and} \quad \frac{-k_Y}{2} < \deg(I_{Y^c}) - e_{Y^c} \quad \text{if } P \notin Y.$$

Note that I is P-quasi-stable over Y if and only if it is over Y^c.

We say that I is *P-quasi-stable* over C if it is P-quasi-stable over every proper subcurve of C. Since the conditions are additive on connected components, it is enough to check them over connected subcurves. In fact, it is easy to see that it suffices to check on connected subcurves with connected complement.

Given the map of curves $\phi : C(d) \to C$ and a polarization $\underline{e}$ over C, we define the polarization $\underline{e}(d)$ over $C(d)$ simply by $e(d)_Y = e_{\phi(Y)}$ if $\phi(Y)$ is not a point and $e(d)_Y = 0$ otherwise, where Y is a irreducible component of $C(d)$. From now on fix a polarization $\underline{e}$ of degree e on C and its induced polarizations $\underline{e}(d)$.

Let $\pi : \mathcal{C} \to B$ be a pointed regular local family of nodal curves with section $\sigma : B \to \mathcal{C}$. We say that a sheaf $\mathcal{I}$ over $\mathcal{C}$ is *σ-quasi-stable* if it restricts to a torsion-free rank-1 sheaf over each fiber of π and if its restriction to the special fiber C of π is $\sigma(0)$-quasi-stable. The *degree-e compactified Jacobian of π* is the scheme $\overline{\mathcal{J}_{\underline{e}}}$ parameterizing σ-quasi-stable sheaves over $\mathcal{C}$ of degree e. This scheme is proper and of finite type (see [10, Thms. A and B]), and it represents the contravariant functor $\mathbf{J}$ from the category of locally Noetherian B-schemes to sets, defined on a B-scheme S by

$$\mathbf{J}(S) := \{\sigma_S\text{-quasi-stable sheaves of degree } e \text{ over } \mathcal{C} \times_B S \xrightarrow{\pi_S} S\}/\sim,$$

where σ_S is the pullback of the section σ, and $\sim$ is the equivalence relation given by $I_1 \sim I_2$ if and only if there exists an invertible sheaf M on S such that $I_1 \cong I_2 \otimes \pi_S^* M$.

Proposition 2.1. *Let (C, P) be a pointed nodal curve and consider $\phi : C(d) \to C$. Let L be a line bundle over $C(d)$ that is ϕ-admissible and P-quasi-stable over each proper subcurve Y of $C(d)$ such that Y and Y^c are connected and neither is contracted by ϕ. Then the sheaf $\phi_*(L)$ is P-quasi-stable.*

Proof. Fix $\mathcal{C} \to B$ a smoothing of $(C(d), P)$ and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ such that $\mathcal{L}|_{C(d)} = L$. By [6, Propositions 5.2 and 5.3] it suffices to show that exists a twister $\mathcal{O}_{\mathcal{C}}(Z)$, with Z is a divisor supported on the exceptional components of ϕ, such that $(\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(Z))|_{C(d)}$ is P-quasi-stable.

The divisor Z is effective and can be algorithmically computed as follows. Recall that $\mathcal{L}$ is admissible if and only if its degree on each chain of ϕ-exceptional components is -1, 0, or 1. We define invertible sheaves $\mathcal{L}_i$ inductively. Set $\mathcal{L}_0 := \mathcal{L}$. For a maximal chain of ϕ-exceptional components E over some node of C, let $W_{E,i}$ be the (possibly empty) maximal subchain of E such that $\deg(\mathcal{L}_{i-1}|_{W_{E,i}}) = 1$. Define

$$Z_i := \bigcup_E W_{E,i}$$

and $\mathcal{L}_i := \mathcal{L}_{i-1}(Z_i)$.

We claim that $\mathcal{L}_i$ is admissible and that Z_{i+1} is empty or strictly contained in Z_i. Indeed, fix a maximal chain $E = E_1 \cup \cdots \cup E_d$ and assume that $W_{E,i} = E_\ell \cup \cdots \cup E_h$ is nonempty; otherwise, the result is clear. We have that $\deg(\mathcal{L}_{i-1}|_{W_{E,i}}) = 1$, and the maximality of $W_{E,i}$ implies that either $\ell = 1$ or

$$\deg(\mathcal{L}_{i-1}|_{E_{\ell-1}}) = -1 \quad \text{and} \quad \deg(\mathcal{L}_{i-1}|_{E_k}) = 0 \quad \text{for every } 1 \le k \le \ell - 2;$$

also, either $h = d$ or

$$\deg(\mathcal{L}_{i-1}|_{E_{h+1}}) = -1 \quad \text{and} \quad \deg(\mathcal{L}_{i-1}|_{E_k}) = 0 \quad \text{for every } h+2 \le k \le d.$$

This implies that either $\ell = 1$ or

$$\deg(\mathcal{L}_i|_{E_k}) = 0 \quad \text{for every } 1 \le k \le \ell - 1;$$

also, either $h = d$ or

$$\deg(\mathcal{L}_i|_{E_k}) = 0 \quad \text{for every } h+1 \le k \le d.$$

Therefore, we see that $\mathcal{L}_i$ is admissible.

Moreover, we have

$$\deg(\mathcal{L}_i|_{W_{E,i}}) = -1,$$

meaning that there exist ℓ' and h' such that

$$\deg(\mathcal{L}_i|_{E_{\ell'}}) = -1 \quad \text{and} \quad \deg(\mathcal{L}_i|_{E_k}) = 0 \quad \text{for every } \ell \le k \le \ell' - 1$$

and

$$\deg(\mathcal{L}_i|_{E_{h'}}) = -1 \quad \text{and} \quad \deg(\mathcal{L}_i|_{E_k}) = 0 \quad \text{for every } h'+1 \le k \le h.$$

Clearly, $W_{E,i+1}$ is a subchain of $E_{\ell'+1} \cup \cdots \cup E_{h'-1}$ (which may be empty if $\ell' = h'$), and then $W_{E,i+1}$ is strictly contained in $W_{E,i}$. This concludes the proof of the claim.

Define

$$Z := \sum_{i \ge 1} Z_i.$$

Now it is enough to prove that $\mathcal{N} := \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(Z)$ restricted to $C(d)$ is P-quasi-stable. Let Y be a proper connected subcurve of $C(d)$ with connected complement. If Y is contracted by the map ϕ, then Y is a chain of exceptional components; hence, since $\mathcal{N}$ is admissible and there is no chain of exceptional components over which $\mathcal{N}$ has degree 1, it follows that $\deg(\mathcal{N}|_Y) \in \{-1, 0\}$. This proves that $\mathcal{N}$ is P-quasi-stable over Y and Y^c.

Now assume that neither Y nor Y^c is contracted by ϕ. For every $N \in \Sigma_{\phi(Y)}$, we define

$$Y^\circ := \overline{Y \setminus \bigcup_{N \in \Sigma_{\phi(Y)}} \phi^{-1}(N)}, \qquad N^\circ := \phi^{-1}(N) \cap Y^\circ,$$

$$E_N := \overline{(\phi^{-1}(N) \setminus \{N^\circ\}) \cap Y}, \quad \text{and} \quad E_Y := \bigcup_{N \in \Sigma_{\phi(Y)}} E_N.$$

Note that $Y^\circ = \overline{Y \setminus E_Y}$ and, hence,

$$\deg(\mathcal{N}|_Y) = \deg(\mathcal{N}|_{Y^\circ}) + \deg(\mathcal{N}|_{E_Y}).$$

Moreover, we have

$$\deg(\mathcal{N}|_{Y^\circ}) = \deg(\mathcal{L}|_{Y^\circ}) + \sum_{N \in \Sigma_{\phi(Y)}} \varepsilon_N,$$

where ε_N is 1 if $N^\circ \in Z$ and 0 otherwise. Note that if $\varepsilon_N = 0$, then either there exists a chain of exceptional components E'_N such that $\deg(\mathcal{L}|_{E'_N}) = -1$ and $E'_N \cap Y^\circ \neq \emptyset$, or the degree of $\mathcal{L}$ over every chain of exceptions components contained in $\phi^{-1}(N)$ is zero, and in this case, define $E'_N = \emptyset$. Define

$$Y' := Y^\circ \cup \bigcup_{\substack{N \in \Sigma_{\phi(Y)} \\ \varepsilon_N = 0}} E'_N.$$

Then

$$\begin{aligned}
\deg(\mathcal{N}|_Y) &= \deg(\mathcal{L}|_{Y^\circ}) + \sum_{N \in \Sigma_{\phi(Y)}} \varepsilon_N + \deg(\mathcal{N}|_{E_Y}) \\
&= \deg(\mathcal{L}|_{Y^\circ}) + \sum_{\varepsilon_N = 0} \deg(\mathcal{N}|_{E_N}) + \sum_{\varepsilon_N = 1} (\varepsilon_N + \deg(\mathcal{N}|_{E_N})) \\
&\geq \deg(\mathcal{L}|_{Y'}),
\end{aligned}$$

implying that

$$\deg(\mathcal{N}|_Y) - e(d)_Y \geq \deg(\mathcal{L}|_{Y'}) - e(d)_{Y'}.$$

We can repeat the same process for Y^c, obtaining a subcurve $Y^{c\prime}$ satisfying

$$\deg(\mathcal{N}|_{Y^c}) - e(d)_{Y^c} \geq \deg(\mathcal{L}|_{Y^{c\prime}}) - e(d)_{Y^{c\prime}}.$$

Since both Y' and $Y^{c\prime}$ are connected with connected complement and are not contracted by ϕ, it follows that $\mathcal{L}$ is P-quasi-stable over Y' and $Y^{c\prime}$, and therefore $\mathcal{N}$ is P-quasi-stable over Y. The proof is complete. □

Let $\pi : \mathcal{C} \to B$ be a pointed regular local family of nodal curves with section $\sigma : B \to \mathcal{C}$. Let C be the special fiber of π with irreducible components $C_1, \dots, C_p$. We define $\dot{\mathcal{C}}$ as the smooth locus of π and $\dot{C}_j := C_j \cap \dot{\mathcal{C}}$. Set $\dot{\mathcal{C}}^d := \dot{\mathcal{C}} \times_B \dot{\mathcal{C}} \times_B \cdots \times_B \dot{\mathcal{C}}$, the product of d copies of $\dot{\mathcal{C}}$ over B. Note that the special fiber of $\dot{\mathcal{C}}^d \to B$ is

$$\coprod_{1 \le j_1, \dots, j_d \le p} \dot{C}_{j_1} \times \cdots \times \dot{C}_{j_d}.$$

For each d-tuple $\underline{j} = (j_1, \dots, j_d)$, define $\dot{C}_{\underline{j}} := \dot{C}_{j_1} \times \cdots \times \dot{C}_{j_d}$. Let $\mathcal{L}$ be a degree-e invertible sheaf over $\mathcal{C}$. There exists a degree-d Abel map from $\dot{\mathcal{C}}^d$ to the degree-e Jacobian of π simply sending the d-tuple $(Q_1, \dots, Q_d)$ over b to the invertible sheaf

$$\mathcal{L}|_{\mathcal{C}_b}(d \cdot \sigma(b) - Q_1 - \cdots - Q_d). \tag{1}$$

We want to extend this Abel map to $\mathcal{C}^d$, and it is convenient to consider the degree-e compactified Jacobian $\overline{\mathcal{J}_{\underline{e}}}$ as target. However, the sheaf (1) may not be $\sigma(b)$-quasi-stable, and thus we do not even have a map from $\dot{\mathcal{C}}^d$ to $\overline{\mathcal{J}_{\underline{e}}}$ defined as before. To solve this, we use twisters and the fact that $\overline{\mathcal{J}_{\underline{e}}}$ represents the functor $\mathbf{J}$.

Indeed, form the fiber diagram

$$\begin{array}{ccc} \dot{\mathcal{C}}^d \times_B \mathcal{C} & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle \pi_d}\downarrow & & \downarrow{\scriptstyle \pi} \\ \dot{\mathcal{C}}^d & \longrightarrow & B \end{array}$$

By [10, Thm. 32, (4)], for each $\underline{j}$, there exists a divisor

$$Z_{\underline{j}} = \sum_{j=1}^{p} \ell_{\underline{j},i} \cdot \dot{C}_{\underline{j}} \times C_i \tag{2}$$

of $\dot{\mathcal{C}}^d \times_B \mathcal{C}$ such that the invertible sheaf $\mathcal{M}$ defined as

$$\mathcal{M} := f^*\mathcal{L} \otimes \mathcal{O}_{\dot{\mathcal{C}}^d \times_B \mathcal{C}}\Big(d \cdot f^*\sigma(B) - \sum_{i=1}^{d} \Delta_{i,d+1}\Big) \otimes \mathcal{O}_{\dot{\mathcal{C}}^d \times_B \mathcal{C}}\Big(-\sum_{\underline{j}} Z_{\underline{j}}\Big)$$

is $f^*\sigma$-quasi-stable, where $\Delta_{i,d+1}$ is the preimage of the diagonal via the projection map $\dot{\mathcal{C}}^d \times_B \mathcal{C} \to \mathcal{C} \times_B \mathcal{C}$ onto the ith and $(d+1)$th factors. This $f^*\sigma$-quasi-stable sheaf $\mathcal{M}$ induces the Abel map

$$\alpha^d_{\mathcal{L}} : \dot{\mathcal{C}}^d \longrightarrow \overline{\mathcal{J}_{\underline{e}}}.$$

In this paper, we give conditions to determine when this map extends to a suitable desingularization of $\mathcal{C}^d$.

3. Desingularizations

Given a smoothing $\pi : \mathcal{C} \to B$ of a curve C and N a node of C, we can write the completion of the local ring of $\mathcal{C}$ at N as

$$\widehat{\mathcal{O}}_{\mathcal{C},N} \simeq K[[x, y]].$$

The map $\pi : \mathcal{C} \to B$ is, locally around N, given by $xy = t$. In this section, we study the geometry of this local map and its formation with base change. In Figure 1, we collect all the relevant results in an explicit example.

Recall that we defined $S = \mathrm{Spec}(K[[u_1, \ldots, u_{d+1}]])$ and a map $S \to B$ given by $t = u_1 \cdot \ldots \cdot u_{d+1}$. Define $T := \mathrm{Spec}(K[[x, y]])$ and the map $T \to B$ given by $t = xy$. Let $T_S := T \times_B S$. Clearly, we have

$$T_S = \mathrm{Spec}\left(\frac{K[[u_1, \ldots, u_{d+1}, x, y]]}{(xy - u_1 \cdots u_{d+1})}\right).$$

Given a subset A of $\{1, \ldots, d + 1\}$, we define

$$u_A := \prod_{j\in A} u_j.$$

To desingularize T_S, we will blowup Weil divisors of type $D_A := V(x, u_A)$, where A is a proper nonempty subset of $\{1, \ldots, d + 1\}$. More precisely, given a collection of proper nonempty subsets $\mathcal{A} := (A_1, \ldots, A_k)$ of $\{1, \ldots, d + 1\}$, we will perform a sequence of blowups

$$\phi : \widetilde{T}_S^{\mathcal{A}} := \widetilde{T}_S^k \xrightarrow{\phi_k} \widetilde{T}_S^{k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_2} \widetilde{T}_S^1 \xrightarrow{\phi_1} \widetilde{T}_S^0 := T_S, \tag{3}$$

where the map ϕ_i is the blowup of the strict transform $\widetilde{D}_{A_i}$ of D_{A_i} via the composition map $\phi_1 \circ \cdots \circ \phi_{i-1}$.

Remark 3.1. Note that the local equations of the blowup of T_S along D_A are given by $\alpha x - \alpha' u_A = 0$ and $\alpha' y - \alpha u_{A^c} = 0$, where $(\alpha : \alpha')$ are the coordinates of $\mathbb{P}^1$. It is easy to see that if we blow up $V(y, u_{A^c})$, then we will obtain the same equations. Therefore, blowing up $V(y, u_{A^c})$ is equivalent to blowing up D_A.

The same property holds for the blowup along $V(x - u_{A^c}, y - u_A)$. Indeed, the local equation of such a blowup is

$$\alpha(x - u_{A^c}) - \alpha'(y - u_A) = 0. \tag{4}$$

Nevertheless, we know that the relation $xy = u_A u_{A^c}$ holds, and this relation is equivalent to $x(y - u_A) = u_A(u_{A^c} - x)$. Hence, we can simplify equation (4) to the equations

$$\alpha x + \alpha' u_A = 0 \quad \text{and} \quad \alpha' y + \alpha u_{A^c} = 0,$$

which, up to sign, are the same equations for the blowup along D_A. This justifies why in the sequel we only consider blowups along divisors of type D_A.

Let $\mathcal{A} = (A_1, \ldots, A_k)$ be a collection of subsets of $\{1, \ldots, d + 1\}$, and A be a subset of $\{1, \ldots, d + 1\}$. Assume that $\widetilde{T}_S^{\mathcal{A}}$ is obtained by a sequence of blowups of T_S as in (3). Also, let S_A be the complement of $V(u_A)$ in S. We have $S_A =$

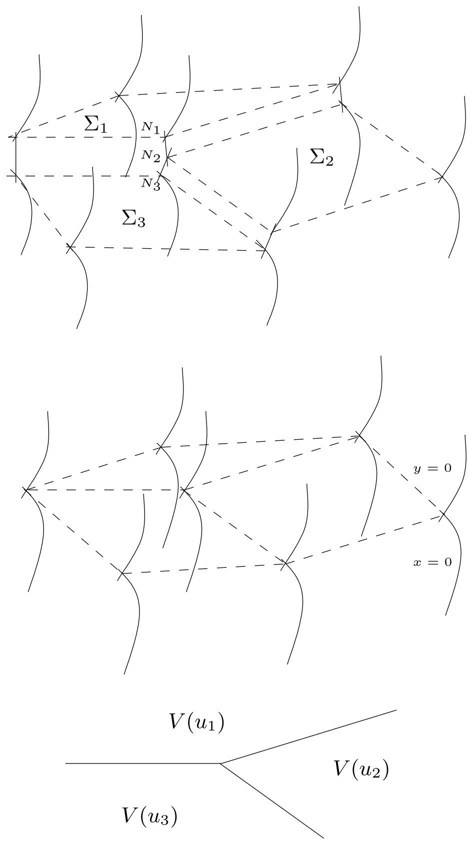

Figure 1 In this picture, we describe the maps

$$\widetilde{\pi}_S : \widetilde{T}_S^{\mathcal{A}} \xrightarrow{\phi} T_S \xrightarrow{\pi_S} S$$

over the special point of B in the case $d = 2$ for $\mathcal{A} = (\{1\}, \{2\})$.

At the bottom, we have depicted the variety S with its divisors $V(u_i)$.

At the middle, the variety T_S with the branch $y = 0$ being the top one. The inverse image of each $V(u_i)$ via the map π_S is the union of the Weil divisors $V(x, u_i)$ and $V(y, u_i)$. The map ϕ is the blowup of T_S along $V(x, u_1)$ and then $V(x, u_2)$.

At the top, we have the variety $\widetilde{T}_S^{\mathcal{A}}$. The dotted lines bound the singular loci Σ_1, Σ_2, and Σ_3 of the map $\widetilde{\pi}_S$. The permutation η in this case is the identity; thus, the node N_i belongs to Σ_i, see Proposition 3.5 and Corollary 3.6. Note that in the central fiber we have two ϕ-exceptional curves, see Proposition 3.2 and Corollary 3.3. The node N_1 belongs to the strict transforms of $V(x, u_1)$, $V(y, u_1)$, $V(y, u_2)$, and $V(y, u_3)$, whereas N_2 belongs to the ones of $V(x, u_1)$, $V(x, u_2)$, $V(y, u_2)$, and $V(y, u_3)$, and finally N_3 belongs to the ones of $V(x, u_1)$, $V(x, u_2)$, $V(x, u_3)$, and $V(y, u_3)$, see Corollary 3.6

$\mathrm{Spec}(K[[u_1,\dots,u_{d+1}]]_{u_A})$. Define $T_{S_A} := T \times_B S_A$ and $\widetilde{T}^{\mathcal{A}}_{S_A} := \widetilde{T}^{\mathcal{A}}_S \times_S S_A$. We have the fiber diagram

$$\begin{tikzcd}
\widetilde{T}^{\mathcal{A}}_{S_A} \arrow[r] \arrow[d] & \widetilde{T}^{\mathcal{A}}_{S} \arrow[d] \\
T_{S_A} \arrow[r] \arrow[d] & T_S \arrow[d] \\
S_A \arrow[r] & S
\end{tikzcd}$$

We call a collection $\mathcal{A} := (A_1,\dots,A_k)$ of subsets of a finite set F a *smooth collection for* F if for every distinct $i, j \in F$, there exists ℓ such that either $j \in A_\ell$ and $i \notin A_\ell$, or $i \in A_\ell$ and $j \notin A_\ell$.

PROPOSITION 3.2. *The scheme $\widetilde{T}^{\mathcal{A}}_S$ is smooth if and only if $\mathcal{A}$ is a smooth collection for $\{1,\dots,d+1\}$. Moreover, in that case, the inverse image of the closed point of T_S in $\widetilde{T}^{\mathcal{A}}_S$ is a chain of d rational curves.*

Proof. First assume that $\widetilde{T}^{\mathcal{A}}_S$ is smooth. Consider the open subscheme $S_{\{i,j\}^c}$ with $i, j = 1,\dots,d+1$ distinct. Note that $\widetilde{T}_{S_{\{i,j\}^c}}$ is smooth and $T_{S_{\{i,j\}^c}}$ is not. Hence, there exists one of the divisors $D_{A_\ell} = V(x, u_{A_\ell})$ such that the restriction to $T_{S_{\{i,j\}^c}}$ is not Cartier. However, the equation of $T_{S_{\{i,j\}^c}}$ is $xy - u_i u_j = 0$, and hence the restriction of D_{A_ℓ} is not Cartier only if $i \in A_\ell$ and $j \notin A_\ell$ or $i \notin A_\ell$ and $j \in A_\ell$.

Assume now that $\mathcal{A}$ is smooth. An open covering for $\widetilde{T}^1_S$ is given by

$$\begin{aligned}
U &:= \mathrm{Spec}\left(\frac{K[[x,y,u_1,\dots,u_{d+1}]][\alpha]}{(\alpha x - u_{A_1}, y - \alpha u_{A_1^c})}\right)\\
&= \mathrm{Spec}\left(\frac{K[[x,u_1,\dots,u_{d+1}]][\alpha]}{(\alpha x - u_{A_1})}\right), \qquad (5)\\
V &:= \mathrm{Spec}\left(\frac{K[[x,y,u_1,\dots,u_{d+1}]][\alpha']}{(x - \alpha' u_{A_1}, \alpha' y - u_{A_1^c})}\right)\\
&= \mathrm{Spec}\left(\frac{K[[y,u_1,\dots,u_{d+1}]][\alpha']}{(\alpha' y - u_{A_1^c})}\right). \qquad (6)
\end{aligned}$$

We claim that the strict transform $\widetilde{D}_{A_2}$ in $\widetilde{T}^1_S$ is given locally, up to Cartier divisors, by

$$V(x, u_{A_1\cap A_2}) \subset U \quad \text{and} \quad V(\alpha', u_{A_1^c\cap A_2}) \subset V.$$

To see this, just note that

$$(x, u_{A_2}) = \bigcap_{j\in A_2} (x, u_j).$$

Hence, we need only analyze the strict transforms $\widetilde{V}(x,u_j)$ of $V(x,u_j)$. Since the strict transform is contained in the inverse image, we get

$$(x, u_j) \subset I(\widetilde{V}(x,u_j) \cap U).$$

(Here, note that we are using the same notation for the coordinates in both U and T_S.) Therefore, by equation (5), we readily see that (x, u_j) has codimension 1 in U if and only if $j \in A_1$, which implies that $\widetilde{V}(x, u_j)$ is empty if $j \in A_1^c$. Arguing similarly for V, we see that

$$(\alpha' u_{A_1}, u_j) \subset I(\widetilde{V}(x, u_j) \cap V).$$

Therefore, if $j \in A_1$, using equation (6), we get that $I(\widetilde{V}(x, u_j)) = (u_j)$ in V, and hence $\widetilde{V}(x, u_j)$ is Cartier in V. Otherwise, if $j \in A_1^c$, then we have

$$(\alpha' u_{A_1}, u_j) = (\alpha', u_j) \cap \bigcap_{i \in A_1} (u_i, u_j).$$

Since (u_i, u_j) has codimension 2 in V, we conclude that

$$I(\widetilde{V}(x, u_j) \cap V) = (\alpha', u_j).$$

To sum up, the strict transform $\widetilde{V}(x, u_j)$ of $V(x, u_j)$ has empty intersection with U (resp. is a Cartier divisor in V) if $j \in A_1^c$ (resp. if $j \in A_1$). Otherwise, if $j \in A_1$ (resp. if $j \in A_1^c$), then this intersection is given by (x, u_j) in U (resp., by (α', u_j) in V). The proof of the claim is complete, and this also proves the proposition in the case $d = 1$.

We proceed now by induction on d. First, we split sequence (3) into two using the open covering $\widetilde{T}_S^1 = U \cup V$. Define

$$U_\ell := (\phi_2 \circ \cdots \circ \phi_\ell)^{-1}(U) \quad \text{and} \quad V_\ell := (\phi_2 \circ \cdots \circ \phi_\ell)^{-1}(V)$$

and the sequence

$$U_k \xrightarrow{\phi_k} U_{k-1} \xrightarrow{\phi_{k-1}} \cdots \xrightarrow{\phi_3} U_2 \xrightarrow{\phi_2} U,$$

where the map ϕ_ℓ is the blowup along the intersection $\widetilde{D}_{A_\ell} \cap U_\ell$. Second, we note that the singular locus of $\widetilde{T}_S^1 = U \cup V$ is contained in the locus defined by

$$(\alpha:\ \alpha') = (1:\ 0) \text{ and } (0:\ 1).$$

Hence, we can argue locally, that is, we can assume that

$$U = \operatorname{Spec}\left(\frac{K[[x, u_1, \ldots, u_{d+1}, \alpha]]}{(\alpha x - u_{A_1})}\right)$$

and

$$V = \operatorname{Spec}\left(\frac{K[[y, u_1, \ldots, u_{d+1}, \alpha']]}{(\alpha' y - u_{A_1^c})}\right).$$

By the previous claim, the strict transforms of $D_{A_2}, D_{A_3}, \ldots, D_{A_k}$ via the map $U \to T_S$ are given by the equations

$$(x, u_{A_1 \cap A_2}), (x, u_{A_1 \cap A_3}), \ldots, (x, u_{A_1 \cap A_k}).$$

Now, we just observe that the collection

$$\mathcal{A}_U := (A_1 \cap A_2, A_1 \cap A_3, \ldots, A_1 \cap A_k)$$

is a smooth collection for A_1. Since $|A_1| \leq d$, by the induction hypothesis, U_k is smooth.

For V, the argument is similar; just note that Cartier divisors may appear as components of the strict transforms, but they do not give contributions to the blowups. This proves the smoothness.

To prove the second statement, we still proceed by induction on d. By the induction hypothesis, the inverse image of the closed point in U via the map $U_k \to U$ is a chain of $|A_1| - 1$ rational curves, and the inverse image of the closed point in V via the map $V_k \to V$ is a chain of $|A_1^c| - 1$ rational curves. Since the blowup of $V(x, u_{A_1})$ adds exactly one rational curve, we get the result. □

COROLLARY 3.3. *Let $\mathcal{A}$ be a smooth collection for $\{1, \ldots, d+1\}$. Then, the inverse image of the closed point in T_{S_A} via the map $\widetilde{T}_{S_A}^{\mathcal{A}} \to T_{S_A}$ is a chain of $d - |A|$ rational curves.*

Proof. Just note that $(A_1 \cap A, \ldots, A_k \cap A)$ is a smooth collection for A. □

Let $\mathcal{A} = (A_1, \ldots, A_k)$ be a smooth collection for a finite set F. We define the *$\mathcal{A}$-ordering* of F as follows.

Let m, n be distinct elements of F. We say that $m <_{\mathcal{A}} n$ if there exists j such that $m \in A_j$, $n \notin A_j$ and for every $i < j$, we have that either $\{m, n\} \subset A_i$ or $\{m, n\} \subset A_i^c$. Since $\mathcal{A}$ is smooth, the ordering $<_{\mathcal{A}}$ is a complete ordering of F.

Fix a smooth collection $\mathcal{A}$ of $\{1, \ldots, d+1\}$. Let $\widetilde{T}_S^{\mathcal{A}}$ be the desingularization of T_S obtained via $\mathcal{A}$. The inverse image of the closed point in S via the map $T_S \to S$ is the germ of nodal curve given by two branches $x = 0$ and $y = 0$. It follows from Proposition 3.2 that the inverse image of the closed point of S via the map $\widetilde{T}_S^{\mathcal{A}} \to S$ is the union of these two branches, but with the singular point replaced with a chain of d rational curves. We denote by $N_1, \ldots, N_{d+1}$ the nodes lying on the chain, where N_1 is the one in $y = 0$, and N_{d+1} is the one in $x = 0$, and by $E_1, \ldots, E_d$ the rational curves, where $\{N_i, N_{i+1}\} \subset E_i$.

From now on all the strict transforms will be via the map $\phi : \widetilde{T}_S^{\mathcal{A}} \to T_S$.

LEMMA 3.4. *Let $\mathcal{A}$ be a smooth collection for $\{1, \ldots, d+1\}$. If $i <_A j$, then the strict transforms via $\widetilde{T}_S^{\mathcal{A}} \to T_S$ of $V(x, u_j)$ and $V(y, u_i)$ do not intersect.*

Proof. Keep the notation of the proof of Proposition 3.2. We proceed by induction on d. We first analyze the case where $i \in A_1$ and $j \notin A_1$. It follows from the proof of Proposition 3.2 that the strict transform of $V(x, u_j)$ is empty in U. Similarly, the strict transform of $V(y, u_i)$ is empty in V. Therefore, there is no intersection in this case.

On the other hand, if $i, j \in A_1$, then the equations of the strict transforms of $V(x, u_j)$ and $V(y, u_i)$ in U become (x, u_j) and (α, u_i). By the induction hypothesis the intersection of these strict transforms in U_k is empty. Since the strict transform of $V(y, u_i)$ is empty in V, we are done also in this case. The case $i, j \in A_1^c$ is similar. □

The singular locus of the map $\widetilde{T}_S^{\mathcal{A}} \to S$ consists of $d+1$ connected components, each of which dominates one region of type $V(u_j) \subset S$ for some $j = 1, \dots, d+1$. Indeed, if we keep the same notation of Proposition 3.2, then the singular locus lying over $V(u_j)$ of the map $T_S \to S$ is contained in both $V(x, u_j)$ and $V(y, u_j)$, and if $i \neq j$, then either the strict transforms of $V(x, u_j)$ and $V(y, u_i)$ do not intersect, or the strict transforms of $V(x, u_i)$ and $V(y, u_j)$ do not intersect. We will denote by Σ_j the connected component of the singular locus of the map $\widetilde{T}_S^{\mathcal{A}} \to S$ that dominates $V(u_j)$.

In the sequel, we will often use the following fact: If $N_i \in \Sigma_j$, then the rational curves $E_i, \dots, E_d$ are contained in the strict transform of $V(x, u_j)$.

Proposition 3.5. *Let $\mathcal{A}$ be a smooth collection for $\{1, \dots, d+1\}$. If η is a permutation of $1, \dots, d+1$ such that $\eta(1) <_{\mathcal{A}} \eta(2) <_{\mathcal{A}} \cdots <_{\mathcal{A}} \eta(d+1)$, then $N_i \in \Sigma_{\eta(i)}$.*

Proof. Since the Σ_j are disjoint and each node belongs to at least one of them, it is clear that each node N_i is contained in exactly one Σ_j; we denote such an index by $j := \tau(i)$.

Without loss of generality, we may assume that η is the identity. This means that the $\mathcal{A}$-ordering is the usual one. Since $N_i \in \Sigma_{\tau(i)}$, we get that the strict transform of $V(x, u_{\tau(i)})$ contains the rational curves $E_i, \dots, E_d$. Hence, the strict transform of $V(y, u_{\tau(i)})$ contains the rational curves $E_1, \dots, E_{i-1}$. Given a $k > i$, the strict transform of $V(y, u_{\tau(k)})$ contains the rational curves $E_1, \dots, E_{k-1}$. Hence, the intersection of the strict transforms of $V(x, u_{\tau(i)})$ and $V(y, u_{\tau(k)})$ contains E_i and therefore is nonempty. By Lemma 3.4 we have $\tau(i) < \tau(k)$ for every $i < k$, and we conclude that τ is the identity. □

Corollary 3.6. *Let $\mathcal{A}$ be a smooth collection for $\{1, \dots, d+1\}$. Let η be a permutation of $1, \dots, d+1$ such that $\eta(1) <_{\mathcal{A}} \eta(2) <_{\mathcal{A}} \cdots <_{\mathcal{A}} \eta(d+1)$. Then $E_i, \dots, E_{d+1}$ is contained in the strict transform of $V(x, u_{\eta(i)})$ via $\phi : \widetilde{T}_S^{\mathcal{A}} \to T_S$. Furthermore, the intersection of the strict transforms of the divisors*

$$V(x, u_{\eta(1)}), \dots, V(x, u_{\eta(i)}), V(y, u_{\eta(i)}), \dots, V(y, u_{\eta(d+1)})$$

is exactly the node N_i.

Remark 3.7. Using Proposition 3.2, we see that for any given regular local family of curves with smooth irreducible components $\pi : \mathcal{C} \to B$, there exists a desingularization of $\mathcal{C}_S := \mathcal{C} \times_B S$ obtained by blowing up Weil divisors. Moreover, the map $\widetilde{\pi}_S : \widetilde{\mathcal{C}}_S \to S$ is a regular family of curves since $\widetilde{\mathcal{C}}_S$ is smooth. Also note that Corollary 3.3 implies that the fiber of $\widetilde{\pi}_S$ over the special point 0 of S is $C(d)$; more generally, the fibers of $\widetilde{\pi}_S$ are either the smooth curve $\mathcal{C}_\eta$ over the generic point η of S or curves of the form $C(k)$ for some $k \in \{0, \dots, d\}$.

4. Local Conditions

Recall that $B = \mathrm{Spec}(K[[t]])$, $S = \mathrm{Spec}(K[[u_1, \dots, u_{d+1}]])$, and consider the map $S \to B$ given by $t = u_1 \cdot u_2 \cdot \dots \cdot u_{d+1}$. Let $\pi : \mathcal{C} \to B$ be a pointed regular family of nodal curves with special fiber C and section $\sigma : B \to \mathcal{C}$ through the smooth locus of π. We let $P := \sigma(0)$. Consider $\mathcal{C}_S := \mathcal{C} \times_B S$ and let $\pi_S : \mathcal{C}_S \to S$ be the induced map; form the fiber diagram

$$\begin{tikzcd} \mathcal{C}_S \arrow[r, "f"] \arrow[d, "\pi_S"'] & \mathcal{C} \arrow[d, "\pi"] \\ S \arrow[r] & B \end{tikzcd}$$

Any section $S \to \mathcal{C}_S$ of the map π_S induces a B-map $S \to \mathcal{C}$ by composition; conversely, every B-map $S \to \mathcal{C}$ induces a section of π_S. We will abuse notation using the same name for both the section and the B-map.

Let $\delta : S \to \mathcal{C}$ be a B-map. Assume that $\delta(0) = N$, where N is a node of C. We can write the completion of the local ring of $\mathcal{C}$ at N as

$$\widehat{\mathcal{O}}_{\mathcal{C},N} \simeq K[[x, y]].$$

The map $\pi : \mathcal{C} \to B$ is given by $xy = t$ locally around N. Up to multiplication by an invertible element, the map δ is given by

$$x = u_A \quad \text{and} \quad y = u_{A^c}, \tag{7}$$

where A is a proper nonempty subset of $\{1, \dots, d+1\}$. Note that, geometrically, this means that $\delta(Q_j) \subset V(x)$ if and only if $j \in A$, where Q_j is the generic point of $V(u_j) \subset S$.

Given sections $\delta_1, \dots, \delta_m$ of π_S passing through nodes of C, a subcurve Y of C, and a node N of C, we define

$$a_j^N(Y) := \#\{k \mid \delta_k(0) = N \text{ and } \delta_k(Q_j) \subset Y^c\}. \tag{8}$$

Note that if $N \notin \Sigma_Y$, then the index j plays no role; in this case, we simply write $a^N(Y)$. Also note that if $N \in Y^{\mathrm{sing}}$, then $a^N(Y) = 0$.

Recall that $S_{\{j\}^c}$ is the complement of $\bigcup_{i \neq j} V(u_i)$ in S and it is given by

$$S_{\{j\}^c} = \mathrm{Spec}(K[[u_1, \dots, u_{d+1}]]_{u_{\{j\}^c}}). \tag{9}$$

Hence, there exists a map $S_{\{j\}^c} \to S$. Let $\mathcal{C}_{S_{\{j\}^c}} := \mathcal{C}_S \times_S S_{\{j\}^c}$ and denote by $g_j : \mathcal{C}_{S_{\{j\}^c}} \to \mathcal{C}_S$ the projection onto the first factor. Form the following fiber diagram:

$$\begin{tikzcd} \mathcal{C}_{S_{\{j\}^c}} \arrow[r, "g_j"] \arrow[d, "\pi_j"'] & \mathcal{C}_S \arrow[d, "\pi_S"] \\ S_{\{j\}^c} \arrow[r] & S \end{tikzcd} \tag{10}$$

Let $f_j := f \circ g_j$, and let $\delta_1, \dots, \delta_m$ be sections of π_S passing through nodes of C. If we restrict these sections to $S_{\{j\}^c}$, then we obtain the sections $S_{\{j\}^c} \to \mathcal{C}_{S_{\{j\}^c}}$ passing through the smooth locus of π_j.

Let $\mathcal{L}$ be a degree-e invertible sheaf over $\mathcal{C}$. Denote by $\mathcal{L}_S$ the pullback of $\mathcal{L}$ to $\mathcal{C}_S$ and define

$$\mathcal{M} := \mathcal{L}_S \otimes f^*\mathcal{O}_{\mathcal{C}}(m \cdot \sigma(B)) \otimes \mathcal{I}_{\delta_1(S)|\mathcal{C}_S} \otimes \cdots \otimes \mathcal{I}_{\delta_m(S)|\mathcal{C}_S}.$$

Note that the sheaf $\mathcal{M}$ induces a rational map $S \dashrightarrow \overline{J}$ since the generic fiber of π_S is smooth. We also define

$$\mathcal{M}_j := g_j^*\mathcal{M} \quad \text{for every } j = 1, \dots, d+1. \tag{11}$$

Since the restrictions of the sections $\delta_1, \dots, \delta_m$ to $S_{\{j\}^c}$ are sections passing through the smooth locus of C, the sheaf $\mathcal{M}_j$ is invertible. Also, since $S_{\{j\}^c}$ is the spectrum of a DVR, there exists an invertible sheaf $\mathcal{O}_{\mathcal{C}_{S_{\{j\}^c}}}(-Z_j)$, where

$$Z_j = \sum_{i=1}^{p} \ell_{j,i} \cdot f_j^* C_i, \tag{12}$$

such that

$$\mathcal{M}_j \otimes \mathcal{O}_{\mathcal{C}_{S_{\{j\}^c}}}(-Z_j) \tag{13}$$

is σ-quasi-stable.

Given a degree-e invertible sheaf $\mathcal{L}$, sections $\delta_1, \dots, \delta_m$ of π_S, a subcurve Y of C, and a node N in the intersection of C_r and C_s, we define

$$b_j^N(Y, \mathcal{L}) := \begin{cases} \ell_{j,s} - \ell_{j,r} & \text{if } C_r \subset Y \text{ and } C_s \not\subset Y, \\ \ell_{j,r} - \ell_{j,s} & \text{if } C_r \not\subset Y \text{ and } C_s \subset Y, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Recall that since we are working with curves with smooth irreducible components, for every node N, there exists $r \neq s$ such that N is in the intersection of C_r and C_s.

Proposition 4.1. *Let $\mathcal{L}$ be a degree-e invertible sheaf on $\mathcal{C}$, let $\delta_1, \dots, \delta_m$ be sections of π_S passing through nodes of C, and let Y be a subcurve of C containing P. Then, for every $h \in \{1, \dots, d+1\}$, we have*

$$-\frac{k_Y}{2} < \deg(\mathcal{L}|_Y) - e_Y + \sum_{N \in C^{\text{sing}}} (a_h^N(Y) - b_h^N(Y, \mathcal{L})) \leq \frac{k_Y}{2}.$$

Proof. Let Q_h be the generic point of $V(u_h)$. Identify Y with $Y \times_B Q_h$. By the definition of a^N and by the fact that each section δ_i goes through some node N, we clearly have

$$\deg(\mathcal{M}_j|_Y) = \deg(\mathcal{L}|_Y) + \sum_{N \in C^{\text{sing}}} a_h^N(Y).$$

Indeed, the sum $\sum a_h^N(Y)$ is the number of sections δ_i such that $\delta_i(Q_h) \in Y^c$. It follows that

$$\deg(\mathcal{M}_j \otimes \mathcal{O}_{\mathcal{C}_{S_{\{h\}^c}}}(-Z_h)|_Y) = \deg(L|_Y) + \sum_{N \in C^{\text{sing}}} a_h^N(Y) - Z_h \cdot Y.$$

However, we have

$$Z_h \cdot Y = \sum_{N \in C^{\mathrm{sing}}} b_h^N(Y, \mathcal{L}),$$

which concludes the proof. □

Let $\phi\colon \widetilde{\mathcal{C}}_S \to \mathcal{C}_S$ be a fixed desingularization of $\mathcal{C}_S$ as in Remark 3.7. Let $\widetilde{\Delta}_j$ be the strict transform of $\delta_j(S)$. Since $\widetilde{\mathcal{C}}_S$ is regular, it follows that $\widetilde{\Delta}_j$ is a Cartier divisor. We define

$$\widetilde{C} := (\pi_S \circ \phi)^{-1}(0),$$

the special fiber of the map $\pi_S \circ \phi$. Recall that, by Proposition 3.2, we have an identification of $\widetilde{C}$ with $C(d)$. Let $\mathcal{C}_{[u_j\ i]}$ be the closure of $\widetilde{g}_j(f_j^{-1}(C_i))$ in $\widetilde{\mathcal{C}}_S$, where $\widetilde{g}_j$ is the induced map $\widetilde{g}_j : \mathcal{C}_{S_{\{j\}^c}} \to \widetilde{\mathcal{C}}_S$ and $f_j = f \circ g_j$. Finally, we let

$$\overline{Z}_j := \sum_{i=1}^{p} \ell_{j,i} \cdot \mathcal{C}_{[u_j\ i]} \tag{15}$$

and define the invertible sheaf on $\widetilde{\mathcal{C}}_S$

$$\widetilde{\mathcal{M}}_\phi := \phi^*(\mathcal{L}_S) \otimes \phi^* f^*(\mathcal{O}_{\mathcal{C}}(m \cdot \sigma(B))) \otimes \mathcal{O}_{\widetilde{\mathcal{C}}_S}\Bigg(-\sum_{j=1}^{m} \widetilde{\Delta}_j - \sum_{j=1}^{d+1} \overline{Z}_j\Bigg). \tag{16}$$

THEOREM 4.2. *Let $\mathcal{L}$ be a degree-e invertible sheaf on $\mathcal{C}$, and $\delta_1, \ldots, \delta_m$ be sections of π_S. There exists a map $S \to \overline{\mathcal{J}_{\underline{e}}}$ extending the rational map defined by $\mathcal{M}$ if the following two conditions hold for every subcurve $Y \subset C$ containing P:*

1. *For every $j_1, j_2 = 1, \ldots, d+1$ and every node $N \in \Sigma_Y$, we have*

$$|(a_{j_1}^N(Y) - b_{j_1}^N(Y, \mathcal{L})) - (a_{j_2}^N(Y) - b_{j_2}^N(Y, \mathcal{L}))| \leq 1.$$

2. *For every function $j : C^{\mathrm{sing}} \to \{1, \ldots, d+1\}$, we have*

$$-\frac{k_Y}{2} < \deg(\mathcal{L}|_Y) - e_Y + \sum_{N \in C^{\mathrm{sing}}} (a_{j(N)}^N(Y) - b_{j(N)}^N(Y, \mathcal{L})) \leq \frac{k_Y}{2}.$$

In this case, if $\phi : \widetilde{\mathcal{C}}_S \to \mathcal{C}_S$ is any desingularization of $\mathcal{C}_S$ as in Remark 3.7, then this map is induced by the invertible sheaf $\widetilde{\mathcal{M}}_\phi$.

Proof. Throughout the proof, we fix a desingularization $\phi : \widetilde{\mathcal{C}}_S \to \mathcal{C}_S$ of $\mathcal{C}_S$ as in Remark 3.7 and set $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_\phi$. It is enough to prove that under the hypothesis the sheaf $\phi_*(\widetilde{\mathcal{M}})$ is σ-quasi-stable. By Proposition 2.1 it suffices to check that $\widetilde{\mathcal{M}}$ is admissible and σ-quasi-stable over each connected subcurve Y of $\widetilde{C}$ with connected complement such that Y and Y^c are not contracted by the map ϕ.

We begin by computing the degrees of the restriction of $\widetilde{\mathcal{M}}$ to the components of the special fiber. Let E be a chain of ϕ-exceptional components, and let $N = \phi(E)$. Clearly, the degree of

$$\phi^*(\mathcal{L}_S) \otimes \phi^* f^*(\mathcal{O}_{\mathcal{C}}(m \cdot \sigma(B)))|_E$$

is zero. The same property holds for $\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_j)|_E$ if the section δ_j does not pass through the node N. Since E is contracted, we can look locally around the node N. Let $T := \operatorname{Spec}\widehat{\mathcal{O}}_{C,N}$; the subcurve E can be seen as a subcurve of the special fiber of the map $\widetilde{T}_S^{\mathcal{A}} \to S$ for some collection $\mathcal{A}$, as in Section 3. Let $\{N_i, N_j\} = E \cap E^c$ be the extremal nodes of the chain. It follows from Proposition 3.5 that $N_i \in \Sigma_{\eta(i)}$ and $N_j \in \Sigma_{\eta(j)}$. Without loss of generality, we will assume that η is the identity.

Let Y be a fixed subcurve of C containing P and admitting N as an extremal node. Let δ_k be a section through N. Up to renaming i and j, we can assume that the strict transform of $Y \times V(u_i)$ does not contain the node N_j. Let Q_i be the generic point of $V(u_i)$. Set $Y_i := Y \times Q_i \subset \widetilde{\mathcal{C}}_S$ and $Y_{i,0} := \overline{Y}_i \cap \widetilde{C}$, where the bar denotes the closure in $\widetilde{\mathcal{C}}_S$.

The degree of $\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)|_{Y_i}$ is -1 if $\delta_k(Q_i) \in Y_i$ and 0 otherwise. Since the degree of $\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)|_{Y_{i,0}}$ is the same as the degree of $\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)|_{Y_i}$, we have

$$\deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)|_E) = \begin{cases} 1 & \text{if } \delta_k(Q_i) \in Y_i \text{ and } \delta_k(Q_j) \notin Y_j, \\ -1 & \text{if } \delta_k(Q_i) \notin Y_i \text{ and } \delta_k(Q_j) \in Y_j, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, notice that $\overline{Y_{j,0} \setminus Y_{i,0}} \cup \overline{Y_{i,0} \setminus Y_{j,0}}$ consists of E and other chains of rational curves contracted by ϕ; since the section δ_k goes through the node N, it follows that the line bundle $\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)$ restricted to these other chains has degree 0. Similarly, we can compute:

$$\deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\mathcal{C}_{[u_k\ r]}|_E)) = \begin{cases} 1 & \text{if } k = j \text{ and } N \in C_r \subset Y, \\ 1 & \text{if } k = i \text{ and } N \in C_r \subset Y^c, \\ -1 & \text{if } k = j \text{ and } N \in C_r \subset Y^c, \\ -1 & \text{if } k = i \text{ and } N \in C_r \subset Y, \\ 0 & \text{otherwise.} \end{cases}$$

Summing up all the contributions, we get that the degree of $\widetilde{\mathcal{M}}|_E$ is

$$(a_j^N(Y) - b_j^N(Y, \mathcal{L})) - (a_i^N(Y) - b_i^N(Y, \mathcal{L})),$$

and this shows that the admissibility of $\widetilde{\mathcal{M}}$ is equivalent to condition (l).

Let Z be a connected subcurve of $\widetilde{C}$ with connected complement such that neither Z nor Z^c are contracted by ϕ and set $Y := \phi(Z) \subset C$. We can assume that $P \in Y$. We want to compute the degree of $\widetilde{\mathcal{M}}|_Z$. Again by Proposition 3.5 each extremal node of Z belongs to only one Σ_j. Let $j_Z : \Sigma_Z \to \{1, \dots, d+1\}$ be the induced function; note that the extremal nodes of Z map bijectively onto the extremal nodes of Y, and hence we can also consider Σ_Y as a domain for the function j_Z. We have

$$\deg(\phi^*(\mathcal{L}_S) \otimes \phi^* f^*(\mathcal{O}_{\mathcal{C}}(m \cdot \sigma(B)))|_Z) = \deg(\mathcal{L}|_Y) + m.$$

If we fix k such that $\delta_k(0) = N$, then we also have

$$\deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\widetilde{\Delta}_k)|_Z) = \begin{cases} -1 & \text{if } N \in \Sigma_Y \text{ and } \delta_k(Q_{j_Z(N)}) \in Y_{j_Z(N)}, \\ -1 & \text{if } N \in Y \setminus \Sigma_Y, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have

$$\deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\mathcal{C}_{[u_k\ r]})|_Z) = \begin{cases} -\#(j_Z^{-1}(k) \cap \Sigma_{C_r}) & \text{if } C_r \subset Y^c, \\ \#(j_Z^{-1}(k) \cap \Sigma_{C_r}) & \text{if } C_r \subset Y. \end{cases}$$

Indeed, $j_Z^{-1}(k)$ is the collection of extremal nodes of Z that belong to Σ_k. This means that over each node $N \in \Sigma_Y \setminus j_Z^{-1}(k)$ the intersection of the divisor $\mathcal{C}_{[u_k\ r]}$ with Z is either empty or a (not necessarily maximal) chain of rational curves contracted by ϕ with image the node N. In either case, the contribution to the intersection number $\mathcal{C}_{[u_k\ r]} \cdot Z$ is zero. On the other hand, if $N \in j_Z^{-1}(k)$ and $C_r \subset Y^c$, then the intersection $\mathcal{C}_{[u_k\ r]} \cap Z \cap \phi^{-1}(N)$ is a single point, and in this case, the contribution to the intersection number is 1. The case where $C_r \subset Y$ is analogous.

To sum up, if we define

$$\begin{aligned} c := & \sum_{N \in \Sigma_Y} \#\{k \mid n\delta_k(0) = N \text{ and } \delta_k(Q_{j_Z(N)}) \in Y_{j_Z(N)}\} \\ & + \sum_{N \in Y^{\text{sing}}} \#\{k \mid \delta_k(0) = N\}, \end{aligned} \tag{17}$$

then the degree of $\widetilde{\mathcal{M}}|_Z$ is

$$\deg(\mathcal{L}|_Y) + m - c + \sum_{k=1}^{d+1} \sum_{r=1}^{p} \ell_{r,k} \deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\mathcal{C}_{[u_k\ r]})|_Z).$$

We note now that

$$m - c = \sum_{N \in \Sigma_Y} a^N_{j_Z(N)}(Y) + \sum_{N \in C^{\text{sing}} \setminus \Sigma_Y} a^N(Y). \tag{18}$$

In fact, we have a total of m sections, and hence $m - c$ is the number of sections that do not satisfy the conditions in equation (17), that is, the number of sections that satisfy either $\delta_k(0) \in \Sigma_Y$ and $\delta_k(Q_{j_Z(N)}) \in Y^c$ or $\delta_k(0) \in (Y^c)^{\text{sing}}$ and $\delta_k(Q_{j_Z(N)}) \in Y^c$. This is clearly equal to the right-hand side of equation (18).

Let ε_r be 1 if $C_r \subset Y$ and -1 otherwise. We have

$$\begin{aligned} \sum_{k=1}^{d+1} \sum_{r=1}^{p} \ell_{r,k} \deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\mathcal{C}_{[u_k\ r]})|_Z) &= \sum_{k=1}^{d+1} \sum_{r=1}^{p} \sum_{\substack{j_Z(N)=k \\ N \in C_r}} \varepsilon_r \ell_{r,k} \\ &= \sum_{N \in C^{\text{sing}}} \sum_{\substack{N \in C_r \\ r=1,\dots,p}} \varepsilon_r \ell_{r, j_Z(N)}, \end{aligned}$$

and since N only belongs to two components, we also have

$$\sum_{\substack{N\in C_r\\ r=1,\dots,p}} \varepsilon_r \ell_{r,j_Z(N)} = -b^Y_{j_Z(N)}(Y,\mathcal{L}).$$

Therefore, we conclude that

$$\begin{aligned}\sum_{k=1}^{d+1}\sum_{r=1}^{p} \ell_{r,k} \deg(\mathcal{O}_{\widetilde{\mathcal{C}}_S}(-\mathcal{C}_{[u_k r]})|_Z) &= -\sum_{N\in\Sigma_Y} b^Y_{j_Z(N)}(Y,\mathcal{L})\\ &= -\sum_{N\in C^{\text{sing}}} b^Y_{j_Z(N)}(Y,\mathcal{L}),\end{aligned}$$

and the proof is complete. □

5. Curves with Two Components

Let $\pi : \mathcal{C} \to B$ be a pointed smoothing of a nodal curve C with section $\sigma : B \to \mathcal{C}$ through the smooth locus of π. Let $\mathcal{L}$ be a invertible sheaf of degree e over $\mathcal{C}$. From now on, we assume that C has two smooth components C_1 and C_2 meeting at q nodes $N_1, \dots, N_q$, with the marked point $P := \sigma(0)$ on the component C_1. Locally around each node N_ℓ, the completion of the local ring of $\mathcal{C}$ at N_ℓ is given by

$$\widehat{\mathcal{O}}_{\mathcal{C},N} \simeq K[[x,y]],$$

where $x = 0$ is the local equation of C_1, and $y = 0$ is that of C_2. Hence, if we let $T = \operatorname{Spec}(K[[x,y]])$, then, for each node N, there exists a map $T \to \mathcal{C}$ taking the closed point of T to N. Moreover, the composition map $T \to B$ is given by $t = xy$.

Our goal is to resolve the rational map $\alpha^d_{\mathcal{L}} : \mathcal{C}^d \dashrightarrow \overline{\mathcal{J}_{\underline{e}}}$. Let $\widetilde{\mathcal{C}}^d$ be the blowup of $\mathcal{C}^d$ obtained inductively as follows. First, define $\widetilde{\mathcal{C}}^1 := \mathcal{C}^1$. Then assume that $\widetilde{\mathcal{C}}^d$ is given and let $\widetilde{\mathcal{C}}^{d+1} \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ be the sequence of blowups along the strict transforms of the following Weil divisors in the stated order

$$\Delta_{d,d+1}, \Delta_{d-1,d+1}, \dots, \Delta_{1,d+1}, \tag{19}$$

and then

$$\begin{aligned}&C_1^{d+1}, C_1^d \times C_2, C_1^{d-1} \times C_2 \times C_1, C_1^{d-1} \times C_2^2,\\ &\dots, C_2^{d-1} \times C_1 \times C_2, C_2^d \times C_1, C_2^{d+1},\end{aligned} \tag{20}$$

where $\Delta_{i,d+1}$ is the image of the section $\widetilde{\mathcal{C}}^d \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ induced by the composition $\delta_i : \widetilde{\mathcal{C}}^d \to \mathcal{C}^d \to \mathcal{C}$, where the last map is the projection onto the ith factor.

Lemma 5.1. *The following properties hold.*

1. *The scheme* $\widetilde{\mathcal{C}}^d$ *is smooth.*
2. *For each closed point* $R \in \widetilde{\mathcal{C}}^d$, *there exists a map* $\iota_R : S_R \to \widetilde{\mathcal{C}}^d$, *where* $S_R = \operatorname{Spec}(K[[u_1,\dots,u_{d+1}]])$, *such that the image of the closed point of* S_R *is the point* R *and the composed map* $S_R \to \widetilde{\mathcal{C}}^d \to B$ *is given by* $t = u_1 \cdot \dots \cdot u_k$ *for*

some $k = 1, \dots, d+1$. *In particular, defining* $S'_R = \mathrm{Spec}(K[[u_1, \dots, u_k]])$, *we have a natural map* $S'_R \to S_R \to \widetilde{\mathcal{C}}^d$.

3. *For each closed point* $R \in \widetilde{\mathcal{C}}^d$ *and each node* $N \in C$, *there exists a smooth collection* $\mathcal{A}$ *for* $\{1, \dots, k\}$ *and a fiber diagram*

$$\begin{tikzcd} \widetilde{T}^{\mathcal{A}}_{S'_R} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}^{d+1} \arrow[d] & \\ T_{S'_R} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}^d \times_B \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ S'_R \arrow[r] & \widetilde{\mathcal{C}}^d \arrow[r] & B \end{tikzcd}$$

where the map $T_{S'_R} = S'_R \times_B T \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ *is induced by the map* $T = \mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{C},N}) \to \mathcal{C}$, *and* $\widetilde{T}^{\mathcal{A}}_{S'_R}$ *is constructed as in equation* (3).

We postpone the proof to Section 5.2.

Let $\phi : \widetilde{\mathcal{C}}^{d+1} \to \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ be the desingularization previously given. The projection $\widetilde{\pi} : \widetilde{\mathcal{C}}^{d+1} \to \widetilde{\mathcal{C}}^d$ onto the first factor is a regular family of nodal curves. As in (16), we define the sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathcal{C}}^{d+1}$ as

$$\widetilde{\mathcal{M}} := \phi^* f^*(\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(d \cdot \sigma(B))) \otimes \mathcal{O}_{\widetilde{\mathcal{C}}^{d+1}}\left(-\sum_{i=1}^{d} \widetilde{\Delta}_{i,d+1}\right) \otimes \mathcal{O}_{\widetilde{\mathcal{C}}^{d+1}}(-\mathcal{Z}),$$

where $\mathcal{Z}$ is defined as the sum of the strict transforms of the divisors $Z_{\underline{j}}$, defined in equation (2), via the map ϕ.

Note that there is a relation between the divisors $Z_{\underline{j}}$ and the divisors Z_i defined in equation (12). Indeed, recalling equation (9), for each closed point $R \in \widetilde{\mathcal{C}}^d$ such that $k = d+1$ in condition (2) of Lemma 5.1, one can form the following commutative diagram:

$$\begin{tikzcd} S_{\{i\}^c} \arrow[r] \arrow[d] & \dot{\mathcal{C}}^d \arrow[d] \\ S_R \arrow[r, "\iota_R"] & \widetilde{\mathcal{C}}^d \end{tikzcd}$$

where the image of the closed point of $S_{\{i\}^c}$ belongs to some $\dot{C}_{\underline{j}}$. Then, the pullback of $Z_{\underline{j}}$ via the map $\mathcal{C}_{S_{\{i\}^c}} \to \dot{\mathcal{C}}^d \times_B \mathcal{C}$ (recall the diagram in equation (10)) is the divisor Z_i.

We can now state our main result.

THEOREM 5.2. *There exists a map* $\overline{\alpha}^d_{\mathcal{L}} : \widetilde{\mathcal{C}}^d \to \overline{\mathcal{J}_{\underline{e}}}$ *induced by* $\widetilde{\mathcal{M}}$ *extending the map* $\alpha^d_{\mathcal{L}}$.

We devote the rest of this section to prove Theorem 5.2.

5.1. Special Points

Define U_i, $i=1,\dots,d$, as the locus in $\widetilde{\mathcal{C}}^d$ such that the fiber of $\widetilde{\pi}$ is the curve $C(i)$ (see Corollary 3.3 for a description of these loci), and $U_0 := \dot{\mathcal{C}}^d$. Then $U_0,\dots,U_d$ define a stratification of $\widetilde{\mathcal{C}}^d$ by locally closed subschemes. Furthermore, note that each neighborhood of U_d intersects every irreducible component of U_i. Note that the sheaf $\widetilde{\mathcal{M}}$ is σ-quasi-stable over U_0.

In order to prove the theorem, we can argue locally on the base. Recall that σ-quasi-stability is a numerical condition. Also note that the restrictions of $\widetilde{\mathcal{M}}$ to the fibers over points in a connected component of U_i for $i\geq 1$ have constant multidegree because the restriction of the family $\widetilde{\pi}$ to U_i is the product $U_i\times C(i)$ for $i\geq 1$. Therefore, it follows that if $\widetilde{\mathcal{M}}$ is σ-quasi-stable over a point in a connected component of U_i, then it is so over the whole connected component. Since σ-quasi-stability is an open condition by [10, Prop. 34] and since every neighborhood of U_d intersects every connected component of U_i, it suffices to prove that there exists a map extending $\alpha^d_{\mathcal{L}}$ locally around U_d.

Let R be a point in U_d. We call the point R a *special point.* Locally around R, the scheme $\widetilde{\mathcal{C}}^d$ is given by $S_R=\operatorname{Spec}(K[[u_1,\dots,u_{d+1}]])$. Let

$$\iota_R : S_R\to\widetilde{\mathcal{C}}^d \tag{21}$$

be the natural map. Let also $S_R\to B$ be the restriction of the map $\widetilde{\mathcal{C}}^d\to B$; hence, $S_R\to B$ is given by $t=u_1\cdot\ldots\cdot u_{d+1}$.

We can associate to the special point R a d-tuple $(\ell_1,\ell_2,\dots,\ell_d)$, with $\ell_k\in\{1,\dots,q\}$ by the rule $\delta_k(R)=N_{\ell_k}$. Also, we can associate to u_j a d-tuple $[\varepsilon_1,\dots,\varepsilon_d]$ with $\varepsilon_j\in\{1,2\}$, where u_j is the local equation of the strict transform $\mathcal{C}_{[u_j]}$ of $C_{\varepsilon_1}\times\cdots\times C_{\varepsilon_d}$ via $\widetilde{\mathcal{C}}^d\to\mathcal{C}^d$. Abusing notation, we will denote this d-tuple by $[u_j]$, and we set $u_j(k):=\varepsilon_k$. We define a *special point data* as a set

$$\mathcal{R}:=\{(\ell_1,\dots,\ell_d),[u_1],\dots,[u_{d+1}]\}.$$

We call such a data a *constructible special point data* if it arises from a special point R of $\widetilde{\mathcal{C}}^d$. In this case, we may simply refer to it as a *special point* and denote it by R.

Let $R=\{(\ell_1,\dots,\ell_d),[u_1],\dots,[u_{d+1}]\}$ be a special point of $\widetilde{\mathcal{C}}^d$, and N_ℓ be a node of C. We have maps $S:=S_R\to\widetilde{\mathcal{C}}^d$ and $T\to\mathcal{C}$ associated to these points. Then equations (19) and (20) induce a collection $\mathcal{A}_{R,\ell}$ of subsets of $\{1,\dots,d+1\}$ that gives the desingularization of T_S as in equation (3). We will call the $\mathcal{A}_{R,\ell}$-ordering of $[u_1],\dots,[u_{d+1}]$ simply the *ℓ-ordering of* $[u_1],\dots,[u_{d+1}]$.

We proceed now to determine what special point data are constructible. For $d=1$, the only constructible special point data are of the form

$$\{(\ell),[1],[2]\}.$$

For $d=2$, we use Corollary 3.6. We just need to find each collection $\mathcal{A}_{R,\ell}$ associated with the blowup described by equations (19) and (20). First, note that each special point $\{(\ell_1),[1],[2]\}$ in $\widetilde{\mathcal{C}}^1$ is locally given by $t=[1]\cdot[2]$ and each node N_{ℓ_2} of $\mathcal{C}$ is given locally by $t=xy$. Therefore, we just need to compute the local equations of the diagonal of $\mathcal{C}^2$ and of each of the divisors $C_1\times C_1$,

$C_1 \times C_2$, $C_2 \times C_1$, and $C_2 \times C_2$. If $\ell_2 \neq \ell_1$, then the diagonal is empty, otherwise the equation of the diagonal is $(x - [1], y - [2])$. On the other hand, the equation of $C_\varepsilon \times C_1$ is $(x, [\varepsilon])$ and of $C_\varepsilon \times C_2$ is $(y, [\varepsilon])$ for $\varepsilon \in \{1, 2\}$. By Remark 3.1, the blowup of the diagonal is locally given by the blowup of $V(x, [2])$, whereas the blowup of $C_\varepsilon \times C_2$ is locally given by the blowup of $V(x, [3 - \varepsilon])$. It follows that the ℓ_2-ordering of [1], [2] is

$$\begin{aligned} &[1], [2] \quad \text{if } \ell_2 \neq \ell_1, \\ &[2], [1] \quad \text{if } \ell_2 = \ell_1. \end{aligned}$$

Note that in Corollary 3.6 the nodes N_i are the special points. Moreover, the strict transform of the divisor $V(x, [\varepsilon])$ becomes $[\varepsilon\ \ 1]$, and that of $V(y, [\varepsilon])$ becomes $[\varepsilon\ \ 2]$. Therefore, the constructible special point data for $\widetilde{\mathcal{C}^2}$ are

$$\{(\ell, \ell), [21], [22], [12]\} \quad \text{and} \quad \{(\ell, \ell), [21], [11], [12]\}$$

and

$$\{(\ell_1, \ell_2), [11], [12], [22]\} \quad \text{and} \quad \{(\ell_1, \ell_2), [11], [21], [22]\}$$

for $\ell_1 \neq \ell_2$.

For $d = 3$, we proceed in a similar fashion. First, fix a special point R with special point data $\{(\ell_1, \ell_2), [u_1], [u_2], [u_3]\}$ in $\widetilde{\mathcal{C}^2}$ and choose a node N_{ℓ_3} of C. The equation of the diagonal $\Delta_{k,3}$ is of the form

$$(x - u_{A'_k}, y - u_{(A'_k)^c}),$$

where

$$A'_k := \{j \mid u_j(k) = 1 \text{ and } \ell_3 = \ell_k\}$$

since $x = 0$ is the equation of C_1. Note that if $\ell_3 \neq \ell_k$, then A'_k is empty, and then so is $\Delta_{k,3}$. On the other hand, the local equation of $C_{[u_j]} \times C_1$ is (x, u_j), and the one of $C_{[u_j]} \times C_2$ is (y, u_j). It follows that the blowup of the diagonal $\Delta_{k,3}$ is locally given by the blowup of $V(x, u_{(A'_k)^c})$ and the blowup of $C_{[u_j]} \times C_2$ is locally given by the blowup of $V(x, u_{\{j\}^c})$.

Now, for $R = \{(\ell, \ell), [11], [12], [21]\}$ and $\ell_3 = \ell$, we see that the collection $\mathcal{A}_{R,\ell_3}$ is given by $A_1 = (A'_2)^c = \{[12]\}$, $A_2 = (A'_1)^c = \{[21]\}$, $A_3 = \{[11]\}$, $A_4 = \{[12], [21]\}$, and so on. However, (A_1, A_2) is a smooth collection. Then the ℓ_3-ordering of $[11], [12], [21]$ is $[12], [21], [11]$, and hence, by Corollary 3.6, we have three special points in $\widetilde{\mathcal{C}^3}$ lying over R:

$$\begin{aligned} &\{(\ell, \ell, \ell), [121], [122], [212], [112]\}, \\ &\{(\ell, \ell, \ell), [121], [211], [212], [112]\}, \\ &\{(\ell, \ell, \ell), [121], [211], [111], [112]\}. \end{aligned} \tag{22}$$

Similarly, for $R = \{(\ell, \ell), [12], [21], [22]\}$ and $\ell_3 = \ell$, the ℓ_3-ordering of [12], [21], [22] is [22], [12], [21], then we get three special points:

$$\begin{aligned} &\{(\ell, \ell, \ell), [221], [222], [122], [212]\}, \\ &\{(\ell, \ell, \ell), [221], [121], [122], [212]\}, \\ &\{(\ell, \ell, \ell), [221], [121], [211], [212]\}. \end{aligned}$$

As for the case $R=\{(\ell,\ell),[11],[12],[21]\}$ and $\ell_3\neq\ell$, we see that the two diagonals are empty. Therefore, $A_1=\{[11]\}$, $A_2=\{[12],[21]\}$, $A_3=\{[12]\}$, $A_4=\{[11],[21]\}$, and so on. We see that (A_1,A_2,A_3) is a smooth collection, and in fact the given desingularization is the same as that given by the collection (A_1,A_3). The ℓ_3-ordering of $[11],[12],[21]$ is $[11],[12],[21]$. It follows again from Corollary 3.6 that the special points of $\widetilde{\mathcal{C}}^3$ over R are

$$\{(\ell,\ell,\ell_3),[111],[112],[122],[212]\},$$
$$\{(\ell,\ell,\ell_3),[111],[121],[122],[212]\},$$
$$\{(\ell,\ell,\ell_3),[111],[121],[211],[212]\}.$$

Similarly, for $R=\{(\ell,\ell),[12],[21],[22]\}$ and $\ell_3\neq\ell$, the ℓ_3-ordering of $[12]$, $[21]$, $[22]$ is $[12],[21],[22]$, and then we get three special points:

$$\{(\ell,\ell,\ell_3),[121],[122],[212],[222]\},$$
$$\{(\ell,\ell,\ell_3),[121],[211],[212],[222]\},$$
$$\{(\ell,\ell,\ell_3),[121],[211],[221],[222]\}.$$

As for the case $R=\{(\ell_1,\ell_2),[11],[12],[22]\}$ with $\ell_1\neq\ell_2$ and $\ell_3=\ell_1$, we see that the diagonal $\Delta_{2,3}$ is empty. Therefore, $A_1=\{[22]\}$ (it comes from the diagonal $\Delta_{1,3}$), $A_2=\{[11]\}$, $A_3=A_2^c$, $A_4=\{[12]\}$, and so on. We see that (A_1,A_2) is a smooth collection. The ℓ_3-ordering of $[11],[12],[22]$ is $[22],[11],[12]$. It follows again from Corollary 3.6 that the special points of $\widetilde{\mathcal{C}}^3$ over R are

$$\{(\ell_1,\ell_2,\ell_1),[221],[222],[112],[122]\},$$
$$\{(\ell_1,\ell_2,\ell_1),[221],[111],[112],[122]\},$$
$$\{(\ell_1,\ell_2,\ell_1),[221],[111],[121],[122]\}.$$

Similarly for $R=\{(\ell_1,\ell_2),[11],[21],[22]\}$ with $\ell_1\neq\ell_2$ and $\ell_3=\ell_1$, the ℓ_3-ordering of $[11],[21],[22]$ is $[21],[22],[11]$, and then we get three special points:

$$\{(\ell_1,\ell_2,\ell_1),[211],[212],[222],[112]\},$$
$$\{(\ell_1,\ell_2,\ell_1),[211],[221],[222],[112]\},$$
$$\{(\ell_1,\ell_2,\ell_1),[211],[221],[111],[112]\}.$$

As for the case $R=\{(\ell_1,\ell_2),[11],[12],[22]\}$ with $\ell_1\neq\ell_2$ and $\ell_3=\ell_2$, we see that the diagonal $\Delta_{1,3}$ is empty. Therefore, $A_1=\{[12],[22]\}$ (it comes from the diagonal $\Delta_{2,3}$), $A_2=\{[11]\}$, $A_3=A_2^c$, $A_4=\{[12]\}$, and so on. We see that (A_1,A_2,A_3,A_4) is a smooth collection, and in fact the given desingularization is the same as that given by the collection (A_1,A_4). The ℓ_3-ordering of $[11],[12],[22]$ is $[12],[22],[11]$. It follows again from Corollary 3.6 that the special points of $\widetilde{\mathcal{C}}^3$ over R are

$$\{(\ell_1,\ell_2,\ell_2),[121],[122],[222],[112]\},$$
$$\{(\ell_1,\ell_2,\ell_2),[121],[221],[222],[112]\},$$
$$\{(\ell_1,\ell_2,\ell_2),[121],[221],[111],[112]\}.$$

Similarly for $R = \{(\ell_1, \ell_2), [11], [21], [22]\}$ with $\ell_1 \neq \ell_2$ and $\ell_3 = \ell_2$, the ℓ_3-ordering of $[11], [21], [22]$ is $[22], [11], [21]$, and then we get three special points:

$$\{(\ell_1, \ell_2, \ell_2), [221], [222], [112], [212]\},$$
$$\{(\ell_1, \ell_2, \ell_2), [221], [111], [112], [212]\},$$
$$\{(\ell_1, \ell_2, \ell_2), [221], [111], [211], [212]\}.$$

As for the case $R = \{(\ell_1, \ell_2), [11], [12], [22]\}$ with $\ell_1 \neq \ell_2$ and $\ell_3 \neq \ell_1, \ell_2$, we see that the diagonals are empty. Therefore $A_1 = \{[11]\}$, $A_2 = A_1^c$, $A_3 = \{[12]\}$, $A_4 = A_3^c$, and so on. We see that (A_1, A_2, A_3) is a smooth collection, and in fact the given desingularization is the same as that given by the collection (A_1, A_3). The ℓ_3-ordering of $[11], [12], [22]$ is $[11], [12], [22]$. It follows from Corollary 3.6 that the special points of $\widetilde{\mathcal{C}}^3$ over R are

$$\{(\ell_1, \ell_2, \ell_3), [111], [112], [122], [222]\},$$
$$\{(\ell_1, \ell_2, \ell_3), [111], [121], [122], [222]\},$$
$$\{(\ell_1, \ell_2, \ell_3), [111], [121], [221], [222]\}.$$

Similarly for $R = \{(\ell_1, \ell_2), [11], [21], [22]\}$ with $\ell_1 \neq \ell_2$ and $\ell_3 \neq \ell_1, \ell_2$, the ℓ_3-ordering of $[11], [21], [22]$ is $[11], [21], [22]$, and then we get three special points:

$$\{(\ell_1, \ell_2, \ell_3), [111], [112], [212], [222]\},$$
$$\{(\ell_1, \ell_2, \ell_3), [111], [211], [212], [222]\},$$
$$\{(\ell_1, \ell_2, \ell_3), [111], [211], [221], [222]\}.$$

Let $R = \{(\ell_1, \dots, \ell_d), [u_1], \dots, [u_{d+1}]\}$ be a special point of $\widetilde{\mathcal{C}}^d$, and let $N_{\ell_{d+1}}$ be a node of C. The special points of $\widetilde{\mathcal{C}}^{d+1}$ over $(R, N_{\ell_{d+1}}) \in \widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ are of the form

$$\{(\ell_1, \dots, \ell_d, \ell_{d+1}), [v_1\ \ 1], [v_2\ \ 1], \dots, [v_h\ \ 1], [v_h\ \ 2], \dots, [v_{d+1}\ \ 2]\}, \tag{23}$$

where $[v_1], [v_2], \dots, [v_{d+1}]$ is the ℓ_{d+1}-ordering of $[u_1], \dots, [u_{d+1}]$ for each $h = 1, \dots, d+1$. Recall that $\mathcal{A}_{R,\ell_{d+1}}$ is the collection associated to the blowup given by equations (19) and (20). As in the case $d = 3$, we see that the equation of the diagonal $\Delta_{k,d+1}$ is of the form

$$(x - u_{A'_k}, y - u_{(A'_k)^c}),$$

where

$$A'_k = \{j \mid u_j(k) = 1 \text{ and } \ell_{d+1} = \ell_k\}.$$

In particular, if $\ell_{d+1} \neq \ell_k$, then A'_k is empty, and so is $\Delta_{k,d+1}$. If $[u_1], \dots, [u_{d+1}]$ is written in lexicographical order, then $A_1 = (A'_d)^c$, $A_2 = (A'_{d-1})^c, \dots, A_d = (A'_1)^c$, $A_{d+1} = \{[u_1]\}$, $A_{d+2} = \{[u_1]\}^c$, $A_{d+3} = \{[u_2]\}$, and so on. Note that some of $A_1, \dots, A_d$ might be empty, and in the previous examples, we omitted such sets. We sum up what we have shown in the following lemma.

LEMMA 5.3. *We have $u_{j_1} <_{\ell_{d+1}} u_{j_2}$ if and only if one of the following conditions holds.*

1. *There exists k_0 such that $u_{j_1}(k_0) = 2$ and $u_{j_2}(k_0) = 1$ with $\ell_{k_0} = \ell_{d+1}$; moreover, $u_{j_1}(k) = u_{j_2}(k)$ for each $k > k_0$ such that $\ell_k = \ell_{d+1}$.*

2. *For all k such that $\ell_k = \ell_{d+1}$, we have $u_{j_1}(k) = u_{j_2}(k)$, and there exists k_0 such that $\ell_{k_0} \neq \ell_{d+1}$ with $u_{j_1}(k_0) = 1$ and $u_{j_2}(k_0) = 2$; moreover, for all $k < k_0$, we have $u_{j_1}(k) = u_{j_2}(k)$.*

5.2. Proof of the Main Theorem

We begin by proving Lemma 5.1.

Proof of Lemma 5.1. We proceed by induction on d. Of course, $\widetilde{\mathcal{C}}^1$ is smooth and satisfies condition (2). Indeed, note that if R is a node of C, then $k = 2$, otherwise, if R is in the smooth locus of C, then $k = 1$.

To prove condition (3) for $d = 1$, we assume that R is a node of C; if R is not a node, then the computations are similar. In this setting, the map $\widetilde{\mathcal{C}}^1 \to B$ is locally given at R by the equation $t = u_1 \cdot u_2$, where $u_j = 0$ is the local equation of C_j for $j = 1, 2$. Hence, the local equation of $\widetilde{\mathcal{C}}^1 \times_B \mathcal{C}$ at (R, N) is $u_1 u_2 = xy$, which is the same as the local equation of $T_{S'_R}$ at the closed point. Moreover, each Weil divisor in equation (20) has equations of the form (x, u_j) or (y, u_j) for some $j = 1, 2$, whereas, by Remark 3.1, the blowups of the Weil divisors in equation (19), if they contain (R, N), are the same as the blowups of the divisors with equations (x, u_j) for some $j = 1, 2$. Note that, if a Weil divisor in equation (19) does not contain (R, N), then its blowup is an isomorphism locally around (R, N). Hence, the blowups defining $\widetilde{\mathcal{C}}^2$ are locally the blowups constructed in equation (3), and the induced collection $\mathcal{A}$ is smooth. This concludes the proof of condition (3) for $d = 1$.

Assume now that conditions (1), (2), and (3) holds for $d - 1$. First, note that, by Proposition 3.2 and its proof, condition (3) for $d - 1$ implies conditions (1) and (2) for d.

To prove condition (3) for d, we note that the local coordinates of $\widetilde{\mathcal{C}}^d$ at R are $u_1, \dots, u_{d+1}$, where each $u_j = 0$ is the local equation of the strict transform $\mathcal{C}_{[u_j]}$ of some $C_{\varepsilon_1} \times \cdots \times C_{\varepsilon_d}$ via the map $\widetilde{\mathcal{C}}^d \to \mathcal{C}^d$ for $j = 1, \dots, k$ and some $\varepsilon_1, \dots, \varepsilon_d \in \{1, 2\}$. We may assume that $k = d + 1$, the other cases being analogous. Hence, the local equations of $\widetilde{\mathcal{C}}^d \times_B \mathcal{C}$ at the point (R, N) are the same as the local equations of $T_{S'_R}$ at the closed point. Moreover, the strict transform of a Weil divisor in equation (20) has equations of the form (x, u_j) or (y, u_j) for some $j = 1, \dots, d+1$. Indeed, the strict transform of $C_{\varepsilon_1} \times \cdots \times C_{\varepsilon_d} \times C_1$ is $\mathcal{C}_{[u_j]} \times C_1$, which has equations (x, u_j). Furthermore, by Remark 3.1, the blowups of the Weil divisors in equation (19) are the same as the blowups of the divisors with equations (x, u_A) for some (possibly empty) $A \subset \{1, 2, \dots, d + 1\}$. Hence, the blowups defining $\widetilde{\mathcal{C}}^d$ are locally the blowups constructed in equation (3). Moreover, the collection $\mathcal{A}$ induced by the sequence of blowups (19) and (20) is smooth because there exists a set $A_\ell \in \mathcal{A}$ with only j as an element for each j. This concludes the proof. □

We now start to check the conditions in Theorem 4.2. Given a special point $R = \{(\ell_1, \dots, \ell_d), [u_1], \dots, [u_{d+1}]\}$ in $\widetilde{\mathcal{C}}^d$, recall the map $\iota_R : S_R \to \widetilde{\mathcal{C}}^d$ introduced in equation (21). We only need to compute the numbers $a_j^N(C_1)$ and

$b_j^N(C_1, \mathcal{L})$, where N is a node of C, and $j = 1, \dots, d+1$. We observe that $a_j^N(C_1)$ is the number of sections δ_k such that $\delta_k(R) = N$ and $\delta_k(Q_j) \in C_2$, where Q_j is the generic point of $V(u_j)$. Thus, by the definition of δ_k, we see that $a_j^N(C_1)$ is the number of k such that $\delta_k(R) = N$ and $u_j(k) = 2$. Thus, it is convenient to define

$$\begin{aligned} a^\ell_{[u_j],R} &:= a_j^{N_\ell}(C_1) = \#\{k \mid \ell_k = \ell \text{ and } u_j(k) = 2\},\\ a_{[u_j],R} &:= (a^1_{[u_j],R}, a^2_{[u_j],R}, \dots, a^q_{[u_j],R}),\\ |a_{[u_j],R}| &:= \sum_{\ell=1}^{q} a^\ell_{[u_j],R}. \end{aligned}$$

Now, in order to compute the numbers $b_j^N(C_1, \mathcal{L})$, we have to compute the degree of the restriction to C_2 of the sheaf $\mathcal{M}_j$ defined in equation (11). We have that

$$\deg(\mathcal{M}_j|_{C_2}) = \deg L|_{C_2} - |a_{[u_j],R}|.$$

On the other hand, we can assume that the divisor Z_j (defined in equation (12)) is supported on C_2, and hence we can write $Z_j = \ell_{j,2} \cdot f_j^* C_2$. Imposing the quasi-stability conditions to the sheaf defined in equation (13), we get

$$-\frac{q}{2} \le \deg L|_{C_2} - |a_{[u_j],R}| + q\ell_{j,2} - e_{C_2} < \frac{q}{2},$$

and using that $b_j^N(C_1, \mathcal{L}) = \ell_{j,2}$, we finally arrive at

$$b_{u_j,R} := b_j^N(C_1, \mathcal{L}) = \left\lceil \frac{|a_{[u_j],R}| - \deg(L|_{C_2}) + e_{C_2}}{q} - \frac{1}{2} \right\rceil. \qquad (24)$$

Given two q-tuples $x = (x_1, \dots, x_q)$ and $y = (y_1, \dots, y_q)$ of integers, we write $x \le y$ if $x_i \le y_i$ for every $i = 1, \dots, q$.

PROPOSITION 5.4. *Let $R = \{(\ell_1, \dots, \ell_d), [u_1], \dots, [u_{d+1}]\}$ be a constructible special point data with $[u_1], \dots, [u_{d+1}]$ written in lexicographical order. Let also N_ℓ be a node of C, and $[v_1], \dots, [v_{d+1}]$ be the ℓ-ordering of $[u_1], \dots, [u_{d+1}]$ with respect to the node N_ℓ. The following conditions hold.*

1. *The permutation $[v_1], \dots, [v_{d+1}]$ of $[u_1], \dots, [u_{d+1}]$ is cyclic.*
2. $a_{[u_j],R} \le a_{[u_{j+1}],R}$ *for each $j = 1, \dots, d$.*
3. $a^\ell_{[v_j],R} \ge a^\ell_{[v_{j+1}],R}$ *for each $j = 1, \dots, d$.*
4. $a^\ell_{[v_1],R} - a^\ell_{[v_{d+1}],R} \le 1$; *furthermore, the equality holds if and only if there exists $i \in \{1, \dots, d\}$ such that $\ell_i = \ell$.*
5. $|a_{[u_{j+1}],R}| - |a_{[u_j],R}| \le 1$ *for each $j = 1, \dots, d$.*

Proof. We proceed by induction on d. For $d = 1$, the constructible special point data is of the form $\{(\ell), [1], [2]\}$, and hence it satisfies all the stated conditions. Now, assume that these conditions hold for d. Let $N_{\ell_{d+1}}$ be a node of C, and let $R' := \{(\ell_1, \dots, \ell_{d+1}), [w_1], \dots, [w_{d+2}]\}$ be a special point of $\widetilde{\mathcal{C}}^{d+1}$ over $(R, N_{\ell_{d+1}})$.

We begin by proving item (1). Let $[\widetilde{v}_1], \dots, [\widetilde{v}_{d+1}]$ be the ℓ_{d+1}-ordering of $[u_1], \dots, [u_{d+1}]$. It follows from equation (23) that $[w_1], \dots, [w_{d+2}]$ is a permutation of

$$[\widetilde{v}_1 \ \ 1], \dots, [\widetilde{v}_h \ \ 1], [\widetilde{v}_h \ \ 2], \dots, [\widetilde{v}_{d+1} \ \ 2]. \tag{25}$$

By Lemma 5.3 the ℓ_{d+1}-ordering of equation (25) is

$$[\widetilde{v}_h \ \ 2], \dots, [\widetilde{v}_{d+1} \ \ 2], [\widetilde{v}_1 \ \ 1], \dots, [\widetilde{v}_h \ \ 1]. \tag{26}$$

Let $[v_1], \dots, [v_{d+1}]$ be the ℓ-ordering of $[u_1], \dots, [u_{d+1}]$ for $\ell \neq \ell_{d+1}$. By induction hypothesis, the following relations hold for some h_0:

$$\begin{aligned} \widetilde{v}_{h_0} = v_1, \qquad & \dots, \qquad \widetilde{v}_{d+1} = v_{d+2-h_0}, \\ \widetilde{v}_1 = v_{d+3-h_0}, \qquad & \dots, \qquad \widetilde{v}_{h_0-1} = v_{d+1}. \end{aligned}$$

If $h_0 \leq h$, then it follows from Lemma 5.3 that the ℓ-ordering of equation (25) is

$$\begin{aligned} &[\widetilde{v}_{h_0} \ \ 1], [\widetilde{v}_{h_0+1} \ \ 1], \dots, [\widetilde{v}_h \ \ 1], [\widetilde{v}_h \ \ 2], \dots, \\ &\qquad [\widetilde{v}_{d+1} \ \ 2], [\widetilde{v}_1 \ \ 1], \dots, [\widetilde{v}_{h_0-1} \ \ 1]. \end{aligned} \tag{27}$$

If $h_0 > h$, then the ℓ-ordering of equation (25) is

$$\begin{aligned} &[\widetilde{v}_{h_0} \ \ 2], [\widetilde{v}_{h_0+1} \ \ 2], \dots, [\widetilde{v}_{d+1} \ \ 2], [\widetilde{v}_1 \ \ 1], \dots, \\ &\qquad [\widetilde{v}_h \ \ 1], [\widetilde{v}_h \ \ 2], \dots, [\widetilde{v}_{h_0-1} \ \ 2]. \end{aligned} \tag{28}$$

Since, by induction hypothesis, $[\widetilde{v}_1], \dots, [\widetilde{v}_{d+1}]$ is a cyclic permutation of $[u_1], \dots, [u_{d+1}]$ and the latter is in lexicographical order, the following relations hold for some h':

$$\begin{aligned} \widetilde{v}_{h'} = u_1, \qquad & \dots, \qquad \widetilde{v}_{d+1} = u_{d+2-h'}, \\ \widetilde{v}_1 = u_{d+3-h'}, \qquad & \dots, \qquad \widetilde{v}_{h'-1} = u_{d+1}. \end{aligned}$$

If $h' \leq h$, then the lexicographical ordering of equation (25) is

$$\begin{aligned} &[\widetilde{v}_{h'} \ \ 1], [\widetilde{v}_{h'+1} \ \ 1], \dots, [\widetilde{v}_h \ \ 1], [\widetilde{v}_h \ \ 2], \dots, \\ &\qquad [\widetilde{v}_{d+1} \ \ 2], [\widetilde{v}_1 \ \ 1], \dots, [\widetilde{v}_{h'-1} \ \ 1]. \end{aligned} \tag{29}$$

If $h' > h$, then the lexicographical ordering of equation (25) is

$$\begin{aligned} &[\widetilde{v}_{h'} \ \ 2], [\widetilde{v}_{h'+1} \ \ 2], \dots, [\widetilde{v}_{d+1} \ \ 2], [\widetilde{v}_1 \ \ 1], \dots, \\ &\qquad [\widetilde{v}_h \ \ 1], [\widetilde{v}_h \ \ 2], \dots, [\widetilde{v}_{h'-1} \ \ 2]. \end{aligned} \tag{30}$$

We conclude that if $[w_1], \dots, [w_{d+2}]$ is in lexicographical order (see equations (29) and (30)), then their ℓ-ordering (see equations (27) and (28)) is obtained by a cyclic permutation. The proof of the first item is complete.

Recall that, by induction hypothesis, items (2), (3), and (4) hold for d. We want to prove that these items also hold for $d + 1$. We can rewrite equations (29) and (30) as follows:

$$\begin{aligned} &[u_1 \ \ 1], \dots, [u_{h+1-h'} \ \ 1], [u_{h+1-h'} \ \ 2], \dots, \\ &\qquad [u_{d+2-h'} \ \ 2], [u_{d+3-h'} \ \ 1], \dots, [u_{d+1} \ \ 1] \end{aligned} \tag{31}$$

and

$$\begin{aligned}&[u_1\ \ 2],\dots,[u_{d+2-h'}\ \ 2],[u_{d+3-h'}\ \ 1],\dots,\\&\quad[u_{d+2+h-h'}\ \ 1],[u_{d+2+h-h'}\ \ 2],\dots,[u_{d+1}\ \ 2].\end{aligned} \tag{32}$$

If $\ell \neq \ell_{d+1}$, we have that $a^{\ell}_{[u_j\ \varepsilon],R'} = a^{\ell}_{[u_j],R}$ for $\varepsilon = 1,2$; moreover, the ℓ-ordering of $[w_1],\dots,[w_{d+2}]$ is essentially the same of $[u_1],\dots,[u_{d+1}]$, as we can see in equations (27) and (28). Therefore, in this case, we have nothing to prove.

Consider now the case $\ell = \ell_{d+1}$. We have

$$a^{\ell_{d+1}}_{[u_j\ 1],R'} = a^{\ell_{d+1}}_{[u_j],R} \quad \text{and} \quad a^{\ell_{d+1}}_{[u_j\ 2],R'} = a^{\ell_{d+1}}_{[u_j],R} + 1.$$

We have now two cases.

Case 1. Assume that there exists $i \in \{1,\dots,d\}$ such that $\ell_i = \ell_{d+1}$. Using the induction hypothesis, it is easy to see that the following relations hold:

$$a^{\ell_{d+1}}_{[u_1],R} + 1 = \cdots = a^{\ell_{d+1}}_{[u_{d+2-h'}],R} + 1 = a^{\ell_{d+1}}_{[u_{d+3-h'}],R} = \cdots = a^{\ell_{d+1}}_{[u_{d+1}],R}.$$

Note that the following relations also hold:

$$a^{\ell_{d+1}}_{[u_{d+2-h'}\ 2],R'} = a^{\ell_{d+1}}_{[u_{d+2-h'}],R} + 1 = a^{\ell_{d+1}}_{[u_{d+3-h'}],R} = a^{\ell_{d+1}}_{[u_{d+3-h'}\ 1],R'}.$$

Therefore, with respect to equation (31), we have

$$a^{\ell_{d+1}}_{[u_1\ 1],R'} + 1 = \cdots = a^{\ell_{d+1}}_{[u_{h+1-h'}\ 1],R'} + 1 = a^{\ell_{d+1}}_{[u_{h+1-h'}\ 2],R'} = \cdots = a^{\ell_{d+1}}_{[u_{d+1}\ 1],R'},$$

whereas with respect to equation (32), we have

$$\begin{aligned}a^{\ell_{d+1}}_{[u_1\ 2],R'} + 2 = \cdots &= a^{\ell_{d+1}}_{[u_{d+2+h-h'}\ 1],R'} + 2\\&= a^{\ell_{d+1}}_{[u_{d+2+h-h'}\ 2],R'} + 1 = \cdots = a^{\ell_{d+1}}_{[u_{d+1}\ 2],R'} + 1.\end{aligned}$$

This proves item (2). To prove items (3) and (4), we just note that the ℓ_{d+1}-ordering of $[w_1],\dots,[w_{d+2}]$ is given by equation (26); moreover, in the case of equation (31), we have $\widetilde{v}_h = u_{h+1-h'}$, whereas in the case of equation (32), we have $\widetilde{v}_h = u_{d+2+h-h'}$. In particular, we have equality in item (4).

Case 2. Assume that there is no $i \in \{1,\dots,d\}$ with $\ell_i = \ell_{d+1}$. Then the ℓ_{d+1}-ordering of $[u_1],\dots,[u_{d+1}]$ is simply $[u_1],\dots,[u_{d+1}]$. In this case, using equation (25) (with $\widetilde{v}_i = u_i$) and the fact that $a^{\ell_{d+1}}_{[u_j],R} = 0$, we see that items (2), (3), and (4) readily hold.

Finally, item (5) follows from equations (31) and (32), observing that

$$|a_{[u_j\ \varepsilon],R'}| = |a_{[u_j],R}| + \varepsilon - 1. \qquad \square$$

Proof of Theorem 5.2. To conclude the proof of Theorem 5.2, we have to check the conditions of Theorem 4.2 for every special point R of $\widetilde{\mathcal{C}}^d$. With the notation of this section, these conditions become:

(1) For every $j_1, j_2 = 1, \dots, d+1$ and every node N_ℓ of C, we have

$$|(a^\ell_{[u_{j_1}],R} - b_{[u_{j_1}],R}) - (a^\ell_{[u_{j_2}],R} - b_{[u_{j_2}],R})| \le 1.$$

(2) For every $j_1, \dots, j_q \in \{1, \dots, d+1\}$, we have

$$-\frac{q}{2} < \deg(\mathcal{L}|_{C_1}) - e_{C_1} + \sum_{\ell=1}^{q}(a^\ell_{[u_{j_\ell}],R} - b_{[u_{j_\ell}],R}) \le \frac{q}{2}.$$

First, we note that by item (2) of Proposition 5.4 and by equation (24) we have

$$b_{[u_i],R} \le b_{[u_{i+1}],R} \quad \text{for every } i = 1, \dots, d.$$

Moreover, by item (4) of Proposition 5.4, we get $|a_{[u_{d+1}],R}| - |a_{[u_1],R}| \le q$, and hence $b_{[u_{d+1}],R} - b_{[u_1],R} \le 1$.

We now prove condition (1). Assume without loss of generality that $j_1 > j_2$. By items (2) and (4) of Proposition 5.4 we see that

$$0 \le a^\ell_{[u_{j_1}],R} - a^\ell_{[u_{j_2}],R} \le 1$$

and, by the previous observation, that

$$0 \le b_{[u_{j_1}],R} - b_{[u_{j_2}],R} \le 1.$$

Therefore, condition (1) holds.

As for condition (2), we just have to compute the minimum and maximum of the function

$$F(j_1, \dots, j_q) = \sum_{\ell=1}^{q}(a^\ell_{[u_{j_\ell}],R} - b_{[u_{j_\ell}],R}).$$

Clearly it is enough to find the minimum and maximum of each function

$$F_\ell(j) := a^\ell_{[u_j],R} - b_{[u_j],R}.$$

Since $b_{[u_i],R} \le b_{[u_{i+1}],R}$ and $b_{[u_{d+1}],R} - b_{[u_1],R} \le 1$, we have two cases. In the first case, we have $b_{[u_i],R} = b_{[u_j],R}$ for every $i, j \in \{1, \dots, d+1\}$; in the second case, there exists h such that the following relations hold:

$$b_{[u_1],R} + 1 = \cdots = b_{[u_h],R} + 1 = b_{[u_{h+1}],R} = \cdots = b_{[u_{d+1}],R}.$$

In the first case, it follows from item (2) of Proposition 5.4 that the minimum of F_ℓ is attained at $j = 1$, whereas the maximum is attained at $j = d+1$. On the other hand, in the second case, we claim that the minimum of F_ℓ is attained at $j = h+1$. Indeed, using item (4) of Proposition 5.4, we see that for every $j \le h$, we have

$$a^\ell_{[u_{h+1}],R} - b_{[u_{h+1}],R} = (a^\ell_{[u_{h+1}],R} - 1) - b_{[u_j],R} \le a^\ell_{[u_j],R} - b_{[u_j],R}.$$

On the other hand, using item (2) of Proposition 5.4, we see that for every $j > h+1$, we have

$$a^\ell_{[u_{h+1}],R} - b_{[u_{h+1}],R} = a^\ell_{[u_{h+1}],R} - b_{[u_j],R} \le a^\ell_{[u_j],R} - b_{[u_j],R}.$$

Similarly, one can show that the maximum of F_ℓ is attained at $j = h$.

By the previous arguments, there exists some h such that the minimum (resp. maximum) of $F(j_1, \ldots, j_q)$ is attained at $(h, h, \ldots, h)$. It follows from Proposition 4.1 that the sum

$$\deg(\mathcal{L}_{C_1}) - e_{C_1} + \sum_{\ell=1}^{q}(a^{\ell}_{[u_h],R} - b_{[u_h],R})$$

satisfies condition (2). This concludes the proof of Theorem 5.2. □

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