Grothendieck Classes of Quiver Cycles as Iterated Residues

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ABSTRACT. In the case of Dynkin quivers, we establish a formula for the Grothendieck class of a quiver cycle as the iterated residue of a certain rational function, for which we provide an explicit combinatorial construction. Moreover, we utilize a new definition of the double stable Grothendieck polynomials due to Rimányi and Szenes in terms of iterated residues to exhibit that the computation of quiver coefficients can be reduced to computing the coefficients in a combinatorially prescribed Laurent expansion of the aforementioned rational function.

1. Introduction

Let Q be a quiver with a finite vertex set $Q_0 = \{1, \ldots, N\}$ and finite set of arrows Q_1. For each arrow $a \in Q_1$, denote the vertex at its *head* by $h(a)$ and the vertex at its *tail* by $t(a)$. Throughout the sequel, we will refer also to the set

$$T(i) = \{j \in Q_0 \mid \exists a \in Q_1 \text{ with } h(a) = i \text{ and } t(a) = j\}. \tag{1}$$

Given a *dimension vector* of nonnegative integers $\mathbf{v} = (v_1, \ldots, v_N)$, define the vector spaces $E_i = \mathbb{C}^{v_i}$ and the affine *representation space* $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(E_{t(a)}, E_{h(a)})$ with a natural action of the algebraic group $\mathbb{G} = GL(E_1) \times \cdots \times GL(E_N)$ given by

$$(g_i)_{i \in Q_0} \cdot (f_a)_{a \in Q_1} = (g_{h(a)} f_a g_{t(a)}^{-1})_{a \in Q_1}. \tag{2}$$

A *quiver cycle* $\Omega \subset V$ is a $\mathbb{G}$-stable, closed, irreducible subvariety and, as such, has a well-defined structure sheaf $\mathcal{O}_\Omega$. The goal of this paper is the calculation of the class

$$[\mathcal{O}_\Omega] \in K_{\mathbb{G}}(V),$$

in the $\mathbb{G}$-equivariant Grothendieck ring of V. To accomplish this, we reformulate the problem in an equivalent setting; we realize $[\mathcal{O}_\Omega]$ as the K-class associated to a certain degeneracy locus of a quiver of vector bundles over a smooth complex projective base variety X.

Formulas for this class exist already in the literature, the most general of which is due to Buch [Buc08], and which we now explain. Buch's result is given in terms of the stable version of Grothendieck polynomials first invented by Lascoux and Schützenberger [LS82] as representatives of structure sheaves of Schubert varieties in a flag manifold, which are applied to the E_i in an appropriate way. For details specific to this context, see [Buc08], Section 3.2, and for a comprehensive

introduction to the role of Grothendieck polynomials in K-theory, we refer the reader to [Buc05].

The stable Grothendieck polynomials G_λ are indexed by partitions, that is, nonincreasing sequences of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with only finitely many nonzero parts. The number of nonzero parts is called the *length* of the partition and denoted $\ell(\lambda)$. For each $i \in Q_0$, form the vector space $M_i = \bigoplus_{j \in T(i)} E_j$. With this notation, Buch shows that for *unique* integers $c_\mu(\Omega) \in \mathbb{Z}$, we have

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega) G_{\mu_1}(E_1 - M_1) \cdots G_{\mu_N}(E_N - M_N) \in K_{\mathbb{G}}(V), \qquad (3)$$

where the sum is taken over all sequences of partitions $\mu = (\mu_1, \ldots, \mu_N)$ subject to the constraint $\ell(\mu_i) \leq v_i$ for all $1 \leq i \leq N$. The integers $c_\mu(\Omega)$ are called the *quiver coefficients*. In the case that Q is a *Dynkin quiver*, that is, its underlying nonoriented graph is one of the simply laced Dynkin diagrams (of type A, D, or E), Buch shows that the sum (3) is finite. The central question in the theory is: are the quiver coefficients *alternating*? In this setting, alternating means that $(-1)^{|\mu| - \mathrm{codim}(\Omega)} c_\mu(\Omega) \geq 0$ for all μ, where $|\mu| = \sum_i |\mu_i|$, and $|\mu_i|$ is the area of the corresponding Young diagram. An answer to this question supersedes many of the other positivity conjectures in this vein, in particular, whether or not the cohomology class $[\Omega] \in H^*_{\mathbb{G}}(V)$ is Schur positive, since the leading term of G_λ is the Schur function s_λ, and the coholomology class $[\Omega]$ can be interpreted as a certain leading term of the K-class $[\mathcal{O}_\Omega]$. For this reason, the quiver coefficients $c_\mu(\Omega)$ for which $|\mu| = \mathrm{codim}(\Omega)$ are called the *cohomological quiver coefficients*.

The goal of this paper is to give a new formula for $[\mathcal{O}_\Omega]$ in terms of iterated residue operations. The motivation is plain – namely there has been some considerable recent success in attacking positivity and stability results in analogous settings once armed with such a formula.

Fehér and Rimányi [FR07] discover that the Thom polynomials of singularities share unexpected stability properties, and this is made evident through nonconventional generating sequences. The ideas of [FR07] are further developed and organized in [BS12], [FR12], and [Kaz10b], where the generating sequence formulas appear under the name *iterated residue*. In particular, Bérczi and Szenes [BS12] prove new positivity results for certain Thom polynomials, and Kazarian [Kaz10b] is able to calculate new classes of Thom polynomials through iterated residue machinery developed in [Kaz10a].

Even more recently, a new formula for the cohomology class of the quiver cycle in $H^*_{\mathbb{G}}(V)$ as an iterated residue has been reported in [Rim14], and some new promising initial results on Schur positivity have been obtained from this formula in [Kal13]. Moreover, Rimányi [Rim13] describes an explicit connection between the iterated residue formula for cohomological quiver coefficients of [Rim14] and certain structure constants in the cohomological Hall algebra (COHA) of Kontsevich and Soilbelman [KS11].

The organization of the paper is as follows. In Section 2, we describe quiver representations in some more detail and define the degeneracy loci associated to

them. In Section 3, we discuss an algorithm of Reineke to resolve the singularities of the degeneracy loci in question, which produces a sequence of well-understood maps that we eventually utilize for our calculations. In Section 4, we define our iterated residue operations and provide some illustrative examples of their application. In Section 5, we present the statement of the main result and by example compare our method to previous formulas, most notably that of [Buc08] and the cohomological iterated residue formula from [Rim14]. In Section 6, we describe how the push-forward (or Gysin) maps associated to Grassmannian fibrations are calculated with equivariant localization and translated to the language of iterated residues, and in Section 7, we provide the proof of the main theorem. In Section 8, we use a new definition of Grothendieck polynomials proposed by Rimányi and Szenes to exhibit that our formula produces an explicit rational function whose coefficients, once expanded as a multivariate Laurent series, correspond to the quiver coefficients. Finally, in Section 9, we pose several questions regarding the rational functions of Section 8. We expect that further analysis of these rational functions will produce new positivity results regarding the quiver coefficients.

2. Quiver Representations and Degeneracy Loci

2.1. Quiver Cycles for Dynkin Quivers

In this paper, we consider only Dynkin quivers, which always have finite sets of vertices and arrows and contain no cycles. Throughout the sequel, Q denotes a Dynkin quiver with vertices $Q_0 = \{1, \ldots, N\}$ and arrows Q_1, $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{N}^N$ denotes a dimension vector, and V denotes the corresponding representation space.

Let Ω be a quiver cycle. For technical reasons, we henceforth assume that Ω is Cohen–Macaulay with rational singularities. In the case of Dynkin quivers, Gabriel's theorem [Gab72] implies that there are only finitely many stable $\mathbb{G}$-orbits and, as a consequence, every quiver cycle must be a $\mathbb{G}$-orbit closure (and conversely). Moreover, the orbits have an explicit description as follows.

Let $\{\phi_i : \ 1 \leq i \leq N\}$ denote the set of simple roots of the corresponding root system, and Φ^+ the set of positive roots. For any positive root ϕ, one obtains the integers $d_1(\phi), \ldots, d_N(\phi)$ defined uniquely by $\phi = \sum_{i=1}^{N} d_i(\phi)\phi_i$. The $\mathbb{G}$-orbits in V are in one-to-one correspondence with the vectors

$$m = (m_\phi) \in \mathbb{N}^{\Phi^+} \quad \text{such that} \quad \sum_{\phi \in \Phi^+} m_\phi d_i(\phi) = v_i \quad \text{for each } 1 \leq i \leq N.$$

Observe that the list of orbits does not depend on the orientation of the arrows of Q but only on the underlying nonoriented graph. Throughout the sequel, we will denote the orbit-closure corresponding to $m \in \mathbb{N}^{\Phi^+}$ by Ω_m.

2.2. *Degeneracy Loci Associated to Quivers*

Let X be a smooth complex projective variety, and let $K(X)$ denote the Grothendieck ring of algebraic vector bundles over X. A *Q-bundle* $(\mathcal{E}_\bullet, f_\bullet) \to X$ is the following data:

- for each $i \in Q_0$, a vector bundle $\mathcal{E}_i \to X$ with $\mathrm{rank}(\mathcal{E}_i) = v_i$, and
- for each arrow $a \in Q_1$, a map of vector bundles $f_a : \mathcal{E}_{t(a)} \to \mathcal{E}_{h(a)}$ over X.

Let $(\mathcal{E}_\bullet, f_\bullet)_x$ denote the *fiber* of the Q-bundle at the point $x \in X$. This consists of the fibers of the vector bundles $(\mathcal{E}_1)_x, \ldots, (\mathcal{E}_N)_x$ and also a linear map $(f_a)_x : (\mathcal{E}_{t(a)})_x \to (\mathcal{E}_{h(a)})_x$ for each $a \in Q_1$. Corresponding to the quiver cycle $\Omega \subset V$, define the *degeneracy locus*

$$\Omega(\mathcal{E}_\bullet) = \{x \in X \mid (\mathcal{E}_\bullet, f_\bullet)_x \in \Omega\}. \tag{4}$$

Observe that the fiber $(\mathcal{E}_\bullet, f_\bullet)_x$ only belongs to $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(E_{t(a)}, E_{h(a)})$ once one specifies a basis in each vector space $(\mathcal{E}_i)_x$. However, the degeneracy locus (4) is well defined since the action of $\mathbb{G}$ on V described by equation (2) can interchange any two choices for bases, and Ω is $\mathbb{G}$-stable. The relevance of the degeneracy locus $\Omega(\mathcal{E}_\bullet)$ is as follows.

PROPOSITION 2.1 (Buch). *If X and Ω are both Cohen–Macaulay and the codimension of $\Omega(\mathcal{E}_\bullet)$ in X is equal to the codimension of Ω in V, then*

$$[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] = \sum_\mu c_\mu(\Omega) G_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots G_{\mu_N}(\mathcal{E}_N - \mathcal{M}_N) \in K(X),$$

where $\mathcal{M}_i = \bigoplus_{j \in T(i)} \mathcal{E}_j$, and the $c_\mu(\Omega)$ are exactly the quiver coefficients defined by equation (3).

The hypothesis of the above result is the reason for our technical assumption that Ω be Cohen–Macaulay. The goal of this paper is to give a new formula for the class corresponding to the structure sheaf of $\Omega(\mathcal{E}_\bullet)$ in the Grothendieck ring $K(X)$ and hence, by the uniqueness of the quiver coefficients, a new formula for $[\mathcal{O}_\Omega] \in K_\mathbb{G}(V)$.

REMARK 2.2 (Notation and genericity). A choice of maps $f_\bullet$ for a Q-bundle amounts to a section of $\mathcal{V} = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathcal{E}_{t(a)}, \mathcal{E}_{h(a)})$. When choices $f_\bullet$ for which the degeneracy locus $\Omega(\mathcal{E}_\bullet)$ has its expected codimension in X actually exist, we call $(\mathcal{E}_\bullet, f_\bullet) \to X$ a *generic Q-bundle*, and in this case, the K-class of the degeneracy locus is independent of the maps.

Our purpose is to calculate the quiver coefficients, which one defines in terms of the equivariant K-theory as in equation (3), and, moreover, we are allowed complete freedom in choosing the base variety X, so we are free to assume the generic case. We will consider only this situation and therefore are justified in omitting any decoration referring to $f_\bullet$ in our notation, for example, as in the definition of equation (4).

3. Resolution of Singularities

In general, the degeneracy locus $\Omega(\mathcal{E}_\bullet)$ defined by (4) is singular, though in the case of Dynkin quivers some "worst-case scenario" results have been established. For example, it is known [BZ01] that over any algebraically closed field, $\Omega(\mathcal{E}_\bullet)$ has at worst rational singularities when Q is of type A, and when one assumes additionally that the field has characteristic zero, the same is true for type D [BZ02]. We work exclusively over $\mathbb{C}$, so the additional technical assumption that Ω have rational singularities is necessary only when Q is of exceptional type (i.e., its underlying nonoriented graph is the Dynkin diagram for E_6, E_7, or E_8).

The proof of our main theorem depends on a construction originally due to Reineke [Rei03] to resolve the singularities, but we follow a slightly more general approach as in [Buc08] and adapt it specifically for Q-bundles. The original idea for using the Reineke resolution to compute cohomology classes of degeneracy loci for quivers comes from the (unfortunately unpublished) work [KS06] of Knutson and Shimozono, who give the name "Kempf collapsing" to this process. For still more details, see also [Rim14].

Let $(\mathcal{E}_\bullet, f_\bullet) \to X$ be a generic Q-bundle. Given $i \in Q_0$ and an integer $0 \le r \le v_i$, we construct the *Grassmannization* $\mathrm{Gr}_{v_i - r}(\mathcal{E}_i) \to X$ with tautological exact sequence of bundles $\mathcal{S} \to \mathcal{E}_i \to \mathcal{Q}$. Here $\mathcal{S}$ is the *tautological subbundle* (whose rank is $s = v_i - r$), and $\mathcal{Q}$ is the *tautological quotient bundle* (whose rank is r). Define $X_{i,r}(\mathcal{E}_\bullet, f_\bullet) = X_{i,r}$ to be the zero scheme $Z(\mathcal{M}_i \to \mathcal{Q}) \subset \mathrm{Gr}_s(\mathcal{E}_i)$ where $\mathcal{M}_i = \bigoplus_{j \in T(i)} \mathcal{E}_j$. Here the map $\mathcal{M}_i \to \mathcal{Q}$ is understood to be the composition of the map $\mathcal{M}_i \to \mathcal{E}_i$, given explicitly by $\sum_{j \in T(i)} f_j$, followed by the natural projection $\mathcal{E}_i \to \mathcal{Q}$.

Observe that over $X_{i,r} \subset \mathrm{Gr}_s(\mathcal{E}_i)$ we obtain an *induced Q-bundle* $(\tilde{\mathcal{E}}_\bullet, \tilde{f}_\bullet)$ defined by the following:

- for $j \neq i$, set $\tilde{\mathcal{E}}_j = \mathcal{E}_j$,
- set $\tilde{\mathcal{E}}_i = \mathcal{S}$,
- if $a \in Q_1$ such that $h(a) \neq i$ and $t(a) \neq i$, then $\tilde{f}_a = f_a$,
- if $t(a) = i$, set $\tilde{f}_a = f_a|_{\mathcal{S}}$,
- if $h(a) = i$, set also $\tilde{f}_a = f_a$.

The last bullet is well defined (and this is the key point) since $y \in Z(\mathcal{M}_i \to \mathcal{Q})$ implies that in the fiber over y, the image of $(f_a)_y : (\mathcal{E}_{t(a)})_y \to (\mathcal{E}_i)_y$ must lie in $\mathcal{S}_y$. Let $\rho_i^r : X_{i,r} \to X$ denote the natural map given by the composition $X_{i,r} = Z(\mathcal{M}_i, \mathcal{Q}) \hookrightarrow \mathrm{Gr}_s(\mathcal{E}_i) \to X$.

More generally, let $\mathbf{i} = (i_1, \dots, i_p)$ be a sequence of quiver vertices, and $\mathbf{r} = (r_1, \dots, r_p)$ a sequence of nonnegative integers subject to the restriction that for each $i \in Q_0$, we have $v_i \ge \sum_{i_\ell = i} r_\ell$. We can now inductively apply the schemes $X_{i,r}$ to obtain the new variety

$$X_{\mathbf{i}, \mathbf{r}} = (\cdots ((X_{i_1, r_1})_{i_2, r_2}) \cdots)_{i_p, r_p}.$$

Let $\rho_{\mathbf{i}}^{\mathbf{r}} : X_{\mathbf{i}, \mathbf{r}} \to X$ denote the natural mapping obtained from the composition $\rho_{i_1}^{r_1} \circ \cdots \circ \rho_{i_p}^{r_p}$.

Now identify each simple root $\phi_i \in \Phi^+$ for $1 \le i \le N$ with the standard unit vector in $\mathbb{N}^N$ with 1 in position i and 0 elsewhere. For dimension vectors $\mathbf{u}, \mathbf{w} \in \mathbb{N}^N$, let

$$\langle \mathbf{u}, \mathbf{w} \rangle = \sum_{i \in Q_0} u_i w_i - \sum_{a \in Q_1} u_{t(a)} w_{h(a)}$$

denote the *Euler form* associated to the quiver Q. If $\Phi' \subset \Phi^+$ is any subset of positive roots, then a partition $\Phi' = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ is called *directed* if for every $1 \le j \le \ell$, we have:

- $\langle \alpha, \beta \rangle \ge 0$ for all $\alpha, \beta \in \mathcal{I}_j$, and
- $\langle \alpha, \beta \rangle \ge 0 \ge \langle \beta, \alpha \rangle$ whenever $i < j$ and $\alpha \in \mathcal{I}_i$, $\beta \in \mathcal{I}_j$.

For Dynkin quivers, a directed partition always exists [Rei03].

Now choose $m = (m_\phi)_{\phi \in \Phi^+}$, a vector of nonnegative integers corresponding to the quiver cycle Ω_m. Let $\Phi' \subset \Phi^+$ be a subset containing $\{\phi \mid m_\phi \ne 0\}$, and let $\Phi' = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_\ell$ be a directed partition. For each $1 \le j \le \ell$, compute the vector

$$\sum_{\phi \in \mathcal{I}_j} m_\phi \phi = (p_1^{(j)}, \ldots, p_N^{(j)}) \in \mathbb{N}^N.$$

From this data construct the sequence $\mathbf{i}_j = (i_1, \ldots, i_n)$ to be any list of the vertices $i \in Q_0$ for which $p_i^{(j)} \ne 0$, with no vertices repeated and ordered so that for every $a \in Q_1$, the vertex $t(a)$ comes before $h(a)$. From this information, set $\mathbf{r}_j = (p_{i_1}^{(j)}, \ldots, p_{i_n}^{(j)})$. Finally, let $\mathbf{i}$ and $\mathbf{r}$ be the concatenated sequences $\mathbf{i} = \mathbf{i}_1 \cdots \mathbf{i}_\ell$ and $\mathbf{r} = \mathbf{r}_1 \cdots \mathbf{r}_\ell$. A pair of sequences $(\mathbf{i}, \mathbf{r})$ constructed in this way is called a *resolution pair* for Ω_m.

PROPOSITION 3.1 (Reineke). *Let Q be a Dynkin quiver, Ω_m a quiver cycle, and $(\mathbf{i}, \mathbf{r})$ a resolution pair for Ω_m. Then in our notation, the natural map $\rho_{\mathbf{i}}^{\mathbf{r}} : X_{\mathbf{i}, \mathbf{r}} \to X$ is a resolution of $\Omega_m(\mathcal{E}_\bullet)$, that is, it has the image $\Omega_m(\mathcal{E}_\bullet)$ and is a birational isomorphism onto this image.*

The important consequence of Reineke's theorem is the following corollary.

COROLLARY 3.2. *With $\rho_{\mathbf{i}}^{\mathbf{r}}$ as before, $(\rho_{\mathbf{i}}^{\mathbf{r}})_*(1) = [\mathcal{O}_{\Omega_m(\mathcal{E}_\bullet)}] \in K(X)$.*

In the above statement, $1 \in K(X_{\mathbf{i}, \mathbf{r}})$ is the class $[\mathcal{O}_{X_{\mathbf{i}, \mathbf{r}}}]$. As we will see in Section 7, this provides an inductive recipe to give a formula for our desired K-class, which has been used previously by Buch (e.g., in [Buc08]). However, our method of computing push-forward maps by iterated residues, which we explain in Sections 4 and 6, is essentially different, and this technology produces formulas in a more compact form. For an analogous approach to this problem in the cohomological setting, see [Rim14].

4. Iterated Residue Operations

Let $f(x)$ be a rational function in the variable x with coefficients in some commutative ring R. Define the operation

$$\operatorname*{Res}_{x=0,\infty} (f(x)\,dx) = \operatorname*{Res}_{x=0}(f(x)\,dx) + \operatorname*{Res}_{x=\infty} (f(x)\,dx), \qquad (5)$$

where $\operatorname{Res}_{x=0}(f(x)\,dx)$ is the usual residue operation from complex analysis (i.e., take the coefficient of x^{-1} in the corresponding Laurent series about $x=0$), and furthermore one recalls that

$$\operatorname*{Res}_{x=\infty} (f(x)\,dx) = \operatorname*{Res}_{x=0}\left(-\frac{1}{x^2} f(1/x)\,dx \right).$$

The idea of using the operation $\operatorname{Res}_{x=0,\infty}$ in K-theory is due to Rimányi and Szenes [RS14].

More generally, let $\mathbf{z} = \{z_1, \ldots, z_n\}$ be an alphabet of ordered commuting indeterminants, and $F(\mathbf{z})$ a rational function in these variables with coefficients in R. Then we define

$$\operatorname*{Res}_{\mathbf{z}=0,\infty} (F(\mathbf{z})\,d\mathbf{z}) = \operatorname*{Res}_{z_n=0,\infty} \cdots \operatorname*{Res}_{z_1=0,\infty} (F(\mathbf{z})\,dz_1 \cdots dz_n).$$

EXAMPLE 4.1. Consider the function $g(a) = \frac{1}{(1-a/b)a}$. Using the convention that $a \ll b$ (which we use throughout the sequel), we obtain that

$$\operatorname*{Res}_{a=0}(g(a)\,da) = \operatorname*{Res}_{a=0}\left(\frac{1}{a}\left(1 + \frac{a}{b} + \frac{a^2}{b^2} + \cdots \right) da \right) = 1.$$

On the other hand,

$$-\frac{1}{a^2} g(1/a) = b\left(\frac{1}{1-ab} \right),$$

and so $\operatorname{Res}_{a=\infty}(g(a)\,da) = 0$. Thus $\operatorname{Res}_{a=0,\infty}(g(a)\,da) = 1$. However, it is more convenient to do the calculation by using the fact that for any meromorphic differential form, the sum of all residues (including the point at infinity) is zero. Since the only other pole of g occurs at $a = b$, we see easily that

$$\operatorname*{Res}_{a=0,\infty} (g(a)\,da) = -\operatorname*{Res}_{a=b}\left(\frac{da}{(1-a/b)a} \right) = 1.$$

EXAMPLE 4.2. Consider the meromorphic differential form

$$F(z_1, z_2) = \frac{(1-\beta_1/z_2)(1-\beta_2/z_2)(1-z_2/z_1)}{(1-z_1/\alpha_1)(1-z_2/\alpha_1)(1-z_1/\alpha_2)(1-z_2/\alpha_2)z_1 z_2} dz_1\,dz_2.$$

Functions of this type will occur often in our analysis, where the result of the operation $\operatorname{Res}_{\mathbf{z}=0,\infty}(F)$ is a certain (Laurent) polynomial in the variables α_i and β_j, separately symmetric in each. We begin by factoring $F = F_1 F_2$, where

$$F_1 = \frac{(1-z_2/z_1)}{(1-z_1/\alpha_1)(1-z_1/\alpha_2)z_1} dz_1 \quad \text{and}$$

$$F_2 = \frac{(1-\beta_1/z_2)(1-\beta_2/z_2)}{(1-z_2/\alpha_1)(1-z_2/\alpha_2)z_2} dz_2.$$

We first use the residue theorem as in the previous example to write that

$$\operatorname*{Res}_{z_1=0,\infty}(F) = -\left(\operatorname*{Res}_{z_1=\alpha_1}(F) + \operatorname*{Res}_{z_1=\alpha_2}(F)\right),$$

and we compute that

$$-\operatorname*{Res}_{z_1=\alpha_1}(F) = -F_2\left(\operatorname*{Res}_{z_1=\alpha_1}(F_1)\right) = F_2\left(\frac{(1-z_2/\alpha_1)}{(1-\alpha_1/\alpha_2)}\right) = F',$$

$$-\operatorname*{Res}_{z_1=\alpha_2}(F) = -F_2\left(\operatorname*{Res}_{z_1=\alpha_2}(F_1)\right) = F_2\left(\frac{(1-z_2/\alpha_2)}{(1-\alpha_2/\alpha_1)}\right) = F''.$$

It is not difficult to see that $\operatorname{Res}_{z_2=\alpha_1}(F') = \operatorname{Res}_{z_2=\alpha_2}(F'') = 0$, so it remains only to compute

$$\begin{aligned}
\operatorname*{Res}_{\mathbf{z}=0,\infty}(F) &= -\operatorname*{Res}_{z_2=\alpha_2}(F') - \operatorname*{Res}_{z_2=\alpha_1}(F'') \\
&= \frac{(1-\beta_1/\alpha_2)(1-\beta_2/\alpha_2)}{(1-\alpha_1/\alpha_2)} + \frac{(1-\beta_1/\alpha_1)(1-\beta_2/\alpha_1)}{(1-\alpha_2/\alpha_1)} = 1 - \frac{\beta_1\beta_2}{\alpha_1\alpha_2}.
\end{aligned}$$

The last line bears resemblance to a Berline–Vergne–Atiyah–Bott-type formula for equivariant localization, adapted for K-theory. This is not accidental, a connection that we explain in Section 6.

5. The Main Theorem

Choose an element $m = (m_\phi) \in \mathbb{N}^{\Phi^+}$ corresponding to the $\mathbb{G}$-orbit closure $\Omega_m \subset V$ having only rational singularities. Let $\mathbf{i} = (i_1, \ldots, i_p)$ and $\mathbf{r} = (r_1, \ldots, r_p)$ be a resolution pair for Ω_m. Let $(\mathcal{E}_\bullet, f_\bullet) \to X$ be a generic Q-bundle over the smooth complex projective base variety X. For each $k \in \{1, \ldots, p\}$, define the alphabets of ordered commuting variables

$$\mathbf{z}_k = \{z_{k1}, \ldots, z_{kr_k}\}$$

and the *discriminant* factors

$$\Delta(\mathbf{z}_k) = \prod_{1 \le i < j \le r_k} \left(1 - \frac{z_{kj}}{z_{ki}}\right).$$

For each $i \in Q_0$, recall the definition of the set $T(i)$ from equation (1) and define the alphabets of commuting variables

$$\mathbb{E}_i = \{\varepsilon_{i1}, \ldots, \varepsilon_{iv_i}\}, \qquad \mathbb{M}_i = \bigcup_{j \in T(i)} \mathbb{E}_j,$$

where the degree d elementary symmetric function $e_d(\mathbb{E}_i) = e_d(\varepsilon_{i1}, \ldots, \varepsilon_{iv_i})$ is interpreted as the class $[\bigwedge^d(\mathcal{E}_i)] \in K(X)$. Consequently, we conclude that $e_d(\varepsilon_{i1}^{-1}, \ldots, \varepsilon_{iv_i}^{-1}) = [\bigwedge^d(\mathcal{E}_i^\vee)]$. Henceforth, we will call such a set of formal commuting variables the *Grothendieck roots* of $\mathcal{E}_i$. Finally, for each $k \in \{1, \ldots, p\}$, define:

- the *residue factors*

$$R_k = \prod_{y \in \mathbf{z_k}} \frac{\prod_{x \in \mathbb{M}_{i_k}} (1 - xy)}{\prod_{x \in \mathbb{E}_{i_k}} (1 - xy)},$$

- the *interference factors*

$$I_k = \prod_{y \in \mathbf{z_k}} \frac{\prod_{\substack{\ell < k : i_\ell = i_k \\ x \in \mathbf{z_\ell}}} (1 - y/x)}{\prod_{\substack{\ell < k : i_\ell \in T(i_k) \\ x \in \mathbf{z_\ell}}} (1 - y/x)}, \quad \text{and}$$

- the *differential factors*

$$D_k = \Delta(\mathbf{z}_k) \cdot d \log(\mathbf{z}_k) = \Delta(\mathbf{z}_k) \prod_{i=1}^{r_k} \frac{dz_{ki}}{z_{ki}}.$$

THEOREM 5.1. *With the notation just given, the class $[\mathcal{O}_{\Omega_m(\mathcal{E}_\bullet)}] \in K(X)$ is given by the iterated residue*

$$\operatorname*{Res}_{\mathbf{z}_1 = 0, \infty} \cdots \operatorname*{Res}_{\mathbf{z}_p = 0, \infty} \left(\prod_{k=1}^{p} R_k I_k D_k \right). \tag{6}$$

REMARK 5.2. The result of the operation in (6) is certainly a Laurent polynomial in the Grothendieck roots of the bundles $\mathcal{E}_i$. Moreover, the argument of the iterated residue operation is separately symmetric in each set of Grothendieck roots since they appear only in the residue factors R_k and do so symmetrically. This ensures that the result is actually an element of $K(X)$ since it can be written as a polynomial in the classes $[\bigwedge^d \mathcal{E}_i]$ and $[\bigwedge^d \mathcal{E}_i^\vee]$.

EXAMPLE 5.3. Consider the "inbound A_3" quiver $\{1 \to 2 \leftarrow 3\}$. Let ϕ_1, ϕ_2, and ϕ_3 be the corresponding simple roots, so that the positive roots of the underlying root system can be represented by $\phi_{ij} = \sum_{i \le \ell \le j} \phi_\ell$ for $1 \le i \le j \le 3$. Consider now the orbit closure $\Omega_m \subset V = \operatorname{Hom}(E_1, E_2) \oplus \operatorname{Hom}(E_3, E_2)$ corresponding to $m_{11} = m_{23} = 0$, but all other $m_{ij} = 1$, so that the resulting dimension vector is $\mathbf{v} = (2, 3, 2)$. Set $\Phi' = \{\phi_{12}, \phi_{13}, \phi_{22}, \phi_{33}\}$ and choose the directed partition

$$\Phi' = \{\phi_{22}\} \cup \{\phi_{12}, \phi_{13}\} \cup \{\phi_{33}\}$$

with corresponding resolution pair $\mathbf{i} = (2, 1, 3, 2, 3)$ and $\mathbf{r} = (1, 2, 1, 2, 1)$. Let $\mathcal{E}_\bullet \to X$ be a generic Q-bundle. Set

$$\mathbb{E}_1 = \{\alpha_1, \alpha_2\}, \qquad \mathbb{E}_2 = \{\beta_1, \beta_2, \beta_3\}, \qquad \mathbb{E}_3 = \{\gamma_1, \gamma_2\}$$

to be the Grothendieck roots of $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, respectively. In particular, this means that $\mathbb{M}_1 = \mathbb{M}_3 = \{\}$, whereas $\mathbb{M}_2 = \{\alpha_1, \alpha_2, \gamma_1, \gamma_2\}$. Following the recipe of the theorem and equation (6), we form the alphabets $\mathbf{z_k}$ for $1 \le k \le 5$, which we rename as

$$\mathbf{z}_1 = \{v\}, \qquad \mathbf{z}_2 = \{w_1, w_2\}, \qquad \mathbf{z}_3 = \{x\}, \qquad \mathbf{z}_4 = \{y_1, y_2\}, \qquad \mathbf{z}_5 = \{z\}$$

and construct the differential form

$$\frac{\prod_{s\in\mathbb{M}_2, t\in\mathbf{z}_1\cup\mathbf{z}_4}(1-st)}{\prod_{\substack{s\in\mathbb{E}_2\\t\in\mathbf{z}_1\cup\mathbf{z}_4}}(1-st)\prod_{\substack{s\in\mathbb{E}_1\\t\in\mathbf{z}_2}}(1-st)\prod_{\substack{s\in\mathbb{E}_3\\t\in\mathbf{z}_3\cup\mathbf{z}_5}}(1-st)}$$

$$\times\;\frac{(1-z/x)\prod_{i=1}^{2}(1-y_i/v)}{\prod_{\substack{s\in\mathbf{z}_4\\t\in\mathbf{z}_2\cup\mathbf{z}_3}}(1-s/t)}\prod_{k=1}^{5}D_k. \tag{7}$$

A calculation in *Mathematica* shows that the result of applying the iterated residue operation $\mathrm{Res}_{\mathbf{z}_1=0,\infty}\cdots\mathrm{Res}_{\mathbf{z}_5=0,\infty}$ to the latter form gives

$$\begin{aligned}
[\mathcal{O}_{\Omega_m(\mathcal{E}_\bullet)}] = 1 &- \frac{\alpha_1\alpha_2\gamma_1^2\gamma_2^2}{\beta_1^2\beta_2^2\beta_3^2} + \frac{\alpha_1\alpha_2\gamma_1\gamma_2}{\beta_1\beta_2\beta_3^2} + \frac{\alpha_1\alpha_2\gamma_1\gamma_2}{\beta_1\beta_2^2\beta_3} + \frac{\alpha_1\alpha_2\gamma_1\gamma_2}{\beta_1^2\beta_2\beta_3}\\
&- \frac{\gamma_1\gamma_2}{\beta_1\beta_2} - \frac{\gamma_1\gamma_2}{\beta_1\beta_3} - \frac{\gamma_1\gamma_2}{\beta_2\beta_3} - \frac{\alpha_1\alpha_2\gamma_1}{\beta_1\beta_2\beta_3}\\
&- \frac{\alpha_1\alpha_2\gamma_2}{\beta_1\beta_2\beta_3} + \frac{\gamma_1^2\gamma_2}{\beta_1\beta_2\beta_3} + \frac{\gamma_1\gamma_2^2}{\beta_1\beta_2\beta_3}.
\end{aligned} \tag{8}$$

Following Buch's combinatorial description of the inbound A_3 case (cf. Section 7.1 of [Buc08]), we obtain in terms of double stable Grothendieck polynomials that

$$[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] = G_{21}(\mathcal{E}_2-\mathcal{M}_2) + G_2(\mathcal{E}_2-\mathcal{M}_2)G_1(\mathcal{E}_1) - G_{21}(\mathcal{E}_2-\mathcal{M}_2)G_1(\mathcal{E}_1), \tag{9}$$

which, as one can check, agrees with equation (8) once expanded (note that in the expression above, the subscript "21" is the partition whose Young diagram has two rows, the first with two boxes and the second with one box). The leading term (see [Buc08], Corollary 4.5) is given by $s_{21}(\mathcal{E}_2-\mathcal{M}_2) + s_2(\mathcal{E}_2-\mathcal{M}_2)s_1(\mathcal{E}_1)$, which agrees with the result of [Rim14], Section 6.2.

We wish also to check this example against the cohomological iterated residue formula of Rimányi [Rim14]. From the K-class $[\mathcal{O}_{\Omega_m(\mathcal{E}_\bullet)}]$ we obtain the cohomology class $[\Omega_m(\mathcal{E}_\bullet)]$ by the following method, which we explain in general.

Let $\mathcal{E}_1,\ldots,\mathcal{E}_n$ be vector bundles over X with ranks $e_1,\ldots,e_n$, respectively, and

$$\mathbb{E}_1 = \{\varepsilon_{11},\ldots,\varepsilon_{1e_1}\}, \qquad \ldots, \qquad \mathbb{E}_n = \{\varepsilon_{n1},\ldots,\varepsilon_{ne_n}\}$$

the respective sets of Grothendieck roots. If $f(\varepsilon_{ij})$ is a Laurent polynomial, separately symmetric in each set of variables $\mathbb{E}_i$, then f represents a well-defined element in $K(X)$, and for such a class, we replace each ε_{ij} with the exponential $\exp(\varepsilon_{ij}\xi)$. Then a class in $H^*(X)$ is given by taking the lowest degree nonzero term in the Taylor expansion (with respect to ξ about zero) of $f(\exp(\varepsilon_{ij}\xi))$ where, once in the cohomological setting, the variables ε_{ij} are interpreted as *Chern roots* of the corresponding bundles. In particular, applying this process to the class $[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}]$ yields the class $[\Omega(\mathcal{E}_\bullet)] \in H^*(X)$. This is actually the leading term of the *Chern character* $K(X) \to H^*(X)$. For more details, see Section 4 of [Buc08].

Applying the latter algorithm to the Laurent polynomial (8) gives that the corresponding class in $H^*(X)$ must be

$$
\begin{aligned}
[\Omega_m(\mathcal{E}_\bullet)] = {}& 2\beta_1\beta_2\beta_3 + \beta_1^2\beta_2 + \beta_1\beta_2^2 + \beta_1^2\beta_3 + \beta_2^2\beta_3 + \beta_1\beta_3^2 + \beta_2\beta_3^2 \\
& - \alpha_1\beta_1\beta_2 - \alpha_2\beta_1\beta_2 - \alpha_1\beta_2\beta_3 - \alpha_2\beta_2\beta_3 - \alpha_1\beta_1\beta_3 - \alpha_2\beta_1\beta_3 \\
& - 2(\beta_1\beta_2\gamma_1 + \beta_1\beta_2\gamma_2 + \beta_2\beta_3\gamma_1 + \beta_2\beta_3\gamma_2 + \beta_1\beta_3\gamma_1 + \beta_1\beta_3\gamma_2) \\
& - \beta_1^2\gamma_1 - \beta_1^2\gamma_2 - \beta_2^2\gamma_1 - \beta_2^2\gamma_2 - \beta_3^2\gamma_1 - \beta_3^2\gamma_2 \\
& + \beta_1\gamma_1^2 + \beta_2\gamma_1^2 + \beta_3\gamma_1^2 + \beta_1\gamma_2^2 + \beta_2\gamma_2^2 + \beta_3\gamma_2^2 \\
& + 2(\beta_1\gamma_1\gamma_2 + \beta_2\gamma_1\gamma_2 + \beta_3\gamma_1\gamma_2) - \gamma_1^2\gamma_2 - \gamma_1\gamma_2^2 \\
& - \alpha_1\gamma_1^2 - \alpha_2\gamma_1^2 - \alpha_1\gamma_2^2 - \alpha_2\gamma_2^2 - \alpha_1\gamma_1\gamma_2 - \alpha_2\gamma_1\gamma_2 \\
& + \alpha_1\beta_1\gamma_1 + \alpha_2\beta_1\gamma_1 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_2\gamma_1 + \alpha_1\beta_3\gamma_1 + \alpha_2\beta_3\gamma_1 \\
& + \alpha_1\beta_1\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_1\beta_2\gamma_2 + \alpha_2\beta_2\gamma_2 + \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_2,
\end{aligned}
$$

where the variables $\{\alpha_i\}$, $\{\beta_i\}$, and $\{\gamma_i\}$ are now interpreted as the Chern roots of $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$, respectively. If one sets $A_i = c_i(\mathcal{E}_1)$, $B_i = c_i(\mathcal{E}_2)$, and $C_i = c_i(\mathcal{E}_3)$ to be the corresponding Chern classes, then the latter expression becomes

$$
[\Omega_m] = (B_1 - A_1)(B_2 + C_1^2) - C_1(B_1^2 + C_2) + A_1(B_1C_1 + C_2) - B_3. \quad (10)
$$

In [Rim14], equation (9), this class is computed to be

$$
-c_3(\mathcal{M}_2^\vee - \mathcal{E}_2^\vee) + c_2(\mathcal{M}_2^\vee - \mathcal{E}_2^\vee)c_1(\mathcal{M}_2^\vee - \mathcal{E}_2^\vee) + c_2(\mathcal{M}_2^\vee - \mathcal{E}_2^\vee)c_1(-\mathcal{E}_1^\vee), \quad (11)
$$

where the *relative Chern classes* $c_n(\mathcal{V}^\vee - \mathcal{W}^\vee)$ are defined by the formal expression

$$
\sum_{n \geq 0} c_n(\mathcal{V}^\vee - \mathcal{W}^\vee)\xi^n = \frac{\sum_{k \geq 0} c_k(\mathcal{V})(-\xi)^k}{\sum_{\ell \geq 0} c_\ell(\mathcal{W})(-\xi)^\ell}
$$

for bundles $\mathcal{V}$ and $\mathcal{W}$ with respective Chern classes $c_k(\mathcal{V})$ and $c_\ell(\mathcal{W})$. Using the Chern classes A_i, B_i, and C_i as before, we substitute into the expression (11) to obtain

$$
\begin{aligned}
[\Omega_m] = {}& -[(B_1^3 + B_3 - 2B_1B_2) - (B_1^2 - B_2)(A_1 + C_1) \\
& + B_1(A_2 + A_1C_1 + C_2) - (A_1C_2 + A_2C_2)] \\
& + [(B_1^2 - B_2) - B_1(A_1 + C_1) + (A_2 + A_1C_1 + C_2)][B_1 - (A_1 + C_1)] \\
& + [(B_1^2 - B_2) - B_1(A_1 + C_1) + (A_2 + A_1C_1 + C_2)]A_1,
\end{aligned}
$$

and a little high-school algebra shows that this is identical to (10). We will give a different computation of this class in Section 8 using iterated residues to explicitly write the K-class above as a polynomial in double stable Grothendieck polynomials. Further, in Section 9, we will compare directly to Rimanyi's iterated residue operation producing the Schur expansion in cohomology.

REMARK 5.4. The leading term of the class (9) is, according to Buch, denoted by $s_{21}(\mathcal{E}_2 - \mathcal{M}_2) + s_2(\mathcal{E}_2 - \mathcal{M}_2)s_1(\mathcal{E}_1)$. In [Rim14], the same Schur functions
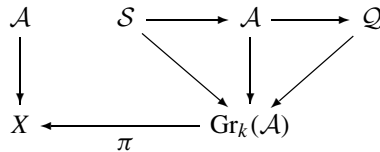
are instead evaluated on $\mathcal{M}_i^\vee - \mathcal{E}_i^\vee$, but both authors' notations are interpreted to mean

$$s_\lambda = \det(h_{\lambda_i + j - i}),$$

where the h_ℓ are the appropriate relative Chern classes defined above.

6. Equivariant Localization and Iterated Residues

Let X be a smooth complex projective variety, and $\mathcal{A} \to X$ a vector bundle of rank n. Choose an integer $1 \le k \le n$ and set $q = n - k$. The integers n, k, and q will be fixed throughout the section. Form the Grassmannization of $\mathcal{A}$ over X, $\pi : \mathrm{Gr}_k(\mathcal{A}) \to X$, with tautological exact sequence of vector bundles $\mathcal{S} \to \mathcal{A} \to \mathcal{Q}$ over $\mathrm{Gr}_k(\mathcal{A})$. By convention we suppress the notation of pullback bundles. The following diagram is useful to keep in mind:

$$
\begin{array}{ccccc}
\mathcal{A} & & \mathcal{S} \longrightarrow \mathcal{A} \longrightarrow \mathcal{Q} \\
\downarrow & & \searrow \quad \downarrow \quad \swarrow \\
X & \xleftarrow{\;\;\pi\;\;} & \mathrm{Gr}_k(\mathcal{A})
\end{array}
$$

Let $\{\sigma_1, \ldots, \sigma_k\}$ and $\{\omega_1, \ldots, \omega_q\}$ be sets of Grothendieck roots for $\mathcal{S}$ and $\mathcal{Q}$, respectively. Set $R = K(X)$ and let f be a Laurent polynomial in $R[\sigma_i^{\pm 1}; \omega_j^{\pm 1}]$ separately symmetric in the σ and ω variables (where $1 \le i \le k$ and $1 \le j \le q$). The symmetry of f implies that it represents a K-class in $K(\mathrm{Gr}_k(\mathcal{A}))$. The purpose of this section is to give an explanation of the push-forward map $\pi_* : K(\mathrm{Gr}_k(\mathcal{A})) \to K(X)$ applied to f.

Many formulas for π_* exist in the literature. For example, Buch [Buc02a, Theorem 7.3] has given a formula in terms of stable Grothendieck polynomials and the combinatorics of integer sequences. We utilize the method of equivariant localization. The following formula is well known to experts, deeply embedded in the folklore of the subject, and, as such, a single (or original) reference is unknown to the author. Following the advice of [FS12], we refer the reader to various sources, namely [KR99] and [CG97].

PROPOSITION 6.1. *Let $\{\alpha_1, \ldots, \alpha_n\}$ be Grothendieck roots for $\mathcal{A}$ and set $[n] = \{1, \ldots, n\}$. Let $[n, k]$ denote the set of all k-element subsets of $[n]$, and for any subset $J = \{j_1, \ldots, j_r\} \subset [n]$, let α_J denote the collection of variables $\{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$. With this notation, π_* acts by*

$$f(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_q) \mapsto \sum_{I \in [n, k]} \frac{f(\alpha_I; \alpha_{\bar{I}})}{\prod_{i \in I, j \in \bar{I}} (1 - \alpha_i / \alpha_j)},$$

where $\bar{I}$ denotes the complement $[n] \setminus I$.

EXAMPLE 6.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both vector bundles of rank 2, and let $\{\alpha_1, \alpha_2\}$ be as before. Let $\{\beta_1, \beta_2\}$ be Grothendieck roots of $\mathcal{B}$. Form the Grassmannization $\mathrm{Gr}_1(\mathcal{A}) = \mathbb{P}(\mathcal{A})$ and consider the class

$$f(\sigma, \omega) = \left(1 - \frac{\beta_1}{\omega}\right)\left(1 - \frac{\beta_2}{\omega}\right) \in K(\mathbb{P}(\mathcal{A})).$$

The expert will recognize this expression as the K-class associated to the structure sheaf of the subvariety in $\mathbb{P}(\mathcal{A})$ defined by the vanishing of a generic section $\mathbb{P}(\mathcal{A}) \to \mathrm{Hom}(\mathcal{B}, \mathcal{Q})$. In any event, applying Proposition 6.1 gives that

$$\pi_*(f(\sigma, \omega)) = \frac{(1 - \beta_1/\alpha_2)(1 - \beta_2/\alpha_2)}{(1 - \alpha_1/\alpha_2)} + \frac{(1 - \beta_1/\alpha_1)(1 - \beta_2/\alpha_1)}{(1 - \alpha_2/\alpha_1)},$$

an expression which we concluded was equal to $1 - \beta_1\beta_2/(\alpha_1\alpha_2)$ in Example 4.2. In comparison to Buch's formula (see [Buc02a], Theorem 7.3), we have set $f = G_2(\mathcal{Q} - \mathcal{B})$ and obtained that $\pi_*(f) = G_1(\mathcal{A} - \mathcal{B})$.

Observe that, in general, the expression obtained from applying Proposition 6.1 has many terms (the binomial coefficient $\binom{n}{k}$ to be precise) and by this measure is quite complicated. Hence, we seek to encode the expression in a more compact form, and this is accomplished by the following proposition, which is just a clever rewriting of the localization formula, pointed out to the author by Rimányi in correspondence with Szenes.

PROPOSITION 6.3. *Let* $\mathbf{z} = \{z_1, \dots, z_n\}$ *be an alphabet of ordered, commuting variables. If* f *has no poles in* $R = K(X)$ *(aside from zero and the point at infinity), then in the setting of Proposition 6.1 we have that* π_* *acts by*

$$f(\sigma_1, \dots, \sigma_k; \omega_1, \dots, \omega_q) \mapsto \operatorname*{Res}_{\mathbf{z}=0,\infty} \left(f(\mathbf{z}) \frac{\prod_{1 \leq i < j \leq n}(1 - z_j/z_i)}{\prod_{i,j=1}^n (1 - z_i/\alpha_j)} d\log \mathbf{z} \right) m,$$

where $d\log \mathbf{z} = \prod_{i=1}^n d\log(z_i) = \prod_{i=1}^n (dz_i)/z_i.$

Proof. The proof is a formal application of the fact that the sum of the residues at all poles (including infinity) vanishes. We leave the details to the reader, but for a similar proof in the case of equivariant localization and proper push-forward in cohomology, see [Zie14]. $\square$

If the class represented by f depends only on the variables σ_i, then the expression above can be dramatically simplified – namely, one needs to utilize only the variables z_i for $1 \leq i \leq k$.

COROLLARY 6.4. *If* $f = f(\sigma_1, \dots, \sigma_k)$ *depends only on the Grothendieck roots of* $\mathcal{S}$, *then setting* $\mathbf{z} = \{z_1, \dots, z_k\}$, π_* *acts by*

$$f(\sigma_1, \dots, \sigma_k) \mapsto \operatorname*{Res}_{\mathbf{z}=0,\infty} \left(f(\mathbf{z}) \frac{\prod_{1 \leq i < j \leq k}(1 - z_j/z_i)}{\prod_{1 \leq i \leq k, 1 \leq j \leq n}(1 - z_i/\alpha_j)} d\log \mathbf{z} \right).$$

Proof. We will prove the result in the case $n = 2$ and $s = q = 1$; the general case is analogous. Let $f(\sigma)$ represent a class in $K(\mathrm{Gr}_s(\mathcal{A}))$. Proposition 6.3 implies that $\pi_*(f)$ is

$$\operatorname*{Res}_{\mathbf{z}=0,\infty}\left(f(z_1)\frac{(1-z_2/z_1)\,d\log\mathbf{z}}{\prod_{i,j=1}^2(1-z_i/\alpha_j)}\right).$$

Taking the "finite" residues of $z_1 = \alpha_1$ and $z_1 = \alpha_2$, we obtain that this is equal to

$$\operatorname*{Res}_{z_2=0,\infty}\left(\frac{f(\alpha_1)\cancel{(1-z_2/\alpha_1)}\,dz_2}{\cancel{(1-z_2/\alpha_1)}(1-\alpha_1/\alpha_2)(1-z_2/\alpha_2)z_2}\right.$$
$$\left.+\frac{f(\alpha_2)\cancel{(1-z_2/\alpha_2)}\,dz_2}{\cancel{(1-z_2/\alpha_2)}(1-\alpha_2/\alpha_1)(1-z_2/\alpha_1)z_2}\right).$$

In both terms of this expression, the only parts that depend on z_2 have the form $1/((1-z_2/\alpha_i)z_2)$, and Example 4.1 implies that residues of this type always evaluate to 1. Observe then that the expression is equivalent to what we would have obtained by removing all the factors involving z_2 at the beginning. □

We can obtain a similar expression for classes depending only on the variables ω_j, which requires only $n - k = q$ residue variables.

COROLLARY 6.5. *If $f = f(\omega_1, \ldots, \omega_q)$ depends only on the Grothendieck roots of $\mathcal{Q}$, then setting $\mathbf{z} = \{z_1, \ldots, z_q\}$, π_* acts by*

$$f(\omega_1,\ldots,\omega_q)\mapsto\operatorname*{Res}_{\mathbf{z}=0,\infty}\left(f(z_1^{-1},\ldots,z_q^{-1})\frac{\prod_{1\le i<j\le k}(1-z_j/z_i)}{\prod_{1\le i\le q,1\le j\le n}(1-\alpha_j z_i)}\,d\log\mathbf{z}\right).$$

Proof. We use the fact that $\mathrm{Gr}_s(\mathcal{A})$ is homeomorphic to the Grassmannian fibration $\mathrm{Gr}_q(\mathcal{A}^\vee)$, over which the tautological exact sequence $\mathcal{Q}^\vee \to \mathcal{A}^\vee \to \mathcal{S}^\vee$ lies. We are now in a situation to apply the previous corollary, once we recognize that for any bundle $\mathcal{B}$, if $\{\beta_i\}_{1\le i\le\mathrm{rank}\,\mathcal{B}}$ is a set of Grothendieck roots, then the corresponding Grothendieck roots of $\mathcal{B}^\vee$ are supplied by $\{\beta_i^{-1}\}_{1\le i\le\mathrm{rank}\,\mathcal{B}}$. □

7. Proof of the Main Theorem

In this section, we prove Theorem 5.1 and use the notation of Section 5 except where otherwise specified. We will need the language and notation of Reineke's construction, which is detailed in Section 3. We also introduce the following notation. If $\mathbb{A} = \{a_1, \ldots, a_n\}$ and $\mathbb{B} = \{b_1, \ldots, b_m\}$, then we write

$$\left(1-\frac{\mathbb{A}}{\mathbb{B}}\right)=\prod_{\substack{1\le i\le n\\1\le j\le m}}\left(1-\frac{a_i}{b_j}\right),\qquad(1-\mathbb{A}\mathbb{B})=\prod_{\substack{1\le i\le n\\1\le j\le m}}(1-a_ib_j).$$

In the special case that $\mathbb{A}$ and $\mathbb{B}$ are the respective sets of Grothendieck roots of vector bundles $\mathcal{A}$ and $\mathcal{B}$, we will write $\mathcal{A}_\bullet = \mathbb{A}$ and $\mathcal{B}_\bullet = \mathbb{B}$. We can also mix

these notations and write, for example,

$$\left(1 - \frac{\mathcal{A}_\bullet}{\mathbb{B}}\right) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left(1 - \frac{a_i}{b_j}\right), \qquad \left(1 - \frac{\mathbb{A}}{\mathcal{B}_\bullet}\right) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left(1 - \frac{a_i}{b_j}\right),$$

$$(1 - \mathcal{A}_\bullet\mathbb{B}) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (1 - a_i b_j), \qquad (1 - \mathbb{A}\mathcal{B}_\bullet) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (1 - a_i b_j),$$

provided that $\mathbb{A}$ corresponds to a set of Grothendieck roots, and $\mathbb{B}$ represents a set of some other formal variables (as on the left) or vice versa (as on the right). This is not to be confused with the notation $\mathcal{E}_\bullet \to X$ used to denote a Q-bundle. The context should always make clear the intended meaning of the "bullet" symbol as a subscript to calligraphic letters.

We will prove Theorem 5.1 by iteratively understanding the sequence of maps $\rho_{i_k}^{r_k}$ in the Reineke resolution, which break up into a natural inclusion followed by a natural projection from a Grassmannization (cf. Section 3). Our first step is the following lemma, which provides a formula for the natural inclusion.

LEMMA 7.1. *Let X be a smooth base variety, and $\mathcal{M} \to \mathcal{E}$ a map of vector bundles over X. Let $0 \leq s \leq \mathrm{rank}(\mathcal{E})$ and form the Grassmannization $\pi : \mathrm{Gr}_s(\mathcal{E}) \to X$ with tautological exact sequence $\mathcal{S} \to \mathcal{E} \to \mathcal{Q}$. Set $Z = Z(\mathcal{M} \to \mathcal{Q}) \subset \mathrm{Gr}_s(\mathcal{E})$ and let $\iota : Z \hookrightarrow \mathrm{Gr}_s(\mathcal{E})$ denote the natural inclusion. If $f \in K(Z)$ is a class expressed entirely in terms of bundles pulled back from $\mathrm{Gr}_s(\mathcal{E})$, then $\iota_* : K(Z) \to K(\mathrm{Gr}_s(\mathcal{E}))$ acts on f by*

$$f \mapsto f \cdot \left(1 - \frac{\mathcal{M}_\bullet}{\mathcal{Q}_\bullet}\right).$$

Proof. Set $r = \mathrm{rank}(\mathcal{Q}) = \mathrm{rank}(\mathcal{E}) - s$ and $m = \mathrm{rank}(\mathcal{M})$. Because of our assumption on f, we know that $\iota_*(f) = \iota_*(\iota^*(f))$, and therefore the adjunction formula implies that $\iota_*(f) = f \cdot \iota_*(1)$. The image of $\iota_*(1)$ is exactly the class $[\mathcal{O}_{Z(\mathcal{M} \to \mathcal{Q})}] \in K(\mathrm{Gr}_s(\mathcal{E}))$, which is given by the K-theoretic Giambelli–Thom–Porteous theorem, proved in [Buc02a], Theorem 2.3. Explicitly,

$$\iota_*(1) = G_R(\mathcal{Q} - \mathcal{M}),$$

where G_R denotes the double stable Grothendieck polynomial associated to the rectangular partition $R = (m)^r$, that is, the partition whose Young diagram has r rows, each containing m boxes. The result of evaluating G_R on the bundles in question is given, for example, by [Buc02b], equation (7.1)

$$G_R(\mathcal{Q} - \mathcal{M}) = G_R(x_1, \ldots, x_r; y_1, \ldots, y_m) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq m}} (x_i + y_j - x_i y_j)$$

with the specializations $x_i = 1 - \omega_i^{-1}$ and $y_j = 1 - \mu_j$, where $\mathcal{Q}_\bullet = \{\omega_i\}_{i=1}^r$ and $\mathcal{M}_\bullet = \{\mu_j\}_{j=1}^m$ denote the respective Grothendieck roots. The result of this substitution is exactly the statement of the lemma. □

For the Dynkin quiver Q, smooth complex projective variety X, and quiver cycle Ω, let $\mathcal{E}_\bullet \to X$ be a generic Q-bundle, and $\mathbf{i} = (i_1, \ldots, i_p)$ and $\mathbf{r} = (r_1, \ldots, r_p)$ be a resolution pair for Ω. We will show that at each step in the Reineke resolution, the result can be written as an iterated residue entirely in terms of residue variables (i.e., the alphabets $\mathbf{z}_k$) and Grothendieck roots only of the bundles $\mathcal{E}_i$ or the tautological quotient bundles constructed at previous steps. Moreover, the form of this result is arranged in such a way to evidently produce the formula of the main theorem.

By Corollary 3.2 we must begin with the image of $(\rho_{i_p}^{r_p})_*(1)$. Set $i = i_p \in Q_0$ and $\mathcal{A} = \mathcal{E}_i$. Write $T(i) = \{t_1, \ldots, t_\ell\} \subset Q_0$ and denote $\mathcal{E}_{t_j} = \mathcal{B}_j$. Recall that whenever $j \in Q_0$ appears in the Reineke resolution sequence $\mathbf{i}$, it is subsequently replaced with a tautological subbundle. For any bundle $\mathcal{F}$ and Grassmannization $\mathrm{Gr}_s(\mathcal{F})$, we will denote the tautological subbundle by $\mathcal{S}\mathcal{F}$. If this is done multiple times, we let $\mathcal{S}^n\mathcal{F}$ denote the tautological subbundle over $\mathrm{Gr}_{s'}(\mathcal{S}^{n-1}\mathcal{F})$. Similarly, we denote the tautological quotient over $\mathrm{Gr}_s(\mathcal{F})$ by $\mathcal{Q}\mathcal{F}$.

Suppose that the vertex $i \in Q_0$ appears n times in $\mathbf{i}$ and, moreover, that each tail vertex t_j appears n_j times. Set

$$Y = (\cdots (X)_{i_1, r_1} \cdots)_{i_{p-1}, r_{p-1}}, \qquad \mathcal{M} = \bigoplus_{j=1}^{\ell} \mathcal{S}^{n_j}\mathcal{B}_j,$$

$$Z = Z(\mathcal{M} \to \mathcal{Q}\mathcal{S}^{n-1}\mathcal{A}).$$

Then the composition $\rho_{i_p}^{r_p} = \pi_p \circ \iota_p$ is depicted diagrammatically as follows:
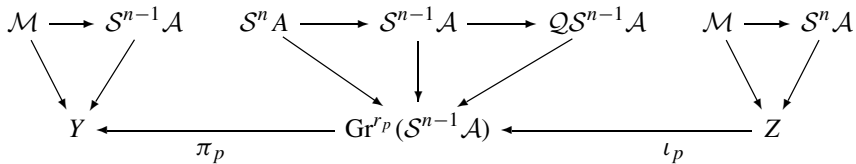
where the notation $\mathrm{Gr}^r(\mathcal{F})$ denotes that the rank of the tautological quotient is r.

Starting with the class $1 \in K(Z)$, Lemma 7.1 implies that $(\iota_p)_*(1)$ is the product $(1 - \mathcal{M}_\bullet/(\mathcal{Q}\mathcal{S}^{n-1}\mathcal{A})_\bullet)$. Now for any family of variables $\mathbb{T}$, bundle $\mathcal{F}$, and Grassmannization $\mathrm{Gr}_s(\mathcal{F})$, we have the formal identity

$$(1 - \mathcal{F}_\bullet\mathbb{T}) = (1 - (\mathcal{S}\mathcal{F})_\bullet\mathbb{T})(1 - (\mathcal{Q}\mathcal{F})_\bullet\mathbb{T}), \tag{12}$$

and applying this many times, we can rewrite $(\iota_p)_*(1)$ as

$$\prod_{j=1}^{\ell} \frac{(1 - (\mathcal{B}_j)_\bullet/(\mathcal{Q}\mathcal{S}^{n-1}\mathcal{A})_\bullet)}{\prod_{k=1}^{n_j}(1 - (\mathcal{Q}\mathcal{S}^{n_j-k}\mathcal{B}_j)_\bullet/(\mathcal{Q}\mathcal{S}^{n-1}\mathcal{A})_\bullet)}.$$

Using Corollary 6.5 to compute $(\pi_p)_*$ of the latter, we obtain that $(\rho_{i_p}^{r_p})_*(1)$ is given by

$$\operatorname*{Res}_{\mathbf{z}_p = 0, \infty} \left(\prod_{j=1}^{\ell} \frac{(1 - (\mathcal{B}_j)_\bullet\mathbf{z}_p)}{(1 - (\mathcal{S}^{n-1}\mathcal{A})_\bullet\mathbf{z}_p)} \frac{D_p}{\prod_{k=1}^{n_j}(1 - (\mathcal{Q}\mathcal{S}^{n_j-k}\mathcal{B}_j)_\bullet\mathbf{z}_p)} \right),$$

but using equation (12) on the denominator factors $(1 - (\mathcal{S}^{n-1}\mathcal{A})_\bullet \mathbf{z}_p)$, this can also be rewritten as

$$\operatorname*{Res}_{\mathbf{z}_p=0,\infty} \left(R_p D_p \frac{\prod_{w=1}^{n}(1 - (Q\mathcal{S}^{n-w}\mathcal{A})_\bullet \mathbf{z}_p)}{\prod_{j=1}^{\ell} \prod_{k=1}^{n_j}(1 - (Q\mathcal{S}^{n_j-k}\mathcal{B}_j)_\bullet \mathbf{z}_p)} \right). \tag{13}$$

Now observe that when the alphabets $\mathbf{z}_u$ for $u < p$ are utilized as residue variables to push-forward classes containing only Grothendieck roots corresponding to tautological quotient bundles (as in Corollary 6.5) through the rest of the Reineke resolution, the remaining rational function will produce exactly the interference factor I_p. Expression (13) depends only on bundles pulled back to Y from earlier iterations of the Reineke construction, and so Lemma 7.1 again applies. Furthermore, the formal algebraic manipulations required to compute each subsequent step in the resolution are completely analogous to those previously given, and therefore the result of the composition $(\rho_{\mathbf{i}}^{\mathbf{r}})_* = (\rho_{i_1}^{r_1})_* \circ \cdots \circ (\rho_{i_p}^{r_p})_*(1)$ is exactly the expression of equation (6). This proves Theorem 5.1.

8. Expansion in Terms of Grothendieck Polynomials

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be an integer sequence (not necessarily a partition), and $\mathcal{A}$ and $\mathcal{B}$ vector bundles of respective ranks n and p. Let $\mathbb{A} = \{\alpha_i\}_{i=1}^n$ and $\mathbb{B} = \{\beta_j\}_{j=1}^p$ be sets of Grothendieck roots for $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\mathbf{z} = \{z_1, \ldots, z_r\}$ and set $l = p - n$. Now define the factors

$$\mu_\lambda(\mathbf{z}) = \prod_{i=1}^{r}(1 - z_i)^{\lambda_i - i},$$

$$\Delta(\mathbf{z}) = \prod_{1 \le i < j \le r} \left(1 - z_j/z_i\right),$$

$$P(\mathbb{A}, \mathbb{B}, \mathbf{z}) = \prod_{i=1}^{r} \frac{\prod_{b\in\mathbb{B}}(1 - bz_i)}{(1 - z_i)^l \prod_{a\in\mathbb{A}}(1 - az_i)}.$$

The *double stable g-polynomial* $g_\lambda(\mathcal{A} - \mathcal{B})$ corresponding to the integer sequence λ is defined to be

$$g_\lambda(\mathcal{A} - \mathcal{B}) = \operatorname*{Res}_{\mathbf{z}=0,\infty} (\mu_\lambda(\mathbf{z}) \cdot \Delta(\mathbf{z}) \cdot P(\mathbb{A}, \mathbb{B}, \mathbf{z}) \cdot d\log \mathbf{z}). \tag{14}$$

This definition was pointed out to the author by Rimányi and Szenes, who have proven the following theorem in the upcoming paper [RS14].

THEOREM 8.1. *For any integer sequence* $\lambda = (\lambda_1, \ldots, \lambda_r)$ *and any vector bundles* $\mathcal{A}$ *and* $\mathcal{B}$, *the double stable g-polynomial* $g_\lambda(\mathcal{A} - \mathcal{B})$ *defined by equation* (14) *agrees with the double stable Grothendieck polynomial* $G_\lambda(\mathcal{A} - \mathcal{B})$ *defined by* [Buc08], *equation* (7).

As a result, we henceforth use only the notation G_λ for the (double) stable Grothendieck polynomials and take equation (14) as their definition. Combining this with our main theorem, we obtain the following steps to expand the class $[\mathcal{O}_\Omega]$ in terms of the appropriate Grothendieck polynomials. Using the notation of Theorem 5.1:

- For each $i \in Q_0$, collect families of residue variables $\mathbf{z}_k$ such that $i_k = i$, say $\mathbf{z}_{j_1}, \ldots, \mathbf{z}_{j_\ell}$.
- Combine these into the new families $\mathbf{u}_i = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\} = \mathbf{z}_{j_1} \cup \cdots \cup \mathbf{z}_{j_\ell}$ where $j_1 < \cdots < j_\ell$ and observe that the numerators of the interference factors I_k multiplied with the discriminant factors D_k produce exactly the products $\Delta(\mathbf{u}_i)$.
- For each $i \in Q_0$, let $l_i = \mathrm{rank}(\mathcal{E}_i) - \mathrm{rank}(\mathcal{M}_i)$ and form the rational function $F(\mathbf{u}_i)$ whose denominator is exactly the same as that of the product of all interference factors, but whose numerator is the product

$$\prod_{i \in Q_0} \prod_{u \in \mathbf{u}_i} (1 - u)^{-l_i}.$$

- For all i and j, substitute $u_{ij} = 1 - v_{ij}$ into F and multiply by the factor $\prod_{i \in Q_0} \prod_{j=1}^{n_i} v_{ij}^j$ to form a new rational function F'.
- Expand F' as a Laurent series according to the convention that for any arrow $a \in Q_1$, $v_{t(a)j} \ll v_{h(a)k}$ for any j or k.
- Finally, the expansion of $[\mathcal{O}_\Omega]$ in Grothendieck polynomials is obtained by interpreting the monomial

$$\prod_{i \in Q_0} \mathbf{v}_i^{\lambda_i} \rightsquigarrow \prod_{i \in Q_0} G_{\lambda_i}(\mathcal{E}_i - \mathcal{M}_i),$$

where for the integer sequence $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{in_i})$, $\mathbf{v}_i^{\lambda_i}$ denotes the multiindex notation $\prod_{j=1}^{n_i} v_{ij}^{\lambda_{ij}}$, which we adopt throughout the sequel.

EXAMPLE 8.2. Consider the A_2 quiver with vertices labeled $\{1 \to 2\}$. Consider the orbit closure $\Omega_m(\mathcal{E}_\bullet)$ corresponding to $m_{11} = m_{12} = m_{22} = 1$ and hence having dimension vector $(2, 2)$. From the directed partition $\Phi^+ = \{\phi_{22}\} \cup \{\phi_{12}, \phi_{11}\}$ we obtain the resolution pair $\mathbf{i} = (2, 1, 2)$ and $\mathbf{r} = (1, 2, 1)$. Following the recipe of Theorem 5.1, set

$$\mathbf{z}_1 = \{x\}, \qquad \mathbf{z}_1 = \{y_1, y_2\}, \qquad \mathbf{z}_3 = \{z\}.$$

Let $\mathcal{E}_\bullet \to X$ be the corresponding generic Q-bundle and set $\mathcal{E}_1 = \mathcal{A}$, $\mathcal{E}_2 = \mathcal{B}$, $\mathbb{E}_1 = \{\alpha_1, \alpha_2\}$, $\mathbb{E}_2 = \{\beta_1, \beta_2\}$. Notice that this implies that $\mathbb{M}_1 = \{\}$ and $\mathbb{M}_2 = \mathbb{E}_1 = \{\alpha_1, \alpha_2\}$. Applying the main theorem, we obtain that $[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}]$ is equal to applying the operation

$$\operatorname*{Res}_{x=0,\infty} \operatorname*{Res}_{y_2=0,\infty} \operatorname*{Res}_{y_1=0,\infty} \operatorname*{Res}_{z=0,\infty}$$

to the differential form

$$\left(\prod_{i=1}^{2}\frac{1-\alpha_i x}{1-\beta_i x}\right)\frac{(1-y_2/y_1)}{\prod_{i,j=1}^{2}(1-\alpha_i y_j)}\left(\prod_{i=1}^{2}\frac{1-\alpha_i z}{1-\beta_i z}\right)\frac{(1-z/x)}{\prod_{j=1}^{2}(1-z/y_j)}\prod_{k=1}^{3}d\log \mathbf{z}_k.$$

Renaming $x = u_1$ and $z = u_2$ and setting $\mathbf{u} = \{u_1, u_2\}$ and $\mathbf{y} = \mathbf{z}_2 = \{y_1, y_2\}$, this is further equal to

$$P(\mathbb{E}_1, \mathbb{M}_1, \mathbf{y})P(\mathbb{E}_2, \mathbb{M}_2, \mathbf{u})\Delta(\mathbf{y})\Delta(\mathbf{u})(d\log\mathbf{y})(d\log\mathbf{u})$$

times the rational function

$$\frac{1}{\prod_{i=1}^{2}(1-y_i)^2\prod_{j=1}^{2}(1-u_2/y_j)}.$$

Setting $a_i = 1 - y_i$ and $b_i = 1 - u_i$ for $1 \le i \le 2$ and multiplying this rational function by $a_1 a_2^2 b_1 b_2^2$ produces the rational function

$$\frac{b_1(1-a_1)(1-a_2)}{a_1(1-a_1/b_2)(1-a_2/b_2)}, \tag{15}$$

and according to the previous itemized steps, once this is expanded as a Laurent series, we can read off the quiver coefficients by interpreting $a^I b^J \rightsquigarrow G_I(\mathcal{A})G_J(\mathcal{B} - \mathcal{A})$. Since $G_{I,J} = G_I$ whenever J is a sequence of nonpositive integers and $G_\emptyset = 1$ (see [Buc02a], Section 3), the rational function (15) is equivalent (for our purposes) to the one obtained by setting $b_2 = 1$, namely the function $a_1^{-1}b_1$ and hence simply to b_1. This corresponds to the Grothendieck polynomial $G_1(\mathcal{B} - \mathcal{A})$, and we conclude that the quiver efficient $c_{(\emptyset,(1))}(\Omega_m) = 1$, whereas all others are zero.

EXAMPLE 8.3. Consider the inbound A_3 quiver $\{1 \to 2 \leftarrow 3\}$ and the same orbit and notation of Example 5.3. Following the previous itemized list, in equation (7), set $t_1 = x$, $t_2 = z$; $u_1 = v$, $u_2 = y_1$; and $u_3 = y_2$ to obtain the families $\mathbf{w} = \{w_1, w_2\}$, $\mathbf{u} = \{u_1, u_2, u_3\}$, and $\mathbf{t} = \{t_1, t_2\}$, associated to the vertices 1, 2, and 3, respectively. In the new variables, we check that $[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}]$ is given by applying the iterated residue operation $\mathrm{Res}_{\mathbf{w}=0,\infty}\,\mathrm{Res}_{\mathbf{t}=0,\infty}\,\mathrm{Res}_{\mathbf{u}=0,\infty}$ to

$$P(\mathbb{E}_1, \mathbb{M}_1, \mathbf{w})P(\mathbb{E}_2, \mathbb{M}_2, \mathbf{u})P(\mathbb{E}_3, \mathbb{M}_3, \mathbf{t})$$
$$\times \Delta(\mathbf{w})\Delta(\mathbf{u})\Delta(\mathbf{t})(d\log\mathbf{w})(d\log\mathbf{u})(d\log\mathbf{t})$$

times the rational function

$$\frac{\prod_{i=1}^{3}(1-u_i)}{\prod_{i=1}^{2}(1-w_i)^2\prod_{i=1}^{2}(1-t_i)^2}\prod_{i=2}^{3}\frac{1}{\prod_{s\in\{t_1\}\cup\mathbf{w}}(1-u_i/s)}.$$

The order of the residues above is important; in particular, the residues with respect to $\mathbf{u}$ must be done first. In general, for each $a \in Q_1$, the residues with respect to variables corresponding to the vertex $t(a)$ must be computed before those corresponding to the vertex $h(a)$. Comparing the above with equation (14) and setting

$a_i = 1 - w_i$, $b_j = 1 - u_j$, and $c_i = 1 - t_i$ for $1 \le i \le 2$ and $1 \le j \le 3$, observe that the quiver coefficients can be obtained by expanding the rational function

$$\frac{(\prod_{i=1}^2 a_i^i)(\prod_{i=1}^3 b_i^i)(\prod_{i=1}^2 c_i^i)b_1 b_2 b_3 (1-a_1)^2 (1-a_2)^2 (1-c_1)^2}{a_1^2 a_2^2 c_1^2 c_2^2 (b_2 - a_1)(b_2 - a_2)(b_2 - c_1)(b_3 - a_1)(b_3 - a_2)(b_3 - c_1)}$$

as a Laurent series and using the convention that

$$a^I b^J c^K \rightsquigarrow G_I(\mathcal{E}_1) G_J(\mathcal{E}_2 - \mathcal{E}_1 \oplus \mathcal{E}_3) G_K(\mathcal{E}_3).$$

We recommend rewriting this Laurent series in the form

$$\frac{b_1^2 b_3 (1-a_1)^2 (1-a_2)^2 (1-c_1)^2}{a_1 c_1 \prod_{s \in \{b_2, b_3\}} (1 - a_1/s)(1 - a_2/s)(1 - c_1/s)}$$

and expanding in the domain $a_j, c_1 \ll b_i$. In the preceding algebra, we repeatedly use the identity that if $p \mapsto (1 - f)$ and $q \mapsto (1 - g)$, then

$$\frac{1}{1 - p/q} \mapsto \frac{1}{f} \cdot \frac{(1-g)}{(1 - g/f)}. \tag{16}$$

In this example, the codimension of Ω_m is 3 (cf. equation (9)), and we note that rational factor $b_1^2 b_3/(a_1 c_1)$ has odd degree. Thus, when the remaining factors are expanded, the signs alternate as desired. The difficulty is that most monomials in this expansion do not correspond to partitions, and, as in the previous example, we must use a recursive recipe (see equation (3.1) of [Buc02a]) to expand these in the basis $\{G_\lambda\}$ for partitions λ, introducing new signs in a complicated way. Nonetheless, a computation in *Mathematica* confirms that the quiver coefficients are

$$c_{(\emptyset,(2,1),\emptyset)}(\Omega_m) = 1, \qquad c_{((1),(2),\emptyset)}(\Omega_m) = 1, \qquad c_{((1),(2,1),\emptyset)}(\Omega_m) = -1,$$

and all others are zero, which agrees with equation (9).

EXAMPLE 8.4 (Giambelli–Thom–Porteous formula). Consider again the A_2 quiver with vertices labeled $\{1 \to 2\}$. Only now consider the general orbit closure $\Omega_m(\mathcal{E}_\bullet)$ corresponding to $m = (m_{11}, m_{12}, m_{22})$ and hence having dimension vector $(m_{11} + m_{12}, m_{12} + m_{22})$. Let $\mathcal{E}_\bullet$ be a generic Q-bundle and write $e_1 = \text{rank}(\mathcal{E}_1)$ and $e_2 = \text{rank}(\mathcal{E}_2)$. From the directed partition $\Phi^+ = \{\phi_{22}\} \cup \{\phi_{12}, \phi_{11}\}$ we obtain the resolution pair $\mathbf{i} = (2, 1, 2)$ and $\mathbf{r} = (m_{22}, e_1, m_{12})$. Observe that the composition of the first two mappings of the Reineke resolution $\rho_1^{e_1} \circ \rho_2^{m_{22}}$ is a homeomorphism since in the notation of Section 7, it represents the sequence of maps

$$Z(\mathcal{SE}_1 \to \mathcal{QSE}_2) \longrightarrow \text{Gr}_0(\mathcal{SE}_2) \longrightarrow Z(0 \to \mathcal{QE}_1)$$

$$\longrightarrow \text{Gr}_0(\mathcal{E}_1) \longrightarrow Z(\mathcal{E}_1 \to \mathcal{QE}_2),$$

and $\mathcal{SE}_1$ has rank zero. Hence, we need only to compute the image $(\rho_2^{m_{22}})_*(1)$, and this is equivalent to applying Theorem 5.1 to the updated resolution pair $\mathbf{i} = (2)$, $\mathbf{r} = (m_{22})$. The fact that this computation simplifies is related to the fact that

in Example 8.2, the rational function (15) can be simplified to a monomial by setting $b_2 = 1$. We obtain that

$$[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] = \operatorname*{Res}_{\mathbf{z}=0,\infty} \left(\frac{(1 - (\mathcal{E}_1)_\bullet \mathbf{z}_\bullet)}{(1 - (\mathcal{E}_2)_\bullet \mathbf{z}_\bullet)} \Delta(\mathbf{z}) \, d \log \mathbf{z} \right),$$

where $\mathbf{z} = (z_1, \ldots, z_{m_{22}})$.

We set $l = e_2 - e_1$ and consider the product $\prod_{i=1}^{m_{22}} (1 - u_i)^{-l}$ and therefore finally the monomial $\prod_{i=1}^{m_{22}} v_i^{i-l}$ to obtain that

$$[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] = G_{(1-l,2-l,\ldots,m_{22}-l)}(\mathcal{E}_2 - \mathcal{E}_1).$$

Notice that the integer sequence $1 - l, 2 - l, \ldots, m_{22} - l$ is strictly increasing and therefore not a partition. However, $G_{I,p-1,p,J} = G_{I,p,p,J}$ for any integer sequences I and J and any integer p (see Section 3 of [Buc02a]), and so applying this iteratively yields the Grothendieck polynomial $G_R(\mathcal{E}_2 - \mathcal{E}_1)$ where R is the rectangular partition $(m_{22} - l)^{m_{22}}$. Finally, if one sets $r = m_{12}$, then this has the pleasing form $(e_1 - r)^{(e_2 - r)}$ (cf. [Buc02a], Theorem 2.3). One thinks of "r" denoting the rank of the map $f : \mathcal{E}_1 \to \mathcal{E}_2$ since, after all, $\Omega(\mathcal{E}_\bullet)$ is actually the degeneracy locus $\{x \in X : \operatorname{rank}(f) \leq m_{12}\}$. We conclude that the quiver coefficient $c_{(\emptyset, R)}(\Omega) = 1$ and all others are zero.

9. Comments Regarding Rimanyi's Formula and Buch's Conjecture

As in Example 5.3, we again wish to compare our method to Rimanyi's iterated residue formula in cohomology. In particular, we will produce the formula of [Rim14], Example 6.2, which gives a Schur expansion for the cohomology class.

9.1. Another Inbound A_3 Example

We consider again the inbound A_3 quiver and the same orbit as in Examples 5.3 and 8.3. However, this time we choose the directed partition

$$\Phi' = \{\phi_{22}, \phi_{12}\} \cup \{\phi_{13}, \phi_{33}\}$$

and therefore the resolution pair $\mathbf{i} = (1, 2, 1, 3, 2)$ and $\mathbf{r} = (1, 2, 1, 2, 1)$. Applying Theorem 5.1 and following the itemized steps of Section 8 along with the identity of equation (16), we obtain that the desired K-class is given by applying the operation

$$a^I b^J c^K \rightsquigarrow G_I(\mathcal{E}_1) G_J(\mathcal{E}_2 - \mathcal{E}_1 \oplus \mathcal{E}_3) G_K(\mathcal{E}_3) \qquad (17)$$

to the expansion of the rational function

$$\frac{b_1 b_2^2}{a_1 c_1} \cdot \frac{(1 - a_1)^3 (1 - a_2)(1 - c_1)(1 - c_2)}{(1 - a_1/b_1)(1 - a_1/b_2)(1 - a_1/b_3)(1 - a_2/b_3)(1 - c_1/b_3)(1 - c_2/b_3)}. \qquad (18)$$

Here we have again rearranged the expression so that it can be easily expanded in the domain $a_i, c_j \ll b_k$. Notice that when expression (18) is expanded, b_3 appears

with only nonpositive exponents. Thus, as in the analysis of Example 8.2, we set $b_3 = 1$ to obtain that the K-class corresponds to the simplified rational function

$$\frac{b_1 b_2^2}{a_1 c_1} \cdot \frac{(1 - a_1)^2}{(1 - a_1/b_1)(1 - a_1/b_2)},$$

in which c_1 appears with only negative exponents. Thus, we further simplify to the rational function

$$\frac{b_1 b_2^2}{a_1} \cdot \frac{(1 - a_1)^2}{(1 - a_1/b_1)(1 - a_1/b_2)}.$$

Finally, the following lemma allows another simplification.

LEMMA 9.1 (Allman [All14], Lemma C.6(a)). *If $f(b_1, b_2, \ldots)$ is a rational function symmetric in b_i and b_{i+1}, then $f \cdot (1 - b_i^{-1})$ is equivalent to zero under the exponent-to-subscript operation of equation* (17).

The lemma is equivalent to relation (3.1) of [Buc02a], namely that

$$G_{I,p,q,J} - G_{I,p+1,q,J} = G_{I,q,p+1,J} - G_{I,q-1,p+1,J}$$

for any integer sequences I and J and any integers p and q. In our example, setting f to be the rational function

$$\frac{b_1^2 b_2^2}{a_1} \cdot \frac{(1 - a_1)^2}{(1 - a_1/b_1)(1 - a_1/b_2)},$$

the lemma implies that we may use f to compute the desired K-class. Finally, the leading (lowest degree) term of f is given by the rational function

$$\frac{b_1^2 b_2^2}{a_1(1 - a_1/b_1)(1 - a_1/b_2)},$$

and this is precisely the function obtained by Rimányi as an argument for the iterated residue operation that produces the Schur expansion of the desired cohomology class via the convention

$$a^I b^J c^K \rightsquigarrow s_I(\mathcal{E}_1) s_J(\mathcal{E}_2 - (\mathcal{E}_1 \oplus \mathcal{E}_2)) s_K(\mathcal{E}_3).$$

9.2. Concluding Remarks

We believe that Rimanyi's cohomological iterated residue formula can always be obtained from a process analogous to the previous one. However, to do so in general, one needs to develop and organize operations on the level of rational functions like (18) to systematically produce the correct expression. This entails proving more general results in the spirit of Lemma 9.1. We have taken some initial steps in this area; for example, see [All14] and [AR14].

Furthermore, similar machinery is required to attack the Buch conjecture. One needs operations that alter rational functions to plainly produce expansions having partitions in their exponents and alternating in total degree. For example, to compute the K-class of the orbit from the example in the previous subsection

(and therefore that of Examples 5.3 and 8.3), one would prefer to see the rational function

$$\frac{b_1^2 b_2(1-a_1)}{(1-a_1/b_2)} = b_1^2 b_2(1-a_1) + \frac{a_1 b_1^2(1-a_1)}{(1-a_1/b_2)}. \tag{19}$$

Since b_2 appears with only nonpositive exponents in the second term on the right-hand side, this expression is equivalent to the *polynomial*

$$b_1^2 b_2(1-a_1) + a_1 b_1^2,$$

and applying our exponent-to-subscript convention, we immediately obtain the quiver coefficients from Example 8.3. To date, the author has not been able to directly obtain the rational function of equation (19) from the methods of this paper. This raises at least two questions. Is there a "best" resolution pair for each quiver, dimension vector, and orbit; and if so, is it canonical in some way? Second, is there a different resolution, akin to the process of Reineke, which can automatically produce such a "good" rational function?

References

[All14] J. Allman, *K-classes of quiver cycles, Grothendieck polynomials, and iterated residues,* PhD thesis, UNC–Chapel Hill, 2014, http://dc.lib.unc.edu/cdm/ref/collection/etd/id/6000.

[AR14]   J. Allman and R. Rimányi, *An iterated residue perspective on stable Grothendieck polynomials,* 2014, preprint, arXiv:1408.1911.

[BS12]   G. Bérczi and A. Szenes, *Thom polynomials of Morin singularities,* Ann. of Math. (2) 175 (2012), no. 2, 567–629.

[Buc02a] A. S. Buch, *Grothendieck classes of quiver varieties,* Duke Math. J. 115 (2002), no. 1, 75–103.

[Buc02b] ———, *A Littlewood-Richardson rule for the K-theory of Grassmannians,* Acta Math. 189 (2002), no. 1, 37–78.

[Buc05] ———, *Combinatorial K-theory,* Topics in cohomological studies of algebraic varieties, Trends Math., pp. 87–103, Birkhäuser, Basel, 2005.

[Buc08] ———, *Quiver coefficients of Dynkin type,* Michigan Math. J. 57 (2008), 93–120, Special volume in honor of Melvin Hochster.

[BZ01]   G. Bobiński and G. Zwara, *Normality of orbit closures for Dynkin quivers of type $\mathbb{A}_n$ (English summary),* Manuscripta Math. 105 (2001), no. 1, 103–109.

[BZ02]   ———, *Schubert varieties and representations of Dynkin quivers (English summary),* Colloq. Math. 94 (2002), no. 2, 285–309.

[CG97]   N. Chriss and V. Ginzburg, *Representation theory and complex geometry,* Birkhäuser Boston, Inc., Boston, MA, 1997.

[FR07]   L. M. Fehér and R. Rimányi, *On the structure of Thom polynomials of singularities,* Bull. Lond. Math. Soc. 39 (2007), no. 4, 541–549.

[FR12]   _____, *Thom series of contact singularities,* Ann. of Math. (2) 176 (2012), no. 3, 1381–1426.

[FS12]   A. Fink and D. E. Speyer, *K-classes for matroids and equivariant localization,* Duke Math. J. 161 (2012), no. 14, 2699–2723.

[Gab72] P. Gabriel, *Unzerlegbare Darstellungen. I (German. English summary),* Manuscripta Math. 6 (1972), 71–103, correction, ibid. 6 (1972), 309.

[Kal13] R. Kaliszewski, *Structure of quiver polynomials and Schur positivity,* Ph.D. thesis, UNC–Chapel Hill, 2013.

[Kaz10a] M. Kazarian, *Gysin homomorphism and degeneracies,* unpublished, 2010.

[Kaz10b] _____, *Non-associative Hilbert scheme and Thom polynomials,* unpublished, 2010.

[KR99]   A. Knutson and I. Rosu, *Equivariant K-theory and equivariant cohomology,* Math. Z. 243 (1999), no. 3, 423–448, appendix.

[KS06]   A. Knutson and M. Shimozono, *Kempf collapsing and quiver loci,* 2006, preprint, arXiv:math/0608327.

[KS11]   M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants,* Commun. Number Theory Phys. 5 (2011), no. 2, 231–352.

[LS82]   A. Lascoux and M. P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux (French. English summary),* C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 11, 629–633.

[Rei03] M. Reineke, *Quivers, desingularizations and canonical bases,* Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., 210, pp. 325–344, Birkhäuser, Boston, MA, 2003.

[Rim13] R. Rimányi, *On the cohomological Hall algebra of Dynkin quivers,* 2013, preprint, arXiv:1303.3399.

[Rim14] _____, *Quiver polynomials in iterated residue form,* J. Algebraic Combin. 40 (2014), no. 2, 527–542.

[RS14]   R. Rimányi and A. Szenes, *K-theoretic Thom polynomials and their expansions in Grothendieck polynomials,* 2014, in preparation.

[Zie14] M. Zielenkiewicz, *Integration over homogenous spaces for classical Lie groups using iterated residues at infinity,* Cent. Eur. J. Math. 12 (2014), 574–583.

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