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Four-Dimensional Compact Manifolds with Nonnegative Biorthogonal Curvature

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Abstract. The goal of this article is to study the pinching problem proposed by S.-T. Yau in 1990 replacing sectional curvature by a weaker condition on biorthogonal curvature. Moreover, we classify four-dimensional compact oriented Riemannian manifolds with nonnegative biorthogonal curvature. In particular, we obtain a partial answer to the Yau conjecture on pinching theorem for four-dimensional compact manifolds.

1. Introduction

In the last century, very much attention has been given to four-dimensional compact Riemannian manifolds with positive scalar curvature. A classical problem in geometry is to classify such manifolds in the category of either topology, diffeomorphism, or isometry. This subject have been studied extensively because of their connections with general relativity and quantum theory. For comprehensive references on such a theory, we indicate, for instance, [1; 3; 5; 7; 12; 15; 17; 20], and [22]. Arguably, classifying four-dimensional compact Riemannian manifolds or understanding their geometry is definitely an important issue.

In 1990, S.-T. Yau collected some important open problems. Here, we call attention to the paragraph where he wrote:

> *"The famous pinching problem says that on a compact simply connected manifold if $K_{\min} > \frac{1}{4}K_{\max} > 0$, then the manifold is homeomorphic to a sphere. If we replace $K_{\max}$ by normalized scalar curvature, can we deduce similar pinching results?"* (See [24], problem 12, page 369; see also [27].)

In other words, Yau's conjecture on pinching theorem can be rewritten as follows (see [13]).

Conjecture 1 (Yau, 1990). *Let (M^n, g) be a compact simply connected Riemannian manifold. Denote by s_0 the normalized scalar curvature of M^n. If $K_{\min} > \frac{n-1}{n+2}s_0$, then M^n is diffeomorphic to a standard sphere $\mathbb{S}^n$.*

A classical example obtained in [13] shows that $\frac{n-1}{n+2}$ is the best possible pinching for this conjecture (see Example 3.1 in [13]). We also notice that if s is the scalar curvature of a Riemannian manifold M^n, then the normalized scalar curvature of M^n is given by $s_0 = \frac{s}{n(n-1)}$.

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From this point on, M^4 will denote a compact oriented four-dimensional manifold, and $\mathcal{M}$ the set of Riemannian metrics g on M^4 with scalar curvature s and sectional curvature K. Before we state our first theorem, we introduce some definitions. First, let us recall that for each plane $P \subset T_pM$ at a point $p \in M^4$, we define *the biorthogonal (sectional) curvature* of P by the following average of the sectional curvatures:

$$K^{\perp}(P) = \frac{K(P) + K(P^{\perp})}{2}, \tag{1.1}$$

where $P^{\perp}$ is the orthogonal plane to P.

The sum of two sectional curvatures on two orthogonal planes appeared previously in works of Seaman [21] and Noronha [18]. It should be remarked that a compact manifold M^4 is Einstein if and only if $K^{\perp} = K$. Moreover, M^4 is locally conformally flat if and only if $K^{\perp} = s/12$. We also notice that $\mathbb{S}^1 \times \mathbb{S}^3$ with its canonical metric shows that positive biorthogonal curvature does not imply positive Ricci curvature. Indeed, the positivity of the biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature.

Next, we recall that the biorthogonal curvature of a Riemannian manifold M^4 is called *weakly* $1/4$*-pinched* if there exists a positive function $f \in C^{\infty}(M)$ satisfying a suitable pinching condition involving the biorthogonal curvature. Seaman [21] showed that this pinching condition implies nonnegative isotropic curvature, whereas the first author observed in [6] that if $K_1^{\perp}$ is the minimum of the biorthogonal curvature in each point, then $12K_1^{\perp}$ is a modified scalar curvature with the corresponding modified Yamabe invariant $Y_1^{\perp}(M)$. In particular, Costa used the notion of biorthogonal curvature to show a relationship between these invariants and Hopf's conjecture. We recall that Hopf's conjecture asks if $\mathbb{S}^2 \times \mathbb{S}^2$ admits a metric with positive sectional curvature. Costa was able to show Hopf's conjecture, provided that $Y_1^{\perp}(\mathbb{S}^2 \times \mathbb{S}^2) \leq 0$; see [6]. However, Bettiol proved that $Y_1^{\perp}(\mathbb{S}^2 \times \mathbb{S}^2) > 0$, which implies that $\mathbb{S}^2 \times \mathbb{S}^2$ admits metrics of positive biorthogonal curvature; for more details, see Theorem 1 in [4]. In particular, Bettiol showed that the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ admits metrics with positive biorthogonal curvature.

To fix notation, we now consider for each point $p \in M^4$ the following functions:

$$K_1^{\perp}(p) = \min\{K^{\perp}(P);\, P \text{ is a 2-plane in } T_pM\}, \tag{1.2}$$

$$K_3^{\perp}(p) = \max\{K^{\perp}(P);\, P \text{ is a 2-plane in } T_pM\}, \tag{1.3}$$

and

$$K_2^{\perp}(p) = \frac{s(p)}{4} - K_1^{\perp}(p) - K_3^{\perp}(p). \tag{1.4}$$

These functions appeared previously in [6]. In the next section, we will collect some their properties. It is perhaps worth mentioning that the canonical metrics of the manifolds $\mathbb{S}^4$, $\mathbb{CP}^2$, and $\mathbb{S}^1 \times \mathbb{S}^3$ have $K_1^\perp = s/12$, $K_1^\perp = s/24$, and $K_1^\perp = s/12$, respectively.

Our aim is to investigate the pinching problem on four-dimensional compact manifolds replacing sectional curvature by biorthogonal curvature conditions. To do this, we start by replacing the assumption $K > s/24$ of the sectional curvature in Conjecture 1 by a weaker condition on biorthogonal curvature. With this setting, we now announce our first result.

THEOREM 1. *Let (M^4, g) be a compact oriented Riemannian manifold with positive scalar curvature s satisfying $K_1^\perp \geq s/24$. Then one of the following assertions holds*:

(1) *(M^4, g) is diffeomorphic to a connected sum $\mathbb{S}^4 \sharp (\mathbb{R} \times \mathbb{S}^3)/G_1 \sharp \cdots \sharp (\mathbb{R} \times \mathbb{S}^3)/G_n$, where each G_i is a discrete subgroup of the isometry group of $\mathbb{R} \times \mathbb{S}^3$;*
(2) *or (M^4, g) is isometric to a complex projective space $\mathbb{CP}^2$ with the Fubini–Study metric.*

It is worth pointing out that Theorem 1 remains true if the assumption $K_1^\perp \geq s/24$ is replaced by $K_3^\perp \leq s/6$; this comment will be clarified in the next section. As an application of Theorem 1, we deduce the following result under the finite fundamental group hypothesis.

COROLLARY 1. *Let (M^4, g) be a compact oriented Riemannian manifold satisfying $K_1^\perp \geq s/24$. We assume that M^4 has a finite fundamental group. Then we have*:

(1) *Either M^4 is diffeomorphic to a sphere $\mathbb{S}^4$,*
(2) *or (M^4, g) is isometric to a complex projective space $\mathbb{CP}^2$ with the Fubini–Study metric.*

Now recall that the space of harmonic 2-forms $H^2(M^4, \mathbb{R})$ can be split as

$$H^2(M^4, \mathbb{R}) = H^+(M^4, \mathbb{R}) \oplus H^-(M^4, \mathbb{R}),$$

where $H^\pm(M^4, \mathbb{R})$ is the space of positive and negative harmonic 2-forms, respectively. Moreover, the second Betti number b_2 of M^4 is $b_2 = b^+ + b^-$, where $b^\pm = \dim H^\pm(M^4, \mathbb{R})$. We recall that M^4 is called *positive definite* (resp. *negative definite*) if $b^- = 0$ (resp. $b^+ = 0$). Otherwise, M^4 will be called *indefinite*. According to Donaldson [10] and Freedman [11], if M^4 is simply connected and definite, then M^4 is homeomorphic to sphere $\mathbb{S}^4$, provided that $b_2 = 0$ or M^4 is homeomorphic to the connected sum of complex projective spaces $\mathbb{CP}^2 \sharp \cdots \sharp \mathbb{CP}^2$ (b_2 times).

In fact, an elegant argument of Seaman [22] shows that a compact oriented Riemannian manifold M^4 with positive sectional curvature admitting a harmonic 2-form of constant length must be definite. Later, this result was improved by Noronha [18]; for more details, see Theorems 3.5 and 3.6 therein. More precisely, Noronha proved the following result.

THEOREM 2 (Noronha [18]). *Let* (M^4, g) *be a compact oriented Riemannian manifold with positive scalar curvature. Then the following assertions hold*:

(1) *If* (M^4, g) *admits a nontrivial harmonic* 2*-form of constant length and* $K^{\perp} > 0$, *then* M^4 *is definite.*
(2) *If* (M^4, g) *admits a nontrivial parallel* 2*-form and* $K^{\perp} \geq 0$, *then* M^4 *is biholomorphic to* $\mathbb{CP}^2$, *or its universal covering* $\widetilde{M}$ *is isometric to* $M_1^2 \times M_2^2$, *where each* M_i^2 *is diffeomorphic to sphere* $\mathbb{S}^2$.

We now recall that a Riemannian manifold (M^4, g) is called *geometrically formal* if the wedge product of two harmonic forms is again harmonic; for more details, we refer the reader to [2]. This concept appeared recently in a work of Kotschick [14], where he showed that for this class of metrics, harmonic forms have constant length (see also Theorems B and C in [2]). In particular, Kotschick proved that if M^4 is formal and has a finite fundamental group, then M^4 has second Betti number $b_2 \leq 2$. We call attention to the following result of Kotschick (see Corollary 3 in [14]).

THEOREM 3 (Kotschick [14]). *Let* M^4 *be a compact simply connected manifold. If* M^4 *is formal and admits a metric* (*possibly nonformal*) *with nonnegative scalar curvature*, *then one of the following assertions occurs*:

(1) M^4 *is homeomorphic to a sphere* $\mathbb{S}^4$;
(2) M^4 *is diffeomorphic to a complex projective space* $\mathbb{CP}^2$;
(3) *or* M^4 *is diffeomorphic to a product of two spheres* $\mathbb{S}^2 \times \mathbb{S}^2$.

More recently, Bär [2] was able to prove the following classification under nonnegative sectional curvature assumption.

THEOREM 4 (Bär [2]). *Let* (M^4, g) *be a compact oriented geometrically formal Riemannian manifold.*

(1) *If* M^4 *is simply connected and* (M^4, g) *has nonnegative sectional curvature, then*:
    (a) M^4 *is homeomorphic to a sphere* $\mathbb{S}^4$;
    (b) M^4 *is diffeomorphic to a complex projective space* $\mathbb{CP}^2$;
    (c) *or* $(M^4, g) = M_1^2 \times M_2^2$, *where each* M_i^2 *is diffeomorphic to a sphere* $\mathbb{S}^2$ *and has nonnegative sectional curvature.*
(2) *If* (M^4, g) *has positive sectional curvature. Then*:
    (a) *Either* M^4 *is homeomorphic to a sphere* $\mathbb{S}^4$;
    (b) *or* M^4 *is diffeomorphic to a complex projective space* $\mathbb{CP}^2$.

As it was previously mentioned, we are interested in classifying four-dimensional manifolds under biorthogonal curvature hypotheses. To this end, we notice that from Micallef and Moore work [16] nonnegative sectional curvature implies $K_3^{\perp} \leq s/4$, and nonnegative isotropic curvature implies $K_3^{\perp} \leq s/4$. For more details, see the next section. Based on these observations and inspired by the ideas of Seaman [22], Noronha [18], Kotschick [14], and Bär [2], we now announce our second theorem.

THEOREM 5. *Let* (M^4, g) *be a compact oriented Riemannian manifold with positive scalar curvature. Then we have the following assertions*:

(1) *If* (M^4, g) *admits a nontrivial harmonic* 2*-form of constant length and* $K_3^{\perp} < \frac{s}{4}$, *then* M^4 *is definite.*
(2) *If* (M^4, g) *admits a nontrivial parallel* 2*-form and* $K_3^{\perp} \leq \frac{s}{4}$, *then* M^4 *is biholomorphic to a complex projective space* $\mathbb{CP}^2$ *with the Fubini–Study metric*, *or its universal covering* $\widetilde{M}$ *is isometric to* $M_1^2 \times M_2^2$, *where each* M_i^2 *is homeomorphic to a sphere* $\mathbb{S}^2$.

We point out that Theorem 5 can be seen as an improvement to Theorems 2 and 4. In particular, we obtain the following characterization under an integral condition involving the biorthogonal curvature.

COROLLARY 2. *Let* (M^4, g) *be a compact oriented simply connected Riemannian manifold with positive scalar curvature* s *and satisfying*

$$\int_M (s - 4K_3^{\perp})\, dV_g \geq 0.$$

If all harmonic forms of (M^4, g) *have constant length*, *then one of the following assertions holds*:

(1) M^4 *is homeomorphic to* $\mathbb{S}^4$;
(2) M^4 *is diffeomorphic to* $\mathbb{CP}^2$;
(3) *or* M^4 *is isometric to* $M_1^2 \times M_2^2$, *where each* M_i^2 *is diffeomorphic to a sphere* $\mathbb{S}^2$.

In order to state the next result, we adopt the following notation. For an oriented manifold M^4, we consider Λ^2 be the bundle of 2-forms $\alpha \in M^4$ and let $* : \Lambda^2 \to \Lambda^2$ be the Hodge star operator. Thus, there is a invariant decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, where $\Lambda^{\pm} = \{\alpha \in \Lambda^2; *\alpha = \pm\alpha\}$, depending only on the orientation and the conformal class of the metric. Therefore, the Weyl curvature tensor W is an endomorphism of Λ^2 such that $W = W^+ \oplus W^-$, where $W^{\pm} : \Lambda^{\pm} \longrightarrow \Lambda^{\pm}$ are called of the self-dual and anti-self-dual parts of W. Half conformally flat metrics are also known as self-dual or anti-self-dual if W^- or $W^+ = 0$, respectively. These metrics are, in a certain sense, analogous to anti-self-dual connections in Yang–Mills theory.

The formal divergence δ for any $(0,4)$-tensor T is given by

$$\delta T(X_1, X_2, X_3) = -\operatorname{trace}_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

where g is the metric of M^4. We say that the Weyl tensor of M^4 is harmonic when $\delta W = 0$.

One fundamental inequality in Riemannian geometry is *Kato's inequality.* Namely, let $s \in \Gamma(E)$, where $E \to M$ is a vector bundle over M; then $|\nabla|s|| \leq$

$|\nabla s|$ away from the zero locus of s. In a famous article, LeBrun and Gursky proved a *refined Kato's inequality.* More precisely, they showed that if W^+ is harmonic, then away from the zero locus of W^+, we have

$$|d|W^+|| \leq \sqrt{\frac{3}{5}}|\nabla W^+|. \tag{1.5}$$

Moreover, (1.5) holds in the distributional sense on M^4; for more details, see Lemma 2.1 in [12].

On the basis of these observations and inspired by ideas developed in [26], we use improved Kato's inequality jointly with a classical theorem due to Hitchin [3] to prove the following result.

THEOREM 6. *Let (M^4, g) be a compact oriented Riemannian manifold with harmonic Weyl tensor and positive scalar curvature. We assume that g is analytic and*

$$K_1^{\perp} \geq \frac{s^2}{8(3s + 5\lambda_1)},$$

where λ_1 stands for the first eigenvalue of the Laplacian with respect to g. Then one of the following assertions holds:

(1) *M^4 is diffeomorphic to a connected sum $\mathbb{S}^4 \sharp (\mathbb{R}\times\mathbb{S}^3)/G_1 \sharp \cdots \sharp (\mathbb{R}\times\mathbb{S}^3)/G_n$, where each G_i is a discrete subgroup of the isometry group of $\mathbb{R}\times\mathbb{S}^3$. In this case*, *$g$ is locally conformally flat*;
(2) *or M^4 is isometric to a complex projective space $\mathbb{CP}^2$ with the Fubini–Study metric.*

As an immediate consequence, we obtain the following corollary.

COROLLARY 3. *Let (M^4, g) be a compact oriented Riemannian manifold with harmonic Weyl tensor and analytic metric g. We assume that $Ric \geq \rho > 0$ and $K_1^{\perp} \geq \frac{3s^2}{8(9s^2+20\rho)}$. Then we have*:

(1) *Either M^4 is isometric to $\mathbb{S}^4$ with its canonical metric*,
(2) *or M^4 is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric*.

We already know that a compact manifold M^4 is Einstein if and only if $K^{\perp} = K$. Moreover, by Theorem 5.26 in [3] Einstein metrics are analytic. It should be emphasized that there are regularity results which could be used to show that the harmonic self-dual Weyl tensor implies that the metric is analytic choosing appropriate coordinates (e.g., harmonic one); for more details, see [9]. Finally, we deduce the following corollary, which was first obtained by Yang [26].

COROLLARY 4. *Let (M^4, g) be a compact oriented Einstein manifold satisfying $Ric = \rho > 0$. Suppose that*

$$K \geq \frac{2\rho^2}{12\rho + 5\lambda_1}.$$

Then either M^4 is isometric to $\mathbb{S}^4$ with its canonical metric, or M^4 is isometric to $\mathbb{CP}^2$ with Fubini–Study metric.

2. Preliminaries

Throughout this section, we collect a couple of formulae that will be useful in the proofs of our results. As it was previously commented, the Weyl tensor W is an endomorphism of Λ^2 such that $W = W^+ \oplus W^-$, where $W^\pm : \Lambda^\pm \longrightarrow \Lambda^\pm$ are called the self-dual and anti-self-dual parts of W, respectively. Furthermore, if $\mathcal{R}$ denotes the curvature of M^4, then we get a matrix

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12}Id & B \\ \hline B^* & W^- + \frac{s}{12}Id \end{array} \right), \tag{2.1}$$

where $B : \Lambda^- \to \Lambda^+$ stands for the Ricci traceless operator of M^4 given by $B = Ric - \frac{s}{4}g$. For more details in this subject, we recommend the famous "Besse's book" [3].

We now consider $w_1^\pm \leq w_2^\pm \leq w_3^\pm$ be the eigenvalues of the tensors $W^\pm$, respectively. In [6], the first author proved some formulae involving the biorthogonal curvatures and the eigenvalues of $W^\pm$ that will be important in the proofs of our results. Here we present their proofs for the sake of completeness.

First, we consider a point $p \in M^4$ and $X, Y \in T_pM$ orthonormal. Therefore, the unitary 2-form $\alpha = X \wedge Y$ can be uniquely written as $\alpha = \alpha^+ + \alpha^-$, where $\alpha^\pm \in \Lambda^\pm$ with $|\alpha^+|^2 = \frac{1}{2}$ and $|\alpha^-|^2 = \frac{1}{2}$. Moreover, under these notations, the sectional curvature $K(\alpha)$ is given by

$$K(\alpha) = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+) \rangle + \langle \alpha^-, W^-(\alpha^-) \rangle + 2\langle \alpha^+, B\alpha^- \rangle. \tag{2.2}$$

In particular, we have

$$K(\alpha^\perp) = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+) \rangle + \langle \alpha^-, W^-(\alpha^-) \rangle - 2\langle \alpha^+, B\alpha^- \rangle, \tag{2.3}$$

where $\alpha^\perp = \alpha^+ - \alpha^-$. Combining (2.2) with (2.3), we arrive at

$$\frac{K(\alpha) + K(\alpha^\perp)}{2} = \frac{s}{12} + \langle \alpha^+, W^+(\alpha^+) \rangle + \langle \alpha^-, W^-(\alpha^-) \rangle. \tag{2.4}$$

Hence, we can use (1.2) to obtain

$$\begin{aligned} K_1^\perp = \frac{s}{12} &+ \min\left\{ \langle \alpha^+, W^+(\alpha^+) \rangle; |\alpha^+|^2 = \frac{1}{2} \right\} \\ &+ \min\left\{ \langle \alpha^-, W^-(\alpha^-) \rangle; |\alpha^-|^2 = \frac{1}{2} \right\}. \end{aligned}$$

However, by Proposition 2.1 of [19] there exists an orthonormal basis of Λ^2 given by

$$\{X_1 \wedge Y_1, X_2 \wedge Y_2, X_3 \wedge Y_3\},$$

where $X_i, Y_i \in T_pM$ for all $i = 1, 2, 3$. In particular, we invoke Proposition 2.5 also in [19] to get

$$K_1^{\perp} - \frac{s}{12} = \frac{w_1^+ + w_1^-}{2}. \tag{2.5}$$

Arguing in the same way, we obtain

$$K_3^{\perp} - \frac{s}{12} = \frac{w_3^+ + w_3^-}{2}. \tag{2.6}$$

Finally, from (1.4) we have

$$K_2^{\perp} - \frac{s}{12} = \frac{w_2^+ + w_2^-}{2}. \tag{2.7}$$

From the results of Micallef and Moore M^4 has nonnegative isotropic curvature if and only if $w_3^{\pm} \le s/6$; for more details, see [16]. For that reason, nonnegative sectional curvature implies $K_3^{\perp} \le s/4$, and nonnegative isotropic curvature implies $K_3^{\perp} \le s/4$. Moreover, we notice that $K \ge s/24$ implies that $K_1^{\perp} \ge s/24$, as well as $K_1^{\perp} \ge s/24$ implies that $K_3^{\perp} \le s/6$.

3. Proof of the Results

3.1. Proof of Theorem 1

Let (M^4, g) be a compact oriented Riemannian manifold with positive scalar curvature s. Since $K_1^{\perp} \ge \frac{s}{24}$ implies $K_3^{\perp} \le \frac{s}{6}$, it is enough to assume that $K_3^{\perp} \le s/6$. Now, from (2.6) we arrive at

$$w_3^+ + w_3^- = 2K_3^{\perp} - \frac{s}{6}.$$

From this, we may use that $w_1^{\pm} \le w_2^{\pm} \le w_3^{\pm}$ and $w_1^{\pm} + w_2^{\pm} + w_3^{\pm} = 0$ jointly with our assumption to conclude that $w_3^+ \le s/6$. Similarly, we conclude that $w_3^- \le s/6$, which implies that M^4 has nonnegative isotropic curvature. Assume that M^4 admits a metric with positive isotropic curvature; then M^4 is diffeomorphic to a connected sum

$$\mathbb{S}^4 \sharp (\mathbb{R} \times \mathbb{S}^3)/G_1 \sharp \cdots \sharp (\mathbb{R} \times \mathbb{S}^3)/G_n,$$

where each G_i is a discrete subgroup of the isometry group of $\mathbb{R} \times \mathbb{S}^3$.

On the other hand, we assume that M^4 does not admit a metric with positive isotropic curvature. Hence, if M^4 is irreducible, we can apply Theorem 1.1 of [23] to deduce that (M^4, g) either is locally symmetric or is Kähler. We now suppose that M^4 is irreducible and locally symmetric, which implies that (M^4, g) is an Einstein manifold. Therefore, we may use Theorem 4.4 of [17] to conclude that (M^4, g) is isometric to a complex projective space $\mathbb{CP}^2$. In the Kähler case, it is known that $w_3^+ = \frac{s}{6}$. For this, we invoke (2.6) to obtain

$$w_3^- \le -\frac{s}{6} + 2K_3^{\perp} - \frac{s}{6} \le 0,$$

which implies that $w_3^- = 0$ in M^4. From this it follows that $W^- = 0$. Now, we apply Theorem 1.1 in [7] to conclude that (M^4, g) is locally symmetric, and then (M^4, g) is isometric to a complex projective space $\mathbb{CP}^2$.

Finally, we consider (M^4, g) locally reducible. Since $K_3^{\perp} \leq \frac{s}{6}$, it is not difficult to check that this case can not occur; for more details, see Theorem 3.1 in [17]. So, we conclude the proof of the theorem. □

3.2. *Proof of Theorem 5*

The first part of the proof will follow from Theorem 3.6 in Noronha [18]. First of all, we consider (M^4, g) be a compact oriented Riemannian manifold with positive scalar curvature s. Moreover, let $\alpha^+ \in \Lambda^+$ (resp. $\alpha^- \in \Lambda^-$) be a positive (resp. negative) nondegenerate differentiable 2-form. From this we have two Weitzenböch formulae given by

$$\langle \Delta\alpha^{\pm}, \alpha^{\pm}\rangle = \frac{1}{2}\Delta|\alpha^{\pm}|^2 + |\nabla\alpha^{\pm}|^2 + \left\langle \left(\frac{s}{3} - 2W^{\pm}\right)\alpha^{\pm}, \alpha^{\pm}\right\rangle. \tag{3.1}$$

We now denote $w_3^{\pm}$ be the largest eigenvalues of $W^{\pm}$, respectively. Under these conditions, we have

$$\langle W^{\pm}(\alpha^{\pm}), \alpha^{\pm}\rangle \leq w_3^{\pm}\langle \alpha^{\pm}, \alpha^{\pm}\rangle.$$

It then follows from (3.1) that

$$\langle \Delta\alpha^{\pm}, \alpha^{\pm}\rangle \geq \frac{1}{2}\Delta|\alpha^{\pm}|^2 + |\nabla\alpha^{\pm}|^2 + \left(\frac{s}{3} - 2w_3^{\pm}\right)|\alpha^{\pm}|^2. \tag{3.2}$$

Now we are ready to prove Theorem 5.

Proof of Theorem 5. First, we assume that M^4 has a nondegenerate harmonic 2-form α with constant length. Suppose that M^4 is not definite. This means that $b^{\pm} > 0$, which gives the following possibilities:

(1) α is a negative 2-form.
(2) α is a positive 2-form.
(3) $\alpha = \alpha^+ + \alpha^-$, where $\alpha^{\pm}$ are nondegenerate positive and negative 2-forms, respectively.

We suppose that the first case occurs (the second case has a similar argument). Thus, we may use (3.2) for $\alpha = \alpha^-$ to deduce

$$0 \geq |\nabla\alpha|^2 + \left(\frac{s}{3} - 2w_3^-\right)|\alpha|^2.$$

From this it follows that $w_3^- \geq \frac{s}{6}$ in M^4. Next, from (2.6) we have

$$w_3^- + w_3^+ = 2K_3^{\perp} - \frac{s}{6} < \frac{s}{3}$$

and then $w_3^+ + w_3^- < \frac{s}{3}$, which implies $w_3^+ < \frac{s}{6}$.

On the other hand, insofar as $b^+ > 0$, there exists a harmonic nondegenerate positive 2-form γ. Furthermore, (3.2) with respect to γ ensures

$$0 \geq \frac{1}{2}\Delta|\gamma|^2 + |\nabla\gamma|^2 + \left(\frac{s}{3} - 2w_3^+\right)|\gamma|^2.$$

We integrate the last expression and use the Stokes theorem to obtain

$$0 \geq \int_M |\nabla\gamma|^2\, dV_g + \int_M \left(\frac{s}{3} - 2w_3^+\right)|\gamma|^2\, dV_g > 0,$$

which is a contradiction. This proves the first possibility.

We now treat the third case. To this end, we use that $\alpha = \alpha^+ + \alpha^-$ jointly with (3.2) to infer

$$\begin{aligned} 0 \geq |\nabla\alpha^+|^2 + |\nabla\alpha^-|^2 + \left(\frac{s}{3} - 2w_3^+\right)|\alpha^+|^2 \\ + \left(\frac{s}{3} - 2w_3^-\right)|\alpha^-|^2. \end{aligned}$$

After a straightforward computation we get

$$\begin{aligned} 0 \geq |\nabla\alpha^+|^2 + |\nabla\alpha^-|^2 + (s - 4K_3^{\perp})|\alpha^+|^2 \\ + \left(2w_3^- - \frac{s}{3}\right)(|\alpha^+|^2 - |\alpha^-|^2). \end{aligned} \tag{3.3}$$

Next, if there exists a point $p \in M^4$ such that $|\alpha^+|^2 = |\alpha^-|^2$, then we use (3.3) to deduce

$$0 \geq |\nabla\alpha^+|^2 + |\nabla\alpha^-|^2 + (s - 4K_3^{\perp})|\alpha^+| > 0,$$

which is again a contradiction. Therefore, without loss of generality we can assume that $|\alpha^+|^2 > |\alpha^-|^2$. From this it follows that $w_3^- \leq s/6$ in M^4.

On the other hand, on integrating (3.2) for α^- we obtain

$$0 \geq \int_M |\nabla\alpha^-|^2\, dV_g + \int_M \left(\frac{s}{3} - 2w_3^-\right)|\alpha^-|^2\, dV_g \geq 0, \tag{3.4}$$

which implies that $\nabla\alpha^- = 0$, and then $|\alpha^-|$ is constant. By using once more (3.4) we conclude that $w_3^- = s/6$ and whereas $w_3^+ + w_3^- < s/3$, we infer $w_3^+ < s/6$. Finally, we take the integral in (3.2) to get

$$0 \geq \int_M |\nabla\alpha^+|^2\, dV_g + \int_M \left(\frac{s}{3} - 2w_3^+\right)|\alpha^+|^2\, dV_g > 0,$$

which is once more a contradiction. So, we have proved the first assertion of the theorem.

Continuing, we suppose that (M^4, g) admits a nontrivial parallel 2-form. Under this condition, it is well known that M^4 is Kähler, in particular, $w_3^+ = \frac{s}{6}$. Since $K_3^{\perp} \leq \frac{s}{4}$, we conclude that $w_3^- \leq s/6$ and then, from Micallef–Moore work, M^4 has a nonnegative isotropic curvature (see [16]). Next, if M^4 is locally irreducible, then we use Theorem 1.2 of Seshadri [23] to conclude that M^4 is biholomorphic to $\mathbb{CP}^2$ or isometric to a compact Hermitian symmetric space. In the last case, M^4

is Einstein, and then it is isometric to $\mathbb{CP}^2$. To finish, it suffices to argue as in the proof of Theorem 1 to deduce that M^4 cannot be locally reducible. □

3.3. Proof of Corollary 2

Proof. We assume that M^4 is simply connected and that their harmonic forms have constant length. We now assume the unpublished theorem (at present) of Kotschick (see Theorem 3) to deduce that M^4 is either homeomorphic to a sphere $\mathbb{S}^4$, diffeomorphic to a complex projective space $\mathbb{CP}^2$, or diffeomorphic to the product of two spheres $\mathbb{S}^2 \times \mathbb{S}^2$. In the last case, we have the second Betti number $b_2^{\pm} = 1$. Therefore, we consider two harmonic forms with constant length $\alpha^{\pm} \in H^{\pm}(M, \mathbb{R})$, and without loss of generality, we may assume that the length is equal to one. Whence, on integrating (3.1) we obtain

$$0 \geq \int_M |\nabla \alpha^+|^2 \, dV_g + \int_M |\nabla \alpha^-|^2 \, dV_g + \int_M (s - 4K_3^{\perp}) \, dV_g \geq 0.$$

From this it follows that $\alpha^{\pm}$ are parallel, and by using once more (3.1) for $\alpha^{\pm}$ we infer $w^{\pm} = s/6$ and $K_3^{\perp} = s/4$. Finally, we invoke Theorem 5 to conclude the proof of the corollary. □

3.4. Proof of Theorem 6

Proof. Since $\delta W = 0$, we have the following Weitzenböck formulae (cf. Section 16.73 in [3]):

$$\frac{1}{2}\Delta |W^{\pm}|^2 + |\nabla W^{\pm}|^2 + \frac{s}{2}|W^{\pm}|^2 - 18 \det W^{\pm} = 0. \tag{3.5}$$

Moreover, by use of Lagrange multipliers we infer

$$\det W^{\pm} \leq \frac{\sqrt{6}}{18}|W^{\pm}|^3. \tag{3.6}$$

However, our hypothesis implies that $|W^{\pm}|^2$ are analytic. So far, the set

$$\Sigma = \{p \in M;\, |W^+|(p) = 0 \text{ or } |W^-|(p) = 0\}$$

is finite, provided that $W^{\pm} \not\equiv 0$. Suppose by contradiction that (M^4, g) is not half conformally flat. For this, there is a constant $t > 0$ such that

$$\int_M (|W^+| - t|W^-|) \, dV_g = 0.$$

Choosing W^- in (3.5) and multiplying by t^2 and adding the result to (3.5) with respect to W^+, we deduce

$$\begin{aligned} 0 \geq (|\nabla W^+|^2 + t^2|\nabla W^-|^2) + \frac{s}{2}(|W^+|^2 + t^2|W^-|^2) \\ - 18(\det W^+ + t^2 \det W^-). \end{aligned} \tag{3.7}$$

Applying refined Kato's inequality (1.5) jointly with (3.6) in the previous inequality, we get

$$0 \geq \frac{5}{3}\int_M (|d|W^+||^2 + t^2|d|W^-||^2)\,dV_g + \int_M \frac{s}{2}(|W^+|^2 + t^2|W^-|^2)\,dV_g - \sqrt{6}\int_M (|W^+|^3 + t^2|W^-|^3)\,dV_g. \tag{3.8}$$

On the other hand, we notice that

$$\begin{aligned}(|d|W^+||^2 + t^2|d|W^-||^2) &= \frac{1}{2}(|d(|W^+| - t|W^-|)|^2 + |d(|W^+| + t|W^-|)|^2)\\ &\geq \frac{1}{2}|d(|W^+| - t|W^-|)|^2.\end{aligned}$$

Moreover, from the Poincaré inequality we have

$$\frac{1}{2}\int_M |d(|W^+| - t|W^-|)|^2\,dV_g \geq \frac{\lambda_1}{2}\int_M (|W^+| - t|W^-|)^2\,dV_g,$$

so that, from the two preceding inequalities we obtain

$$\int_M (|d|W^+||^2 + t^2|d|W^-||^2)\,dV_g \geq \frac{\lambda_1}{2}\int_M (|W^+| - t|W^-|)^2\,dV_g. \tag{3.9}$$

Therefore, comparing (3.9) with (3.8), we obtain

$$\begin{aligned}0 \geq{}& \frac{5}{6}\lambda_1\int_M (|W^+|^2 - 2t|W^+||W^-| + t^2|W^-|^2)\,dV_g\\ &+ \int_M \frac{s}{2}(|W^+|^2 + t^2|W^-|^2)\,dV_g - \sqrt{6}\int_M (|W^+|^3 + t^2|W^-|^3)\,dV_g,\end{aligned}$$

which can be written as

$$\begin{aligned}0 \geq \int_M \Big\{&|W^-|^2\Big(\frac{5}{6}\lambda_1 + \frac{s}{2} - \sqrt{6}|W^-|\Big)t^2 - \Big(\frac{5}{3}\lambda_1|W^+||W^-|\Big)t\\ &+ |W^+|^2\Big(\frac{5}{6}\lambda_1 + \frac{s}{2} - \sqrt{6}|W^+|\Big)\Big\}\,dV_g.\end{aligned} \tag{3.10}$$

For simplicity, we can write the integrand of (3.10) as

$$\mathcal{P}(t) = |W^-|^2\big(a - \sqrt{6}|W^-|\big)t^2 - \frac{5}{3}\lambda_1|W^+||W^-|t + |W^+|^2\big(a - \sqrt{6}|W^+|\big), \tag{3.11}$$

where $a = \frac{5}{6}\lambda_1 + \frac{s}{2}$. We notice that (3.11) is a quadratic function of t and its discriminant Δ is given by

$$\Delta = \frac{25}{9}\lambda_1^2|W^+|^2|W^-|^2 - 4|W^+|^2|W^-|^2\big(a - \sqrt{6}|W^+|\big)\big(a - \sqrt{6}|W^-|\big). \tag{3.12}$$

On the other hand, we recall that $|W^{\pm}| \leq 6(w_1^{\pm})^2$, and then we use (2.5) to deduce

$$|W^+| + |W^-| \leq \sqrt{6}\Big(\frac{s}{6} - 2K_1^{\perp}\Big).$$

A straightforward computation shows that our assumption on biorthogonal curvature implies

$$\sqrt{6}\left(\frac{s}{6}-2K_1^{\perp}\right)\leq\frac{4a^2-(25/9)\lambda_1^2}{4\sqrt{6}a}.$$

Whence,

$$|W^+|+|W^-|\leq\frac{4a^2-(25/9)\lambda_1^2}{4\sqrt{6}a}. \tag{3.13}$$

Therefore, we combine (3.13) with (3.12) to conclude that Δ is less than or equal to zero. Hence, we use once more (3.10) to deduce $|W^+||W^-|=0$ in M^4. But since Σ is finite, we arrive at a contradiction.

Therefore, we conclude that $W^+=0$ or $W^-=0$. Finally, we define the following sets:

$$A=\left\{p\in M^4; Ric(p)\neq\frac{s(p)}{4}g\right\}$$

and

$$B=\{p\in M^4; |W^+|(p)=|W^-|(p)\},$$

where $(Ric-\frac{s}{4}g)$ stands for the traceless Ricci tensor of (M^4,g). If A is empty, then we conclude that M^4 is an Einstein manifold. In this case, we invoke Hitchin's theorem [3] to conclude that M^4 is either isometric to $\mathbb{S}^4$ with its canonical metric or isometric to $\mathbb{CP}^2$ with Fubini–Study metric. Otherwise, if A is not empty, then there exist a point $p\in M^4$ and an open set U such that $p\in U\subset A$. So, we use Corollary 1 of [8] to conclude that $U\subset A\subset B$. So far, since the function $f=|W^+|^2-|W^-|^2$ is analytic, we conclude that f is identically zero. For this, $|W^+|^2=|W^-|^2$, and then M^4 is locally conformally flat. This implies that M^4 has positive isotropic curvature, and then we use once more the Chen–Tang–Zhu theorem [5] to conclude that M^4 is diffeomorphic to a connected sum $\mathbb{S}^4\sharp(\mathbb{R}\times\mathbb{S}^3)/G_1\sharp\cdots\sharp(\mathbb{R}\times\mathbb{S}^3)/G_n$, where each G_i is a discrete subgroup of the isometry group of $\mathbb{R}\times\mathbb{S}^3$. This finishes the proof of the theorem. □

3.5. *Proof of the Corollary 3*

Proof. Since $Ric\geq\rho>0$ implies $\lambda_1\geq\frac{4\rho}{3}$, we combine Theorem 6 with Tani's theorem in [25] and then get the promised result. □

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