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Singular Rationally Connected Surfaces with Nonzero Pluri-Forms

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Abstract. Let X be a projective rationally connected surface with canonical singularities carrying a nonzero reflexive pluri-form, that is, the reflexive hull of $(\Omega^1_X)^{\otimes m}$ has a nonzero global section for some positive integer m. We show that any such surface X can be obtained from a rational ruled surface by a very explicit sequence of blow-ups and blow-downs. Moreover, we interpret the existence of nonzero pluri-forms in terms of semistable reduction.

1. Introduction

Recall that a projective variety X is said to be rationally connected if for any two general points in X, there exists a rational curve passing through them; see [Kol96, Def. 3.2 and Prop. 3.6]. It is known that for a smooth projective rationally connected variety X, $H^0(X, (\Omega^1_X)^{\otimes m}) = \{0\}$ for $m > 0$; see [Kol96, Cor. IV.3.8]. In [GKKP11, Thm. 5.1], it is shown that if a pair (X, D) is klt and X is rationally connected, then $H^0(X, \Omega^{[m]}_X) = \{0\}$ for $m > 0$, where $\Omega^{[m]}_X$ is the reflexive hull of Ω^m_X. By [GKP12, Thm. 3.3], if X is factorial, rationally connected and has canonical singularities, then $H^0(X, (\Omega^1_X)^{[\otimes m]}) = \{0\}$ for $m > 0$, where $(\Omega^1_X)^{[\otimes m]}$ is the reflexive hull of $(\Omega^1_X)^{\otimes m}$. However, this is not true without the assumption of being factorial; see [GKP12, Example 3.7]. In this paper, our aim is to classify rationally connected surfaces with canonical singularities that have nonzero reflexive pluri-forms. We will give two methods to construct such surfaces (see Construction 1.2 and Construction 1.6), and we will also prove that every such surface can be constructed by both of these methods (see Theorem 1.3 and Theorem 1.5). This gives an affirmative answer to [GKP12, Remark and Question 3.8]

The following example is given in [GKP12, Example 3.7].

Example 1.1. Let $\pi' : X' \to \mathbb{P}^1$ be any smooth ruled surface. Choose four distinct points q_1, q_2, q_3, q_4 in $\mathbb{P}^1$. For each point q_i, perform the following sequence of birational transformations of the ruled surface:

(i) Blow up a point x_i in the fiber over q_i. Then we get two (-1)-curves that meet transversely at x'_i.
(ii) Blow up the point x'_i. Over q_i, we get two disjoint (-2)-curves and one (-1)-curve. The (-1)-curve appears in the fiber with multiplicity two.

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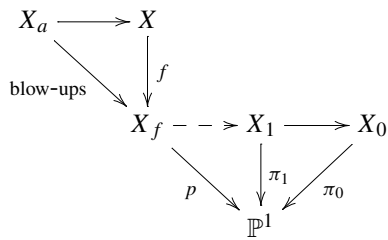
(iii) Blow down the two (-2)-curves. We get two singular points on the fiber, and each of them is of type A_1.

In the end, we get a rationally connected surface $\pi : X \to \mathbb{P}^1$ with canonical singularities such that $H^0(X, (\Omega^1_X)^{[\otimes 2]}) \neq \{0\}$.

We will prove that every projective rationally connected surface X with canonical singularities and having nonzero pluri-forms can be constructed by a similar method (see Construction 1.2) from a smooth ruled surface over $\mathbb{P}^1$.

CONSTRUCTION 1.2. Take a smooth ruled surface $X_0 \xrightarrow{\pi_0} \mathbb{P}^1$ and choose distinct points $q_1, \ldots, q_r$ in $\mathbb{P}^1$ with $r \geq 4$. We perform a sequence of birational transformations as follows.

(i) For each q_i, perform the same sequence of birational transformations as in Example 1.1. We get a fiber surface $\pi_1 : X_1 \to \mathbb{P}^1$. The nonreduced fibers of π_1 are $\pi_1^* q_1, \ldots, \pi_1^* q_r$.
(ii) Perform finitely many times this birational transformation: blow up a smooth point on a nonreduced fiber and then blow down the strict transform of the initial fiber. We obtain another fiber surface $p : X_f \to \mathbb{P}^1$ (see Lemma 4.6).
(iii) Starting from X_f, perform a sequence of blow-ups of smooth points. We get a surface X_a.
(iv) Blow down some chains of exceptional (-2)-curves for $X_a \to X_f$ (this is always possible; see Section 6). We obtain a rational surface X.



THEOREM 1.3. *The surface obtained by Construction 1.2 is a rationally connected surface that carries nonzero pluri-forms. Conversely, if X is a projective rationally connected surface with canonical singularities such that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, then X can be constructed by the method described in Construction 1.2.*

Note that if X is a rational surface obtained by Construction 1.2, then there is a fibration $\pi : X \to \mathbb{P}^1$ induced by π_0. This fibration has multiple fibers over the points $q_1, \ldots, q_r$ that we have chosen at the beginning of the construction. In fact, these multiple fibers are exactly the source of nonzero forms on X by the following theorem.

THEOREM 1.4. *Let X be a projective rationally connected surface with canonical singularities and having nonzero pluri-forms. If X_f is the result of an MMP, then*

X_f is a Mori fiber space over $\mathbb{P}^1$. Let $p : X_f \to \mathbb{P}^1$ be the fibration. If r is the number of points over which p has nonreduced fibers, then we have $r \geq 4$ and

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}\left(-2m + \left[\frac{m}{2}\right]r\right)\right)$$

for $m > 0$. In particular, for fixed m, the number of m-pluri-forms depends only on the number of multiple fibers.

We note that both in Theorem 1.4 and in Construction 1.2, we meet a surface named X_f. We will see later (in the proof of Theorem 1.3 in Section 6) that, by choosing a good MMP, these two surfaces are identical. The points $q_1, \ldots, q_r$ are exactly the points over which $p : X_f \to \mathbb{P}^1$ has multiple fibers. By the semistable reduction we can find a Galois cover $\gamma : E \to \mathbb{P}^1$ such that $Z \to E$ has only reduced fibers, where Z is the normalization of $X_f \times_{\mathbb{P}^1} E$. Let Y be the normalization of $X \times_{\mathbb{P}^1} E$. The following theorem shows that we can always choose a finite Galois cover γ that has degree 4 and the pluri-forms on X are exactly the G-invariant pluri-forms on Y, where G is the Galois group of γ.

THEOREM 1.5. *Let X be a projective rationally connected surface with canonical singularities and having nonzero pluri-forms. Let π be the composition of $X \to X_f \to \mathbb{P}^1$. Then there is a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow[4:1 \text{ cover}]{\Gamma} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[4:1 \text{ cover}]{\gamma} & \mathbb{P}^1 \end{array}$$

such that E is a smooth curve of positive genus and Y is a projective surface with canonical singularities. Both γ and Γ are Galois covers with Galois group $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and Γ is étale in codimension 1. Moreover, for all $m \geq 0$, we have $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(Y, (\Omega^1_Y)^{[\otimes m]})^G \cong H^0(E, (\Omega^1_E)^{\otimes m})^G$.

Note that Y is not rationally connected since E is not rationally connected. This theorem shows that every projective rationally connected surface with canonical singularities that has nonzero pluri-forms can be constructed by the following method.

CONSTRUCTION 1.6. Let Y be a projective surface with canonical singularities, and let G be a finite subgroup of $\mathrm{Aut}(Y)$ whose action is étale in codimension 1. Assume that there is a G-invariant fibration π' from Y to a smooth curve E of positive genus such that $E/G = \mathbb{P}^1$ and that general fibers of π' are smooth rational curves. Let $X = Y/G$. Then X is rationally connected (see [GHS03, Thm. 1.1]), and $H^0(X, (\Omega^1_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$.

2. Notation and Outline of Paper

Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. Unless otherwise specified, every variety is an integral $\mathbb{C}$-scheme of finite type. A curve is a variety of dimension 1, and a surface is a variety of dimension 2. For a variety X, we denote the sheaf of Kähler differentials by Ω^1_X. We denote $\bigwedge^p \Omega^1_X$ by Ω^p_X for $p \in \mathbb{N}$.

For a coherent sheaf $\mathscr{F}$ on a variety X, we denote by $\mathscr{F}^{**}$ the reflexive hull of $\mathscr{F}$. There is an important property for reflexive sheaves.

LEMMA 2.1 ([Har80, Prop. 1.6]). *Let $\mathscr{F}$ be a coherent sheaf on a normal variety V. Then $\mathscr{F}$ is reflexive if and only if $\mathscr{F}$ is torsion-free and for each open $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension at least* 2, *the restriction map $\mathscr{F}(U) \to \mathscr{F}(U \setminus Y)$ is an isomorphism.*

Let V be a normal variety, and let V_0 be its smooth locus. We denote a canonical divisor by K_V. Moreover, let $\Omega^{[p]}_V$ (resp. $(\Omega^1_V)^{[\otimes p]}$) be the reflexive hull of Ω^p_V (resp. $(\Omega^1_V)^{\otimes p}$). By Lemma 2.1, it is the push-forward of the locally free sheaf $\Omega^p_{V_0}$ (resp. $(\Omega^1_{V_0})^{\otimes p}$) to V since V is smooth in codimension 1.

Let S be a normal surface. A smooth rational curve C in S is a $(-k)$-curve if S is smooth along C and the intersection number $C \cdot C = -k$. A projective birational morphism $r : \widetilde{S} \to S$ is called the minimal resolution of singularities (or minimal resolution for short) if $\widetilde{S}$ is smooth and $K_{\widetilde{S}}$ is r-nef. There is a unique minimal resolution of singularities for a normal surface and any resolution of singularities factors through the minimal resolution.

Let S be a normal surface, and let $r : \widetilde{S} \to S$ be the minimal resolution of singularities of S. We say that S has *canonical singularities* if the intersection number $K_{\widetilde{S}} \cdot C$ is zero for every r-exceptional curve C. Canonical surface singularities are also called Du Val singularities. We know all of these singularities, they are A_i, D_j, E_k where $i \geq 1$, $j \geq 3$, and $k = 6, 7, 8$. For more details on Du Val singularities, see [KM98, §4.1].

Let $p : V \to B$ be a fibration from a normal variety to a smooth curve. If the nonreduced fibers of p are $p^*z_1, \ldots, p^*z_r$, then the *ramification divisor* of p is the divisor defined by

$$R = p^*z_1 + \cdots + p^*z_r - \operatorname{Supp}(p^*z_1 + \cdots + p^*z_r).$$

Let X be a projective rationally connected surface with canonical singularities that carries nonzero pluri-forms. Then we can run a minimal model program for X (for more details on MMP, see [KM98, Sections 1.4 and 3.7]). We obtain a sequence of divisorial contractions

$$X = X_0 \to X_1 \to \cdots \to X_n.$$

Since K_X is not pseudoeffective, neither is K_{X_n}. Thus X_n is a Mori fiber space. We have a Mori fibration $p : X_n \to B$. Therefore we have two possibilities: either $\dim B = 0$ or $\dim B = 1$. If $\dim B = 0$, then X_n is a Fano variety with Picard number 1. Here, a Fano variety is a normal projective variety whose anticanonical

divisor is an ample $\mathbb{Q}$-Cartier divisor. In Section 3, we will prove that X cannot have any nonzero pluri-form in this case. Hence we only need to deal with the case that $\dim B = 1$. In Section 4, we will study some properties for Mori fiber surfaces over a curve. In the last three sections, we will prove Theorems 1.4, 1.3, and 1.5 in this order.

3. Vanishing Theorem for Fano Varieties with Picard Number 1

The aim of this section is to prove the following theorem.

THEOREM 3.1. *Let V be a $\mathbb{Q}$-factorial klt Fano variety with Picard number* 1. *Then $H^0(V, (\Omega^1_V)^{[\otimes m]}) = \{0\}$ for any $m > 0$.*

Before proving the theorem, we recall the notion of slopes. Let V be a normal projective $\mathbb{Q}$-factorial variety of dimension d. Let A be an ample divisor in V. Then for a coherent sheaf $\mathscr{F}$, we can define the slope of $\mathscr{F}$ with respect to A, $\mu_A(\mathscr{F})$, by

$$\mu_A(\mathscr{F}) := \frac{\det \mathscr{F} \cdot A^{d-1}}{\operatorname{rank} \mathscr{F}},$$

where $\det \mathscr{F}$ is the reflexive hull of $\bigwedge^{\operatorname{rank} \mathscr{F}} \mathscr{F}$. Moreover, let

$$\mu_A^{\max}(\mathscr{F}) = \sup\{\mu_A(\mathscr{G}) \mid \mathscr{G} \subseteq \mathscr{F} \text{ a coherent subsheaf}\}.$$

For any coherent sheaf $\mathscr{F}$, there is a saturated coherent subsheaf $\mathscr{G} \subseteq \mathscr{F}$ such that $\mu_A^{\max}(\mathscr{F}) = \mu_A(\mathscr{G})$; see [GKP12, Prop. A.2].

PROPOSITION 3.2. *Let V be a projective normal variety that is $\mathbb{Q}$-factorial, and let H be an ample divisor in V. Then for any two coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on V,*

$$\mu_H^{\max}((\mathscr{F} \otimes \mathscr{G})^{**}) = \mu_H^{\max}(\mathscr{F}) + \mu_H^{\max}(\mathscr{G}).$$

For a proof of this proposition, see [GKP12, Prop. A.14]. Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We may assume that $\dim V > 1$. We will argue by contradiction. Assume that there is a positive integer m such that $H^0(V, (\Omega^1_V)^{[\otimes m]}) \neq \{0\}$. Let H be an ample divisor on V.

Since $H^0(V, (\Omega^1_V)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$, we have an injective morphism of sheaves

$$\mathscr{O}_V \hookrightarrow (\Omega^1_V)^{[\otimes m]}.$$

This shows that

$$\mu_H^{\max}((\Omega^1_V)^{[\otimes m]}) \geq \mu_H(\mathscr{O}_V) = 0.$$

By Proposition 3.2 we have $\mu_H^{\max}(\Omega^{[1]}_V) = m^{-1}\mu_H^{\max}((\Omega^1_V)^{[\otimes m]}) \geq 0$.

Therefore, there is a nonzero saturated coherent sheaf $\mathscr{F} \subseteq \Omega^{[1]}_V$ such that $\mu_H(\mathscr{F}) \geq 0$. Observe that $\operatorname{rank} \mathscr{F} < \dim V$; otherwise, $\mathscr{F} = \Omega^{[1]}_V$ and $\det \mathscr{F} \cong K_V$. Thus, $\mu_H(\mathscr{F}) < 0$, a contradiction.

We have two possibilities, either $\mu_H(\mathscr{F}) > 0$ or $\mu_H(\mathscr{F}) = 0$.

Case 1. Assume that $\mu_H(\mathscr{F}) > 0$. Since V has Picard number 1, $\det \mathscr{F}$ is ample, and its Kodaira–Iitaka dimension is $\dim V$. However, this contradicts the Bogomolov–Sommese vanishing theorem (see [Gra13, Cor. 1.3]).

Case 2. Assume that $\mu_H(\mathscr{F}) = 0$. Let $\mathscr{G} = \det \mathscr{F}$. Then $\mathscr{G} \cdot H^{(\dim V - 1)} = 0$. Since V is $\mathbb{Q}$-factorial and klt, by [AD12, Lem. 2.6] there exists an integer l such that $(\mathscr{G}^{\otimes l})^{**}$ is isomorphic to $\mathscr{O}_V$. Let m be the smallest positive integer such that $(\mathscr{G}^{\otimes m})^{**} \cong \mathscr{O}_V$. We can construct the cyclic cover $q : Z \to V$ of V corresponding to $\mathscr{G}$; see [KM98, Def. 2.52]. Then $(q^*\mathscr{G})^{**} \cong \mathscr{O}_Z$. Since q is étale in codimension 1, Z is also klt by [Kol97, Prop. 3.16], and $-K_Z = q^*(-K_V)$ is ample. Thus, Z is rationally connected by [HM07, Cor. 1.3 and 1.5], and there are natural injective morphisms

$$(q^*\mathscr{G})^{**} \hookrightarrow (q^*\Omega_V^{[\operatorname{rank}\mathscr{F}]})^{**} \hookrightarrow \Omega_Z^{[\operatorname{rank}\mathscr{F}]}.$$

Hence, we have an injection $\mathscr{O}_Z \hookrightarrow \Omega_Z^{[\operatorname{rank}\mathscr{F}]}$, but this contradicts [GKKP11, Thm. 5.1]. □

4. Mori Fiber Surfaces over a Curve

Recall that a Mori fibration $V \to W$ is a projective fibration such that $-K_V$ is relatively ample and the relative Picard number is 1. A Mori fiber space V is just a variety endowed with a Mori fibration $V \to W$. In this section, we study a Mori fibration from a quasi-projective surface with canonical singularities to a smooth curve. In the first subsection, we will give some properties of the fibers. In the second subsection, we will classify the singularities on a nonreduced fiber.

We would like to introduce some notation for this section first. Let $p : S \to B$ be a Mori fibration, where B is a smooth curve, and S is a normal surface with canonical singularities. Let $r : \widetilde{S} \to S$ be the minimal resolution, and let $\tilde{p} = p \circ r : \widetilde{S} \to B$.

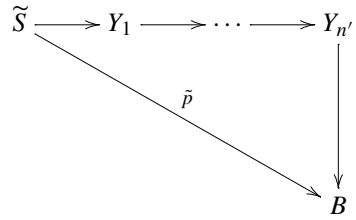
Since S is singular at only finitely many points, p is smooth over general points of B, and general fibers are all isomorphic to $\mathbb{P}^1$. Note that a point in a smooth curve can also be regarded as a Cartier divisor, and since any two fibers of p are numerically equivalent, we have $K_S \cdot p^*z = -2$ and $p^*z \cdot p^*z = 0$ for any $z \in B$ by the adjunction formula.

We recall the definition of dual graph. Let $E = \bigcup E_i$ be a collection of proper curves on a normal surface V such that V is smooth along E. The *dual graph* Γ of E is defined as follows:

(1) The vertices of Γ are the curves E_i.
(2) Two vertices $E_i \neq E_j$ are connected with $(E_i \cdot E_j)$ edges.

4.1. Some Properties of Fibers

Running a $\tilde{p}$-MMP for $\widetilde{S}$, we obtain a sequence of divisorial contractions



LEMMA 4.1. *With the notation in the diagram above, the surface $Z = Y_{n'}$ is a ruled surface over B. Moreover, the support of $\tilde{p}^*b$ is an snc tree, that is, it is an snc divisor, and its dual graph is a tree, for all points $b \in B$.*

Proof. Since $p_Z : Z \to B$ is the result of a $\tilde{p}$-relative MMP, Z is a smooth surface. Note that K_Z has negative intersection number with general fibers of p_Z. Hence, the next extremal contraction in the MMP is a contraction of fiber type. This contraction gives Z the ruled surface structure over B.

Note that $\widetilde{S}$ can be obtained by a sequence of blow-ups from Z. Thus, the dual graph of the support of any fiber of $\tilde{p}$ is an snc tree. □

We collect some properties for the fiber of $p : S \to B$.

PROPOSITION 4.2. *Let z be a point in B. Then*:

(1) *the support C of p^*z is an irreducible Weil divisor for every $z \in B$*;
(2) *the coefficient of C in p^*z is at most equal to* 2;
(3) *S is smooth along the support of p^*z if and only if p^*z is reduced*;
(4) *there exist at most two singular points of S on C.*

Proof. (1) Assume the opposite and let D, D' be two distinct components in p^*z that meet. Then $D \cdot D' > 0$ and $D \cdot D < 0$ since $p^*z \cdot D = 0$. However, there is a positive number λ such that λD and D' are numerically equivalent by the definition of Mori fibration. Hence, $D \cdot D > 0$. This is a contradiction.

(2) Let $\alpha \in \mathbb{N}$ be the coefficient of C in p^*z. Then

$$-2 = K_S \cdot p^*z = \alpha K_S \cdot C.$$

However, since K_S is a Cartier divisor, $K_S \cdot C \in \mathbb{Z}$. Thus, $-2 \in \alpha\mathbb{Z}$, which means that $\alpha \leq 2$.

(3) Note that S is Cohen–Macaulay since it is a normal surface. Therefore, the Cartier divisor p^*z is also Cohen–Macaulay. Hence, it is generically reduced if and only if it is a reduced subscheme. Moreover, since B is a smooth curve, the morphism p is a flat morphism.

First, we assume that p^*z is reduced. Then the arithmetic genus of p^*z is 0 since p is flat and general fibers of p are smooth rational curves. Hence, p^*z is isomorphic to $\mathbb{P}^1$ (cf. [Har77, Ex. IV.1.8(b)]). Hence, p is smooth over z since it is flat. Thus, S is smooth along p^*z.

Conversely, assume that S is smooth along p^*z. Then by adjunction formula we have

$$2h^1(C, \mathcal{O}_C) - 2 = (K_S + C) \cdot C = K_S \cdot C < 0.$$

Therefore, $K_S \cdot C = -2$, which is equal to $K_S \cdot p^*z$. Hence, p^*z is reduced.

(4) Assume that S is not smooth along C. Then $p^*z = 2C$ by (2) and (3). Let $\widetilde{C}$ be the strict transform of C in $\widetilde{S}$, and let $E = \tilde{p}^*z - 2\widetilde{C}$. Since E is r-exceptional, we have

$$K_{\widetilde{S}} \cdot E = r^*K_S \cdot E = 0.$$

Thus,

$$K_{\widetilde{S}} \cdot \widetilde{C} = 2^{-1}(K_{\widetilde{S}} \cdot \tilde{p}^*z) = -1.$$

By the adjunction formula we have $\widetilde{C}^2 = -1$ ($\widetilde{C}$ is smooth by Lemma 4.1). Then

$$-1 = \widetilde{C}^2 = 2^{-1}\widetilde{C} \cdot (\tilde{p}^*z - E) = -2^{-1}\widetilde{C} \cdot E.$$

We obtain $\widetilde{C} \cdot E = 2$. This implies that $\widetilde{C}$ and E meet at most at two points. Hence, S has at most two singular points on C. □

4.2. *Singularities on Nonreduced Fibers*

The aim of this subsection is to give a complete list of possible multiple fibers of $p : S \to B$. The subject was studied in [KM99, Section 11.5], but we will give some elementary proofs of the results here. In the remainder of this section, we will assume that p has a nonreduced fiber over and only over $0 \in B$. By Proposition 4.2 this implies that $p^*0 = 2C$, where C is the support of p^*0. We denote the strict transform of C in $\widetilde{S}$ by $\widetilde{C}$. We will prove the following theorem.

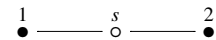
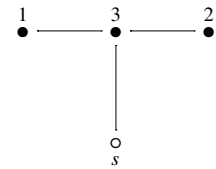
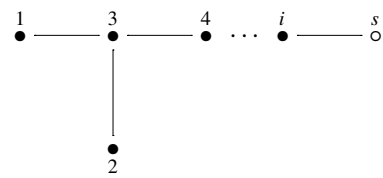
THEOREM 4.3. *Let $p : S \to B$ be a Mori fibration such that S is a quasi-projective surface with canonical singularities and B is a smooth curve. Assume that p^*0 is a multiple fiber, where $0 \in B$. Let $r : \widetilde{S} \to S$ be the minimal resolution of singularities along the fiber p^*0 and let $\tilde{p} = p \circ r$. We have the following table of possibilities for p^*0, and each of these possibilities can occur. In Table 1, the dual graph is the one of the support of $\tilde{p}^*0 \subseteq \widetilde{S}$, the point with label s corresponds to $\widetilde{C}$, and the other points correspond to the r-exceptional divisors.*

In Table 1, we see that a multiple fiber of type $(A_1 + A_1)$ is a multiple fiber that contains exactly two singular points and both of them are of type A_1. A multiple fiber of type (D_i) with $i \geq 3$ is a multiple fiber that contains exactly one singular point that is of type D_i (note that the singularity D_3 is the same as A_3).

We will prove the theorem by proving several lemmas (Lemma 4.4, 4.5, and 4.7). Note that by Proposition 4.2 there exist one or two singular points of S on C. We will first treat the case of two singular points.

LEMMA 4.4. *Assume that there are two singular points on C. Then each of them is of type A_1.*

Table 1

Type of fiber	Dual graph
$(A_1 + A_1)$	
(D_3)	
(D_i) with $i > 3$	

Proof. Let $E = \tilde{p}^*0 - 2\widetilde{C}$. As in the proof of Proposition 4.2.4, we have $\widetilde{C} \cdot E = 2$. Since there are two singular points on C, E has exactly two connected components. Thus, we can decompose E into $D + D' + R$, where D and D' are the two components in E that meet $\widetilde{C}$. Then we have

$$\widetilde{C} \cdot D = \widetilde{C} \cdot D' = 1, \quad D \cdot D' = 0 \quad \text{and} \quad \widetilde{C} \cdot R = 0.$$

Note that both D and D' are (-2)-curves, and hence

$$0 = \tilde{p}^*z \cdot D = 2\widetilde{C} \cdot D + D^2 + D' \cdot D + R \cdot D = R \cdot D.$$

This implies that R and D do not meet since both R and D are effective. By symmetry, R and D' also do not meet. However, since E has exactly two connected components, we obtain that $R = 0$. Hence, both of the singular points on C are of type A_1. □

This type of fiber is the type $(A_1 + A_1)$. Note that this type of fiber does exist by Example 1.1. Next, we will study the case of one singular point. We will prove that this isolated singularity is of type D_i ($i \geq 3$, and the type D_3 is just A_3).

LEMMA 4.5. *The isolated singularity on the fiber over* $0 \in B$ *can only be of type* D_i ($i \geq 3$).

Proof. Let $C_0 = \widetilde{C}$, and let $E_0 = \tilde{p}^*0 - 2C_0$. As in the proof of Proposition 4.2.4, we have

$$C_0^2 = -1 \quad \text{and} \quad E_0 \cdot C_0 = 2.$$

Since $E_0 \cdot p^*0 = 0$, we obtain $E_0^2 = -4$.

Since there is only one singular point on C and the support of $\tilde{p}^*0$ is an snc tree (see Lemma 4.1), we can decompose E_0 into $2C_1 + E_1$, where C_1 is the unique

component in E_0 that meets C_0. Then C_1 is a (-2)-curve. Since $2C_1 \cdot \tilde{p}^*0 = 0$ and $E_0^2 = -4$, we have

$$E_1^2 = -4 \quad \text{and} \quad C_1 \cdot E_1 = 2.$$

Thus, the support of E_1 intersects C_1 at one or two points. If they intersect at two points, then as in Lemma 4.4, $E_1 = D + D'$ where D, D' are smooth rational curves, and we have

$$D \cdot D' = 0, \quad D \cdot C_1 = 1, \quad D' \cdot C_1 = 1.$$

If E_1 and C_1 intersect at one point, then we can decompose E_1 into $2C_2 + E_2$ where C_2 is the unique component of E_1 that meets C_1. As before, we have

$$C_2^2 = -2, \quad E_2^2 = -4 \quad \text{and} \quad E_2 \cdot C_2 = 2.$$

We are in the same situation as before. Hence, by induction we can decompose E_0 into $2(D_1 + \cdots + D_i) + D + D'$ where D, D' and all the D_j are (-2)-curves. Furthermore, we have

$$D \cdot D' = 0, \quad D_i \cdot D = 1, \quad D_i \cdot D' = 1, \quad D_j \cdot D_{j+1} = 1$$

for $1 \le j \le i-1$, and $D_j \cdot D_k = 0$ if $k - j > 1$. This shows that the singular point is of type D_{i+2}. □

These types of fibers are (D_i) $(i \ge 3)$. Now we will prove that these kinds of fibers exist. We will need the following lemma.

LEMMA 4.6. *Let $x \in S$ be a smooth point over $0 \in B$, and let W be the blow-up of S at x with exceptional divisor $E \subseteq W$. Let D be the strict transform of C in W. Then we can blow down D and obtain another Mori fiber surface $q : T \to B$.*

Proof. Let $W \hookrightarrow W_1$ be a projective compactification of W' such that W_1 has canonical singularities. If we can blow down D in W_1, then we can also blow down D in W. Hence, we may assume that W is projective.

We have $C \cdot C = 0$, $K_S \cdot C = -1$, $K_W \cdot E = -1$, $E \cdot E = -1$, and $D \cdot E = 1$. Thus,

$$K_W \cdot D = 0 \quad \text{and} \quad D \cdot D = -1.$$

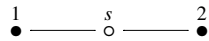
Let H be an ample divisor on W. Then there is a positive integer k such that $(H + kD) \cdot D = 0$. Let $A = H + kD$. Note that A is nef and big and D is the only curve that has intersection number 0 with A. Since $K_W \cdot D = 0$, for large enough positive integer a, the divisor $aA - K_X$ is nef and big. Hence, by the basepoint-free theorem (see [KM98, Thm. 3.3]) there is a positive integer b such that the linear system $|bA|$ is basepoint-free. Let $c : W \to T$ be the fibration induced by the linear system $|bA|$. Then c contracts exactly D. Since D is contracted by $W \to B$, the fibration $W \to B$ induces a fibration $q : T \to B$, which is also a Mori fibration. □

We can use the elementary transformation in Lemma 4.6 to construct every type of multiple fibers mentioned previously.

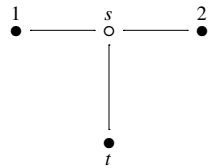
LEMMA 4.7. *If S is of type $(A_1 + A_1)$ over $0 \in B$, then T is of type (D_3) over $0 \in B$. If S is of type (D_i) over $0 \in B$, then T is of type (D_{i+1}) over $0 \in B$ for $i \geq 3$.*

Proof. We will compute the dual graph of the support of the fiber $\tilde{q}^*0$, where $\widetilde{T} \to T$ is the minimal resolution, and $\tilde{q}$ is the composition of $\widetilde{T} \to T \to B$. Let W be the same as in Lemma 4.6. From the construction of T we know that $\widetilde{T} \to T$ factors through $\widetilde{T} \to W$ and the last morphism is also the minimal resolution of W. Since $W \to S$ is a blow-up of a smooth point of S, the surface $\widetilde{T}$ can be obtained by blowing up the same point in $\widetilde{S}$.

If the fiber p^*0 is of type $(A_1 + A_1)$, then the dual graph of the support of $\tilde{p}^*0$ in $\widetilde{S}$ is

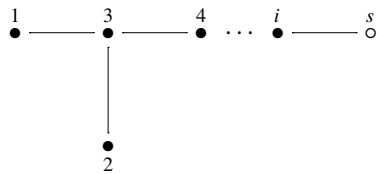


where s represents $\widetilde{C}$. Blow up the point we mentioned before; the new graph is

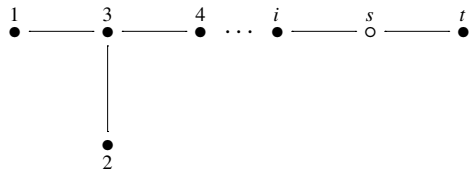


This graph is the dual graph of the support of $\tilde{q}^*0$, and the point with label t corresponds to the strict transform of the support of q^*0 in $\widetilde{T}$. The graph shows that there is only one singular point of T on q^*0 that is of type D_3. Hence, the fiber q^*0 is of type (D_3).

If p^*0 is of type D_i, then from the proof of Lemma 4.5 we know that the dual graph of the support of $\tilde{p}^*0$ is



where the point with label s corresponds to $\widetilde{C}$ (if $i = 3$, then s is just connected to the point with label 3). By blowing up the point, we obtain the dual graph of the support of $\tilde{q}^*0$, which is



The point with label t corresponds to the strict transform of the support of q^*0 in $\widetilde{T}$. This implies that q^*0 is of type (D_{i+1}). □

Proof of Theorem 4.3. We can deduce the theorem from Lemmas 4.4, 4.5, and 4.7. □

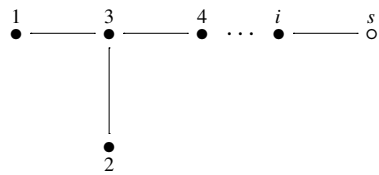
Now we will show that every singular fiber can be obtained from a smooth fiber by the methods we mentioned before.

LEMMA 4.8. *The singular fiber p^*0 of $p : S \to B$ can be obtained from a smooth ruled surface $S_1 \to B$ by the method of Example 1.1 followed by a finite sequence of elementary birational transformations described in Lemma 4.6.*

Proof. Let $\widetilde{S} \to Z$ be the result of a $\tilde{p}$-MMP. Then $Z \to B$ is a ruled surface by Lemma 4.1. Moreover, $\widetilde{S}$ can be obtained from Z by a sequence of blow-ups.

If p^*0 is of type $(A_1 + A_1)$, then $\widetilde{S}$ can be obtained from Z by two blow-ups as in the first two steps of Example 1.1. Blowing down the two (-2)-curves in $\widetilde{S}$, we obtain S. In this case, we take $S_1 = Z$.

If p^*0 is of type (D_i), then the dual graph of $\tilde{p}^*0$ is



Note that the curve corresponding to the point s is a (-1)-curve. Hence, we may blow down this curve and the curves that correspond to the points in the graph that do not meet the point s (this is always possible by the next lemma). Then we will obtain another Mori fiber surface $p_U : U \to B$. The fiber p_U^*0 is of type (D_{i-1}) if $i > 3$ and of type $(A_1 + A_1)$ if $i = 3$. Moreover, U is smooth around the image of the curve corresponding to s. If we perform the birational transformation in Lemma 4.6 for U, then we will obtain S.

We can conclude the lemma by induction. □

The following lemma shows that we can contract some connected collection of (-2)-curves in a surface.

LEMMA 4.9. *Let $E = \bigcup_{1 \le k \le i} E_k$ be a connected collection of (-2)-curves in a smooth surface V whose dual graph is the same as the one of the support of the exceptional set of a minimal resolution for a canonical surface singularity. Then there exists a morphism $c : V \to W$ such that W has canonical singularities and c contracts exactly E.*

Proof. We have $K_V \cdot E_k = 0$ for every k. The intersection matrix $\{E_k \cdot E_j\}$ is negative definite by [KM98, Lem. 3.40]. Thus, there is a contraction $c : V \to W$ contracting exactly E by [KM98, Prop. 4.10]. Note that c is also the minimal resolution of W and $K_V = c^* K_W$. Hence, W has canonical singularities. □

5. Proof of Theorem 1.4

We will first prove Theorem 1.4. Let X be a projective rationally connected surface with canonical singularities that has nonzero pluri-forms. Run an MMP for X. We will get a sequence of divisorial contractions

$$X = X_0 \to X_1 \to \cdots \to X_n = X_f.$$

The rational surface X_f is a Mori fiber surface over $\mathbb{P}^1$ by Theorem 3.1. Let $p : X_f \to \mathbb{P}^1$ be the Mori fibration. Let $f : X \to X_f$ be the composition of the sequence of the birational morphisms above, and let $\pi = p \circ f : X \to \mathbb{P}^1$. Then for any $m \in \mathbb{N}$, there is an injection $H^0(X, (\Omega^1_X)^{[\otimes m]}) \hookrightarrow H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]})$.

5.1. *Source of Nonzero Reflexive Pluri-forms*

In this subsection, we will find out the source of nonzero pluri-forms on X_f. Let U be the smooth locus of X_f. Then the morphism of locally free sheaves on U

$$p^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_U$$

factors through

$$\phi : p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_U(R) \to \Omega^1_U,$$

where R is the ramification divisor of p. Let V be the largest subset of U such that for any point $x \in V$, the evaluation of ϕ at x is injective. Then $\operatorname{codim} X_f \backslash V \geq 2$. By Lemma 2.1 this implies that $H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0(V, (\Omega^1_V)^{\otimes m})$ for any $m \in \mathbb{N}$.

Consider the exact sequence of sheaves on V

$$0 \to p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R) \to \Omega^1_V \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is isomorphic to $\Omega^1_{V/\mathbb{P}^1}/\text{torsion}$. It is an invertible sheaf on V since $\mathscr{G} \otimes k_x$ is of rank 1 at every point x of V, where k_x is the residue field of x (see [Har77, Ex. II.5.8]). Then there is a filtration (see Lemma 5.3 at the end of this subsection) over V

$$(\Omega^1_V)^{\otimes m} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s$$

such that for every $i \in \{0, \ldots, s-1\}$, the quotient $\mathscr{F}_i/\mathscr{F}_{i+1}$ is isomorphic to $\mathscr{G}^{\otimes a_i} \otimes (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))^{\otimes(m-a_i)}$ with $0 < a_i \leq m$ an integer. Moreover, we have $\mathscr{F}_s \cong (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))^{\otimes m} \cong (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_V(mR)$.

Lemma 5.1. *With the previous notation, there is a natural isomorphism*

$$H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_X(mR))$$

for all $m \geq 0$.

Proof. Fix some $m \geq 0$. For a general point $z \in \mathbb{P}^1$, the support C of the fiber p^*z is isomorphic to $\mathbb{P}^1$ and is contained in V. Since p is smooth along C, we have

$$\mathscr{G}|_C \cong \mathscr{O}_C(-2) \quad \text{and} \quad (p^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_V(R))|_C \cong \mathscr{O}_C.$$

Thus, $(\mathscr{F}_i/\mathscr{F}_{i+1})|_C$ is isomorphic $\mathscr{O}_C(-2a_i)$ for $i < s$. Since $a_i > 0$, we have $H^0(V, \mathscr{F}_i/\mathscr{F}_{i+1}) = 0$ and $H^0(V, \mathscr{F}_i) \cong H^0(V, \mathscr{F}_{i+1})$ for $i < s$. This implies that

$$H^0(V, (\Omega^1_V)^{\otimes m}) \cong H^0(V, (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_V(mR)).$$

By Lemma 2.1 this isomorphism induces the isomorphism

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X, (p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_X(mR)).$$

Note that the right-hand side above is isomorphic to $H^0(\mathbb{P}^1, p_*((p^*\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes \mathscr{O}_X(mR)))$. By the projection formula it is isomorphic to $H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1})^{\otimes m} \otimes p_*\mathscr{O}_X(mR))$. Hence,

$$H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes p_*\mathscr{O}_X(mR)). \qquad \square$$

Note that $p_*\mathscr{O}_X(mR)$ is a torsion-free sheaf of rank 1 on $\mathbb{P}^1$. Thus, it is an invertible sheaf, and there is a $k \in \mathbb{Z}$ such that $\mathscr{O}_{\mathbb{P}^1}(k)$ is isomorphic to $p_*\mathscr{O}_X(mR)$. In the following lemma, we will compute the integer k.

LEMMA 5.2. *Assume that the nonreduced fibers of $p : X_f \to \mathbb{P}^1$ are over $z_1, \ldots, z_r$. Then for $m \in \mathbb{N}$, we have $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}](z_1 + \cdots + z_r)) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}]r)$, where $[\cdot]$ is the integer part. In particular, $H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$.*

Proof. Since the problem is local around every point z_i, we may assume that $r = 1$ for simplicity. From Proposition 4.2.1 and Proposition 4.2.2 we know that R is irreducible and $p^*z_1 = 2R$. We may assume that $p_*\mathscr{O}_X(mR) \cong \mathscr{O}_{\mathbb{P}^1}(k \cdot z_1)$, and we have to prove that $k = [\frac{m}{2}]$.

Note that $\gamma \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(k \cdot z_1))$ is just a rational function on $\mathbb{P}^1$, which can only have a pole at z_1 with multiplicity at most k. Its pull-back to X is a rational function, which can only have pole along R with multiplicity at most $2k$. Thus, k is the largest integer such that $2k \le m$, that is, $k = [\frac{m}{2}]$. $\square$

In the following lemma, we will prove the existence of a filtration on $(\Omega^1_V)^{[\otimes m]}$.

LEMMA 5.3. *Let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ be an exact sequence of locally free sheaves on a variety X. Then for any $m > 0$, there is a positive integer s and a filtration*

$$\mathscr{F}^{\otimes m} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s = \mathscr{E}^{\otimes m}$$

such that for every $i \in \{0, \ldots, s-1\}$, the quotient $\mathscr{F}_i/\mathscr{F}_{i+1}$ is isomorphic to $\mathscr{G}^{\otimes a_i} \otimes \mathscr{E}^{\otimes(m-a_i)}$, where $0 < a_i \le m$ is an integer.

Proof. We will prove by induction. If $m = 1$, then the assertion is true ($s = 1$ and $\mathscr{F}_1 = \mathscr{E}$). Assume that the assertion is true for $m = k$. Then we have a filtration

$$\mathscr{F}^{\otimes k} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_s = \mathscr{E}^{\otimes k}$$

such that $\mathscr{F}_i/\mathscr{F}_{i+1}$ is isomorphic to $\mathscr{G}^{\otimes a_i} \otimes \mathscr{E}^{\otimes(k-a_i)}$, where $a_i > 0$ is an integer.

Consider the exact sequence

$$0 \to \mathscr{E} \otimes \mathscr{F}^{\otimes k} \to \mathscr{F}^{\otimes(k+1)} \to \mathscr{G} \otimes \mathscr{F}^{\otimes k} \to 0.$$

By hypothesis of induction we have a filtration

$$\mathscr{E} \otimes \mathscr{F}^{\otimes k} = \mathscr{E} \otimes \mathscr{F}_0 \supseteq \mathscr{E} \otimes \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{E} \otimes \mathscr{F}_s = \mathscr{E}^{\otimes(k+1)}.$$

Since $\mathscr{G} \otimes \mathscr{F}^{\otimes k}$ is the quotient of $\mathscr{F}^{\otimes(k+1)}$ by $\mathscr{E} \otimes \mathscr{F}^{\otimes k}$, we have a filtration, induced by the above filtration on $\mathscr{F}^{\otimes k}$,

$$\mathscr{F}^{\otimes(k+1)} = \mathscr{H}_0 \supseteq \mathscr{H}_1 \supseteq \cdots \supseteq \mathscr{H}_s \supseteq \mathscr{E} \otimes \mathscr{F}^{\otimes k}$$

such that $\mathscr{H}_i/\mathscr{H}_{i+1}$ is isomorphic to $\mathscr{G} \otimes (\mathscr{G}^{\otimes a_i} \otimes \mathscr{E}^{\otimes(k-a_i)})$ and $\mathscr{H}_s/(\mathscr{E} \otimes \mathscr{F}^{\otimes k}) \cong \mathscr{G} \otimes \mathscr{E}^{\otimes k}$. Hence, we obtain the fibration

$$\mathscr{F}^{\otimes(k+1)} = \mathscr{H}_0 \supseteq \mathscr{H}_1 \supseteq \cdots \supseteq \mathscr{H}_s \supseteq \mathscr{E} \otimes \mathscr{F}_0 \supseteq \cdots \supseteq \mathscr{E} \otimes \mathscr{F}_s = \mathscr{E}^{\otimes(k+1)},$$

which satisfies the conditions in the lemma. □

5.2. *Back to the Initial Variety*

We have studied X_f, and now we have to reverse the MMP and pull back pluriforms to the initial variety X. Our aim is to prove that

$$H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}).$$

We will need the following proposition.

PROPOSITION 5.4. *Let S be a projective surface that has at most canonical singularities. Let $c : S \to T$ be a divisorial contraction in an MMP. Let E be the exceptional divisor, and let x be the image of E. Then T is smooth at x.*

Proof. We suppose the opposite. Let $r_S : \widetilde{S} \to S$ and $r_T : \widetilde{T} \to T$ be the minimal resolutions. Let $\widetilde{E}$ be the strict transform of E in $\widetilde{S}$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{c}} & \widetilde{T} \\ {\scriptstyle r_S}\downarrow & & \downarrow{\scriptstyle r_T} \\ S & \xrightarrow{c} & T \end{array}$$

Then $K_{\widetilde{S}} \cdot \widetilde{E} = r^* K_S \cdot \widetilde{E} = K_S \cdot r_{S*}\widetilde{E} = K_S \cdot E < 0$ by the definition of MMP. Since $K_{\widetilde{T}}$ is r_T-nef, $\widetilde{E}$ must be contracted by $\tilde{c}$. Since E is over x, $\tilde{c}(\widetilde{E})$ is contained in an exceptional divisor D of r_T. Let $\widetilde{D}$ be the strict transform of D in $\widetilde{S}$. Then $\widetilde{D}$ is contracted by r_S for $\widetilde{D} \neq \widetilde{E}$, and the image of $\widetilde{D}$ in T is a point. Thus, $\widetilde{D}$ is a (-2)-curve in $\widetilde{S}$ since S has canonical singularities.

Since T has canonical singularities, D is also a (-2)-curve. Note that $\tilde{c}$ is the composition of a sequence of blow-ups of smooth points (see [Har77, Cor. V.5.4]). Moreover, for $\tilde{c}$, we have to blow up the point x that is contained in D. Hence, the self-intersection number of $\widetilde{D}$ is less than (-2). We obtain a contradiction. □

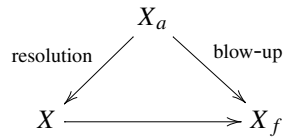
By Proposition 5.4, every exceptional divisor of $f : X \to X_f$ is over a smooth point of X_f. Now we can prove the isomorphism we mentioned at the beginning of this subsection.

LEMMA 5.5. *The natural injection* $H^0(X, (\Omega^1_X)^{[\otimes m]}) \to H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]})$ *is an isomorphism.*

Proof. Let $X_a \to X$ be a projective birational morphism that is the minimal resolution for the singular points of X lying over smooth points of X_f. Then there is a natural injection

$$H^0(X_a, (\Omega^1_{X_a})^{[\otimes m]}) \to H^0(X, (\Omega^1_X)^{[\otimes m]}).$$

By Proposition 5.4, f^{-1} is an isomorphism around the singular points of X_f. Hence, all exceptional divisors of $X_a \to X_f$ are over smooth points of X_f. This implies that X_a can be obtained from X_f by a sequence of blow-ups of smooth points (see [Har77, Cor. V.5.4]).



Then we have a natural isomorphism $H^0(X_a, (\Omega^1_{X_a})^{[\otimes m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]})$, which implies that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]})$. □

We can conclude Theorem 1.4.

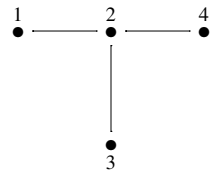
Proof of Theorem 1.4. By Theorem 3.1 we have a Mori fibration $p : X_f \to \mathbb{P}^1$. Lemmas 5.2 and 5.5 show that $H^0(X, (\Omega^1_X)^{[\otimes m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[\otimes m]}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$. □

6. Proof of Theorem 1.3

We will prove Theorem 1.3 in this section. If X is a projective rationally connected surface with canonical singularities such that $H^0(X, (\Omega_X)^{[\otimes m]}) \neq \{0\}$ for some $m > 0$ and X_f is the result of an MMP, then X and X_f are isomorphic around the singular locus of X_f by Proposition 5.4. The proof of Lemma 5.5 gives us an idea of how to reconstruct X from X_f. First, we construct the surface X_a (the surface defined in the proof of Lemma 5.5) that can be obtained from X_f by a sequence of blow-ups of smooth points. Then we blow down some exceptional (-2)-curves for $X_a \to X_f$, and we obtain X. Note that these are just the birational transformations mentioned in steps (iii) and (iv) of Construction 1.2.

In order to contract the (-2)-curves in the transformation above, we want to use Lemma 4.9. Thus, we have to study the structure of the exceptional set of $X_a \to X_f$.

LEMMA 6.1. *Denote a germ of smooth surface by* $(0 \in S)$. *Let* $h : S' \to S$ *be the composition of a sequence of blow-ups of smooth points over* $0 \in S$. *Let* D *be the support of* h^*0. *Then any* (-2)*-curve in* D *meets at most two other* (-2)*-curves. In other words, the dual graph of* D *cannot contain a subgraph as below such that each vertex of the subgraph corresponds a* (-2)*-curve.*



Proof. Assume the opposite. We know that we can reverse the process of blow-ups of smooth points by running an MMP relatively to S. Thus, these four curves will be successively contracted during the MMP. The first one contracted cannot be the curve corresponding to the point with label 2 since after the contraction, the dual graph of the remaining curves is a tree by an analogue result of Lemma 4.1. Without loss of generality, we may assume that the curve corresponding to the point with label 1 is the first one contracted.

If the curve corresponding to the point 3 (or 4) is contracted secondly, then the self-intersection number of the curve corresponding to the point 2 becomes at least 0. If the curve corresponding to the point 2 is contracted secondly, a further contraction will also produce a curve with self-intersection number at least 0.

However, this curve of self intersection at least 0 is over $0 \in S$; it must have negative self-intersection number by the negativity theorem (see [KM98, Lem. 3.40]). This leads to a contradiction. □

In particular, by Lemma 6.1 every connected collection of (-2)-curves in D has a dual graph

This is the dual graph of the exceptional set of the minimal resolution for the singularity of type A_i. By Lemma 4.9 it is possible to contract such a chain of (-2)-curves.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. First, let X be a projective rationally connected surface with canonical singularities that carries nonzero pluri-forms. We will prove that X can be constructed by the method of Construction 1.2. Let $f : X \to X_f$ be the result of an MMP, and let X_a be the surface defined in the proof of Lemma 5.5. The surface X can be obtained from X_a by a contraction of chains of (-2)-curves by Lemmas 5.5, 6.1, and 4.9. By the proof of Lemma 5.5, X_a can be obtained from X_f by a sequence of blow-ups of smooth points. Since $X_f \to \mathbb{P}^1$ is a Mori fibration and X_f has canonical singularities, X_f can be obtained from a smooth ruled surface $X_0 \to \mathbb{P}^1$ by the method of steps (i) and (ii) of Construction 1.2 (see Lemma 4.8). Thus, X can be constructed by the method of Construction 1.2.

Now, let X be a surface constructed by the method of Construction 1.2. We will prove that X carries nonzero pluri-forms. Since $X_f \to \mathbb{P}^1$ is a Mori fibration, by running an f-relative MMP we obtain that X_f is the result of this MMP (this is why we use the same notation X_f). After Lemma 5.2, we know that X_f carries nonzero pluri-forms. By Lemma 5.5 this shows that X carries nonzero pluri-forms. □

7. Proof of Theorem 1.5

We would like to prove Theorem 1.5 in this section. In [GKP12, Remark and Question 3.8], for X in Example 1.1, we can find a smooth elliptic curve E and a smooth ruled surface Y (which is $\widetilde{X}$ in [GKP12]) such that $\mathbb{P}^1$ is the quotient of E by $\mathbb{Z}/2\mathbb{Z}$ and X is the quotient of Y by the same group. In this section, we would like to construct such a surface Y for any rationally connected surface X with canonical singularities and having nonzero pluri-forms.

We will first construct the curve E.

PROPOSITION 7.1. *Let $q_1, \dots, q_r$ be r different points on $\mathbb{P}^1$ with $r \geq 4$. Then there exist a smooth curve E and a $4:1$ Galois cover $\gamma : E \to \mathbb{P}^1$ with Galois group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that γ is exactly ramified over the q_i and the degrees of ramification are all equal to 2.*

Proof. Since $r \geq 4$, we can find an elliptic curve D and a $2:1$ cover $\alpha : D \to \mathbb{P}^1$ such that α is ramified exactly over q_1, q_2, q_3, q_4. Let $\alpha^{-1}(\{q_i\}) = \{s_i, t_i\}$ for $i > 4$, and let $\alpha^{-1}(\{q_i\}) = \{s_i\}$ for $i = 1, 2$.

If $r > 4$, then $\mathscr{O}_D((r-4)s_1)^{\otimes 2}$ is isomorphic to $\mathscr{O}_D(\sum_{i>4} s_i + \sum_{i>4} t_i)$. Thus, we can construct a ramified $2:1$ cyclic cover of E, with respect to the line bundle $\mathscr{O}_D((r-4)s_1)$,

$$\beta : E \to D$$

such that E is smooth and β is ramified exactly over $\{s_i, t_i \mid i > 4\}$ (see [KM98, Def. 2.50]).

If $r = 4$, then $\mathscr{O}_D(s_1 - s_2)^{\otimes 2} \cong \mathscr{O}_D$, and we can construct a $2:1$ cyclic cover of E, with respect to the nontrivial invertible sheaf $\mathscr{O}_D(s_1 - s_2)$,

$$\beta : E \to D$$

such that E is a smooth elliptic curve and β is étale.

Finally, in both cases, the composition

$$\gamma = \alpha \circ \beta : E \to \mathbb{P}^1$$

is a $4:1$ cover that is exactly ramified over the q_i and the degrees of ramification are all equal to 2.

We will show that γ is a Galois cover with Galois group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For simplicity, we assume that $r > 4$. For the case of $r = 4$, the argument is similar. We only need to prove that we can lift the action of $\mathrm{Aut}(D/\mathbb{P}^1) = \mathbb{Z}/2\mathbb{Z}$ on D to E. There is a natural action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. Let $\mathscr{I} = \mathscr{O}_D((4-r)s_1)$. Then $\mathscr{I}$ can be regarded as an ideal sheaf on D. Since s_1 is invariant under the

action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on D, the sheaf $\mathscr{I}$, as a subsheaf of $\mathscr{O}_D$, is stable under the action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. This gives an action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{I}$. Hence, $\mathrm{Aut}(D/\mathbb{P}^1)$ acts (diagonally) on the sheaf $\mathscr{O}_D \oplus \mathscr{I}$. There is a rational function h on D such that the divisor associated to h is

$$\sum_{i>4} s_i + \sum_{i>4} t_i + 2(4-r)s_1.$$

By multiplying h we have a morphism from $\mathscr{I}^{\otimes 2}$ to $\mathscr{O}_D$. This morphism gives an $\mathscr{O}_D$-algebra structure on $\mathscr{O}_D \oplus \mathscr{I}$. By construction, with this $\mathscr{O}_D$-algebra structure, the sheaf $\mathscr{O}_D \oplus \mathscr{I}$ is isomorphic to $\beta_*\mathscr{O}_E$ as $\mathscr{O}_D$-algebras. Note that since

$$\sum_{i>4} s_i + \sum_{i>4} t_i = \alpha^*\left(\sum_{i>4} q_i\right) \quad \text{and} \quad 2(4-r)s_1 = \alpha^*((4-r)q_1),$$

the rational function h is the pullback of a rational function on $\mathbb{P}^1$. Hence, h is invariant under the action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{O}_D$. This shows that the action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{O}_D \oplus \mathscr{I}$ is compatible with the $\mathscr{O}_D$-algebra structure induced by h. Thus, we obtain an extension of the action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on $\mathscr{O}_D$ to $\mathscr{O}_E$. This proves that we can lift the action of $\mathrm{Aut}(D/\mathbb{P}^1)$ on D to E.

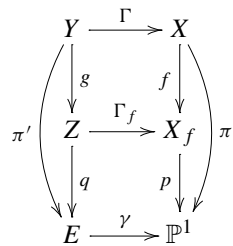
Since γ is a Galois cover, we have $E/G = \mathbb{P}^1$. □

Remark 7.2. What we want in the proposition is to construct a finite morphism $\gamma : E \to \mathbb{P}^1$ that is exactly ramified over the q_i and all of the ramified degrees are equal to 2. Note that the finite cover γ we constructed is of degree four and the one in [GKP12, Remark and Question 3.8] is of degree two. However, if r is odd, then the Hurwitz theorem (see [Har77, Cor. IV.2.4]) shows that it is not possible to have a $2:1$ cover satisfying the condition.

Now we will prove Theorem 1.5.

Proof of Theorem 1.5. Let $q_1, \ldots, q_r$ be all of the points in $\mathbb{P}^1$ over which $p : X_f \to \mathbb{P}^1$ has multiple fibers. Let $\gamma : E \to \mathbb{P}^1$ be the $4:1$ cover constructed in Proposition 7.1. Let Z be the normalization of the fiber product $X_f \times_{\mathbb{P}^1} E$. Let $q : Z \to E$ and $\Gamma_f : Z \to X_f$ be the natural projections. Then Γ_f is étale over the smooth locus of X_f, and q has only reduced fibers.

We know that we can reconstruct X from X_f (see Section 6). Since $\Gamma_f : Z \to X_f$ is étale over the smooth locus of X_f, every operation we do with X_f can be done in the analogue way with Z. After the operations, the surface $Y \xrightarrow{g} Z$ we obtained is just the normalization of $X \times_{\mathbb{P}^1} E$. We have a commutative diagram



Then Γ is étale over the smooth locus of X, and $X = Y/G$ where G is the Galois group of γ. The sheaf $\Gamma_*(\Omega_Y^{[\otimes m]})$ is a G-sheaf on X (the action of G on X is trivial), which is reflexive (see [Har80, Prop. 1.7]). Then $(\Gamma_*(\Omega_Y^{[\otimes m]}))^G$ is also reflexive (see [GKKP11, Lem. B.4]) and is isomorphic to $\Omega_X^{[\otimes m]}$ since Γ is étale over the smooth locus of X. Thus, we have

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]})^G \cong H^0(X, (\Omega_X^1)^{[\otimes m]}).$$

Moreover, for any $m \geq 0$, the natural morphism

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \to H^0(Z, (\Omega_Z^1)^{[\otimes m]})$$

is an isomorphism by the same argument as in the proof of Lemma 5.5. Since every fiber of q is reduced and general fibers of q are smooth rational curves, by the same argument as Lemma 5.1 we have

$$H^0(Y, (\Omega_Y^1)^{[\otimes m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m}).$$

Hence, we obtain the isomorphisms

$$H^0(X, (\Omega_X^1)^{[\otimes m]}) \cong H^0(Y, (\Omega_Y^1)^{[\otimes m]})^G \cong H^0(E, (\Omega_E^1)^{\otimes m})^G. \qquad \square$$

Now we want to compute the dimension of $H^0(X, (\Omega_X^1)^{[\otimes m]})$ in function of multiple fibers of $X_f \to \mathbb{P}^1$ with the previous formula. We will first prove the following lemma.

LEMMA 7.3. *Let R_γ be the ramification divisor of the finite morphism $\gamma : E \to \mathbb{P}^1$. Then $(\gamma_* \mathscr{O}_E(R_\gamma))^G \cong \mathscr{O}_{\mathbb{P}^1}$.*

Proof. We have $H^0(U, (\gamma_* \mathscr{O}_E(R_\gamma))^G) \cong H^0(\gamma^{-1}(U), \mathscr{O}_E(R_\gamma))^G$ for any open set $U \subseteq \mathbb{P}^1$. Let θ be a rational function on E such that θ represents a nonzero element in $H^0(\gamma^{-1}(U), \mathscr{O}_E(R_\gamma))^G$. Since θ is G-invariant, it can also be regarded as a rational function on U. Since θ can only have simple poles at the support of R on $\gamma^{-1}(U)$, it cannot have any pole on U. Thus, $(\gamma_* \mathscr{O}_E(R_\gamma))^G \cong \mathscr{O}_{\mathbb{P}^1}$. $\square$

With the notation in the proof of Theorem 1.5, we have

$$(\Omega_E^1)^{\otimes m} \cong \left(\gamma^* \mathscr{O}_{\mathbb{P}^1}\left(-2m + \left[\frac{m}{2}\right](q_1 + \cdots + q_r)\right)\right) \otimes \mathscr{O}_E\left(\left(m - 2\left[\frac{m}{2}\right]\right) R_\gamma\right).$$

By the projection formula we have $\gamma_*(\Omega_E^1)^{\otimes m} \cong \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r) \otimes \gamma_* \mathscr{O}_E((m - 2[\frac{m}{2}])R_\gamma)$. By taking the G-invariant part we obtain

$$(\gamma_*(\Omega_E^1)^{\otimes m})^G \cong \mathscr{O}_{\mathbb{P}^1}\left(-2m + \left[\frac{m}{2}\right]r\right) \otimes \left(\gamma_* \mathscr{O}_E\left(\left(m - 2\left[\frac{m}{2}\right]\right) R_\gamma\right)\right)^G.$$

The previous lemma implies that $(\gamma_*(\Omega_E^1)^{\otimes m})^G \cong (\mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$. Hence,

$$\begin{aligned} H^0(X, (\Omega_X^1)^{[\otimes m]}) &\cong H^0(E, (\Omega_E^1)^{\otimes m})^G \cong H^0(\mathbb{P}^1, (\gamma_*(\Omega_E^1)^{\otimes m})^G) \\ &\cong H^0\left(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}\left(-2m + \left[\frac{m}{2}\right]r\right)\right). \end{aligned}$$

We recover the same formula as in Theorem 1.4.

EXAMPLE 7.4. We will give some examples. Let $h(m,r)$ be the dimension of $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m+[\frac{m}{2}]r))$. This is just the number of m-pluri-forms as a function of the number r of multiple fibers of $X_f \to \mathbb{P}^1$.

If $r=4$, then $h(m,4)=1$ if $m>0$ is even and $h(m,4)=0$ if m is odd.

If $r=5$, then $h(2,5)=2$, $h(3,5)=0$ and $h(m,5)>0$ if $m\geq 4$.

If $r\geq 6$, then $h(m,r)>0$ for $m\geq 2$.

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