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Exotic Blowup Solutions for the u^5 Focusing Wave Equation in $\mathbb{R}^3$

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Abstract. For the critical focusing wave equation $\Box u = u^5$ on $\mathbb{R}^{3+1}$ in the radial case, we construct a family of blowup solutions that are obtained from the stationary solutions $W(r)$ by means of a dynamical rescaling $\lambda(t)^{1/2}W(\lambda(t)r) + \texttt{correction}$ with $\lambda(t) \to \infty$ as $t \to 0$. The novelty here lies with the scaling law $\lambda(t)$ that eternally oscillates between various pure-power laws.

1. Introduction

The energy critical focusing wave equation in $\mathbb{R}^3$

$$\Box u = u^5, \quad \Box = \partial_t^2 - \triangle \tag{1.1}$$

has been the subject of intense investigations in recent years. This equation is known to be locally well posed in the space $\mathcal{H} := \dot{H}^1 \times L^2(\mathbb{R}^3)$, meaning that if $(u(0), u_t(0)) \in \mathcal{H}$, then there exists a solution locally in time and continuous in time taking values in $\mathcal{H}$. Solutions need to be interpreted in the Duhamel sense:

$$u(t) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}u^5(s)\,ds. \tag{1.2}$$

These solutions have finite energy:

$$E(u, u_t) = \int_{\mathbb{R}^3}\left[\frac{1}{2}(u_t^2 + |\nabla u|^2) - \frac{u^6}{6}\right]dx = \text{ const.}$$

The remarkable series of papers [2; 3; 4; 5] establishes a complete classification of all *possible* type-II blow up dynamics in the radial case. It remains, however, to investigate the *existence* of *all* allowed scenarios in this classification. Steps in this direction were undertaken in [1; 8; 11], where a constructive approach to actually exhibit and thereby prove the existence of such type-II dynamics was undertaken. Recall that a type-II blow up solution $u(t, x)$ with blowup time T_* is one for which

$$\limsup_{t\to T_*} \|u(t,\cdot)\|_{\dot{H}^1} + \|u_t(t,\cdot)\|_{L^2} < \infty$$

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but of which no extension in the usual sense of well-posedness theory in $\dot{H}^1 \times L^2$ exists beyond time T_*. In [5], it is demonstrated that such solutions can be described as a sum of dynamically rescaled ground states $\pm W$,

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2},$$

plus a radiation term. In particular, for solutions where only one such bulk term is present, we can write the solution (up to a sign) as

$$\begin{aligned} u(t,x) &= W_{\lambda(t)}(x) + \varepsilon(t,x) + o_{\dot{H}^1}(1), \\ W_\lambda(x) &= \lambda^{1/2} W(\lambda x), \varepsilon(t,\cdot) \in \dot{H}^1, \end{aligned} \tag{1.3}$$

where the error is in the sense as $t \to T_*$. Moreover, we have the dynamic condition

$$\lim_{t \to T_*} (T_* - t)\lambda(t) = \infty. \tag{1.4}$$

In [11], it was shown that such solutions with $\lambda(t) = t^{-1-\nu}$ do exist, where $\nu > \frac{1}{2}$ is arbitrary. In [9], the latter condition was relaxed to $\nu > 0$.

It is natural to ask which rescaling functions are admissible for (1.3) – both in general, and in particular within the confines of the method developed in [9; 11]. It seems very difficult (perhaps hopeless) to answer this question in full generality. Nevertheless, important progress has been made in recent years such as in the deep works of Raphaël and Rodnianski [13] and Hillairet and Raphaël [7], who studied stable blowup laws (relative to a suitable topology) for energy critical equations. The blowup speeds they exhibit are not of pure power type.

The purpose of this paper is to exhibit an uncountable family of rates that are not of the pure-power type as above. Our main result, which is in the spirit of [10; 11], is as follows.

THEOREM 1.1. *Let $\nu > 3$ and $|\varepsilon_0| \ll 1$ be arbitrary and define*

$$\lambda(t) := t^{-1-\nu} \exp(-\varepsilon_0 \sin(\log t)), \quad 0 < t < \frac{1}{2}. \tag{1.5}$$

Then there exists a radial energy solution u of (1.1) that blows up precisely at $r = t = 0$ and has the following property: in the cone $|x| = r \leq t$ and for small times t, the solution has the form

$$u(t,r) = \lambda^{1/2}(t) W(\lambda(t) r) + \eta(t,x), \tag{1.6}$$

where

$$\int_{[|x|<t]} [|\nabla \eta(t,x)|^2 + |\eta_t(t,x)|^2 + |\eta(t,x)|^6]\, dx \to 0 \quad \text{as } t \to 0,$$

and outside the cone, $u(t,r)$ satisfies

$$\int_{[|x|\geq t]} [|\nabla u(t,x)|^2 + |u_t(t,x)|^2 + |u(t,x)|^6]\, dx < \delta$$

for all sufficiently small $t > 0$, where $\delta > 0$ is arbitrary but fixed. In particular, the energy of these blowup solutions can be chosen arbitrarily close to $E(W,0)$, that is, the energy of the stationary solution.

We remark that

$$\lambda(t) = t^{-1-\nu(t)}, \quad \nu(t) = \nu + \varepsilon_0 \frac{\sin(\log t)}{\log t} \to \nu \text{ as } t \to 0+.$$

This shows that $\nu(t)$ eternally oscillates around the constant ν but does approach that constant. Currently we do not know if it is possible to have such solutions for which $\nu(t)$ is not asymptotically constant. The specific choice of $\lambda(t)$ is a result of the construction of approximate solutions in Section 2. The method from [11], on which we ultimately rely, is not easily adapted to other settings such as this one. One of the key features of this method is the reduction to ODEs for the various profiles, which is a result of the pure power law $\lambda(t) = t^{-1-\nu}$. Deviating from this choice leads instead to PDEs, which are much harder to analyze. The law (1.5) does allow for an analysis of the resulting PDE. It is completely unclear at this point for which choices of rescaling one might succeed in general. In contrast, the analysis in Section 3, which turn the approximate solutions into exact ones, is more robust and has little to do with (1.5).

We also note that the solutions constructed in Theorem 1.1 are not smooth but lie in the space $H^{5/4} \times H^{1/4}$. This can be improved, though, but with our construction it does not seem feasible to go all the way to C^∞.

It is interesting to contrast our result to recent developments for other critical equations. For instance, in the case of the critical harmonic maps heat flow, Gustafson, Nakanishi, and Tsai proved that (infinite-time) blowup can occur with essentially any law [6]. Furthermore, for the critical Korteweg–de Vries equation, Martel, Merle, and Raphaël [12] exhibited a wide range of possible blowup laws. Our result indicates that there is a large variety of possible blowup laws also for the critical wave equation.

The starting point of our investigation was to adapt the method from [11] to the setting where $\lambda(t)$ is not restricted to the class of pure-power laws. This turns out to run into serious difficulties essentially from the beginning, with the "renormalization construction" of the approximate solution being the first serious obstacle. Recall from [11] that this construction relies on an iterative procedure and involves delicate book-keeping of various asymptotic expansions of the approximate solutions, the corrections, as well as the errors. For the more general rates $\lambda(t)$, this cannot be done in the same fashion, and we succeeded in a much modified fashion for the laws (1.5); however, only two steps of the iteration seem feasible. This then forces one to confront a very major difficulty, which was not present in [11], namely the lack of a suitable smallness parameter that allowed for the ultimate contraction argument yielding an *exact* solution rather than an approximate one to go through. In absence of this small parameter, we are forced to follow a different route. The idea is very simple, but its actual implementation turns out to be quite subtle. Schematically, we have to deal with a fixed point problem on a Banach space of the form

$$x = F(x) + Ax + x_0,$$

where the norm of the linear operator A is not small. However, it turns out that A^n has small operator norm, provided that n is sufficiently large. This implies the existence of $(1 - A)^{-1}$ (via the Neumann series), and thus, we may rewrite the problem at hand as

$$x = (1 - A)^{-1} F(x) + (1 - A)^{-1} x_0,$$

which we then solve by the Banach fixed point theorem. Thus, a large part of the present paper is devoted to the development of a technique that allows us to show smallness of $\|A^n\|$. In order to succeed, we have to exploit the fine-structure of the operator A, in particular, smoothing properties and oscillations.

2. The Approximate Solution for a Modified Power-Law Rescaling

2.1. Generalities

The radial quintic wave equation in $\mathbb{R}^3$ is

$$\mathcal{L}_{\text{quintic}} u := u_{tt} - u_{rr} - \frac{2}{r} u_r - u^5 = 0. \tag{2.1}$$

A special stationary solution is $W(r) = (1 + r^2/3)^{-1/2}$. By scaling, $\lambda^{1/2} W(\lambda r)$ is also a solution for any $\lambda > 0$. We are interested in letting λ depend on time. More precisely, we would like to find solutions $\mathcal{L}_{\text{quintic}} u = 0$ of the form

$$u(t, r) = \lambda(t)^{1/2} W(\lambda(t) r) + \varepsilon(t, r), \quad \lambda(t) \to \infty \text{ as } t \to 0+, \tag{2.2}$$

and ε small in a suitable sense. It suffices to show that ε remains small in energy since this ensures that the solution blows up at time $t = 0$ by the mechanism of "energy concentration" at the tip of the light-cone $(t, r) = (0, 0)$ (think of solving backward in time). In the paper [11], such solutions were found with $\lambda(t) = t^{-1-\nu}$ and constant $\nu > \frac{1}{2}$. The goal here is to allow for more general functions; more specifically, we will set

$$\lambda(t) = t^{-1-\nu} \exp(-\varepsilon_0 \sin(\log t)), \quad \nu > 3, |\varepsilon_0| \ll 1. \tag{2.3}$$

This is of the form $\lambda(t) = t^{-1-\nu(t)}$ with

$$\nu(t) = \nu + \varepsilon_0 \frac{\sin(\log t)}{\log t} \to \nu \quad \text{as } t \to 0+.$$

For future reference, we introduce $\mu(t) := t\lambda(t)$ and

$$\kappa(t) := -\frac{t\dot{\mu}(t)}{\mu(t)},$$

so that for (2.3), we obtain

$$\kappa(t) = \nu + \varepsilon_0 \cos(\log t).$$

Our goal is to prove the following result. In what follows, $R = r\lambda(t)$.

PROPOSITION 2.1. *Let $\lambda(t)$ be as in (2.3) and $t_0 \ll 1$.*

(i) *There exists some $u_2(t,r) \in C^2(\{0 < t < t_0, 0 \le r \le t\})$ such that*

$$u_2(t,r) = \sqrt{\lambda(t)}(W(R) + \mu^{-2}(t)O(R)), \quad 0 < t < t_0, 0 < r < t, \tag{2.4}$$

and $e_2 := \mathcal{L}_{\text{quintic}} u_2$ satisfies

$$\begin{aligned} &t^2\lambda^{-1/2}(t)e_2(t,r) \\ &\quad = \mu^{-2}(t)O\left(\frac{\log(R+2)}{R+1}\right), \quad 0 < t < t_0, 0 < r < t. \end{aligned} \tag{2.5}$$

(ii) *For $0 < t < t_0$, $0 < r < t/2$, and all $k, j \ge 0$, we have*

$$\partial_t^k \partial_r^j u_2(t,r) = \partial_t^k \partial_r^j \sqrt{\lambda(t)} W(R) + t^{-k} r^{-j} \sqrt{\lambda(t)} \mu^{-2}(t) O(R) \tag{2.6}$$

and

$$\partial_t^k \partial_r^j (t^2 \lambda^{-1/2}(t) e_2(t,r)) = t^{-k} r^{-j} \mu^{-2}(t) O\left(\frac{\log(R+2)}{R+1}\right). \tag{2.7}$$

The same bound applies without the restriction $r < \frac{t}{2}$, provided that $k + j \le 2$.

(iii) *The function $u_2(t,r)$ admits a C^2-extension (on fixed time slices) beyond the light cone $r \le t$ with the property that given $\delta > 0$,*

$$\int_{r \ge t} [|\partial_r u_2|^2 + \partial_t u_2^2 + u_2^6](t,r) r^2 \, dr < \delta,$$

provided that $t < t_0$ is sufficiently small.

The proof will be given in Section 2.7.

2.2. *The Bulk Term*

Define

$$u_0(t,r) = \lambda(t)^{1/2} W(r\lambda(t)) = \lambda(t)^{1/2} W(R). \tag{2.8}$$

Whereas u_0 is very far from being an approximate solution, the construction in [11] for $\lambda(t) = t^{-\alpha}$ where $\alpha > 1$ is **constant** shows that one can add successive corrections via an iterative procedure

$$u = u_0 + v_1 + v_2 + v_3 + \cdots + v_k,$$

so that this function approximately solves (2.1) in the light cone $\{r \le t \ll 1\}$. To be specific, we achieved that $\mathcal{L}_{\text{quintic}} u(t)$ goes to zero like t^N in the energy norm as $t \to 0$, where N can be made arbitrarily large by taking k large.

For (2.3), we will content ourselves with two steps of the construction only, that is, $u = u_0 + v_1 + v_2$. Let us first compute the error resulting from u_0.

Define $\mathcal{D} := \frac{1}{2} + r\partial_r = \frac{1}{2} + R\partial_R$. Then

$$\begin{aligned} e_0 &:= \mathcal{L}_{\text{quintic}} u_0 = \lambda^{1/2}(t)\left[\left(\frac{\lambda'}{\lambda}\right)^2(t)(\mathcal{D}^2 W)(R) + \left(\frac{\lambda'}{\lambda}\right)'(t)(\mathcal{D}W)(R)\right], \\ t^2 e_0 &=: \lambda^{1/2}(t)\left[\omega_1(t)\frac{1 - R^2/3}{(1 + R^2/3)^{3/2}} + \omega_2(t)\frac{9 - 30R^2 + R^4}{(1 + R^2/3)^{5/2}}\right]. \end{aligned} \tag{2.9}$$

We note that (2.3) satisfies

$$\frac{t\lambda'(t)}{\lambda(t)}, \frac{t^2\lambda''(t)}{\lambda(t)} = O(1), \quad t \to 0+, \tag{2.10}$$

and analogously for higher derivatives. Moreover, the functions on the left remain bounded under $(t\partial_t)^\ell$ for any ℓ; the same properties hold for $\omega_1(t)$, $\omega_2(t)$.

Then $t^2 e_0 = \lambda(t)^{1/2} O(R^2\langle R\rangle^{-3})$ uniformly in $0 < t \ll 1$ (with derivatives). Clearly, this error blows up as $t \to 0$ like t^{-2}.

2.3. *The First Correction*

Then $t^2 e_0 = \lambda(t)^{1/2} O(R^2\langle R\rangle^{-3})$ as $R \to \infty$. This error blows up as $t \to 0$ like t^{-2}. The goal is now to reduce it – in fact turn it into an error that vanishes as $t \to 0$ – by adding corrections to u_0, the first one being v_1. We will do this by setting $\lambda^2(t) L_0 v_1 = e_0$ where

$$L_0 := \partial_R^2 + \frac{2}{R}\partial_R + 5W^4(R). \tag{2.11}$$

Note that this is the linearized operator obtained by plugging $u_0 + v_1$ into (2.1) and discarding ∂_t altogether. Whereas this may seem strange, the idea is to look first at the regime $0 < r \ll t$ where ∂_t should matter less than ∂_r. We shall see shortly that v_1 has the good property that it decays like $(t\lambda(t))^{-2}$, but it produces errors for the nonlinear PDE that grow in r too strongly. To remove this growth, we carry out a correction at the second stage by solving a suitable differential operator. At this stage, the self-similar variable $a = \frac{r}{t}$ becomes important.

Now we discuss v_1 in more detail. A fundamental system of L_0 is

$$\varphi_1(R) := \frac{1 - R^2/3}{(1 + R^2/3)^{3/2}}, \qquad \varphi_2(R) := \frac{1 - 2R^2 + R^4/9}{R(1 + R^2/3)^{3/2}}. \tag{2.12}$$

The operator

$$\tilde{L}_0 = RL_0R^{-1} = \partial_R^2 + 5W^4(R) \tag{2.13}$$

has a fundamental system

$$\begin{aligned} \tilde{\varphi}_1(R) &:= \frac{R(1 - R^2/3)}{(1 + R^2/3)^{3/2}} = \tilde{\psi}_1(R^{-2}), \\ \tilde{\varphi}_2(R) &:= \frac{1 - 2R^2 + R^4/9}{(1 + R^2/3)^{3/2}} = R\tilde{\psi}_2(R^{-2}). \end{aligned} \tag{2.14}$$

The right-hand sides here are for large R, and the $\tilde{\psi}_j$ are analytic around 0. The Wronskian is

$$\tilde{\varphi}_1'(R)\tilde{\varphi}_2(R) - \tilde{\varphi}_1(R)\tilde{\varphi}_2'(R) = 1.$$

We let $\mu(t) = t\lambda(t)$ as above and

$$\mu^2(t)L_0 v_1 = t^2 e_0, \quad v_1(0) = v_1'(0) = 0. \tag{2.15}$$

We claim that

$$v_1(t,r) = \mu^{-2}(t)L_0^{-1}t^2 e_0 = \lambda^{1/2}(t)\mu^{-2}(t)O(R) \quad \text{as } R \to \infty.$$

To be more specific, write

$$t^2 e_0 = \lambda^{1/2}(t)(\omega_1(t)g_1(R) + \omega_2(t)g_2(R)); \tag{2.16}$$

see (2.9). Note that the g_j are of the form

$$g_j(R) = R^{-1}\phi_j(R^{-2}), \quad R \gg 1, \tag{2.17}$$

where ϕ_j is analytic around 0. Then $L_0 f_j = g_j$ with $f_j(0) = f_j'(0) = 0$ satisfies

$$\begin{aligned} f_j(R) = R^{-1}\Big(&\tilde{\varphi}_1(R)\int_0^R \tilde{\varphi}_2(R')R'g_j(R')\,dR' \\ &- \tilde{\varphi}_2(R)\int_0^R \tilde{\varphi}_1(R')R'g_j(R')\,dR'\Big) \end{aligned} \tag{2.18}$$

for $j = 1, 2$. Then we check that

$$\begin{aligned} f_j(R) &= b_{1j}R + b_{2j} + b_{3j}\frac{\log R}{R} + O\left(\frac{1}{R}\right) \quad \text{as } R \to \infty, \\ f_j(R) &= c_{1j}R^2 + O(R^4) \quad \text{as } R \to 0. \end{aligned} \tag{2.19}$$

In fact, around $R = 0$, the $f_j(R)$ are even analytic functions, whereas around $R = \infty$, one has the representation

$$\begin{aligned} f_j(R) &= R(b_{1j} + b_{2j}R^{-1} + R^{-2}\log R\varphi_{1j}(R^{-2}) + R^{-2}\varphi_{2j}(R^{-1})) \\ &=: R(F_j(\rho) + \rho^2 G_j(\rho^2)\log\rho), \end{aligned} \tag{2.20}$$

where $\varphi_{1j}, \varphi_{2j}$ and F_j, G_j are analytic around zero, with $\rho := R^{-1}$. This follows from (2.14), (2.17), and (2.18). For future reference, we remark that the structure in (2.20) is preserved under application of $\mathcal{D}$. In particular, and abusing notation somewhat, we have

$$\begin{aligned} v_1(t,r) &= \lambda^{1/2}(t)\mu^{-2}(t)(\omega_1(t)f_1(R) + \omega_2(t)f_2(R)) \\ &=: \lambda^{1/2}(t)\mu^{-2}(t)\omega(t)f(R). \end{aligned} \tag{2.21}$$

Define

$$u_1 := u_0 + v_1 = \lambda^{1/2}(t)(W(R) + \mu^{-2}(t)\omega(t)f(R)).$$

In view of (2.19) and $R \le \mu$ (recall that we are inside of the light cone $r \le t$),

$$\begin{aligned} u_1(t,r) &= \lambda^{1/2}(t)O(R^{-1}), \quad R \ge 1, \\ u_1(t,r) &= \lambda^{1/2}(t)O(1), \quad 0 \le R < 1, \end{aligned} \tag{2.22}$$

uniformly in $0 < t < 1$; moreover, we may apply $t\partial_t$ or $r\partial_r = R\partial_R$ any number of times without affecting this asymptotic property. Finally, $\lambda(t)^{-1/2}u_1(t,r)$ is an even analytic function around $R = 0$.

2.4. The Error from u_1

Set $e_1 := \mathcal{L}_{\text{quintic}}(u_1)$. Then

$$e_1 = \partial_t^2 v_1 - 10u_0^3 v_1^2 - 10u_0^2 v_1^3 - 5u_0 v_1^4 - v_1^5. \tag{2.23}$$

We have

$$\begin{aligned} t^2\lambda^{-1/2}(t)e_1 = \lambda^{-1/2}(t)&((t\partial_t)^2 - t\partial_t)(\lambda^{1/2}(t)w_1(t, r\lambda(t))) \\ &- \mu^2(t)(10W^3(R)w_1^2(t,R) + 10W^2(R)w_1^3(t,R) \\ &+ 5W(R)w_1^4(t,R) + w_1^5(t,R)). \end{aligned} \tag{2.24}$$

We write symbolically $w_1(t,R) = \mu^{-2}(t)\omega(t)f(R)$. Then the nonlinearity in (2.24) is

$$\begin{aligned} &\mu^2(t)(10W^3(R)w_1^2(t,R) + 10W^2(R)w_1^3(t,R) + 5W(R)w_1^4(t,R) + w_1^5(t,R)) \\ &\quad = \mu^{-2}(t)(10W^3(R)\omega^2(t)f^2(R) + 10W^2(R)\mu^{-2}(t)\omega^3(t)f^3(R) \\ &\qquad + 5W(R)\mu^{-4}(t)\omega^4(t)f^4(R) + \mu^{-6}(t)\omega^5(t)f^5(R)), \end{aligned} \tag{2.25}$$

whereas

$$\begin{aligned} &\lambda^{-1/2}(t)((t\partial_t)^2 - t\partial_t)(\lambda^{1/2}(t)w_1(t, r\lambda(t))) \\ &\quad = \left(\left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D}\right)^2 - \left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D}\right)\right)w_1(t,R). \end{aligned} \tag{2.26}$$

Now

$$\begin{aligned} &\mu^2(t)\left(t\partial_t + \frac{t\lambda'(t)}{\lambda(t)}\mathcal{D}\right)\mu^{-2}(t)\omega(t)f(R) \\ &\quad = \left(-\frac{2t\dot\mu(t)}{\mu(t)}\omega(t) + t\dot\omega(t) + \frac{t\lambda'(t)}{\lambda(t)}\omega(t)\mathcal{D}\right)f(R). \end{aligned} \tag{2.27}$$

Note that this is schematically of the form $\omega(t)f(R)$ with f as in (2.19) and $\omega(t)$ bounded together with all powers of $t\partial_t$ as $t \to 0+$. Henceforth, we refer to such functions $\omega(t)$ as *admissible.* Thus, we can write

$$\begin{aligned} &t^2\lambda^{-1/2}(t)e_1(t,r) \\ &\quad = \mu^{-2}(t)(\omega(t)f(R) \\ &\qquad - (10W^3(R)\omega^2(t)f^2(R) + 10W^2(R)\mu^{-2}(t)\omega^3(t)f^3(R) \\ &\qquad + 5W(R)\mu^{-4}(t)\omega^4(t)f^4(R) + \mu^{-6}(t)\omega^5(t)f^5(R))). \end{aligned} \tag{2.28}$$

We let $a = \frac{r}{t} = \frac{R}{\mu} = Rb$, $b := \mu^{-1}$ and isolate those terms in (2.28) that do not decay for large R. Since we are working inside of the light cone, we have $0 \le a \le$

1. Now, abusing notation somewhat,

$$
\begin{aligned}
\mu^{-2}(t)f(R) &= b^2R(F(\rho)+\rho^2G(\rho^2)\log\rho)\\
&= ba(F(\rho)+\rho^2G(\rho^2)\log\rho),\\
\mu^{-2}(t)W^3(R)f^2(R) &= b^2R^{-3}\Omega(\rho^2)R^2(F(\rho)+\rho^2G(\rho^2)\log\rho)^2\\
&= b^2R^{-1}(F(\rho)+\rho^2F(\rho)\log\rho\\
&\quad+\rho^4G(\rho^2)\log^2\rho),\\
\mu^{-4}(t)W^2(R)f^3(R) &= b^4R^{-2}\Omega(\rho^2)R^3(F(\rho)+\rho^2G(\rho^2)\log\rho)^3\\
&= b^3a(F(\rho)+\rho^2F(\rho)\log\rho\\
&\quad+\rho^4F(\rho)\log^2\rho+\rho^6G(\rho^2)\log^3\rho),
\end{aligned}
\tag{2.29}
$$

where F, G can change from line to line. Similarly,

$$
\begin{aligned}
\mu^{-6}(t)W(R)f^4(R) &= b^6R^{-1}\Omega(\rho^2)R^4(F(\rho)+\rho^2G(\rho^2)\log\rho)^4\\
&= b^3a^3(F(\rho)+\rho^2F(\rho)\log\rho+\rho^4F(\rho)\log^2\rho\\
&\quad+\rho^6F(\rho)\log^3\rho+\rho^8G(\rho^2)\log^4\rho),\\
\mu^{-8}(t)f^5(R) &= b^8R^5(F(\rho)+\rho^2G(\rho^2)\log\rho)^5\\
&= b^3a^5(F(\rho)+\rho^2F(\rho)\log\rho+\rho^4F(\rho)\log^2\rho\\
&\quad+\rho^6F(\rho)\log^3\rho+\rho^8F(\rho)\log^4\rho\\
&\quad+\rho^{10}G(\rho^2)\log^5\rho).
\end{aligned}
\tag{2.30}
$$

From (2.29) and (2.30) we extract the leading order

$$
t^2\lambda^{-1/2}(t)e_1^0(t,r) := \mu^{-1}(t)(c_1a+c_2b+(c_3a+c_4a^3+c_5a^5)b^2) \tag{2.31}
$$

with $c_j = c_j(t)$ admissible functions. Indeed, from the first line in (2.29) we extract $ba(F(0)+\rho F'(0)) = bac_1 + b^2c_2$, whereas from the fifth we extract $b^3aF(0)$. From the second line in (2.30) we retain $b^3a^3F(0)$, and from the fifth one $b^3a^5F(0)$. The point here is that with this choice of e_1^0 we obtain a decaying error, as $R\to\infty$,

$$
\begin{aligned}
&t^2\lambda^{-1/2}(t)(e_1-e_1^0)(t,r)\\
&\quad= \mu^{-2}(t)\left[\frac{\log R}{R}\Phi_1(t,a,b,\rho\log\rho,\rho)\right.\\
&\qquad\left.+\frac{1}{R}\Phi_2(t,a,b,\rho\log\rho,\rho)\right],
\end{aligned}
\tag{2.32}
$$

where $\Phi_j(t,a,b,u,v)$ are polynomials in a, b and analytic in u, v near $(0,0)$; moreover, their time dependence is polynomial in admissible functions. Writing $b=\frac{a}{R}$, we may delete the terms involving $b^2 = ba/R$ on the right-hand side of (2.31) since they are of the form (2.32). Thus, it suffices to consider the simpler leading error

$$t^2\lambda^{-1/2}(t)e_1^0(t,r) := c_1 a\mu^{-1}(t) + c_2\mu^{-2}(t) = c_1(t)ab + c_2(t)b^2 \tag{2.33}$$

with $c_1(t)$, $c_2(t)$ admissible.

2.5. *The Second Correction*

Now we would like to solve the corrector problem "near $r = t$," that is,

$$t^2\left(v_{tt} - v_{rr} - \frac{2}{r}v_r\right) = -t^2 e_1^0. \tag{2.34}$$

Note that we have discarded the nonlinearity on the left-hand side since it decays near $r = t$. This is designed exactly so as to remove the growth in R. We seek a solution in the form

$$v(t,r) = \lambda(t)^{1/2}(\mu^{-1}(t)q_1(a,t) + \mu^{-2}(t)q_2(a,t)) \tag{2.35}$$

with boundary conditions $q_1(0,t) = 0$, $q_1'(0,t) = 0$, and $q_2(0,t) = 0$, $q_2'(0,t) = 0$. These translate into the boundary conditions $v(t,0) = 0$, $\partial_r v(t,0) = 0$ at $r = 0$. This v will essentially be the function v_2. In view of

$$\lambda(t)^{-1/2}\mu^{\alpha}\partial_t\lambda(t)^{1/2}\mu^{-\alpha} = \partial_t + t^{-1}\left(\frac{1}{2}\frac{t\dot\lambda}{\lambda} - \alpha\frac{t\dot\mu}{\mu}\right),$$

we are reduced to the system

$$t^2\left(-\left(\partial_t + \frac{\beta_1(t)}{t}\right)^2 + \partial_{rr} + \frac{2}{r}\partial_r\right)q_1(a,t) = c_1(t)a \tag{2.36}$$

and

$$t^2\left(-\left(\partial_t + \frac{\beta_2(t)}{t}\right)^2 + \partial_{rr} + \frac{2}{r}\partial_r\right)q_2(a,t) = c_2(t), \tag{2.37}$$

where

$$\beta_j(t) = \frac{1}{2}\frac{t\dot\lambda}{\lambda} - j\frac{t\dot\mu}{\mu} = \left(j - \frac{1}{2}\right)\kappa(t) - \frac{1}{2}, \qquad j = 1,2.$$

We impose the boundary conditions $q_j(0,t) = \partial_a q_j(0,t) = 0$.

LEMMA 2.2. *Let $\lambda(t)$ be as in (2.3) with $|\varepsilon_0|$ sufficiently small. Equations (2.36), (2.37) have bounded solutions $q_j(a,t)$ satisfying $q_j(0,t) = \partial_a q_j(0,t) =$ 0, $q_j(a,t) \in C^2(\{0 < t < t_0, 0 \le a \le 1\})$. Furthermore, for $k \ge 0$ and $0 \le \ell \le 2$, we have*

$$\partial_t^k\partial_a^\ell q_2(a,t) = O(t^{-k}a^{2-\ell}); \qquad \partial_t^k\partial_a^\ell q_1(a,t) = O(t^{-k}a^{3-\ell}). \tag{2.38}$$

Proof. Now, with $\dot q_j = \partial_t q_j$, (2.36) and (2.37) can be written as

$$\begin{aligned} t^2&\left(-\left(\partial_t + \frac{\beta_j(t)}{t}\right)^2 + \partial_{rr} + \frac{2}{r}\partial_r\right)q_j(a,t) \\ &= ((1-a^2)\partial_a^2 + (2(\beta_j(t)-1)a + 2a^{-1})\partial_a \end{aligned}$$

$$-\beta_j^2(t)+\beta_j(t)-t\dot{\beta}_j(t))q_j(a,t)$$
$$-(t^2\ddot{q}_j(a,t)+2\beta_j(t)t\dot{q}_j(a,t))+2at\partial_a\dot{q}_j(a,t)=c_j(t)a^{2-j}. \quad (2.39)$$

We note that the admissible functions c_j are as in (2.31), which are of the form $\omega^k(t)$ in (2.28). These are parts of $t^2\lambda^{-1/2}(t)e_1(t,r)$ with e_1 defined in (2.23), and they come from (2.27) applied no more than twice to v_1 instead of f, where v_1 is as defined in (2.21) with ω_j coming from t^2e_0 in (2.9). Thus, we see that c_j are polynomials of $\kappa(t)$ with the operator $t\partial_t$ applied finitely many times. By (2.3) we can write $\beta_j(t)=\tilde{\nu}_j+2\tilde{\epsilon}_j\cos(\log t)$ with $\tilde{\nu}_j=(j-1/2)\nu-\frac{1}{2}>1$ and

$$c_j(t)=\sum_{n=0}^{N_j}\sum_{m=0}^{n}\tilde{\epsilon}_j^n\tilde{c}_{n,m}^{(j)}t^{(n-2m)i} \quad (2.40)$$

since polynomials of $\kappa(t)$ have this type of expansions and they are preserved by the operator $t\partial_t$.

For convenience, we drop the subscript j and write (2.39) as

$$((1-a^2)\partial_a^2+(2(\beta(t)-1)a+2a^{-1})\partial_a-\beta^2(t)+\beta(t)-t\dot{\beta}(t))q(a,t)$$
$$-(t^2\ddot{q}(a,t)+2\beta(t)t\dot{q}(a,t))+2at\partial_a\dot{q}(a,t)=c(a,t), \quad (2.41)$$

where

$$\beta(t)=\tilde{\nu}+2\tilde{\varepsilon}\cos(\log t)=\tilde{\nu}+\tilde{\varepsilon}t^i+\tilde{\varepsilon}t^{-i}$$

and

$$c(a,t)=\sum_{n=0}^{N}\sum_{m=0}^{n}\tilde{\varepsilon}^n\tilde{c}_{n,m}(a)t^{(n-2m)i},$$

where $\tilde{c}_{n,m}(a)$ is linear in a, and we let

$$\hat{c}_1=\sup_{0\le a\le 1,0\le m\le n\le N}(|\tilde{c}_{n,m}(a)|,|\tilde{c}'_{n,m}(a)|).$$

We seek a solution to (2.41) of the form

$$q(a,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\tilde{\varepsilon}^n g_{n,m}(a)t^{(n-2m)i}, \quad (2.42)$$

where $g_{n,m}(a)=\bar{g}_{n,n-m}(a)$. Plugging (2.42) and (2.40) into (2.41), we see that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\tilde{\varepsilon}^n t^{(n-2m)i}((1-a^2)g''_{n,m}(a)+(2(\tilde{\nu}-1)a+2a^{-1})g'_{n,m}(a)$$
$$+2a(g'_{n-1,m}(a)+g'_{n-1,m-1}(a))+(\tilde{\nu}-\tilde{\nu}^2)g_{n,m}(a)-2g_{n-2,m-1}(a)$$
$$+(1-i-2\tilde{\nu})g_{n-1,m}(a)$$
$$+(1+i-2\tilde{\nu})g_{n-1,m-1}(a)-g_{n-2,m}(a)-g_{n-2,m-2}(a)$$
$$+(n-2m)(n-2m+i-2\tilde{\nu}i)g_{n,m}(a)-2i(n-2m-1)g_{n-1,m}(a)$$
$$-2i(n-2m+1)g_{n-1,m-1}(a)+2ai(n-2m)g'_{n,m}(a))$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\tilde{\varepsilon}^n\tilde{c}_{n,m}(a)t^{(n-2m)i},$$

where $g_{n,m}(a) = 0$ if $n < 0$ or $|n - 2m| > n$. Collecting powers of $\tilde{\varepsilon}$ and t^i, we obtain the equation

$$\begin{aligned}(1-a^2)g''_{n,m}(a) &+ (2(\tilde{\nu}-1)a + 2a^{-1} + 2ai(n-2m))g'_{n,m}(a)\\ &+ (\tilde{\nu} - \tilde{\nu}^2 + (n-2m)(n-2m+i-2\tilde{\nu}i))g_{n,m}(a)\\ &= -2a(g'_{n-1,m}(a) + g'_{n-1,m-1}(a)) - (1+i-2\tilde{\nu}-2i(n-2m))g_{n-1,m}(a)\\ &\quad - (1-i-2\tilde{\nu}-2i(n-2m))g_{n-1,m-1}(a) + 2g_{n-2,m-1}(a)\\ &\quad + g_{n-2,m}(a) + g_{n-2,m-2}(a) + \tilde{c}_{n,m}(a) =: R_{n,m}(a). \end{aligned} \tag{2.43}$$

Note that $R_{0,0}(a) = \tilde{c}_{0,0}(a)$.

The associated homogeneous equation

$$\begin{aligned}(1-a^2)g''_{n,m}(a) &+ (2(\tilde{\nu}-1)a + 2a^{-1} + 2ai(n-2m))g'_{n,m}(a)\\ &+ (\tilde{\nu} - \tilde{\nu}^2 + (n-2m)(n-2m+i-2\tilde{\nu}i))g_{n,m}(a) = 0\end{aligned} \tag{2.44}$$

has two solutions

$$\frac{(1-a)^{\tilde{\nu}+1+(n-2m)i}}{a}, \qquad \frac{(1+a)^{\tilde{\nu}+1+(n-2m)i}}{a},$$

and their Wronskian is $2a^{-2}(\tilde{\nu}+1+(n-2m)i)(1-a^2)^{\tilde{\nu}+(n-2m)i}$. Therefore, by (2.43), $g_{n,m}$ can be defined recursively as

$$\begin{aligned} g_{n,m}(a) &= \frac{(1+a)^{\tilde{\nu}+1+(n-2m)i}}{2a(\tilde{\nu}+1+(n-2m)i)} \int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx\\ &\quad - \frac{(1-a)^{\tilde{\nu}+1+(n-2m)i}}{2a(\tilde{\nu}+1+(n-2m)i)}\\ &\quad \times \int_0^a x(1-x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx, \end{aligned} \tag{2.45}$$

which implies

$$\begin{aligned} (ag_{n,m}(a))' &= \frac{(1+a)^{\tilde{\nu}+(n-2m)i}}{2} \int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx\\ &\quad + \frac{(1-a)^{\tilde{\nu}+(n-2m)i}}{2}\\ &\quad \times \int_0^a x(1-x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx. \end{aligned} \tag{2.46}$$

With $g_{n,m}$ thus defined, (2.42) gives a formal solution to (2.41). In order to show that (2.42) gives a true solution, it is sufficient to show that $g''_{n,m}(a)$ is continuous, and for some $C_0 > 0$, we have

$$\|g^{(k)}_{n,m}\|_\infty := \sup_{a\in[0,1]} |g^{(k)}_{n,m}(a)| \le C_0^n$$

for $0 \le k \le 2$ since this would imply that (2.42) is convergent and twice differentiable in both a and t with continuous second derivatives for $0 < t < t_0$, $0 \le a \le 1$, as long as $\tilde{\varepsilon} < C_0^{-1}$. To show that the initial conditions $q(0,t) = \partial_a q(0,t) = 0$ are satisfied, we only need to show that $g_{n,m}(0) = g'_{n,m}(0) = 0$. By differentiating

(2.42) we see that $\partial_t^k \partial_a^\ell q(a,t) = O(t^{-k}a^{2-\ell})$ for $k \ge 0$ and $0 \le \ell \le 2$. In addition, to show the second estimate of (2.38), we will prove the inequality

$$|g''_{n,m}(a)| \lesssim a\tilde{C}_3^{n+1} \tag{2.47}$$

for some $\tilde{C}_3$ and $0 \le a \le 1/2$. Note that we do not need to show that $g_{n,m}(a) = \bar{g}_{n,n-m}(a)$ since we can simply take the real part of $q(a,t)$ in (2.42) to get a real solution.

Since

$$\begin{aligned}
&\left|(1+a)^{\tilde{\nu}+(n-2m)i}\int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i}R_{n,m}(x)\,dx\right| \\
&\quad \le a\|R_{n,m}\|_\infty (1+a)^{\tilde{\nu}}\int_0^a (1+x)^{-\tilde{\nu}-1}\,dx \\
&\quad = a\tilde{\nu}^{-1}((1+a)^{\tilde{\nu}}-1)\|R_{n,m}\|_\infty \\
&\quad \le a^2(1+a)^{\tilde{\nu}-1}\|R_{n,m}\|_\infty, \\
&\left|(1-a)^{\tilde{\nu}+(n-2m)i}\int_0^a x(1-x)^{-\tilde{\nu}-1-(n-2m)i}R_{n,m}(x)\,dx\right| \\
&\quad \le a\|R_{n,m}\|_\infty (1-a)^{\tilde{\nu}}\int_0^a (1-x)^{-\tilde{\nu}-1}\,dx \\
&\quad = a\tilde{\nu}^{-1}(1-(1-a)^{\tilde{\nu}})\|R_{n,m}\|_\infty \\
&\quad \le a^2\|R_{n,m}\|_\infty,
\end{aligned}$$

and $\sqrt{2}|z| \ge |\Re z| + |\Im z|$ for any z, we have by (2.46)

$$\begin{aligned}
|g_{n,m}(a)| &\le \frac{\sqrt{2}(2^{\tilde{\nu}-1}+2^{-1})a\|R_{n,m}\|_\infty}{(|n-2m|+\tilde{\nu}+1)}, \\
|(ag_{n,m}(a))'| &\le \sqrt{2}(2^{\tilde{\nu}-2}+2^{-1})a^2\|R_{n,m}\|_\infty, \\
|g'_{n,m}(a)| &\le a^{-1}(|(ag_{n,m}(a))'| + |g_{n,m}(a)|) \\
&\le \sqrt{2}(2^{\tilde{\nu}-1}+2^{-1})(1+(\tilde{\nu}+1)^{-1})\|R_{n,m}\|_\infty
\end{aligned} \tag{2.48}$$

for all $a \in [0,1]$. By (2.43) and (2.48) we have

$$\begin{aligned}
|R_{n,m}(a)| &= |-2((ag_{n-1,m}(a))' + (ag_{n-1,m-1}(a))') \\
&\qquad - (-1+i-2\tilde{\nu}-2i(n-2m))g_{n-1,m}(a) \\
&\qquad - (-1-i-2\tilde{\nu}-2i(n-2m))g_{n-1,m-1}(a) \\
&\qquad + 2g_{n-2,m-1}(a) + g_{n-2,m}(a) + g_{n-2,m-2}(a) + \tilde{c}_{n,m}(a)| \\
&\le 2(|(ag_{n-1,m}(a))'| + |(ag_{n-1,m-1}(a))'|) \\
&\qquad + 2(\tilde{\nu}+1+|n-2m|)(|g_{n-1,m}(a)| + |g_{n-1,m-1}(a)|) \\
&\qquad + 4\max_{0\le j\le 2}|g_{n-2,m-j}(a)| + \hat{c}_1 \qquad (2.49) \\
&\le C_1 \max_{1\le j\le 2, 0\le k\le j}\|R_{n-j,m-k}\|_\infty + \hat{c}_1 \qquad (2.50)
\end{aligned}$$

for some $C_1 > 1$. Since $|R_{0,0}(a)| \leq \hat{c}_1$, we have by induction

$$|R_{n,m}(x)| \leq (C_1 + \hat{c}_1)^{n+1}, \tag{2.51}$$

which implies by (2.48) that

$$\|g_{n,m}\|_\infty \leq \frac{\tilde{C}_1^{n+1}}{|n-2m| + \tilde{\nu} + 1} \quad \text{and} \quad \|g'_{n,m}\|_\infty \leq \tilde{C}_1^{n+1} \tag{2.52}$$

for some $\tilde{C}_1 > 1$. Note that by (2.46) $g'_{n,m}$ is differentiable, implying that $R_{n,m}$ is continuous, and thus we know that $g''_{n,m}$ is continuous by differentiating (2.46). To estimate $g''_{n,m}$, we rewrite (2.46) using integration by parts as

$$\begin{aligned}
(ag_{n,m}(a))' &= \frac{(1+a)^{\tilde{\nu}+(n-2m)i}}{2} \int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx \\
&\quad + \frac{aR_{n,m}(a)}{2(\tilde{\nu}+(n-2m)i)} \\
&\quad - \frac{(1-a)^{\tilde{\nu}+(n-2m)i}}{2(\tilde{\nu}+(n-2m)i)} \int_0^a (1-x)^{-\tilde{\nu}-(n-2m)i} (xR_{n,m}(x))'\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&|2(ag_{n,m}(a))''| \\
&\quad \leq \bigg|(\tilde{\nu}+(n-2m)i)(1+a)^{\tilde{\nu}-1+(n-2m)i} \\
&\qquad \times \int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)\,dx\bigg| \\
&\qquad + (1+a)^{-1} a |R_{n,m}(a)| \\
&\qquad + \bigg|(1-a)^{\tilde{\nu}-1+(n-2m)i} \int_0^a (1-x)^{-\tilde{\nu}-(n-2m)i} (xR_{n,m}(x))'\,dx\bigg| \\
&\quad \leq a(1+a)^{-1}(\tilde{\nu}^{-1} 2^{\tilde{\nu}}(\tilde{\nu}+|n-2m|)+1)\|R_{n,m}\|_\infty \\
&\qquad + (\tilde{\nu}-1)^{-1}(1-(1-a)^{\tilde{\nu}-1})\|(xR_{n,m}(x))'\|_\infty.
\end{aligned} \tag{2.53}$$

This, together with (2.48) and (2.51), implies

$$\begin{aligned}
|(aR_{n,m}(a))'| &\leq 2(|(ag_{n-1,m}(a))''| + |(ag_{n-1,m-1}(a))''|) \\
&\qquad + 2(\tilde{\nu}+1+|n-2m|)(|(ag_{n-1,m}(a))'| + |(ag_{n-1,m-1}(a))'|) \\
&\qquad + 4 \max_{0\leq j\leq 2} |(ag_{n-2,m-j}(a))'| + 2\hat{c}_1 \\
&\leq C_2\Big((\tilde{\nu}+1+|n-2m|)(C_1+\hat{c}_1)^n \\
&\qquad + \max_{0\leq k\leq 1} \|(xR_{n-1,m-k}(x))'\|_\infty\Big) + 2\hat{c}_1
\end{aligned} \tag{2.54}$$

for some $C_2 > 1$. In particular, $|(aR_{0,0}(a))'| \leq 2\hat{c}_1$. Thus, we have by induction

$$\|(xR_{n,m}(x))'\|_\infty \leq (\tilde{\nu}+2+n)\tilde{C}_2^{n+1}, \tag{2.55}$$

where $\tilde{C}_2 = 2(C_2 + \hat{c}_1)(C_1 + \hat{c}_1)$. Therefore, by (2.51), (2.53), and (2.55)

$$|(ag_{n,m}(a))''| \le a\hat{C}_2^{n+1} \tag{2.56}$$

for some $\hat{C}_2 > 1$, where we used the fact that in (2.53) we have $0 \le 1 - (1 - a)^{\tilde{\nu}-1} \lesssim a$. Now, integrating the estimate for $(ag_{n,m}(a))'$ in (2.48), we get

$$|g_{n,m}(a)| \le 3^{-1}\sqrt{2}(2^{\tilde{\nu}-2} + 2^{-1})a^2\|R_{n,m}\|_\infty, \tag{2.57}$$

which, together with (2.48) and (2.51), implies

$$\begin{aligned} |g'_{n,m}(a)| &\le a^{-1}(|(ag_{n,m}(a))'| + |g_{n,m}(a)|) \\ &\le \sqrt{2}(2^{\tilde{\nu}-1} + 2^{-1})(1 + 3^{-1})a(C_1 + \hat{c}_1)^{n+1}. \end{aligned} \tag{2.58}$$

By (2.56) and (2.58), for some $C_0 > \tilde{C}_1$, we have

$$|g''_{n,m}(a)| \le a^{-1}(|(ag_{n,m}(a))''| + 2|g'_{n,m}(a)|) \le C_0^{n+1}. \tag{2.59}$$

By (2.52) and (2.59) we have $\|g^{(k)}_{n,m}(a)\|_\infty \le C_0^n$ for $0 \le k \le 2$. Since $R_{n,m}$ is continuous (cf. the discussion below (2.52)), writing $R_{n,m}(x) = R_{n,m}(0) + o(1)$ and expanding (2.45) at $a = 0$, we get $g_{n,m}(a) = o(a)$, implying $g_{n,m}(0) = g'_{n,m}(0) = 0$.

In addition, for q_1, we have $\tilde{c}_{n,m}(a) = \tilde{c}_{n,m}a$. By (2.52) we have $|g'_{n,m}(a)| \le \tilde{C}_1^{n+1}$ and $|g_{n,m}(a)| \le a\tilde{C}_1^{n+1}$, and thus by definition (cf. (2.43))

$$|R_{n,m}(a)| \le aC_3^{n+1}$$

for some $C_3 > 0$. By (2.43) we have

$$\begin{aligned} &|(ag_{n,m}(a))''| \\ &\quad = (1 - a^2)^{-1}|2a(\nu + (n - 2m)i)(ag_{n,m}(a))' \\ &\qquad + (-\tilde{\nu} - \tilde{\nu}^2 + (n - 2m)(n - 2m - i - 2\tilde{\nu}i))ag_{n,m}(a) - aR_{n,m}(a)| \\ &\quad \le a^2\tilde{C}_3^{n+1} \end{aligned}$$

for some $\tilde{C}_3 > 0$ as long as $0 \le a \le 1/2$, which implies (2.47) by direct calculation. Therefore, $q(a,t)$ given by (2.42) is a solution to (2.41) satisfying the stated conditions (see the discussion after (2.46)). □

REMARK 2.1. One can modify the proof of Lemma 2.2 so that the results hold for $\lambda(t) = t^{-1-\nu}F_a(\sin(\log t), \cos(\log t))$, where $F_a(u, v)$ is analytic in u and v at the origin with sufficiently small derivatives. In this case, estimates of the type (2.50) remain valid.

Similarly, the results of Lemma 2.2 hold for $\lambda(t) = t^{-1-\nu}F_b(t^\gamma)$, where F_b is analytic at the origin with sufficiently small derivatives, and $\gamma \in \mathbb{R}^+$. In this case, instead of (2.42), we consider

$$q(a,t) = \sum_{n=0}^{\infty} g_n(a)t^{n\gamma}, \tag{2.60}$$

and the rest of the proof is similar to that of Lemma 2.2.

Using $a = R\mu^{-1}$, we may rewrite (2.35) in the form

$$v_2(t,r) := \frac{\lambda(t)^{1/2}}{\mu^2(t)}(R\tilde{q}_1(a,t) + q_2(a,t)), \tag{2.61}$$

where we have set $\tilde{q}_1(a,t) := a^{-1}q_1(a,t)$. Note that both $\tilde{q}_1$ and q_2 are $O(a^2)$ as $a \to 0$. Thus, by Lemma 2.2 we have the estimate

$$\partial_t^k \partial_a^j v_2(t,at) = O\left(t^{-k}a^{-j}\frac{\lambda(t)^{1/2}a^2(1+R)}{\mu^2(t)}\right) \tag{2.62}$$

for $0 \le j \le 2$. Furthermore, we have the following:

LEMMA 2.3. *For $0 \le r \le t/2$, estimate (2.62) remains valid after we apply $t\partial_t$ or $r\partial_r$ any number of times to the left-hand side, or equivalently,*

$$|\partial_t^k \partial_r^\ell v_2(t,r)| \le C_{k,\ell} t^{-k} r^{-\ell} \frac{\lambda(t)^{1/2}a^2(1+R)}{\mu^2(t)}$$

for all $k, \ell \ge 0$.

Proof. Note that for any differentiable f, we have $r\partial_r f(t,r) = a\partial_a f(t,at)$ and $t\partial_t f(t,r) = (t\partial_t - a\partial_a)f(t,at)$. This implies

$$|t^k r^\ell \partial_t^k \partial_r^\ell v_2(t,r)| \le \tilde{C}_{k,\ell} \max_{0\le m\le k} t^{k-m}a^{\ell+m}|\partial_t^{k-m}\partial_a^{\ell+m}v_2(t,at)|. \tag{2.63}$$

Thus, it is sufficient to show that

$$t^k a^\ell |\partial_t^k \partial_a^\ell v_2(t,at)| \le \hat{C}_{k,\ell} \frac{\lambda(t)^{1/2}a^2(1+R)}{\mu^2(t)} \tag{2.64}$$

for all $k, \ell \ge 0$.

For $\ell \le 2$, this follows from (2.62). For $\ell > 2$, we only need to show

$$\partial_t^k \partial_a^\ell v_2(t,at) = O\left(a^{-1}t^{-k}\frac{\lambda(t)^{1/2}(1+R)}{\mu^2(t)}\right).$$

By (2.61) it is sufficient to show that

$$|\partial_t^k \partial_a^\ell \tilde{q}_1(a,t)| = O(t^{-k}); \qquad |\partial_t^k \partial_a^\ell q_2(a,t)| = O(t^{-k}) \tag{2.65}$$

since $\partial_a R = a^{-1}R$ and $\partial_a^j R = 0$ for $j \ge 2$. By (2.39) we have, for $j = 1, 2$,

$$\begin{aligned}(1-a^2)\partial_a^2(aq_j(a,t)) = &-(2\beta_j(t)a\partial_a - \beta_j^2(t) - \beta_j(t) - t\dot{\beta}_j(t))aq_j(a,t) \\ &+ t^2 a\ddot{q}_j(a,t) + 2(\beta_j(t)+1)ta\dot{q}_j(a,t) \\ &- 2at\partial_a(a\dot{q}_j(a,t)) + a^{3-j}c_j(t).\end{aligned} \tag{2.66}$$

Note that $\partial_t^k \beta_j(t) = O(t^{-k})$, $\partial_t^k c_j(t) = O(t^{-k})$ (cf. (2.40)), and by Lemma 2.2 we have

$$|\partial_t^k \partial_a^{\ell'}(aq_j(a,t))| \lesssim t^{-k}, \quad 0 \le \ell' \le 2. \tag{2.67}$$

Thus, by differentiating (2.66) and using induction on ℓ', we get

$$|\partial_t^k \partial_a^{\ell'}(aq_j(a,t))| \lesssim t^{-k}, \quad \ell' > 2. \tag{2.68}$$

Recall that $q_j(a,t) = O(a^{4-j})$ by Lemma 2.2. Thus, by integrating (2.68) with $\ell' = \ell + 2$ we have

$$a\partial_t^k q_j(a,t) = \sum_{n=5-j}^{\ell+1} c_{j,n}^{(k)}(t)a^n + \hat{q}_j^{(k)}(a,t), \tag{2.69}$$

where $c_{j,n}^{(k)} = O(t^{-k})$ and $\partial_a^m \hat{q}_j^{(k)}(a,t) = O(t^{-k}a^{\ell+2-m})$ for $0 \le m \le \ell+2$. Thus, by (2.69) we have

$$|\partial_t^k \partial_a^\ell \tilde{q}_1(a,t)| \le |\partial_a^\ell(a^{-2}\hat{q}_1^{(k)}(a,t))| \lesssim t^{-k},$$
$$|\partial_t^k \partial_a^\ell q_2(a,t)| \le \ell!\,|\partial_t^k c_{2,\ell+1}(t)| + |\partial_a^\ell(a^{-1}\hat{q}_2^{(k)}(a,t))| \lesssim t^{-k},$$

that is, (2.65) holds. □

We set $u_2 := u_1 + v_2 = u_0 + v_1 + v_2$. Finally, (2.22) remains valid for u_2 as well since $R \le \mu(t)$. In other words, u_0 gives the main shape of the profile as a function of R.

2.6. *The Error from u_2*

We define

$$\begin{aligned} e_2 &:= \mathcal{L}_{\text{quintic}}(u_2) = \mathcal{L}_{\text{quintic}}(u_1 + v_2) \\ &= \mathcal{L}_{\text{quintic}}(u_1) + u_1^5 - (u_1 + v_2)^5 + \left(\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r\right)v_2 \\ &= e_1 - e_1^0 - 5u_1^4 v_2 - 10u_1^3 v_2^2 - 10u_1^2 v_2^3 - 5u_1 v_2^4 - v_2^5. \end{aligned} \tag{2.70}$$

We determine $t^2\lambda(t)^{-1/2}e_2$. First, from (2.32) we have

$$\begin{aligned} &t^2\lambda^{-1/2}(t)(e_1 - e_1^0)(t,r) \\ &\quad = \mu^{-2}(t)\left[\frac{\log R}{R}\Phi_1(a,b,\rho\log\rho,\rho) + \frac{1}{R}\Phi_2(a,b,\rho\log\rho,\rho)\right] \end{aligned} \tag{2.71}$$

for $R \ge 1$. For $|R| < 1$, we read off from (2.28) and (2.33) that

$$t^2\lambda^{-1/2}(t)(e_1 - e_1^0)(t,r) = O(\mu^{-2}(t)). \tag{2.72}$$

This holds uniformly for small times, and $t\partial_t$ and $r\partial_r$ can be applied any number of times without changing this asymptotic behavior as $R \to 0$.

Next, for large R, by (2.22) and (2.62) we have

$$t^2\lambda^{-1/2}(t)u_1^4 v_2 = O(a^2R^{-3}) = O(R^{-1}\mu^{-2}(t)).$$

The final nonlinear term contributes

$$t^2\lambda^{-1/2}(t)v_2^5 = \mu^{-8}(t)O(R^5) = \mu^{-2}(t)R^{-1}O(\mu^{-6}(t)R^6) = O(R^{-1}\mu^{-2}(t)).$$

Thus,

$$t^2\lambda^{-1/2}(t)u_1^k v_2^{5-k} = O(R^{-1}\mu^{-2}(t)) \quad (R \ge 1, 0 \le k \le 4). \tag{2.73}$$

For small R, we have $u_1 = \lambda^{1/2}(t)O(1)$ by (2.22) and $v_2 = \lambda^{1/2}(t)\mu^{-2}(t) \times O(a^2)$ by (2.62). Thus (recall that $a = R\mu^{-1}(t)$),

$$t^2\lambda^{-1/2}(t)u_1^4 v_2 = O(a^2) = O(\mu^{-2}(t)),$$
$$t^2\lambda^{-1/2}(t)u_1 v_2^4 = O(a^2\mu^{-6}(t)) = O(\mu^{-2}(t)),$$

and we have

$$t^2\lambda^{-1/2}(t)u_1^k v_2^{5-k} = O(\mu(t)^{-2}) \quad (R < 1, 0 \le k \le 4). \tag{2.74}$$

By the preceding we gain a factor μ^{-2} for all R, and the decay is at least $\frac{\log R}{R}$ as $R \to \infty$.

Finally, by (2.22) and Lemma 2.3 we see that estimates (2.73) and (2.74) remain valid after we apply $t\partial_t$ or $r\partial_r$ any number of times if $0 \le r \le t/2$. Similarly, by (2.22) and (2.62) we see that if $0 \le r \le t$, then (2.73) and (2.74) remain valid after we apply $t\partial_t$ and $r\partial_r$ no more than twice.

2.7. *Proof of Proposition* 2.1

(i), (ii) Smoothness of u_2 follows from (2.8), (2.21) (where f_j satisfy (2.19), which can be differentiated), and Lemma 2.2, which imply that u_0, v_1, v_2 are all in $C^2(\{0 < t < t_0, 0 \le r \le t\})$.

To show (2.4), it is sufficient to show that $v_{1,2} = \sqrt{\lambda(t)}\mu^{-2}(t)O(R)$ for both small and large R. This follows from (2.19) and (2.21) for v_1 and from (2.62) for v_2. Similarly, (2.6) follows from the fact that $\partial_t^k\partial_r^j v_{1,2} = t^{-k}r^{-j}\sqrt{\lambda(t)}\mu^{-2}(t)O(R)$ by (2.19) (which is clearly differentiable in R), (2.21), and Lemma 2.3. For $k + j \le 2$ and $0 \le r \le t$, we use (2.62) instead of Lemma 2.3.

Finally, (2.5) and (2.7) follow from (2.70), where the different parts are estimated in (2.71), (2.72), (2.73), and (2.74), which remain valid after we apply $t\partial_t$ or $r\partial_r$ any number of times if $0 \le r \le t/2$, or if they are applied no more than twice and $0 \le r \le t$.

(iii) We let

$$\hat{u}_2(t,r) = \begin{cases} u_2(t,r) & \text{if } 0 < t < t_0, 0 \le r \le t, \\ u_2(t,t) + (r-t)u_2^{(0,1)}(t,t) & \\ \quad + \frac{1}{2}(r-t)^2 u_2^{(0,2)}(t,t) & \text{if } 0 < t < t_0, t < r \le (1+2b_1)t, \end{cases}$$

where $b_1 > 0$ and $u_2^{(n,m)}(t,r) := \partial_r^m\partial_t^n u_2(t,r)$. Clearly, $\hat{u}_2$ is C^2 in r for fixed t.

By direct calculation using (2.6) we have, for $0 \le k + j \le 2$,

$$u_2^{(k,j)}(t,r) = O(r^{-j-1}t^{-k}\lambda^{-1/2}(t)).$$

Thus, for $t < r \le (1+2b_1)t$, we have

$$\begin{aligned} \hat{u}_2(t,r) &= O(t^{-1}\lambda^{-1/2}(t)); \qquad \partial_r\hat{u}_2(t,r) = O(t^{-2}\lambda^{-1/2}(t)), \\ \partial_t\hat{u}_2(t,r) &= O\Big(\max_{0\le m\le 1, 0\le n\le 1} |(r-t)^n u_2^{(1-m,n+m)}(t,t)| \\ &\qquad + |(r-t)^2\partial_t u_2^{(0,2)}(t,t)|\Big), \end{aligned} \tag{2.75}$$

where the first term is clearly of order $O(t^{-2}\lambda^{-1/2}(t))$. To estimate the second term, recall that $u_2 = u_1 + v_2$, where by direct calculation $u_1^{(k,j)}(t,r) = O(r^{-j-1}t^{-k}\lambda^{-1/2}(t))$ for all $k, j \geq 0$. Thus,

$$(r-t)^2\partial_t u_1^{(0,2)}(t,t) = O(t^{-2}\lambda^{-1/2}(t)).$$

Since $a = r/t$, we have

$$v_2^{(0,2)}(t,t) = t^{-2}\partial_a^2 v_2(t,t).$$

This and (2.62) imply

$$\partial_t v_2^{(0,2)}(t,t) = O(t^{-4}\lambda^{-1/2}(t)).$$

Therefore,

$$\partial_t \hat{u}_2(t,r) = O(t^{-2}\lambda^{-1/2}(t)). \tag{2.76}$$

Now we let B_1 be a smooth bump function satisfying

$$B_1(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0 & \text{if } x > 1 + b_1, \end{cases}$$

and we let $u_2(t,r) = \hat{u}_2(t,r)B_1(r/t)$ for $r > t$. Clearly, u_2 is C^2 in r for fixed t. By direct calculation using (2.75) and (2.76) we have

$$u_2(t,r) = O(t^{-1}\lambda^{-1/2}(t)); \qquad \partial_r u_2(t,r) = O(t^{-2}\lambda^{-1/2}(t));$$
$$\partial_t u_2(t,r) = O(t^{-2}\lambda^{-1/2}(t)).$$

Therefore, b_1 can be chosen small enough to ensure

$$\int_{t\leq r} u_2^6\,dx \lesssim b_1 t^{-3}\lambda^{-3}(t) < \frac{\delta}{3},$$
$$\int_{t\leq r} (\partial_r u_2)^2\,dx \lesssim b_1 t^{-1}\lambda^{-1}(t) < \frac{\delta}{3},$$
$$\int_{t\leq r} (\partial_t u_2)^2\,dx \lesssim b_1 t^{-1}\lambda^{-1}(t) < \frac{\delta}{3}.$$

Thus, their sum is less than δ.

3. Construction of an Exact Solution

Our aim next is to construct an energy class solution of (2.1) of the form

$$u = u_2 + \varepsilon$$

in the backward light cone $r \leq t$, $0 < t < t_0$. We immediately infer the equation

$$\Box\varepsilon + 5u_0^4\varepsilon = 5(u_0^4 - u_2^4)\varepsilon - N(u_2,\varepsilon) - e_2, \tag{3.1}$$

where we have

$$N(u_2,\varepsilon) = 10u_2^3\varepsilon^2 + 10u_2^2\varepsilon^3 + 5u_2\varepsilon^4 + \varepsilon^5;$$

also, we shall denote by e_2 an extension of e_2 in the preceding section beyond the light cone satisfying the same asymptotics as in Proposition 2.1. Proceeding exactly as in [11], we pass to the variables

$$\tau = \int_t^{t_0} \lambda(s)\, ds,\ R = \lambda(t) r, \qquad v(\tau, R) = R\varepsilon(t(\tau), r(\tau, R))$$

and note that $t \to 0$ means $\tau \to \infty$. Writing[1] $\kappa(\tau) := \lambda(t(\tau))$, $\beta(\tau) := \kappa'(\tau)/\kappa(\tau)$, and

$$\mathcal{D} := \partial_\tau + \beta(\tau)(R\partial_R - 1),$$

we get

$$[\mathcal{D}^2 + \beta(\tau)\mathcal{D} + \mathcal{L}]v = \kappa^{-2}(\tau)\left[5(u_0^4 - u_2^4)v + RN\left(u_2, \frac{v}{R}\right) + Re_2\right], \tag{3.2}$$

where $\mathcal{L} := -\partial_R^2 - 5W^4(R)$, and we interpret u_2, u_0, e_2 as functions of τ, R. In fact, since it suffices to solve this problem in a dilate of the light cone, we replace it by

$$\begin{aligned}[\mathcal{D}^2 + \beta(\tau)\mathcal{D} + \mathcal{L}]v \\ = \kappa^{-2}(\tau)\tilde{\chi}\left(\frac{R}{\nu\tau}\right)\left[5(u_0^4 - u_2^4)v + RN\left(u_2, \frac{v}{R}\right) + Re_2\right]\end{aligned} \tag{3.3}$$

for some smooth cutoff $\tilde{\chi} \in C_0^\infty(\mathbb{R}_+)$ with $\tilde{\chi}|_{r\le 1} = 1$. In fact, the main work consists in solving the linear inhomogeneous problem

$$[\mathcal{D}^2 + \beta(\tau)\mathcal{D} + \mathcal{L}]v = f,$$

where f satisfies bounds like $\kappa^{-2}(\tau)\tilde{\chi}(\frac{R}{\nu\tau})Re_2$. Our approach below is a general framework to solve such problems, applicable to much wider classes of scaling factors $\lambda(t)$. It is the construction of the second correction in Section 2.5 that imposes restrictions on the admissible scaling factors.

3.1. *Strategy of the Proof*

Since the following approach is quite technical, we give a brief sketch of the main steps and ideas.

(1) Following [11], we study equation (3.3) on the "distorted" Fourier side, that is, we apply to equation (3.3) the spectral transformation $\mathcal{U}$ associated to the self-adjoint operator $\mathcal{L}$. By definition, the transformation $\mathcal{U}$ diagonalizes $\mathcal{L}$, that is, it satisfies[2] $\mathcal{U}\mathcal{L}f(\xi) = \xi\mathcal{U}f(\xi)$, and $\mathcal{U}$ is a unitary map from $L^2(0,\infty)$ to $L^2((0,\infty), \rho(\xi)\,d\xi)$ with the spectral measure $\rho(\xi)\,d\xi$. The crucial asymptotic behavior of ρ was obtained by a suitable ODE analysis and Weyl–Titchmarsh theory in [1; 11]. Upon writing $x(\tau,\xi) := \mathcal{U}v(\tau,\cdot)(\xi)$, equation (3.3) transforms into

$$[\hat{\mathcal{D}}^2 + \beta\hat{\mathcal{D}} + \xi]x = \kappa^{-2}\mathcal{N}(x) - 2\beta\mathcal{K}\hat{\mathcal{D}}x - \beta^2\hat{\mathcal{K}}x + \kappa^{-2}\hat{e}_2, \tag{3.4}$$

[1]We warn the reader that here the symbols β and κ have a different meaning from that in Section 2.

[2]We ignore here the fact that $\mathcal{L}$ has a negative eigenvalue. This is inessential for our construction but complicates the notation considerably.

where $\hat{\mathcal{D}} = \partial_\tau - 2\beta(\tau)\xi\partial_\xi + O(\tau^{-1})$, and $\hat{e}_2$ is the term arising from the error e_2. Furthermore, we have absorbed all the terms from the right-hand side containing v into $\mathcal{N}(x)$. The two new operators $\mathcal{K}$ and $\hat{\mathcal{K}}$ are nonlocal, and they emerge as error terms from the operator $R\partial_R$ in $\mathcal{D}$. In a sense, they measure the deviation of $\mathcal{L}$ from the free Schrödinger operator $-\partial_R^2$. Those operators were already studied in [11] and [1].

(2) In order to solve equation (3.4), we consider the inhomogeneous problem $[\hat{\mathcal{D}}^2 + \beta\hat{\mathcal{D}} + \xi]x = y$. The corresponding solution operator (the *parametrix*) is obtained by the method of characteristics (see [9]) and reads

$$x(\tau,\xi) = \mathcal{H}y(\tau,\xi) = \int_\tau^\infty H(\tau,\sigma,\xi)\, y\left(\sigma, \frac{\kappa(\tau)^2}{\kappa(\sigma)^2}\xi\right) d\sigma,$$

where essentially

$$H(\tau,\sigma,\xi) = \xi^{-1/2}\sin\left(\kappa(\tau)\xi^{1/2}\int_\tau^\sigma \kappa(u)^{-1}\,du\right)O(1).$$

Thus, due to the factor $\xi^{-1/2}$, we expect a loss of τ^2 from the parametrix in general.

(3) We study equation (3.4) in the space X defined by the norm

$$\|f\|_X := \||\cdot|^{1/2}\langle\cdot\rangle^{1/8} f\|_{L^2_\rho(0,\infty)} + \|(|\cdot|\langle\cdot\rangle^{-1})^{1/2-\delta} f\|_{L^p(0,\infty)}.$$

Here $\delta > 0$ is supposed to be small and p large (depending on δ). For small frequencies ξ, the L^2-part in X corresponds to the homogeneous Sobolev space[3] $\dot{H}^1(\mathbb{R}^3)$ on the physical side. For large frequencies ξ, the corresponding space is $H^{5/4}(\mathbb{R}^3)$. A main ingredient for estimating the nonlinearity is the fractional Leibniz rule

$$\|f^5\|_{H^{1/4}(\mathbb{R}^3)} \lesssim \|f\|^5_{H^{5/4}(\mathbb{R}^3)}.$$

Thus, the L^p-part in X is introduced to control the small frequencies. The operators $\mathcal{K}$ and $\hat{\mathcal{K}}$ exhibit a smoothing property at small frequencies which amounts to a gain of $\xi^{1/2}$. As a consequence, the parametrix loses only[4] τ for these terms. Thus, the decay of $\beta(\tau)^2 = O(\tau^{-2})$ is strong enough to treat the term $\beta^2\hat{\mathcal{K}}x$ in equation (3.4) perturbatively.

(4) The most difficult term in equation (3.4) is $2\beta\mathcal{K}\hat{\mathcal{D}}x$ where the decay of $\beta(\tau)$ is exactly compensated by the loss of the parametrix. Thus, there is no immediate smallness gain. The treatment of this term is in fact one of the key novel ingredients of the present paper. As it turns out, the operator $2\beta\mathcal{K}\hat{\mathcal{D}}$ displays a certain smallness property after all, but only after sufficiently many reiterations. In other words, we prove smallness of $(2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}})^{n_0}$ for n_0 large enough. This is achieved by a suitable splitting of the operator kernel into diagonal and

---

[3] Note that the *Fourier* variable corresponding to R is $\xi^{1/2}$ (this is the usual convention in the spectral theorem). Thus, the weight $|\cdot|^{1/2}$ corresponds to $\dot{H}^1(\mathbb{R}^3)$ and not to $\dot{H}^{1/2}(\mathbb{R}^3)$ as one might guess at first glance.

[4] In fact, this is only true for the term $\mathcal{K}\hat{\mathcal{D}}$. For $\hat{\mathcal{K}}$, we lose $\tau^{1+2\delta}$ due to the weight in the L^p-component of X, but we ignore this irrelevant technicality here.

off-diagonal components. For the off-diagonal component, we obtain smallness by the aforementioned smoothing property (at small frequencies) and by exploiting oscillations (for large frequencies). For the diagonal component, we distinguish further between small, intermediate, and large frequencies. At small and large frequencies, we can again use the smoothing of the kernel, and at intermediate frequencies, we obtain a smallness gain by exploiting the fact that we integrate over simplices. As a consequence, we infer the existence of $(1+2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}})^{-1}$, and equation (3.4) can be rewritten as

$$x = (1+2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}})^{-1}\mathcal{H}[\kappa^{-2}\mathcal{N}(x) - \beta^2\mathcal{K}x - \kappa^{-2}\hat{e}_2],$$

which is then solved by a fixed point argument.

3.2. The Distorted Fourier Transformation

Here we freely borrow facts from [11] and [1; 9]. We state the following:

THEOREM 3.1 (spectral theory for $\mathcal{L}$).

- *The Schrödinger operator $\mathcal{L}$ is self-adjoint on $L^2(0,\infty)$ with domain*

$$\operatorname{dom}(\mathcal{L}) = \{f \in L^2(0,\infty)\colon\ f, f' \in \mathrm{AC}[0,R]\ \forall R>0,\\ f(0)=0, \mathcal{L}f \in L^2(0,\infty)\},$$

  and its spectrum is given by $\sigma(\mathcal{L}) = \{\xi_d\} \cup [0,\infty)$ with $\xi_d < 0$. The continuous part of the spectrum is absolutely continuous, and the eigenfunction $\phi(R,\xi_d)$ associated to the eigenvalue ξ_d is smooth and decays exponentially as $R \to \infty$.
- *The spectral measure μ is of the form*

$$d\mu(\xi) = \frac{\delta_{\xi_d}(\xi)}{\|\phi(\cdot,\xi_d)\|^2_{L^2(0,\infty)}} + \rho(\xi)\,d\xi,$$

  where δ_{ξ_d} denotes the Dirac measure centered at ξ_d, and the function ρ satisfies $\rho(\xi)=0$ for $\xi<0$ and[5]

$$\rho(\xi) = \frac{1}{3\pi}\xi^{-1/2}[1+O(\xi^{1/5})], \quad 0<\xi\le 1,$$
$$\rho(\xi) = \frac{1}{\pi}\xi^{1/2}[1+O(\xi^{-1/2})], \quad \xi \ge 1,$$

  where the O-terms behave like symbols under differentiation.
- *There exists a unitary operator $\mathcal{U}: L^2(0,\infty) \to L^2(\sigma(\mathcal{L}), d\mu)$ that diagonalizes $\mathcal{L}$, that is, $\mathcal{U}\mathcal{L}f(\xi) = \xi\mathcal{U}f(\xi)$ for all $f \in \operatorname{dom}(\mathcal{L})$. The operator $\mathcal{U}$ is given explicitly by*

$$\mathcal{U}f(\xi) = \lim_{b\to\infty}\int_0^b \phi(R,\xi)f(R)\,dR,$$

  where the limit is understood in $L^2(\sigma(\mathcal{L}), d\mu)$. The function $\phi(\cdot,\xi)$ is smooth and (formally) satisfies $\mathcal{L}\phi(\cdot,\xi) = \xi\phi(\cdot,\xi)$ and $\phi(0,\xi)=0$, $\phi'(0,\xi)=1$.

[5] The conclusion for the asymptotics near $\xi=0$ is not optimal, and we can replace $O(\xi^{1/5})$ by $O(\xi^{1/2})$, but we will not need this.

- *For* $0 < \xi \lesssim 1$, *we have the asymptotics*

$$\phi(R,\xi) = \phi_0(R)[1 + O(\langle R\rangle^2 \xi)], \quad 0 \le R \le \xi^{-1/2},$$

$$\begin{aligned}\phi(R,\xi) = {} & \frac{\sqrt{3}}{2} e^{i\sqrt{\xi}R}[1 + O(\xi^{1/5}) + O(\langle R\rangle^{-3}\xi^{-1/2})] \\ & + \frac{\sqrt{3}}{2} e^{-i\sqrt{\xi}R}[1 + O(\xi^{1/5}) + O(\langle R\rangle^{-3}\xi^{-1/2})], \quad R \ge \xi^{-1/6},\end{aligned}$$

where all O*-terms behave like symbols under differentiation, and*

$$\phi_0(R) := \frac{R(1-(1/3)R^2)}{(1+(1/3)R^2)^{3/2}}.$$

In the case $\xi \gtrsim 1$, *we have*

$$\begin{aligned}\phi(R,\xi) = {} & \frac{1}{2i}\xi^{-1/2} e^{i\sqrt{\xi}R}[1 + O(\xi^{-1/2}) + O(\langle R\rangle^{-3}\xi^{-1/2})] \\ & - \frac{1}{2i}\xi^{-1/2} e^{-i\sqrt{\xi}R}[1 + O(\xi^{-1/2}) + O(\langle R\rangle^{-3}\xi^{-1/2})]\end{aligned}$$

for all $R \ge 0$ *with symbol behavior of all* O*-terms.*

- *The inverse map* $\mathcal{U}^{-1} : L^2(\sigma(\mathcal{L}), d\mu) \to L^2(0,\infty)$ *is given by*

$$\mathcal{U}^{-1} f(R) = \frac{\phi(R,\xi_d)}{\|\phi(\cdot,\xi_d)\|^2_{L^2(0,\infty)}} f(\xi_d) + \lim_{b\to\infty} \int_0^b \phi(R,\xi) f(\xi)\rho(\xi)\, d\xi,$$

where the limit is understood in $L^2(0,\infty)$.

For the following, it turns out to be more convenient to use a vector-valued version of $\mathcal{U}$, which we denote by $\mathcal{F}$ and call the "distorted Fourier transform." Thus, we identify $L^2(\sigma(\mathcal{L}), d\mu)$ with $\mathbb{C} \times L^2_\rho(0,\infty)$ and define the mapping $\mathcal{F} : L^2(0,\infty) \to \mathbb{C} \times L^2_\rho(0,\infty)$ by

$$\mathcal{F} f = \begin{pmatrix} \mathcal{U} f(\xi_d) \\ \mathcal{U} f|_{[0,\infty)} \end{pmatrix}.$$

According to Theorem 3.1, the inverse map $\mathcal{F}^{-1} : \mathbb{C} \times L^2_\rho(0,\infty) \to L^2(0,\infty)$ is then given by

$$\mathcal{F}^{-1}\begin{pmatrix} a \\ f \end{pmatrix} = \frac{\phi(R,\xi_d)}{\|\phi(\cdot,\xi_d)\|^2_{L^2(0,\infty)}} a + \lim_{b\to\infty}\int_0^b \phi(R,\xi) f(\xi)\rho(\xi)\, d\xi.$$

From now on we shall write

$$v(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\, d\xi,$$

where the functions $x(\tau,\xi)$ are the (distorted) Fourier coefficients associated with $v(\tau,\cdot)$. We write

$$\underline{x}(\tau,\xi) := \begin{pmatrix} x_d(\tau) \\ x(\tau,\xi) \end{pmatrix} = \mathcal{F}(v)(\tau,\xi), \qquad \underline{\xi} := \begin{pmatrix} \xi_d \\ \xi \end{pmatrix}.$$

Then precisely as in [9], we obtain the relation

$$(\hat{\mathcal{D}}^2 + \beta(\tau)\hat{\mathcal{D}} + \underline{\xi})\underline{x}(\tau,\xi) = \mathcal{R}(\tau,\underline{x}) + f(\tau,\underline{\xi}), \tag{3.5}$$

where we have

$$\mathcal{R}(\tau,\underline{x})(\xi) = (-4\beta(\tau)\mathcal{K}\hat{\mathcal{D}}\underline{x} - \beta^2(\tau)(\mathcal{K}^2 + [\mathcal{A},\mathcal{K}] + \mathcal{K} + \beta'\beta^{-2}\mathcal{K})\underline{x})(\xi) \tag{3.6}$$

with $\beta(\tau) = \frac{\dot{\kappa}(\tau)}{\kappa(\tau)}$,

$$f(\tau,\xi) = \mathcal{F}(\kappa^{-2}(\tau)[5(u_{2k-1}^4 - u_0^4)v + RN(u_{2k-1},v) + Re_2])(\xi), \tag{3.7}$$

and the operator

$$\hat{\mathcal{D}} = \partial_\tau + \beta(\tau)\mathcal{A}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_c \end{pmatrix}$$

with

$$\mathcal{A}_c = -2\xi\partial_\xi - \left(\frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}\right).$$

Finally, we observe that the "transference operator" $\mathcal{K}$ is given by the following type of expression:

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{dd} & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_{cc} \end{pmatrix}, \tag{3.8}$$

where the matrix elements are certain nonlocal Hilbert-type operators. Then we use the key observation, already made in [9], that the abstract problem (3.5) with $\mathcal{R}(\tau,\underline{x}) = 0$ can be solved *explicitly* for the continuous part $x(\tau,\xi)$. In fact, we have the relation

$$x(\tau,\xi) = \int_\tau^\infty H_c(\tau,\sigma,\xi) f\left(\sigma, \frac{\kappa^2(\tau)}{\kappa^2(\sigma)}\xi\right) d\sigma \tag{3.9}$$

with

$$\begin{aligned} &H_c(\tau,\sigma,\xi) \\ &\quad = \xi^{-1/2}\frac{\kappa^{3/2}(\tau)}{\kappa^{3/2}(\sigma)}\frac{\rho^{1/2}((\kappa^2(\tau)/\kappa^2(\sigma))\xi)}{\rho^{1/2}(\xi)} \\ &\qquad \times \sin\left[\kappa(\tau)\xi^{1/2}\int_\tau^\sigma \kappa^{-1}(u)\,du\right]. \end{aligned} \tag{3.10}$$

Furthermore, letting (as in [1])

$$\hat{\mathcal{D}}_c := \partial_\tau + \beta(\tau)\mathcal{A}_c,$$

we compute from the above parametrix representation that

$$\hat{\mathcal{D}}_c x(\tau,\xi) = \int_\tau^\infty \hat{H}_c(\tau,\sigma,\xi) f\left(\sigma, \frac{\kappa^2(\tau)}{\kappa^2(\sigma)}\xi\right) d\sigma \tag{3.11}$$

with

$$\begin{aligned} &\hat{H}_c(\tau,\sigma,\xi) \\ &\quad = \frac{\kappa^{3/2}(\tau)}{\kappa^{3/2}(\sigma)}\frac{\rho^{1/2}((\kappa^2(\tau)/\kappa^2(\sigma))\xi)}{\rho^{1/2}(\xi)}\cos\left[\kappa(\tau)\xi^{1/2}\int_\tau^\sigma \kappa^{-1}(u)\,du\right]. \end{aligned} \tag{3.12}$$

We can immediately formulate the following:

LEMMA 3.2. *Denoting $\omega_\nu(\tau) := \tau^{1+1/\nu}$ and letting $\kappa(\tau) = \lambda(t(\tau))$ as in the preceding, we have the kernel bounds*

$$|H_c(\tau,\sigma,\xi)| \lesssim \min\left\{\omega_\nu\left(\frac{\tau}{\sigma}\right)\xi^{-1/2},\ \nu\omega_\nu\left(\frac{\tau}{\sigma}\right)\sigma\right\},$$
$$|\hat{H}_c(\tau,\sigma,\xi)| \lesssim \omega_\nu\left(\frac{\tau}{\sigma}\right).$$

Indeed, this is a simple consequence of the fact that

$$\kappa(\tau) \sim \tau^{1+1/\nu}.$$

For the discrete part $x_d(\tau)$ of the solution of (3.5) with $\mathcal{R}(\tau,\underline{x}) = 0$, we obtain the implicit equation

$$x_d(\tau) = \int_\tau^\infty H_d(\tau,\sigma)\tilde{f}_d(\sigma)\,d\sigma, \quad H_d(\tau,\sigma) = -\frac{1}{2}|\xi_d|^{-1/2}e^{-|\xi_d|^{1/2}|\tau-\sigma|},$$
$$\tilde{f}_d(\sigma) = f_d(\sigma) - \beta_\nu(\sigma)\partial_\sigma x_d(\sigma). \tag{3.13}$$

In order to solve problem (3.5) via a fixed point argument, we shall utilize the functional framework developed in [1].

DEFINITION 3.1. For the continuous spectral part $x(\tau,\xi)$, we shall use the following norms:

$$\|f\|_X := \|(|\cdot|\langle\cdot\rangle^{-1})^{1/2-\delta}f\|_{L^p(0,\infty)} + \||\cdot|^{1/2}\langle\cdot\rangle^{1/8}f\|_{L^2_\rho(0,\infty)},$$
$$\|f\|_Y := \|f\|_{L^p(0,\infty)} + \|\langle\cdot\rangle^{1/8}f\|_{L^2_\rho(0,\infty)},$$

and

$$\|u\|_{\mathcal{X}^\beta} := \sup_{\tau>\tau_0}\tau^\beta\|u(\tau,\cdot)\|_X, \qquad \|u\|_{\mathcal{Y}^\beta} := \sup_{\tau>\tau_0}\tau^\beta\|u(\tau,\cdot)\|_Y.$$

Then for the vector valued function $\underline{x}(\tau,\xi)$, we put

$$\|\underline{x}\|_{\mathcal{X}^{\alpha,\beta}} := \sup_{\tau>\tau_0}\tau^\alpha|x_d(\tau)| + \|x(\tau,\cdot)\|_{\mathcal{X}^\beta},$$
$$\|\underline{x}\|_{\mathcal{Y}^{\alpha,\beta}} := \sup_{\tau>\tau_0}\tau^\alpha|x_d(\tau)| + \|x(\tau,\cdot)\|_{\mathcal{Y}^\beta}.$$

We remark that, in the following, $\delta > 0$ is assumed to be small, and $p > 1$ is assumed to be large, depending on δ.

To proceed, we first need to study the linear inhomogeneous problem

$$(\hat{\mathcal{D}}^2 + \beta(\tau)\hat{\mathcal{D}} + \underline{\xi})\underline{x}(\tau,\xi) = \underline{f}(\tau,\underline{\xi}). \tag{3.14}$$

To prepare for this task, we have the following:

LEMMA 3.3. *Let $\kappa(\tau) = \lambda(t(\tau))$ as in the preceding. Let $a, b, \gamma \in \mathbb{R}$, $q \in (1,\infty)$, and*

$$\alpha > 1 + 2\left(\frac{1}{q} + \max(|a|, |a+b|)\right)\left(1 + \frac{1}{\nu}\right) - \gamma.$$

Suppose further that the operator $\mathcal{B}$ is given by

$$\mathcal{B}x(\tau,\xi) = \int_\tau^\infty B(\tau,\sigma,\xi)x(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma,$$

where $\omega(\tau,\sigma) := \kappa(\tau)\kappa^{-1}(\sigma)$, and the kernel B satisfies

$$|B(\tau,\sigma,\xi)| \lesssim \sigma^{-\gamma}$$

for all $0 < \tau_0 \le \tau \le \sigma$, $\xi \ge 0$. Then we have the bound

$$\|\mathcal{B}x(\tau,\cdot)| \cdot |^a\langle\cdot\rangle^b\|_{L^q(0,\infty)} \lesssim \tau^{-\alpha-\gamma+1}\sup_{\sigma>\tau}\sigma^\alpha\|x(\sigma,\cdot)| \cdot |^a\langle\cdot\rangle^b\|_{L^q(0,\infty)}.$$

Proof. First, we consider the case $a = b = 0$. By Hölder's inequality we obtain

$$\begin{aligned}|\mathcal{B}x(\tau,\xi)| &\lesssim \left(\int_\tau^\infty \sigma^{-1-\epsilon}\,d\sigma\right)^{1/q'} \\ &\quad\times \left(\int_\tau^\infty |\sigma^{(1/q')(1+\epsilon)-\gamma}x(\sigma,\omega(\tau,\sigma)^2\xi)|^q\,d\sigma\right)^{1/q}\end{aligned} \tag{3.15}$$

for any $\epsilon > 0$. This implies

$$\begin{aligned}\|\mathcal{B}x(\tau,\cdot)\|_{L^q} &\lesssim \tau^{-\epsilon/q'}\left(\int_\tau^\infty \sigma^{q((1/q')(1+\epsilon)-\gamma-\alpha)}\|\sigma^\alpha x(\sigma,\omega(\tau,\sigma)^2\cdot)\|_{L^q}^q\right)^{1/q} \\ &\lesssim \tau^{-\epsilon/q'}\Big[\sup_{\sigma>\tau}\sigma^\alpha\|x(\sigma,\cdot)\|_{L^q}\Big] \\ &\quad\times\left(\int_\tau^\infty \sigma^{q((1/q')(1+\epsilon)-\gamma-\alpha)}\omega_\nu\left(\frac{\tau}{\sigma}\right)^{-2}d\sigma\right)^{1/q} \\ &\lesssim \tau^{-\alpha-\gamma+1}\sup_{\sigma>\tau}\sigma^\alpha\|x(\sigma,\cdot)\|_{L^q},\end{aligned}$$

provided that $q(1/q' - \gamma - \alpha) + 2(1 + \frac{1}{\nu}) < -1$ and $\epsilon > 0$ is chosen sufficiently small. The case for general a, b follows immediately by noting that

$$\|x(\sigma,\omega(\tau,\sigma)^2\cdot)| \cdot |^a\langle\cdot\rangle^b\|_{L^q}^q \lesssim \omega_\nu\left(\frac{\tau}{\sigma}\right)^{-2-2q\max(|a|,|a+b|)}\|x(\sigma,\cdot)| \cdot |^a\langle\cdot\rangle^b\|_{L^q}^q,$$

which entails the integrability condition

$$q\left(\frac{1}{q'} - \gamma - \alpha\right) + (2 + 2q\max(|a|,|a+b|))\left(1 + \frac{1}{\nu}\right) < -1. \qquad \square$$

We can now solve (3.14) by the following:

LEMMA 3.4. *Let $\alpha_d \in \mathbb{R}$ and $\alpha_c > 1 + \frac{3}{4}(1 + \frac{1}{\nu})$. Then given $\underline{f} \in \mathcal{Y}^{\alpha_d,\alpha_c}$, there exists a solution $\underline{x} \in \mathcal{X}^{\alpha_d,\alpha_c-1-2\delta}$ for (3.14). Denoting this solution by*

$$\underline{x} =: \begin{pmatrix}\mathcal{H}_d f_d \\ \mathcal{H}_c f\end{pmatrix} =: \mathcal{H}\underline{f},$$

we have the estimates

$$\|\mathcal{H}\underline{x}\|_{\mathcal{X}^{\alpha_d,\alpha_c-1-2\delta}} \lesssim \nu^{2\delta}\|\underline{x}\|_{\mathcal{Y}^{\alpha_d,\alpha_c}},$$
$$\|\hat{\mathcal{D}}\mathcal{H}\underline{x}\|_{\mathcal{Y}^{\alpha_d,\alpha_c-1}} \lesssim \|\underline{x}\|_{\mathcal{Y}^{\alpha_d,\alpha_c}},$$

where $\delta > 0$ is the parameter in Definition 3.1.

Proof. We can explicitly define the continuous spectral part $x(\tau,\xi)$ via (3.9), (3.10), and the discrete part $x_d(\tau)$ implicitly via (3.13). Combining (3.11), (3.12), and Lemmas 3.2 and 3.3, we get $\|\hat{\mathcal{D}}_c\mathcal{H}_c x\|_{\mathcal{Y}^{\alpha_c-1}} \lesssim \|x\|_{\mathcal{Y}^{\alpha_c}}$. On the other hand, using

$$|H_c(\tau,\sigma,\xi)| \lesssim \nu^{2\delta}\omega_\nu\left(\frac{\tau}{\sigma}\right)\sigma^{2\delta}\xi^{-1/2+\delta},$$

which follows from Lemma 3.2 by interpolation, and Lemma 3.3, we have

$$\||\cdot|^{1/2-\delta}\mathcal{H}_c x(\tau,\cdot)\|_{L^p} \lesssim \nu^{2\delta}\tau^{-\alpha_c+1+2\delta}\|x\|_{\mathcal{Y}^{\alpha_c}}.$$

Further, Lemma 3.3 gives

$$\||\cdot|^{1/2}\langle\cdot\rangle^{1/8}\mathcal{H}_c x(\tau,\cdot)\|_{L^2_\rho} \lesssim \||\cdot|^{1/2}\mathcal{H}_c x(\tau,\cdot)\|_Y \lesssim \tau^{-\alpha_c+1}\|x\|_{\mathcal{Y}^{\alpha_c}}.$$

This completes the desired bounds for the continuous part $x(\tau,\cdot)$. To control the discrete part, we observe that (see (3.13))

$$\sup_{\tau>\tau_0}\tau^{\alpha_d}\left|\int_\tau^\infty H_d(\tau,\sigma)f(\sigma)\,d\sigma\right| \lesssim \sup_{\tau>\tau_0}\tau^{\alpha_d}|f(\tau)|,$$
$$\sup_{\tau>\tau_0}\tau^{\alpha_d}\left|\partial_\tau\int_\tau^\infty H_d(\tau,\sigma)f(\sigma)\,d\sigma\right| \lesssim \sup_{\tau>\tau_0}\tau^{\alpha_d}|f(\tau)|.$$

In light of the fact that

$$\beta_\nu(\tau) \sim \frac{1}{\tau},$$

the implicit equation

$$x_d(\tau) = \int_\tau^\infty H_d(\tau,\sigma)(f_d(\sigma) - \beta_\nu(\sigma)\partial_\sigma x_d(\sigma))\,d\sigma$$

is then solved via straightforward iteration, provided that $\tau > \tau_0$ with τ_0 sufficiently large, and the limit satisfies

$$\sup_{\tau>\tau_0}\tau^{\alpha_d}|x_d(\tau)| \lesssim \sup_{\tau>\tau_0}\tau^{\alpha_d}|f_d(\tau)|.$$

This completes the proof of the lemma. □

3.3. Solving the Main Equation

Abstractly speaking, equation (3.5) is of the form

$$Lx = x_0 + Ax + F(x), \tag{3.16}$$

where x_0 is a given element in a Banach space X, A is a bounded linear operator on X, and F is a nonlinear mapping from X to X. Furthermore, in light of Lemma 3.4, the operator L is linear and invertible with bounded inverse H. The goal is to find a solution $x \in X$. Compared to equation (3.5), this is a slightly simplified model case, but it captures the essentials. By applying H, we rewrite equation (3.16) as

$$x = Hx_0 + HAx + HF(x). \tag{3.17}$$

The point is to find a method to solve equation (3.17), even if the operator norm of HA is not small, that is, if one cannot apply the Banach fixed point theorem directly. The idea is to perform an iteration procedure. This means that one first proves the existence of $(1 - HA)^{-1}$, which amounts to showing the norm-convergence of the Neumann series

$$\sum_{n=0}^{\infty} (HA)^n.$$

Thus, we have to consider $\|(HA)^n\|$ and prove an appropriate bound that makes the Neumann series convergent. The point is, of course, that only very large n are relevant here, and hence, we may exploit a smallness property, which only shows up after sufficiently many iterations. This is exactly the idea used to solve Volterra equations. Once we have obtained the existence of $(1 - HA)^{-1}$, we rewrite equation (3.17) as

$$x = (1 - HA)^{-1}Hx_0 + (1 - HA)^{-1}HF(x), \tag{3.18}$$

and if the nonlinearity F is suitable, it is possible to solve equation (3.18) by a fixed point argument. This is, roughly speaking, the program we are going to follow in order to solve equation (3.5).

3.4. Time Decay of the Inhomogeneous Term

According to the program outlined at the beginning of Section 3.3, we first focus on the difficult linear terms on the right-hand side of equation (3.5). In fact, the linear term with the least decay is the one containing the derivative $\hat{\mathcal{D}}\underline{x}$ since the other one comes with a prefactor of τ^{-2}, which is sufficient to treat it directly with the Banach fixed point theorem. Thus, for the moment, we focus on the equation

$$[\hat{\mathcal{D}}^2 + \beta\hat{\mathcal{D}} + \underline{\xi}]\underline{x} = \underline{e} - 2\beta\mathcal{K}\hat{\mathcal{D}}\underline{x}, \tag{3.19}$$

where

$$\underline{e}(\tau, \xi) := \kappa(\tau)^{-2}\mathcal{F}[|\cdot|\tilde{\chi}(\tau, \cdot)e_2(\tau, \cdot)](\xi)$$

is the inhomogeneous term on the right-hand side of equation (3.5). The first step, however, is to identify suitable spaces in which we intend to solve equation (3.19). It is clear that we have to solve for the pair $(\underline{x}, \hat{\mathcal{D}}\underline{x})$ since both terms $\underline{x}$ and $\hat{\mathcal{D}}\underline{x}$ appear on the right-hand side of equation (3.5). The estimates in Lemma 3.4 suggest to place $(\underline{x}, \hat{\mathcal{D}}\underline{x})$ in $\mathcal{X}^{\alpha_1,\beta_1} \times \mathcal{Y}^{\alpha_2,\beta_2}$ where the decay rates α_j and β_j, $j = 1, 2$, are dictated by the inhomogeneous term $\underline{e}$. For the latter, we have the following:

LEMMA 3.5. *We have the estimates*

$$|\kappa(\tau)^{-2}\mathcal{U}(|\cdot|\tilde{\chi}(\tau,\cdot)e_2(\tau,\cdot))(\xi_d)| \le C_\nu \tau^{-3+(1/2)(1+1/\nu)+\epsilon}$$

and

$$|\kappa(\tau)^{-2}\mathcal{U}(|\cdot|\tilde{\chi}(\tau,\cdot)e_2(\tau,\cdot))(\xi)| \le C_\nu \tau^{-3+(1/2)(1+1/\nu)+\epsilon}\langle\xi\rangle^{-1}$$

for all $\tau \gtrsim 1$, $\xi \ge 0$ *and any fixed* $\epsilon > 0$.

Proof. According to Proposition 2.1, we have the bound

$$\begin{aligned}|\kappa(\tau)^{-2}\tilde{\chi}(\tau,R)Re_2(\tau,R)| &= \left|\kappa(\tau)^{-2}\chi\left(\frac{R}{\nu\tau}\right)Re_2\left(\frac{\nu\tau}{\kappa(\tau)},\frac{R}{\kappa(\tau)}\right)\right| \\ &\le C_\nu\kappa(\tau)^{1/2}\tau^{-4}\langle R\rangle^{\epsilon}\end{aligned}$$

for some fixed (but arbitrary) $\epsilon > 0$ and with symbol behavior of the derivatives of degree at most two. If $0 < \xi \lesssim 1$ or $\xi = \xi_d$, then we have from Theorem 3.1 the bound $|\phi(R,\xi)| \lesssim 1$ for all $R \ge 0$, and thus,

$$\begin{aligned}|\kappa(\tau)^{-2}\mathcal{U}(|\cdot|\tilde{\chi}(\tau,\cdot)e_2(\tau,\cdot))(\xi)| &\le C_\nu\kappa(\tau)^{1/2}\tau^{-4}\int_0^{3\nu\tau}|\phi(R,\xi)|\langle R\rangle^{\epsilon}\,dR \\ &\le C_\nu\tau^{-3+(1/2)(1+1/\nu)+\epsilon}.\end{aligned}$$

If $\xi \gtrsim 1$, then we exploit the oscillatory behavior of $\phi(R,\xi)$ given in Theorem 3.1 and perform one integration by parts to gain an additional factor $\xi^{-1/2}$. This yields the bound

$$|\kappa(\tau)^{-2}\mathcal{U}(|\cdot|\tilde{\chi}(\tau,\cdot)e_2(\tau,\cdot))(\xi)| \le C_\nu\tau^{-4+(1/2)(1+1/\nu)+\epsilon}\xi^{-1},$$

which implies the claim. □

LEMMA 3.6. *Let* $\alpha, \beta < 3 - \frac{1}{2}(1+\frac{1}{\nu})$. *Then the function* $\underline{e}$ *belongs to the space* $\mathcal{Y}^{\alpha,\beta}$.

Proof. The stated time decay (which implies the conditions on α and β) is an immediate consequence of Lemma 3.5. Based on the estimate in Lemma 3.5, it therefore suffices to prove that $\langle\cdot\rangle^{-1} \in Y$. It is clear that $\langle\cdot\rangle^{-1} \in L^p(0,\infty)$ for p large, and for the L^2-based component, we distinguish between small and large ξ. For small ξ, we recall that $\rho(\xi) \simeq \xi^{-1/2}$ (Theorem 3.1), which is integrable near 0, and for large ξ, we have

$$|\langle\xi\rangle^{-2}\langle\xi\rangle^{1/4}\rho(\xi)| \lesssim \langle\xi\rangle^{-5/4}$$

since $\rho(\xi) \simeq \xi^{1/2}$ for $\xi \gtrsim 1$ again by Theorem 3.1. □

Lemma 3.6 shows that $\underline{e} \in Y^{7/3+\epsilon,7/3+\epsilon}$ for a sufficiently small $\epsilon > 0$, provided that ν is sufficiently large, which we assume from now on. Consequently, Lemma 3.4 yields

$$\mathcal{H}\underline{e} \in \mathcal{X}^{7/3+\epsilon,4/3+\epsilon-2\delta}, \qquad \hat{\mathcal{D}}\mathcal{H}\underline{e} \in \mathcal{Y}^{7/3+\epsilon,4/3+\epsilon},$$

and thus, if we choose $\mathcal{X}^{4/3-2\delta,4/3-2\delta} \times \mathcal{Y}^{4/3,4/3}$ as our solution space, then we even obtain smallness for the inhomogeneous term, that is,

$$\|\mathcal{H}\underline{e}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}} \lesssim \nu^{2\delta}\tau_0^{-\epsilon}, \qquad \|\hat{\mathcal{D}}\mathcal{H}\underline{e}\|_{\mathcal{Y}^{4/3,4/3}} \lesssim \tau_0^{-\epsilon}. \tag{3.20}$$

By applying the operator $\mathcal{H}$ followed by $\hat{\mathcal{D}}$ to equation (3.19) we obtain

$$\hat{\mathcal{D}}\underline{x} = \hat{\mathcal{D}}\mathcal{H}\underline{e} - 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}}\underline{x}. \tag{3.21}$$

Solving this equation for $\hat{\mathcal{D}}\underline{x} \in \mathcal{Y}^{4/3,4/3}$ amounts to proving the existence (and boundedness) of the operator $(1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K})^{-1}$ on $\mathcal{Y}^{4/3,4/3}$. As in [11] and [1], we write

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{dd} & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_{cc} \end{pmatrix}$$

for the matrix components of $\mathcal{K}$. With this notation we have

$$\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K} = \begin{pmatrix} \hat{\mathcal{D}}_d\mathcal{H}_d & 0 \\ 0 & \hat{\mathcal{D}}_c\mathcal{H}_c \end{pmatrix} \begin{pmatrix} \beta\mathcal{K}_{dd} & \beta\mathcal{K}_{dc} \\ \beta\mathcal{K}_{cd} & \beta\mathcal{K}_{cc} \end{pmatrix}, \tag{3.22}$$

where $\hat{\mathcal{D}}_d$ is just ∂_τ. We start by inverting the diagonal elements of $1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K}$. Since $\mathcal{K}_{dd}$ is a linear map from $\mathbb{C}$ to $\mathbb{C}$, it is just given by a number (to be precise, we have $\mathcal{K}_{dd}a = -\frac{3}{2}a$ for all $a \in \mathbb{C}$; see [1]). Furthermore, by Lemma 3.4 we have

$$|\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dd}x_d(\tau)| \lesssim \tau^{-1}|x_d(\tau)| \le \tau_0^{-1}|x_d(\tau)|$$

since $\beta(\tau) \simeq \tau^{-1}$. This shows that $(1 + 2\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dd})^{-1}$ exists.

3.5. Structure and Properties of $\mathcal{K}$

In order to proceed, we need more detailed information on the operator $\mathcal{K}$. The operator $\mathcal{K}$ has been analyzed in detail in [11] and [1]. It is easy to see that $\mathcal{K}_{cd} : \mathbb{C} \to L^2_\rho(0, \infty)$ is given by

$$\mathcal{K}_{cd}a(\xi) = \frac{a}{\|\phi(\cdot, \xi_d)\|^2_{L^2(0,\infty)}} \int_0^\infty \phi(R, \xi)[R\partial_R - 1]\phi(R, \xi_d)\, dR$$

with ϕ from Theorem 3.1. For $\mathcal{K}_{dc}$ and $\mathcal{K}_{cc}$, we recall the following result.

THEOREM 3.7. *The operator $\mathcal{K}_{cc} : L^2_\rho(0, \infty) \to L^2_\rho(0, \infty)$ is given by*

$$\mathcal{K}_{cc}f(\xi) = \int_0^\infty K_{cc}(\xi, \eta) f(\eta)\, d\eta,$$

where the kernel K_{cc} is of the form

$$K_{cc}(\xi, \eta) = \frac{\rho(\eta)}{\xi - \eta} F(\xi, \eta)$$

with a symmetric function $F \in C^2((0, \infty) \times (0, \infty))$. Furthermore, for any $N \in \mathbb{N}$, F satisfies the bounds

$$|F(\xi, \eta)| \le C_N \begin{cases} \xi + \eta, & \xi + \eta \le 1, \\ (\xi + \eta)^{-1}(1 + |\xi^{1/2} - \eta^{1/2}|)^{-N}, & \xi + \eta \ge 1, \end{cases}$$

$$
\begin{aligned}
&|\partial_\xi F(\xi,\eta)| + |\partial_\eta F(\xi,\eta)| \\
&\quad \le C_N \begin{cases} 1, & \xi+\eta \le 1, \\ (\xi+\eta)^{-3/2}(1+|\xi^{1/2}-\eta^{1/2}|)^{-N}, & \xi+\eta \ge 1, \end{cases}
\end{aligned}
$$

$$
\max_{j+k=2} |\partial_\xi^j \partial_\eta^k F(\xi,\eta)| \le C_N \begin{cases} (\xi+\eta)^{-1/2}, & \xi+\eta \le 1, \\ (\xi+\eta)^{-2}(1+|\xi^{1/2}-\eta^{1/2}|)^{-N}, & \xi+\eta \ge 1. \end{cases}
$$

Finally, the operator $\mathcal{K}_{dc} : L^2_\rho(0,\infty) \to \mathbb{C}$ *is of the form*

$$
\mathcal{K}_{dc} f = \int_0^\infty K_{dc}(\xi) f(\xi) \rho(\xi)\, d\xi
$$

with a smooth and rapidly decreasing function K_{dc}.

Proof. See [11], Theorem 5.1. □

As a consequence of Theorem 3.7, we have the following mapping properties of $\mathcal{K}_{cc}$ and the commutator $[\mathcal{A}_c, \mathcal{K}_{cc}]$, where we recall that

$$
\mathcal{A}_c f(\xi) = -2\xi f'(\xi) - \left(\frac{5}{2} + \frac{\xi\rho'(\xi)}{\rho(\xi)}\right) f(\xi).
$$

PROPOSITION 3.8. *We have the bounds*

$$
\begin{aligned}
\|\mathcal{K}_{cc} f\|_X &\lesssim \|f\|_X, & \|[\mathcal{A}_c, \mathcal{K}_{cc}] f\|_X &\lesssim \|f\|_X, \\
\|\mathcal{K}_{cc} f\|_Y &\lesssim \|f\|_X, & \|[\mathcal{A}_c, \mathcal{K}_{cc}] f\|_Y &\lesssim \|f\|_X, \\
\|\mathcal{K}_{cc} g\|_Y &\lesssim \|g\|_Y, & \|[\mathcal{A}_c, \mathcal{K}_{cc}] g\|_Y &\lesssim \|g\|_Y
\end{aligned}
$$

for all $f \in X$ *and* $g \in Y$.

Proof. This follows from the representation in Theorem 3.7 but requires some harmonic analysis. For the proof, we refer the reader to [1], Propositions 5.5 and 5.8. We remark that the bounds for $[\mathcal{A}_c, \mathcal{K}_{cc}]$ can be obtained in the same fashion as the ones for $\mathcal{K}_{cc}$ by noting that the kernel of $[\mathcal{A}_c, \mathcal{K}_{cc}]$ is of the form

$$
\frac{\rho(\eta)}{\xi-\eta} \tilde{F}(\xi,\eta)
$$

with

$$
\tilde{F}(\xi,\eta) = \frac{\eta\rho'(\eta)}{\rho(\eta)} F(\xi,\eta) + [\xi\partial\xi + \eta\partial_\eta] F(\xi,\eta);
$$

see [11], p. 52. □

In what follows, it is necessary to split the operator $\mathcal{K}_{cc}$ into a diagonal and an off-diagonal part. Thus, for $n_0 \in \mathbb{N}$, we set

$$
K^d_{n_0}(\xi,\eta) = \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) K_{cc}(\xi,\eta),
$$

where χ is a standard smooth cut-off with $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Furthermore, we denote by $\mathcal{K}^d_{n_0}$ the corresponding operator and write

$$
\mathcal{K}^{nd}_{n_0} := \mathcal{K}_{cc} - \mathcal{K}^d_{n_0}
$$

for the off-diagonal part. First, we establish a smoothing estimate for the off-diagonal part at small frequencies.

Lemma 3.9. *With p from Definition 3.1, we have the bounds*

$$\||\cdot|^{-1/(2p)}\langle\cdot\rangle^{1/(2p)}\mathcal{K}_{n_0}^{nd} f\|_Y \lesssim n_0^2\|f\|_Y$$

and

$$\||\cdot|^{-1/(2p)}\langle\cdot\rangle^{1/(2p)}\mathcal{K}_{n_0}^{nd} f\|_X \lesssim n_0^4\|f\|_X$$

for all $n_0 \in \mathbb{N}$, $n_0 \geq 100$.

Proof. Explicitly, the operator $\mathcal{K}_{n_0}^{nd}$ is given by

$$\mathcal{K}_{n_0}^{nd} f(\xi) = \int_0^\infty K_{n_0}^{nd}(\xi,\eta) f(\eta)\,d\eta$$

with

$$K_{n_0}^{nd}(\xi,\eta) = \left[1-\chi\left(n_0\left(\frac{\xi}{\eta}-1\right)\right)\right]K_{cc}(\xi,\eta).$$

Note that on the support of $K_{n_0}^{nd}$ we have $|\frac{\xi}{\eta}-1| \geq 1/n_0$, and thus, either $\eta \leq \frac{1}{1+1/n_0}\xi$ or $\eta \geq \frac{1}{1-1/n_0}\xi$. In the former case, we obtain

$$\xi - \eta \geq \left(1-\frac{1}{1+1/n_0}\right)\xi \simeq \frac{1}{n_0}\xi \gtrsim \frac{1}{n_0}(\xi+\eta),$$

and in the latter, $\eta - \xi \gtrsim (1/n_0)(\xi+\eta)$. Thus, we have $|\xi-\eta| \gtrsim (1/n_0)|\xi+\eta|$, and from Theorem 3.7 we obtain the bound $|K_{n_0}^{nd}(\xi,\eta)| \lesssim n_0\eta^{-1/2}$, provided that $\xi+\eta \leq 1$. If $\xi+\eta \geq 1$, then we note that, as before, $|\xi^{1/2}-\eta^{1/2}| \gtrsim n_0^{-1/2}\xi^{1/2}$ and also $|\xi^{1/2}-\eta^{1/2}| \gtrsim n_0^{-1/2}\eta^{1/2}$. Thus, from Theorem 3.7 we obtain the bound $|K_{n_0}^{nd}(\xi,\eta)| \lesssim n_0^2\langle\xi\rangle^{-1}\langle\eta\rangle^{-2}$. We conclude that

$$\begin{aligned}|\tilde{K}(\xi,\eta)| &:= |\xi^{-1/(2p)}\langle\xi\rangle^{1/(2p)}K_{n_0}^{nd}(\xi,\eta)| \\ &\lesssim n_0^2\xi^{-1/(2p)}\langle\xi\rangle^{1/(2p)}\langle\xi\rangle^{-1}\eta^{-1/2}\langle\eta\rangle^{1/2}\langle\eta\rangle^{-3/2}\end{aligned}$$

for all $\xi,\eta \geq 0$, and thus,

$$\|\tilde{K}\|_{L^p(0,\infty)L^{p'}(0,\infty)} \lesssim n_0^2,$$

which implies

$$\||\cdot|^{-1/(2p)}\langle\cdot\rangle^{1/(2p)}\mathcal{K}_{n_0}^{nd} f\|_{L^p(0,\infty)} \lesssim n_0^2\|f\|_{L^p(0,\infty)}.$$

For the weighted L^2-component, we estimate

$$\begin{aligned}|\tilde{K}(\xi,\eta)| &:= |\xi^{-1/(2p)}\langle\xi\rangle^{1/(2p)}\langle\xi\rangle^{1/8}\rho(\xi)^{1/2}K_{n_0}^{nd}(\xi,\eta)\langle\eta\rangle^{-1/8}\rho(\eta)^{-1/2}| \\ &\lesssim n_0^2\xi^{-1/(2p)-1/4}\langle\xi\rangle^{1/(2p)+1/4}\langle\xi\rangle^{-5/8}\eta^{-1/4}\langle\eta\rangle^{1/4}\langle\eta\rangle^{-15/8},\end{aligned}$$

which implies $\|\tilde{K}\|_{L^2(0,\infty)L^2(0,\infty)} \lesssim n_0^2$, and the claim follows.

For the second bound, we proceed completely analogously with the exception that the L^2-component gets estimated in a slightly different way and we use the stronger bound $|K(\xi,\eta)| \lesssim n_0^4 \langle\xi\rangle^{-2}\langle\eta\rangle^{-2}$ from Theorem 3.7. With

$$\tilde{K}(\xi,\eta) := \xi^{-1/(2p)}\langle\xi\rangle^{1/(2p)}\xi^{1/2-\delta}\langle\xi\rangle^{-1/2+\delta}K_{n_0}^{nd}(\xi,\eta)\eta^{-1/2+\delta}\langle\eta\rangle^{1/2-\delta}$$

we obtain $\|\tilde{K}\|_{L^p(0,\infty)L^{p'}(0,\infty)} \lesssim n_0^4$, provided that $(-1+\delta)p' > -1$, which we may safely assume since p is supposed to be large. For the L^2-component in the second bound, we consider the kernel

$$\tilde{K}(\xi,\eta) := \xi^{-1/(2p)}\langle\xi\rangle^{1/(2p)}\xi^{1/2}\langle\xi\rangle^{1/8}\rho(\xi)^{1/2}K_{n_0}^{nd}(\xi,\eta)\eta^{-1/2+\delta}\langle\eta\rangle^{1/2-\delta},$$

which satisfies the bound

$$|\tilde{K}(\xi,\eta)| \lesssim n_0^4\xi^{-1/(2p)-1/4}\langle\xi\rangle^{1/(2p)+1/4}\langle\xi\rangle^{-9/8}\eta^{-1+\delta}\langle\eta\rangle^{1-\delta}\langle\eta\rangle^{-2}.$$

This implies the bound $\|\tilde{K}\|_{L^2(0,\infty)L^{p'}(0,\infty)} \lesssim n_0^4$, which concludes the proof. □

We also need a corresponding smoothing property for the diagonal part. Here it is crucial for the following that the obtained bound does not depend on n_0. We start with an estimate for a truncated version of the Hilbert transform.

LEMMA 3.10. *Let H_n, $n \in \mathbb{N}$, be given by*

$$H_n f(\xi) := \int_0^\infty \frac{\chi(n(\xi/\eta - 1))}{\xi - \eta} f(\eta)\,d\eta, \quad \xi \ge 0,$$

where χ is a smooth cut-off function satisfying $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Then H_n extends to a bounded operator on $L^q(0,\infty)$ for any $q \in (1,\infty)$, and we have

$$\|H_n f\|_{L^q(0,\infty)} \lesssim \|f\|_{L^q(0,\infty)}$$

for all $f \in L^q(0,\infty)$ and all $n \ge 100$.

Proof. We use

$$\chi\left(n\left(\frac{\xi}{\eta} - 1\right)\right) = 1 + \frac{n(\xi-\eta)}{\eta}\int_0^1 \chi'\left(ns\left(\frac{\xi}{\eta} - 1\right)\right)ds$$

to decompose the kernel according to

$$\frac{\chi(n(\xi/\eta - 1))}{\xi - \eta} = \frac{1}{\xi - \eta} + \frac{n}{\eta}O(1). \tag{3.23}$$

Let $I_{jk}^n := [2^{j-1} + 3\frac{k-1}{n}2^{j-1}, 2^{j-1} + 3\frac{k}{n}2^{j-1}]$ and observe that

$$[2^{j-1}, 2^{j+1}] = \bigcup_{k=1}^n I_{jk}^n.$$

Furthermore, set $\Delta_n := \{(\xi,\eta) \in [0,\infty)^2 : \ \chi(n(\frac{\xi}{\eta} - 1)) \ne 0\}$. Since we have $|\xi - \eta| \le \frac{2}{n}\eta$ for all $(\xi,\eta) \in \Delta_n$, $\eta \in I_{jk}^n$ implies $\xi \in \tilde{I}_{jk}^n$ for all $(\xi,\eta) \in \Delta_n$, where

$\tilde{I}^n_{jk}$ are suitable (overlapping) intervals with $|\tilde{I}^n_{jk}| \simeq 2^j/n$ and $[2^{j-2}, 2^{j+2}] = \bigcup_{k=1}^n \tilde{I}^n_{jk}$. As a consequence, we infer

$$\Delta_n \subset \bigcup_{j\in\mathbb{Z}} \bigcup_{k=1}^n \tilde{I}^n_{jk} \times I^n_{jk}$$

for any $n \geq 100$. Thus, we obtain

$$\begin{aligned} H_n f(\xi) &= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{k=1}^n \int_0^\infty \frac{\chi(n(\xi/\eta - 1))}{\xi - \eta} 1_{I^n_{jk}}(\eta) f(\eta)\, d\eta \\ &= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{k=1}^n 1_{\tilde{I}^n_{jk}}(\xi) \int_0^\infty \frac{\chi(n(\xi/\eta - 1))}{\xi - \eta} 1_{I^n_{jk}}(\eta) f(\eta)\, d\eta \\ &= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{k=1}^n 1_{I^n_{jk}}(\xi) H_n(1_{I^n_{jk}} f)(\xi). \end{aligned}$$

Consequently, it suffices to bound the operator $f \mapsto 1_{\tilde{I}^n_{jk}} H_n(1_{I^n_{jk}} f)$ on $L^q := L^q(0, \infty)$, uniformly in $n \geq 100$ and $j \in \mathbb{Z}$, because then we can conclude that

$$\begin{aligned} \|H_n f\|^q_{L^q} &\lesssim \sum_{j\in\mathbb{Z}} \sum_{k=1}^n \|1_{\tilde{I}^n_{jk}} H_n(1_{I^n_{jk}} f)\|^q_{L^q} = \sum_{j\in\mathbb{Z}} \sum_{k=1}^n \|1_{\tilde{I}^n_{jk}} H_n(1_{I^n_{jk}} 1_{I^n_{jk}} f)\|^q_{L^q} \\ &\lesssim \sum_{j\in\mathbb{Z}} \sum_{k=1}^n \|1_{I^n_{jk}} f\|^q_{L^q} \lesssim \|f\|^q_{L^q}. \end{aligned}$$

According to equation (3.23), the kernel of the operator $f \mapsto 1_{\tilde{I}^n_{jk}} H_n(1_{I^n_{jk}} f)$ is of the form

$$1_{\tilde{I}^n_{jk}}(\xi) 1_{I^n_{jk}}(\eta) \frac{\chi(n(\xi/\eta - 1))}{\xi - \eta} = \frac{1_{\tilde{I}^n_{jk}}(\xi) 1_{I^n_{jk}}(\eta)}{\xi - \eta} + n2^{-j} 1_{\tilde{I}^n_{jk}}(\xi) 1_{I^n_{jk}}(\eta) O(1).$$

Thus, we obtain the decomposition

$$1_{\tilde{I}^n_{jk}} H_n(1_{I^n_{jk}} f) = \pi 1_{\tilde{I}^n_{jk}} H(1_{I^n_{jk}} f) + B^n_{jk} f$$

with the standard Hilbert transform H and the kernel of B^n_{jk} pointwise bounded by $Cn2^{-j} 1_{\tilde{I}^n_{jk}}(\xi) 1_{I^n_{jk}}(\eta)$ for some absolute constant $C > 0$. We immediately obtain $\|1_{\tilde{I}^n_{jk}} H(1_{I^n_{jk}} f)\|_{L^q} \lesssim \|f\|_{L^q}$ by the L^q-boundedness of the Hilbert transform for $q \in (1, \infty)$, and the operator norm of B^n_{jk} is bounded by

$$\|B^n_{jk}\|_{L^q} \lesssim n2^{-j} \left(\int_0^\infty 1_{\tilde{I}^n_{jk}}(\xi)\, d\xi \right)^{1/q} \left(\int_0^\infty 1_{I^n_{jk}}(\eta)\, d\eta \right)^{1/q'} \lesssim 1$$

for all $n \geq 100$, $j \in \mathbb{Z}$, and $k \in \{1, 2, \ldots, n\}$ since $|\tilde{I}^n_{jk}| \simeq |I^n_{jk}| \simeq \frac{2^j}{n}$. $\square$

With this result at our disposal, we can now prove the desired smoothing property of $\mathcal{K}^d_{n_0}$.

LEMMA 3.11. *For any* $\epsilon > 0$, $a, b \in \mathbb{R}$, *and* $q \in (1, \infty)$, *we have the bound*

$$\| |\cdot|^{-1/2+\epsilon} \langle\cdot\rangle^{1-2\epsilon} \mathcal{K}^d_{n_0} f |\cdot|^a \langle\cdot\rangle^b \|_{L^q(0,\infty)} \lesssim \| f |\cdot|^a \langle\cdot\rangle^b \|_{L^q(0,\infty)}$$

for all $n_0 \geq 100$.

Proof. Consider the operator $\mathcal{J}$ with kernel

$$\xi^{-1/2+\epsilon} \langle\xi\rangle^{1-2\epsilon} \xi^a \langle\xi\rangle^b K^d_{n_0}(\xi, \eta) \eta^{-a} \langle\eta\rangle^{-b}.$$

In order to prove the assertion, it suffices to show that $\mathcal{J}$ extends to an operator on $L^q := L^q(0, \infty)$ for $q \in (1, \infty)$ that is uniformly bounded in $n_0 \geq 100$. According to Theorem 3.7, the kernel of $\mathcal{J}$ can be written in the form

$$\xi^{-1/2+\epsilon} \langle\xi\rangle^{1-2\epsilon} \xi^a \langle\xi\rangle^b K^d_{n_0}(\xi, \eta) \eta^{-a} \langle\eta\rangle^{-b} = \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \frac{G(\xi, \eta)}{\xi - \eta},$$

where

$$G(\xi, \eta) = \xi^{-1/2+\epsilon} \langle\xi\rangle^{1-2\epsilon} \xi^a \langle\xi\rangle^b \rho(\eta) F(\xi, \eta) \eta^{-a} \langle\eta\rangle^{-b}.$$

We decompose $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$, where

$$\mathcal{J}_1 f(\xi) = \int_0^\infty \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \frac{G(\eta, \eta)}{\xi - \eta} f(\eta)\, d\eta,$$
$$\mathcal{J}_2 f(\xi) = \int_0^\infty \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \frac{G(\xi, \eta) - G(\eta, \eta)}{\xi - \eta} f(\eta)\, d\eta.$$

By setting $g(\eta) := G(\eta, \eta)$ we see that $\mathcal{J}_1 f = H_{n_0}(gf)$ where H_{n_0} is the truncated Hilbert transform from Lemma 3.10. Note that Theorem 3.7 implies $\|g\|_{L^\infty(0,\infty)} \lesssim 1$, and thus,

$$\|\mathcal{J}_1 f\|_{L^q} = \|H_{n_0}(gf)\|_{L^q} \lesssim \|gf\|_{L^q} \lesssim \|f\|_{L^q}$$

for all $n_0 \geq 100$ by Lemma 3.10. Consequently, it suffices to consider the operator $\mathcal{J}_2$.

First, we study the case $\xi, \eta \leq 4$. Since

$$|G(\xi, \eta) - G(\eta, \eta)| \leq |\xi - \eta| \int_0^1 |\partial_1 G(\eta + s(\xi - \eta), \eta)|\, ds,$$

we obtain from Theorem 3.7 the estimate

$$\begin{aligned}
A_1(\xi, \eta) &:= 1_{[0,4]}(\xi) 1_{[0,4]}(\eta) \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \left|\frac{G(\xi, \eta) - G(\eta, \eta)}{\xi - \eta}\right| \\
&\lesssim 1_{[0,4]}(\xi) 1_{[0,4]}(\eta) \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \eta^{-1+\epsilon} \\
&\lesssim 1_{[0,4]}(\xi) 1_{[0,4]}(\eta) \xi^{(1/q)(-1+\epsilon)} \eta^{(1/q')(-1+\epsilon)},
\end{aligned}$$

which yields $\|A_1\|_{L^q L^{q'}} \lesssim 1$ for all $n_0 \geq 100$ and any $q \in (1, \infty)$.

It remains to study the case $\xi, \eta \in \Omega := [0, \infty)^2 \backslash [0, 4]^2$. Here we further distinguish between $|\xi - \eta| \le 1$ and $|\xi - \eta| \ge 1$. In the former case, we obtain from Theorem 3.7 the bound

$$\begin{aligned} A_2(\xi, \eta) &:= 1_{[-1,1]}(\xi - \eta) 1_\Omega(\xi, \eta) \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \left|\frac{G(\xi, \eta) - G(\eta, \eta)}{\xi - \eta}\right| \\ &\lesssim 1_{[-1,1]}(\xi - \eta) 1_\Omega(\xi, \eta) \eta^{-\epsilon}. \end{aligned}$$

We define $J_k := [k + 1, k + 3]$, $\tilde{J}_k := [k, k + 4]$ and note that

$$\Delta \subset \bigcup_{k=1}^{\infty} \tilde{J}_k \times J_k,$$

where $\Delta := \{(\xi, \eta) \in \Omega \colon\ |\xi - \eta| \le 1\}$. Since

$$1_{J_k}(\eta) 1_{[-1,1]}(\xi - \eta) = 1_{\tilde{J}_k}(\xi) 1_{J_k}(\eta) 1_{[-1,1]}(\xi - \eta),$$

it suffices to consider the kernel A_2 on $\tilde{J}_k \times J_k$ (cf. the proof of Lemma 3.10). We obtain $\|A_2\|_{L^q(\tilde{J}_k) L^{q'}(J_k)} \lesssim 1$ for all $k \in \mathbb{N}$ and all $n_0 \ge 100$, which settles the case $|\xi - \eta| \le 1$. Finally, if $|\xi - \eta| \ge 1$, then we define the dyadic intervals $I_N := [2^{N-1}, 2^{N+1}]$, $\tilde{I}_N := [2^{N-2}, 2^{N+2}]$ and consider the kernel on $\tilde{I}_N \times I_N$, $N \in \mathbb{N}$. Thanks to the cut-off $\chi(n_0(\frac{\xi}{\eta} - 1))$, it suffices to bound

$$A_3(\xi, \eta) := 1_{[1,\infty)}(|\xi - \eta|) 1_{\tilde{I}_N}(\xi) 1_{I_N}(\eta) \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) \frac{G(\xi, \eta) - G(\eta, \eta)}{\xi - \eta},$$

uniformly in $N \in \mathbb{N}$. Note that $1 \le |\xi - \eta| \le 2\eta \le 2^{N+2}$ on the support of A_3. We further subdivide this interval by $[1, 2^{N+2}] = \bigcup_{j=1}^{N+1} I_j$, and from Theorem 3.7 we obtain the bound

$$|A_3(\xi, \eta)| \lesssim 2^{-\epsilon N} \sum_{j=1}^{N+1} A_{3j}(\xi, \eta), \tag{3.24}$$

where

$$A_{3j}(\xi, \eta) = 1_{I_j}(|\xi - \eta|) 1_{\tilde{I}_N}(\xi) 1_{I_N}(\eta) \chi\left(n_0\left(\frac{\xi}{\eta} - 1\right)\right) 2^{-j}.$$

Thanks to the cut-off $1_{I_j}(|\xi - \eta|)$, it suffices to bound A_{3j} on squares Q_j of area $\simeq 2^{2j}$, which yields $\|A_{3j}\|_{L^q L^{q'}(Q_j)} \lesssim 1$ for all $j \in \{1, 2, \dots, N + 1\}$ and any $q \in (1, \infty)$. Consequently, by equation (3.24) we obtain

$$\|A_3\|_{L^q(\tilde{I}_N) L^{q'}(I_N)} \lesssim N 2^{-\epsilon N} \lesssim 1$$

for all $N \in \mathbb{N}$, which finishes the proof. $\square$

3.6. *Estimates for the off-Diagonal Part*

Recall that our aim is to prove smallness of $(\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc})^n$ for sufficiently large n. As suggested by the decomposition $\mathcal{K}_{cc} = \mathcal{K}_{n_0}^d + \mathcal{K}_{n_0}^{nd}$, we consider the diagonal and off-diagonal parts separately. In fact, it turns out that for the off-diagonal part, it suffices to consider the operator $\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{n_0}^{nd}\hat{\mathcal{D}}_c\mathcal{H}_c\beta$, that is, $\mathcal{K}_{n_0}^{nd}$ gets "sandwiched" between two copies of $\hat{\mathcal{D}}_c\mathcal{H}_c\beta$. Our goal is to show that the norm (on $\mathcal{Y}^\alpha$) of this operator can be made small by choosing τ_0 in Definition 3.1 large. More precisely, we have the following result.

LEMMA 3.12. *Let* $\alpha > \frac{3}{4}(1+\frac{1}{\nu})$. *Then there exists an* $\epsilon > 0$ *such that*

$$\|\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{n_0}^{nd}\hat{\mathcal{D}}_c\mathcal{H}_c\beta\|_{\mathcal{Y}^\alpha} \lesssim n_0^4\tau_0^{-\epsilon}$$

for all $n_0 \in \mathbb{N}$, *where* τ_0 *is from Definition 3.1.*

Proof. We have

$$\begin{aligned}
&\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{n_0}^{nd}\hat{\mathcal{D}}_c\mathcal{H}_c\beta x(\tau,\xi)\\
&\quad= \int_\tau^\infty \hat{H}_c(\tau,\sigma_1,\xi)\beta(\sigma_1)\int_0^\infty K_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi,\eta)\\
&\qquad\times\int_{\sigma_1}^\infty \hat{H}_c(\sigma_1,\sigma_2,\eta)\beta(\sigma_2)x(\sigma_2,\omega(\sigma_1,\sigma_2)^2\eta)\,d\sigma_2\,d\eta\,d\sigma_1,
\end{aligned} \tag{3.25}$$

where $\omega(s_1,s_2) = \kappa(s_1)\kappa^{-1}(s_2)$, and $K_{n_0}^{nd}$ is the kernel of the operator $\mathcal{K}_{n_0}^{nd}$. We split the integral over σ_1 into two parts by distinguishing between the cases $\sigma_1\xi \lesssim 1$ and $\sigma_1\xi \gtrsim 1$. In the former case, we exploit the smoothing property from Lemma 3.9 in order to gain a small factor. Thus, we write $y(\sigma,\xi) := \hat{\mathcal{D}}_c\mathcal{H}_c\beta x(\sigma,\xi)$ and note that $y \in \mathcal{Y}^\alpha$ by Lemma 3.4. We have to estimate

$$\mathcal{J}_1 y(\tau,\xi) := \int_\tau^\infty \chi(\sigma\xi)\hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)\mathcal{K}_{n_0}^{nd}y(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma.$$

Recall that

$$\begin{aligned}
\hat{H}_c(\tau,\sigma,\xi) &= -\omega(\tau,\sigma)^{3/2}\rho(\xi)^{-1/2}\rho(\omega(\tau,\sigma)^2\xi)^{1/2}\\
&\quad\times\cos\left(\kappa(\tau)\xi^{1/2}\int_\tau^\sigma \kappa^{-1}(u)\,du\right),
\end{aligned}$$

and, since $\omega(\tau,\sigma) \le 1$, we obtain from the asymptotics of ρ in Theorem 3.1 the bound $|\hat{H}_c(\tau,\sigma,\xi)| \lesssim \omega(\tau,\sigma)$. Consequently, with p from Definition 3.1 we have

$$\begin{aligned}
|\mathcal{J}_1 y(\tau,\xi)| &\lesssim \int_\tau^\infty \chi(\sigma\xi)\sigma^{-1}\omega(\tau,\sigma)[\omega(\tau,\sigma)^2\xi]^{1/(2p)}\\
&\quad\times[\omega(\tau,\sigma)^2\xi]^{-1/(2p)}|\mathcal{K}_{n_0}^{nd}y(\sigma,\omega(\tau,\sigma)^2\xi)|\,d\sigma\\
&\lesssim \int_\tau^\infty \sigma^{-1-1/(2p)}\omega(\tau,\sigma)^{1+1/p}\\
&\quad\times[\omega(\tau,\sigma)^2\xi]^{-1/(2p)}|\mathcal{K}_{n_0}^{nd}y(\sigma,\omega(\tau,\sigma)^2\xi)|\,d\sigma,
\end{aligned}$$

and Lemmas 3.3, 3.9 yield

$$\|\mathcal{J}_1 y\|_{\mathcal{Y}^\alpha} \lesssim \tau_0^{-1/(2p)} \sup_{\tau>\tau_0} \tau^\alpha \||\cdot|^{-1/(2p)} \mathcal{K}_{n_0}^{nd} y(\tau,\cdot)\|_Y \lesssim n_0^2 \tau_0^{-1/(2p)} \|y\|_{\mathcal{Y}^\alpha}.$$

Thus, by Lemma 3.4 we obtain $\|\mathcal{J}_1 y\|_{\mathcal{Y}^\alpha} \lesssim n_0^4 \tau_0^{-1/(2p)} \|x\|_{\mathcal{Y}^\alpha}$.

It remains to consider the case $\sigma_1\xi \gtrsim 1$. Unfortunately, this is more complicated, and we have to exploit the oscillation of the kernel. After the change of variable $\eta \mapsto \omega(\sigma_1,\sigma_2)^{-2}\eta$ and an application of the Fubini theorem, it remains to study the operator

$$\begin{aligned}\mathcal{J}_2 x(\tau,\xi) := \int_0^\infty \int_\tau^\infty \int_{\sigma_1}^\infty & [1-\chi(\sigma_1\xi)]\hat{H}_c(\tau,\sigma_1,\xi)\hat{H}_c(\sigma_1,\sigma_2,\omega(\sigma_1,\sigma_2)^{-2}\eta) \\ & \times \beta(\sigma_1)\beta(\sigma_2)\omega(\sigma_1,\sigma_2)^{-2} K_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi, \omega(\sigma_1,\sigma_2)^{-2}\eta) \\ & \times x(\sigma_2,\eta)\, d\sigma_2\, d\sigma_1\, d\eta;\end{aligned}$$

cf. equation (3.25). We have

$$\begin{aligned}&\hat{H}_c(\tau,\sigma_1,\xi)\hat{H}_c(\sigma_1,\sigma_2,\omega(\sigma_1,\sigma_2)^{-2}\eta) \\ &\quad = A(\tau,\sigma_1,\sigma_2,\xi,\eta) \\ &\qquad \times \cos\Big(\kappa(\tau)\xi^{1/2}\int_\tau^{\sigma_1} \kappa^{-1}(u)\,du\Big)\cos\Big(\kappa(\sigma_2)\eta^{1/2}\int_{\sigma_1}^{\sigma_2}\kappa^{-1}(u)\,du\Big),\end{aligned}$$

where

$$\begin{aligned}A(\tau,\sigma_1,\sigma_2,\xi,\eta) = {}& \omega(\tau,\sigma_2)^{3/2}\rho(\xi)^{-1/2} \\ & \times \rho(\omega(\tau,\sigma_1)^2\xi)^{1/2}\rho(\omega(\sigma_1,\sigma_2)^{-2}\eta)^{-1/2}\rho(\eta)^{1/2}.\end{aligned}$$

By the asymptotics of ρ given in Theorem 3.1 and the fact that $\tau \le \sigma_1 \le \sigma_2$, we obtain the estimate $|A(\tau,\sigma_1,\sigma_2,\xi,\eta)| \lesssim \omega(\tau,\sigma_2)$ with symbol behavior under differentiation with respect to each variable. Furthermore, by using the trigonometric identity $2\cos a\cos b = \cos(a+b)+\cos(a-b)$ we observe that the operator in question decomposes as $\mathcal{J}_2 = \mathcal{A}_+ + \mathcal{A}_-$ where

$$\begin{aligned}&\mathcal{A}_\pm x(\tau,\xi) \\ &\quad = \frac{1}{2}\int_0^\infty \int_\tau^\infty \int_{\sigma_1}^\infty [1-\chi(\sigma_1\xi)]\beta(\sigma_1)\beta(\sigma_2)\omega(\sigma_1,\sigma_2)^{-2}A(\tau,\sigma_1,\sigma_2,\xi,\eta) \\ &\qquad \times \cos\Big(\xi^{1/2}\kappa(\tau)\int_\tau^{\sigma_1}\kappa^{-1}(u)\,du \pm \eta^{1/2}\kappa(\sigma_2)\int_{\sigma_1}^{\sigma_2}\kappa^{-1}(u)\,du\Big) \\ &\qquad \times K_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi, \omega(\sigma_1,\sigma_2)^{-2}\eta)x(\sigma_2,\eta)\,d\sigma_2\,d\sigma_1\,d\eta.\end{aligned}$$

It suffices to consider $\mathcal{A}_+$. We abbreviate

$$\mu := \xi^{1/2}\kappa(\tau)\kappa^{-1}(\sigma_1) - \eta^{1/2}\kappa(\sigma_2)\kappa^{-1}(\sigma_1),$$

and since $\partial_{\sigma_1}\mu = -c(\sigma_1)(1+\frac{1}{\nu})\sigma_1^{-1}\mu$, $c(\sigma_1)\in[2^{-1},2]$, we obtain the identity

$$\cos(\Omega) = \partial_{\sigma_1}[\mu^{-1}\sin(\Omega)] - c(\sigma_1)\left(1+\frac{1}{\nu}\right)\sigma_1^{-1}\mu^{-1}\sin(\Omega),$$
$$\Omega = \xi^{1/2}\kappa(\tau)\int_\tau^{\sigma_1}\kappa^{-1}(u)\,du + \eta^{1/2}\kappa(\sigma_2)\int_{\sigma_1}^{\sigma_2}\kappa^{-1}(u)\,du.$$

Then we use the integration by parts formula

$$\begin{aligned}
&\int_\tau^\infty\int_{\sigma_1}^\infty \partial_{\sigma_1} f(\sigma_1,\sigma_2)g(\sigma_1,\sigma_2)\,d\sigma_2\,d\sigma_1\\
&\quad= \int_\tau^\infty f(\sigma_1,\sigma_1)g(\sigma_1,\sigma_1)\,d\sigma_1 - \int_\tau^\infty f(\tau,\sigma_1)g(\tau,\sigma_1)\,d\sigma_1\\
&\qquad - \int_\tau^\infty\int_{\sigma_1}^\infty f(\sigma_1,\sigma_2)\partial_{\sigma_1} g(\sigma_1,\sigma_2)\,d\sigma_2\,d\sigma_1
\end{aligned}$$

to conclude that the operator $\mathcal{A}_+$ decomposes into four types of terms, $\mathcal{A}_+ = \sum_{j=1}^4 \mathcal{A}_j$, of the form

$$\begin{aligned}
\mathcal{A}_1 x(\tau,\xi) &= \int_\tau^\infty\int_0^\infty \tilde{K}_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi,\eta)\\
&\quad\times\int_{\sigma_1}^\infty O(\sigma_1^{-3/2})\sigma_2^{-1}\omega(\tau,\sigma_2)x(\sigma_2,\omega(\sigma_1,\sigma_2)^2\eta)\,d\sigma_2\,d\eta\,d\sigma_1,\\
\mathcal{A}_2 x(\tau,\xi) &= \int_\tau^\infty\int_0^\infty O(\sigma_1^{-3/2})\omega(\tau,\sigma_1)\tilde{K}_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi,\eta)x(\sigma_1,\eta)\,d\eta\,d\sigma_1,\\
\mathcal{A}_3 x(\tau,\xi) &= \int_0^\infty \tilde{K}_{n_0}^{nd}(\xi,\eta)\int_\tau^\infty O(\tau^{-1})\sigma_1^{-1/2}\omega(\tau,\sigma_1)\\
&\quad\times x(\sigma_1,\omega(\tau,\sigma_1)^2\eta)\,d\sigma_1\,d\eta,\\
\mathcal{A}_4 x(\tau,\xi) &= \int_\tau^\infty\int_0^\infty \partial_{\sigma_1}\tilde{K}_{n_0}^{nd}(\omega(\tau,\sigma_1)^2\xi,\omega(\sigma_1,\sigma_2)^{-2}\eta)\\
&\quad\times\int_{\sigma_1}^\infty O(\sigma_1^{-1/2})\sigma_2^{-1}\omega(\tau,\sigma_2)\omega(\sigma_1,\sigma_2)^{-2}x(\sigma_2,\eta)\,d\sigma_2\,d\eta\,d\sigma_1,
\end{aligned}$$

with

$$\tilde{K}_{n_0}^{nd}(\xi,\eta) := \xi^{1/2}\frac{K_{n_0}^{nd}(\xi,\eta)}{\xi^{1/2}-\eta^{1/2}},$$

where we have used the fact that $\sigma_1^{-1/2}\lesssim\xi^{1/2}$ on the support of the cut-off $1-\chi(\sigma_1\xi)$ and performed the change of variable $\eta\mapsto\omega(\sigma_1,\sigma_2)^2\eta$ in the first three terms. Since

$$\left|\frac{\eta^{1/2}}{\xi^{1/2}}-1\right|\gtrsim\frac{1}{n_0}$$

on the support of $K_{n_0}^{nd}$ (cf. the proof of Lemma 3.9), we observe that

$$|\tilde{K}_{n_0}^{nd}(\xi,\eta)|\lesssim n_0|K_{n_0}^{nd}(\xi,\eta)|,$$

and thus, Lemmas 3.3 and 3.9 yield $\|\mathcal{A}_j x\|_{\mathcal{Y}^\alpha} \lesssim n_0^3 \tau_0^{-1/2} \|x\|_{\mathcal{Y}^\alpha}$ for $j = 1, 2, 3$. Finally, after the change of variables $\eta \mapsto \omega(\sigma_1, \sigma_2)^2 \eta$, the operator $\mathcal{A}_4$ can be written as

$$\mathcal{A}_4 x(\tau, \xi) = \int_\tau^\infty \int_0^\infty [\xi \partial_\xi + \eta \partial_\eta] \tilde{K}_{n_0}^{nd}(\omega(\tau, \sigma_1)^2 \xi, \eta) \\ \times \int_{\sigma_1}^\infty O(\sigma_1^{-3/2}) \sigma_2^{-1} \omega(\tau, \sigma_2) x(\sigma_2, \omega(\sigma_1, \sigma_2)^2 \eta) \, d\sigma_2 \, d\eta \, d\sigma_1,$$

and since $|[\xi \partial_\xi + \eta \partial_\eta] \tilde{K}_{n_0}^{nd}(\xi, \eta)| \lesssim n_0^2 |K_{n_0}^{nd}(\xi, \eta)|$ by Theorem 3.1, we obtain

$$\|\mathcal{A}_4 x\|_{\mathcal{Y}^\alpha} \lesssim n_0^4 \tau_0^{-1/2} \|x\|_{\mathcal{Y}^\alpha}$$

as before by Lemmas 3.3 and 3.9. □

3.7. *Estimates for the Diagonal Term*

Next, we consider the diagonal operator $\mathcal{K}_{n_0}^d$. We further decompose $\mathcal{K}_{n_0}^d$ in the following way. We set

$$K_1^\epsilon(\xi, \eta) := 1_{[0,\epsilon)}(\xi) K_{n_0}^d(\xi, \eta)$$

and

$$K_3^\epsilon(\xi, \eta) := 1_{(\epsilon^{-1}, \infty)}(\xi) K_{n_0}^d(\xi, \eta),$$

where $K_{n_0}^d$ is the kernel of $\mathcal{K}_{n_0}^d$. Following our usual scheme, we denote the operator with kernel K_j^ϵ by $\mathcal{K}_j^\epsilon$, $j = 1, 3$. Furthermore, we define $\mathcal{K}_2^\epsilon$ by

$$\mathcal{K}_{n_0}^d = \sum_{j=1}^3 \mathcal{K}_j^\epsilon,$$

which yields the desired decomposition. We bear in mind that the operators $\mathcal{K}_j^\epsilon$ depend on n_0 but suppress this dependence in the notation. Finally, we set

$$\mathcal{A}_\epsilon := 2\hat{\mathcal{D}}_c \mathcal{H}_c \beta \mathcal{K}_1^\epsilon,$$
$$\mathcal{B}_\epsilon := 2\hat{\mathcal{D}}_c \mathcal{H}_c \beta \mathcal{K}_2^\epsilon,$$
$$\mathcal{C}_\epsilon := 2\hat{\mathcal{D}}_c \mathcal{H}_c \beta \mathcal{K}_3^\epsilon.$$

First, we establish some smallness properties. Here and in the following, the product of noncommutative operators A_j is defined as

$$\prod_{j=1}^n A_j := A_1 A_2 \cdots A_n.$$

LEMMA 3.13. *Let $\alpha > \frac{3}{4}(1 + \frac{1}{\nu})$, and let $\epsilon > 0$ be sufficiently small. Then we have the bounds*

$$\|\mathcal{A}_\epsilon\|_{\mathcal{Y}^\alpha} \lesssim \epsilon^{1/4}, \qquad \|\mathcal{C}_\epsilon\|_{\mathcal{Y}^\alpha} \lesssim \epsilon^{1/4},$$

and

$$\Big\|\prod_{j=0}^{n_0-1}\mathcal{B}_{\mu^{-j}\epsilon}\Big\|_{\mathcal{Y}^\alpha}\le\left[\frac{C^{n_0-1}\epsilon^{-(n_0-1)}}{(n_0-1)!}\right]^{1/p}$$

for all sufficiently large $n_0\in\mathbb{N}$, *where* $\mu:=1+4/n_0$, $C>0$ *is some absolute constant, and* p *is from Definition 3.1.*

Proof. From Lemma 3.11 we immediately obtain the estimate

$$\begin{aligned}\|\mathcal{K}_1^\epsilon f\|_Y&=\|1_{[0,\epsilon)}|\cdot|^{1/4}|\cdot|^{-1/4}\mathcal{K}^d_{n_0}f\|_Y\lesssim\|1_{[0,\epsilon)}|\cdot|^{1/4}\|_{L^\infty(0,\infty)}\|f\|_Y\\&=\epsilon^{1/4}\|f\|_Y\end{aligned}$$

and, analogously, $\|\mathcal{K}_3^\epsilon\|_Y\lesssim\epsilon^{1/4}$, uniformly in $n_0\ge 100$. With $\omega(\tau,\sigma):=\kappa(\tau)\kappa(\sigma)^{-1}$ we have

$$\mathcal{A}_\epsilon x(\tau,\sigma)=2\int_\tau^\infty\hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)\mathcal{K}_1^\epsilon x(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma$$

and $|\hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)|\lesssim\omega(\tau,\sigma)\sigma^{-1}$ (cf. the proof of Lemma 3.12). Consequently, Lemma 3.3 yields the stated estimate for $\mathcal{A}_\epsilon$, and the proof for $\mathcal{C}_\epsilon$ is identical.

In order to prove the remaining estimate, note first that

$$\begin{aligned}&\Big(\prod_{j=0}^{n_0-1}\mathcal{B}_{\mu^{-j}\epsilon}\Big)x(\sigma_0,\eta_0)\\&\quad=\int_{\sigma_0}^\infty\int_0^\infty\cdots\int_{\sigma_{n_0-1}}^\infty\int_0^\infty x(\sigma_{n_0},\eta_{n_0})\prod_{j=0}^{n_0-1}[2\hat{H}_c(\sigma_j,\sigma_{j+1},\eta_j)\beta(\sigma_{j+1})\\&\qquad\times K_2^{\mu^{-j}\epsilon}(\omega(\sigma_j,\sigma_{j+1})^2\eta_j,\eta_{j+1})]\,d\eta_{n_0}\,d\sigma_{n_0}\cdots d\eta_1\,d\sigma_1.\end{aligned}\tag{3.26}$$

Now we are going to exploit the following observation. Consider the expression

$$K_2^\epsilon(\omega(\sigma_0,\sigma_1)^2\eta_0,\eta_1)K_2^{\mu^{-1}\epsilon}(\omega(\sigma_1,\sigma_2)^2\eta_1,\eta_2),\tag{3.27}$$

which appears in the integrand of (3.26). Assume σ_0, σ_1 to be fixed and suppose that we want to perform the integration with respect to σ_2. As $\sigma_2\to\infty$, we must have $\eta_1\to\infty$ in order to stay in the support of (3.27). However, since K_2^ϵ is supported near the diagonal, this also entails $\omega(\sigma_0,\sigma_1)^2\eta_0\to\infty$, and we therefore necessarily leave the support of (3.27). Hence, it is not necessary to integrate all the way up to infinity. In order to quantify this argument, we return to equation (3.26) and note that on the support of the integrand we have $\omega(\sigma_j,\sigma_{j+1})^2\eta_j/\eta_{j+1}\ge 1-2/n_0$ for all $j\in\{0,1,\dots,n_0-1\}$. This implies

$$\left(1-\frac{2}{n_0}\right)^{n_0-1}\le\prod_{j=0}^{n_0-2}\omega(\sigma_j,\sigma_{j+1})^2\frac{\eta_j}{\eta_{j+1}}=\omega(\sigma_0,\sigma_{n_0-1})^2\frac{\eta_0}{\eta_{n_0-1}}.\tag{3.28}$$

On the other hand, we have $\omega(\sigma_{n_0-1},\sigma_{n_0})^2\eta_{n_0-1}\geq \mu^{-(n_0-1)}\epsilon$ and $\omega(\sigma_0,\sigma_1)^2\eta_0\leq \epsilon^{-1}$ on the support of the integrand in equation (3.26). By inserting these two estimates into (3.28) we find

$$\left(1-\frac{2}{n_0}\right)^{n_0-1}\leq \omega(\sigma_1,\sigma_{n_0})^2\mu^{n_0-1}\epsilon^{-2},$$

which yields $\sigma_{n_0}^{2(1+1/\nu)}\lesssim \sigma_1^{2(1+1/\nu)}\epsilon^{-2}$ for all $n_0\geq 100$. Hence, we obtain the crude bound $\sigma_{n_0}\leq \sigma_1\epsilon^{-1}$ for all $n_0\geq 100$ (provided that $\epsilon>0$ is sufficiently small), and since $\sigma_0\leq\sigma_1\leq\cdots\leq\sigma_{n_0}$, we have in fact $\sigma_j\leq\sigma_1\epsilon^{-1}$ for all $j\in\{2,3,\dots,n_0\}$. Consequently, upon writing

$$\mathcal{J}_a x(\tau,\xi):=2\int_\tau^a \hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)x(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma$$

and by defining

$$y(\tau,\xi):=\prod_{j=1}^{n_0-1}(\mathcal{J}_{\epsilon^{-1}\tau}\mathcal{K}_2^{\mu^{-j}\epsilon})x(\tau,\xi),$$

we obtain

$$\left(\prod_{j=0}^{n_0-1}\mathcal{B}_{\mu^{-j}\epsilon}\right)x(\tau,\xi)=\mathcal{B}_\epsilon y(\tau,\xi).$$

Therefore, it suffices to prove an appropriate bound for $y(\tau,\xi)$. By Hölder's inequality, Fubini's theorem, and the fact that $|\hat{H}_c(\tau,\sigma,\xi)|\lesssim\omega_\nu(\frac{\tau}{\sigma})$ we infer

$$\|\mathcal{J}_{\epsilon^{-1}\tau}x(\tau,\cdot)\|_{L^p}^p\lesssim \tau^{-1}\int_\tau^{\epsilon^{-1}\tau}\omega_\nu\left(\frac{\tau}{\sigma}\right)^{p-2}\|x(\sigma,\cdot)\|_{L^p}^p\,d\sigma \tag{3.29}$$

and, similarly,

$$\begin{aligned}&\|\mathcal{J}_{\epsilon^{-1}\tau}x(\tau,\cdot)\langle\cdot\rangle^{1/8}\|_{L^2_\rho}^2\\ &\quad\lesssim \tau^{-\delta_0}\int_\tau^{\epsilon^{-1}\tau}\omega_\nu\left(\frac{\tau}{\sigma}\right)^{2-7/2}\sigma^{-1+\delta_0}\|x(\sigma,\cdot)\langle\cdot\rangle^{1/8}\|_{L^2_\rho}^2\,d\sigma,\end{aligned} \tag{3.30}$$

where $\delta_0>0$ is chosen such that $\alpha\geq\frac{3}{4}(1+\frac{1}{\nu})+\delta_0$, that is, such that the integral converges (cf. the proof of Lemma 3.3). By Lemma 3.11 we have the same bounds for $\mathcal{J}_{\epsilon^{-1}\tau}\mathcal{K}_2^{\mu^{-j}\epsilon}$, $j\in\{0,\dots,n_0-1\}$. Consequently, (3.29) implies

$$\begin{aligned}\|y(\sigma_1,\cdot)\|_{L^p}^p&\leq C^{n_0-1}\int_{\sigma_1}^{\epsilon^{-1}\sigma_1}\int_{\sigma_2}^{\epsilon^{-1}\sigma_1}\cdots\int_{\sigma_{n_0-1}}^{\epsilon^{-1}\sigma_1}\prod_{j=1}^{n_0-1}\left(\omega_\nu\left(\frac{\sigma_j}{\sigma_{j+1}}\right)^{p-2}\sigma_j^{-1}\right)\\ &\qquad\times\|x(\sigma_{n_0},\cdot)\|_{L^p}^p\,d\sigma_{n_0}\,d\sigma_{n_0-1}\cdots d\sigma_2\\ &\leq\frac{C^{n_0-1}\epsilon^{-(n_0-1)}}{(n_0-1)!}\sigma_1^{-p\alpha}\|x\|_{\mathcal{Y}^\alpha}^p,\end{aligned}$$

where $C>0$ is some absolute constant. By the same argument we obtain an analogous estimate for $\|y(\sigma_1,\cdot)\|_{L^2_\rho}$, and the proof is finished. □

Next, we prove the following crucial orthogonality relations.

LEMMA 3.14. *For any sufficiently small $\epsilon > 0$ and $n_0 \geq 4$, we have*

$$\mathcal{A}_\epsilon \mathcal{B}_{\mu\epsilon} = 0, \qquad \mathcal{B}_{\mu\epsilon}\mathcal{C}_\epsilon = 0,$$

and

$$\mathcal{A}_\epsilon \mathcal{C}_\epsilon = \mathcal{A}_{\mu\epsilon}\mathcal{C}_\epsilon = \mathcal{A}_\epsilon \mathcal{C}_{\mu\epsilon} = 0,$$

where $\mu := 1 + 4/n_0$.

Proof. Explicitly, we have

$$\mathcal{B}_\epsilon x(\tau,\xi) = 2\int_\tau^\infty \hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)\int_0^\infty K_2^\epsilon(\omega(\tau,\sigma)^2\xi,\eta_2)x(\sigma,\eta_2)\,d\eta_2\,d\sigma,$$

where, as before, $\omega(s_1,s_2) = \kappa(s_1)\kappa^{-1}(s_2)$. Furthermore,

$$\mathcal{K}_1^\epsilon \mathcal{B}_{\mu\epsilon} x(\tau,\xi) = \int_0^\infty K_1^\epsilon(\xi,\eta_1)\mathcal{B}_{\mu\epsilon}x(\tau,\eta_1)\,d\eta_1,$$

and thus, in order to prove $\mathcal{A}_\epsilon \mathcal{B}_{\mu\epsilon} = 0$, it suffices to show that

$$K_1^\epsilon(\xi,\eta_1)K_2^{\mu\epsilon}(\omega(\tau,\sigma)^2\eta_1,\eta_2) = 0 \tag{3.31}$$

for all $\xi, \eta_1, \eta_2 \geq 0$ and $\tau \leq \sigma$. Recall that

$$K_1^\epsilon(\xi,\eta_1) = 1_{[0,\epsilon)}(\xi)\chi\left(n_0\left(\frac{\xi}{\eta_1} - 1\right)\right)K_{cc}(\xi,\eta_1),$$

where $\chi(x) \neq 0$ only if $|x| \leq 2$ and thus, $K_1^\epsilon(\xi,\eta_1) \neq 0$ only if

$$\eta_1 \leq \left(1 - \frac{2}{n_0}\right)^{-1}\xi < \left(1 - \frac{2}{n_0}\right)^{-1}\epsilon. \tag{3.32}$$

On the other hand, we have $K_2^{\mu\epsilon}(\omega(\tau,\sigma)^2\eta_1,\eta_2) \neq 0$ only if $(1 + 4/n_0)\epsilon \leq \omega(\tau,\sigma)^2\eta_1 \leq \eta_1$, and since $1 + 4/n_0 \geq (1 - 2/n_0)^{-1}$ for all $n_0 \geq 4$, this condition is incompatible with (3.32), which proves (3.31).

Similarly, to see that $\mathcal{B}_{\mu\epsilon}\mathcal{C}_\epsilon = 0$, we consider the product kernel

$$K_2^{\mu\epsilon}(\xi,\eta_1)K_3^\epsilon(\omega(\tau,\sigma)^2\eta_1,\eta_2).$$

The second factor is nonvanishing only if $\eta_1 > \epsilon^{-1}$, whereas on the support of the first factor, we have

$$\eta_1 \leq \left(1 - \frac{2}{n_0}\right)^{-1}\xi \leq \left(1 - \frac{2}{n_0}\right)^{-1}\left(1 + \frac{4}{n_0}\right)^{-1}\epsilon^{-1}$$

and $(1 - 2/n_0)^{-1}(1 + 4/n_0)^{-1} \leq 1$ for all $n_0 \geq 4$. This implies the desired $\mathcal{B}_{\mu\epsilon}\mathcal{C}_\epsilon = 0$. The remaining assertions are immediate, provided that $\epsilon > 0$ is sufficiently small. □

Now we can show that $(2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{n_0}^d)^{2n_0}$ has small operator norm on $\mathcal{Y}^\alpha$, provided that n_0 is sufficiently large.

LEMMA 3.15. *Let* $\alpha > \frac{3}{4}(1+\frac{1}{\nu})$ *and* $\delta_0 > 0$. *Then*

$$\|(2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^d_{n_0})^{2n_0}\|_{\mathcal{Y}^\alpha} < \delta_0,$$

provided that $n_0 \in \mathbb{N}$ *is sufficiently large.*

Proof. For brevity, we write $\mathcal{J} := 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^d_{n_0}$. We have

$$\begin{aligned}\mathcal{J}^{2n_0} &= (\mathcal{A}_\epsilon + \mathcal{B}_\epsilon + \mathcal{C}_\epsilon)\mathcal{J}^{2n_0-1} \\ &= \mathcal{A}_\epsilon\mathcal{J}^{2n_0-1} + \mathcal{B}_\epsilon\mathcal{J}^{2n_0-1} + \mathcal{C}_\epsilon\mathcal{J}^{2n_0-1}\end{aligned}$$

and consider each term separately. Furthermore, we set $\mu := 1 + 4/n_0$. For the first term, we note that

$$\mathcal{A}_\epsilon\mathcal{J}^{2n_0-1} = \mathcal{A}_\epsilon(\mathcal{A}_{\mu\epsilon} + \mathcal{B}_{\mu\epsilon} + \mathcal{C}_{\mu\epsilon})\mathcal{J}^{n_0-2} = \mathcal{A}_\epsilon\mathcal{A}_{\mu\epsilon}\mathcal{J}^{2n_0-2}$$

by Lemma 3.14. Thus, inductively we find

$$\mathcal{A}_\epsilon\mathcal{J}^{2n_0-1} = \prod_{j=0}^{2n_0-1}\mathcal{A}_{\mu^j\epsilon}, \tag{3.33}$$

and Lemma 3.13 yields

$$\|\mathcal{A}_\epsilon\mathcal{J}^{2n_0-1}\|_{\mathcal{Y}^\alpha} \le (C\mu^{2n_0/4}\epsilon^{1/4})^{2n_0}.$$

For the second term, we obtain

$$\begin{aligned}\mathcal{B}_\epsilon\mathcal{J}^{2n_0-1} &= \mathcal{B}_\epsilon(\mathcal{A}_{\mu^{-1}\epsilon} + \mathcal{B}_{\mu^{-1}\epsilon} + \mathcal{C}_{\mu^{-1}\epsilon})\mathcal{J}^{n_0-2} \\ &= \mathcal{B}_\epsilon\mathcal{A}_{\mu^{-1}\epsilon}\mathcal{J}^{2n_0-2} + \mathcal{B}_\epsilon\mathcal{B}_{\mu^{-1}\epsilon}\mathcal{J}^{2n_0-2} \\ &= \mathcal{B}_\epsilon\prod_{j=0}^{2n_0-2}\mathcal{A}_{\mu^{-1+j}\epsilon} + \mathcal{B}_\epsilon\mathcal{B}_{\mu^{-1}\epsilon}\mathcal{J}^{2n_0-2}\end{aligned}$$

by Lemma 3.14 and equation (3.33). Inductively we see that this is a sum of $2n_0$ terms that consist of products of consecutive $\mathcal{B}$'s and consecutive $\mathcal{A}$'s. We thus may write $\mathcal{B}_\epsilon\mathcal{J}^{2n_0-1} = \mathcal{S}_1 + \mathcal{S}_2$ where $\mathcal{S}_1$ contains all the terms with at most n_0 $\mathcal{B}$s. From Lemma 3.13 we obtain the bounds

$$\|\mathcal{S}_1\|_{\mathcal{Y}^\alpha} \le (C\mu^{n_0/4}\epsilon^{1/4})^{n_0}, \qquad \|\mathcal{S}_2\|_{\mathcal{Y}^\alpha} \le \left[\frac{C^{n_0}\epsilon^{-n_0}}{n_0!}\right]^{1/p},$$

provided that n_0 is sufficiently large. Finally, we have

$$\mathcal{C}_\epsilon\mathcal{J}^{2n_0-1} = \mathcal{C}_\epsilon\mathcal{A}_\epsilon\mathcal{J}^{2n_0-2} + \mathcal{C}_\epsilon\mathcal{B}_\epsilon\mathcal{J}^{2n_0-2} + \mathcal{C}_\epsilon^2\mathcal{J}^{2n_0-2},$$

and thus, by the exact same token as before we obtain a decomposition $C_\epsilon\mathcal{J}^{2n_0-1} = \mathcal{S}_3 + \mathcal{S}_4$ with the bounds

$$\|\mathcal{S}_3\|_{\mathcal{Y}^\alpha} \le (C\mu^{n_0/4}\epsilon^{1/4})^{n_0}, \qquad \|\mathcal{S}_4\|_{\mathcal{Y}^\alpha} \le \left[\frac{C^{n_0}\epsilon^{-n_0}}{n_0!}\right]^{1/p}$$

for sufficiently large n_0. Hence, by first choosing $\epsilon > 0$ sufficiently small and then n_0 sufficiently large, the claim follows. □

Now we can conclude the existence of $(1 + 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc})^{-1}$.

COROLLARY 3.16. *If* $\alpha > \frac{3}{4}(1+\frac{1}{\nu})$, *then the operator* $1 + 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}$ *has a bounded inverse on* $\mathcal{Y}^\alpha$.

Proof. For brevity, we write $\mathcal{J} := 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}$ and decompose

$$\begin{aligned}\mathcal{J}^{2n_0} &= (2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^{nd}_{n_0} + 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^{d}_{n_0})^{2n_0}\\ &= \mathcal{S} + (2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^{d}_{n_0})^{2n_0},\end{aligned}$$

where $\mathcal{S}$ consists of $2^{2n_0}-1$ terms, each of which containing the operator $\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}^{nd}_{n_0}\hat{\mathcal{D}}_c\mathcal{H}_c\beta$. Hence, by first choosing n_0 sufficiently large and then τ_0 sufficiently large (depending on n_0), we obtain from Lemmas 3.12 and 3.15 the bound

$$\|\mathcal{J}^{2n_0}\|_{\mathcal{Y}^\alpha} < 1.$$

This implies

$$\sum_{n=0}^{\infty}\|\mathcal{J}^n\|_{\mathcal{Y}^\alpha} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{2n_0-1}\|\mathcal{J}^{2n_0k+\ell}\|_{\mathcal{Y}^\alpha} \lesssim \sum_{k=0}^{\infty}\|\mathcal{J}^{2n_0}\|^k_{\mathcal{Y}^\alpha} \lesssim 1,$$

and the claim follows. □

3.8. *The Inverse of* $1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K}$

Finally, we consider the matrix operator $1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K}$, which explicitly reads

$$1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K} = \begin{pmatrix} 1 + 2\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dd} & 2\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dc} \\ 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cd} & 1 + 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}\end{pmatrix};$$

cf. equation (3.22).

LEMMA 3.17. *Let* $\alpha_c > \frac{3}{4}(1+\frac{1}{\nu})$ *and* $\alpha_c \leq \alpha_d < \alpha_c + 1$. *Then the operator* $1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K}$ *has a bounded inverse on* $\mathcal{Y}^{\alpha_d,\alpha_c}$.

Proof. For brevity, we write

$$\begin{aligned}\mathcal{J}_{dd} &:= 1 + 2\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dd}, & \mathcal{J}_{dc} &:= 2\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dc},\\ \mathcal{J}_{cd} &:= 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cd}, & \mathcal{J}_{cc} &:= 1 + 2\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}.\end{aligned}$$

From Corollary 3.16 and the comment following equation (3.22) we know that $\operatorname{diag}(\mathcal{J}_{dd}, \mathcal{J}_{cc})^{-1} = \operatorname{diag}(\mathcal{J}_{dd}^{-1}, \mathcal{J}_{cc}^{-1})$ exists as a bounded operator on $\mathcal{Y}^{\alpha_d,\alpha_c}$. Consequently, the equation

$$\begin{pmatrix}\mathcal{J}_{dd} & \mathcal{J}_{dc}\\ \mathcal{J}_{cd} & \mathcal{J}_{cc}\end{pmatrix}\begin{pmatrix}x_d\\ x\end{pmatrix} = \begin{pmatrix}y_d\\ y\end{pmatrix}$$

implies

$$\begin{aligned}(1 - \mathcal{J}_{dd}^{-1}\mathcal{J}_{dc}\mathcal{J}_{cc}^{-1}\mathcal{J}_{cd})x_d &= \mathcal{J}_{dd}^{-1}(y_d - \mathcal{J}_{dc}\mathcal{J}_{cc}^{-1}y),\\ (1 - \mathcal{J}_{cc}^{-1}\mathcal{J}_{cd}\mathcal{J}_{dd}^{-1}\mathcal{J}_{dc})x &= \mathcal{J}_{cc}^{-1}(y - \mathcal{J}_{cd}\mathcal{J}_{dd}^{-1}y_d),\end{aligned}$$

and it suffices to prove smallness of $\mathcal{J}_{dc}$. From Theorem 3.7 and Lemma 3.4 we obtain

$$|\mathcal{J}_{dc}x(\tau)| = 2|\hat{\mathcal{D}}_d\mathcal{H}_d\beta\mathcal{K}_{dc}x(\tau)| \lesssim \tau^{-\alpha_c-1}\sup_{\tau>\tau_0}\tau^{\alpha_c}\|x(\tau,\cdot)\|_Y,$$

which yields

$$\sup_{\tau>\tau_0}\tau^{\alpha_d}|\mathcal{J}_{dc}x(\tau)| \lesssim \tau_0^{\alpha_d-\alpha_c-1}\|x\|_{\mathcal{Y}^{\alpha_c}},$$

and we obtain smallness by choosing τ_0 sufficiently large since $\alpha_d-\alpha_c-1<0$ by assumption. □

3.9. *The Inverse of* $1+2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}}$

For the following, it is also necessary to invert the operator $1+2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}}$. As before, we first consider the difficult continuous component $\mathcal{H}_c\beta\mathcal{K}_{cc}\hat{\mathcal{D}}_c$, which is explicitly given by

$$\begin{aligned}
&\mathcal{H}_c\beta\mathcal{K}_{cc}\hat{\mathcal{D}}_c x(\tau,\xi)\\
&\quad=\int_\tau^\infty H_c(\tau,\sigma,\xi)\beta(\sigma)\mathcal{K}_{cc}\hat{\mathcal{D}}_c x(\sigma,\omega(\tau,\sigma)\xi)\,d\sigma\\
&\quad=\int_\tau^\infty H_c(\tau,\sigma,\xi)\beta(\sigma)\int_0^\infty K_{cc}(\omega(\tau,\sigma)^2\xi,\eta)\hat{\mathcal{D}}_c x(\sigma,\eta)\,d\eta\,d\sigma
\end{aligned}\tag{3.34}$$

with $\omega(s_1,s_2)=\kappa(s_1)\kappa^{-1}(s_2)$. Now recall that

$$\hat{\mathcal{D}}_c x(\tau,\xi)=\partial_1 x(\tau,\xi)+\beta(\tau)\left[-2\xi\partial_2 x(\tau,\xi)-\left(\frac{5}{2}+\frac{\xi\rho'(\xi)}{\rho(\xi)}\right)x(\tau,\xi)\right]$$

and

$$\hat{\mathcal{D}}_c x(\tau,\xi)=\kappa(\tau)^{5/2}\rho(\xi)^{-1/2}[\partial_\tau-2\beta(\tau)\xi\partial_\xi][\kappa(\tau)^{-5/2}\rho(\xi)^{1/2}x(\tau,\xi)],$$

where

$$\beta(\tau)=\frac{\kappa'(\tau)}{\kappa(\tau)}\sim\tau^{-1}.$$

Hence, if we set $y(\tau,\xi):=\kappa(\tau)^{-5/2}\rho(\xi)x(\tau,\xi)$, then we may write

$$\hat{\mathcal{D}}_c x(\tau,\xi)=\kappa(\tau)^{5/2}\rho(\xi)^{-1/2}\tilde{\mathcal{D}}_c y(\tau,\xi),\tag{3.35}$$

where $\tilde{\mathcal{D}}_c y(\tau,\xi):=\partial_1 y(\tau,\xi)-2\beta(\tau)\partial_2 y(\tau,\xi)$. Now observe that

$$\tilde{\mathcal{D}}_c y(\sigma,\omega(\tau,\sigma)^2\eta)=\partial_\sigma y(\sigma,\omega(\tau,\sigma)^2\eta)$$

since $\partial_\sigma\omega(\tau,\sigma)^2=-2\beta(\sigma)\omega(\tau,\sigma)^2$. Thus, from the definition of y and equation (3.35) we infer the identity

$$\begin{aligned}
&\hat{\mathcal{D}}_c x(\sigma,\omega(\tau,\sigma)^2\eta)\\
&\quad=\kappa(\sigma)^{5/2}\rho(\omega(\tau,\sigma)^2\eta)^{-1/2}\\
&\qquad\times\partial_\sigma[\kappa(\sigma)^{-5/2}\rho(\omega(\tau,\sigma)^2\eta)^{1/2}x(\sigma,\omega(\tau,\sigma)^2\eta)].
\end{aligned}\tag{3.36}$$

Consequently, after the change of variable $\eta \mapsto \omega(\tau,\sigma)^2\eta$, equation (3.34) can be rewritten as

$$\begin{aligned}
&\mathcal{H}_c\beta\mathcal{K}_{cc}\hat{\mathcal{D}}_c x(\tau,\xi)\\
&\quad=\int_\tau^\infty\int_0^\infty H_c(\tau,\sigma,\xi)\beta(\sigma)\omega(\tau,\sigma)^2 K_{cc}(\omega(\tau,\sigma)^2\xi,\omega(\tau,\sigma)^2\eta)\\
&\qquad\times\kappa(\sigma)^{5/2}\rho(\omega(\tau,\sigma)^2\eta)^{-1/2}\\
&\qquad\times\partial_\sigma[\kappa(\sigma)^{-5/2}\rho(\omega(\tau,\sigma)^2\eta)^{1/2}x(\sigma,\omega(\tau,\sigma)^2\eta)]\,d\eta\,d\sigma. \qquad (3.37)
\end{aligned}$$

LEMMA 3.18. *Let $\alpha > 1+\frac{3}{4}(1+\frac{1}{\nu})$. Then the operator $1+2\mathcal{H}_c\beta\mathcal{K}_{cc}\hat{\mathcal{D}}_c$ has a bounded inverse on $\mathcal{X}^\alpha$.*

Proof. Performing an integration by parts with respect to σ in equation (3.37) and noting that

$$\begin{aligned}
&\partial_\sigma\left[\xi^{-1/2}\sin\left(\xi^{1/2}\kappa(\tau)\int_\tau^\sigma\kappa^{-1}(u)\,du\right)\right]\\
&\qquad=\omega(\tau,\sigma)\cos\left(\xi^{1/2}\kappa(\tau)\int_\tau^\sigma\kappa^{-1}(u)\,du\right),
\end{aligned}$$

we obtain the decomposition $\mathcal{H}_c\beta\mathcal{K}_{cc}\hat{\mathcal{D}}_c = \mathcal{A}_1+\mathcal{A}_2$ where

$$\mathcal{A}_1 x(\tau,\xi) = -\int_\tau^\infty \omega(\tau,\sigma)\hat{H}_c(\tau,\sigma,\xi)\beta(\sigma)\mathcal{K}_{cc}x(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma$$

and

$$\mathcal{A}_2 x(\tau,\xi) = \int_\tau^\infty H_c(\tau,\sigma,\xi)O(\sigma^{-2})\tilde{\mathcal{K}}_{cc}x(\sigma,\omega(\tau,\sigma)^2\xi)\,d\sigma.$$

The kernel of the operator $\tilde{\mathcal{K}}_{cc}$ consists of a linear combination of σ-derivatives of $K_{cc}(\omega(\tau,\sigma)^2\xi,\omega(\tau,\sigma)^2\eta)$ and is therefore of the same type as the kernel of the commutator $[\mathcal{A}_c,\mathcal{K}_{cc}]$ (cf. the proof of Proposition 3.8). In particular, $\tilde{\mathcal{K}}_{cc}$ maps the space X to Y (Proposition 3.8), and Lemma 3.4 immediately yields

$$\|\mathcal{A}_2 x\|_{\mathcal{X}^\alpha} \lesssim \nu^{2\delta}\tau_0^{-1+2\delta}\|x\|_{\mathcal{X}^\alpha}.$$

Consequently, smallness can be achieved by choosing τ_0 sufficiently large (depending on ν). The operator $\mathcal{A}_1$, on the other hand, is of the same type as $\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}$ but this time viewed as a map from $\mathcal{X}^\alpha$ to $\mathcal{X}^\alpha$. However, this does not make any difference since the crucial Lemmas 3.3, 3.9 and 3.11 are valid for the space $\mathcal{X}^\alpha$ (respectively X) as well. Note carefully that the stronger requirement $\alpha > 1+\frac{11}{4}(1+\frac{1}{\nu})-\gamma$ in Lemma 3.3 is exactly compensated by the additional factor $\omega(\tau,\sigma)$ in $\mathcal{A}_1$. Hence, the operator $\mathcal{A}_1$ can be treated in the exact same fashion as $\hat{\mathcal{D}}_c\mathcal{H}_c\beta\mathcal{K}_{cc}$, and, in particular, Lemmas 3.12 and 3.15 hold accordingly for $\mathcal{A}_1$ on $\mathcal{X}^\alpha$. $\square$

As in the case of Lemma 3.17, Lemma 3.18 implies the invertibility of the matrix operator $1+2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}}$.

LEMMA 3.19. *Let $\alpha_c > \frac{3}{4}(1+\frac{1}{\nu})$ and $\alpha_c \le \alpha_d < \alpha_c + 1$. Then the operator $1 + 2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}}$ has a bounded inverse on $\mathcal{X}^{\alpha_d,\alpha_c}$.*

Proof. The proof is completely analogous to the proof of Lemma 3.17 and therefore omitted. □

3.10. Solution of the Main Equation

Now we are ready to solve the main equation (3.5). By setting $\mathcal{N} := \sum_{j=1}^{5} \mathcal{N}_j$ and $\hat{\mathcal{K}} := \mathcal{K}^2 + [\mathcal{A}, \mathcal{K}] + \mathcal{K} + (\beta'/\beta^2)\mathcal{K}$ we rewrite equation (3.5) as

$$[\hat{\mathcal{D}}^2 + \beta\hat{\mathcal{D}} + \underline{\xi}]\underline{x} = \underline{e} + \mathcal{N}(\underline{x}) - 2\beta\mathcal{K}\hat{\mathcal{D}}\underline{x} - \beta^2\hat{\mathcal{K}}\underline{x}, \tag{3.38}$$

where $\underline{e}(\tau,\xi) = \kappa(\tau)^{-2}\mathcal{F}[|\cdot|\tilde{\chi}(\tau,\cdot)e_2(\tau,\cdot)](\xi)$. By applying $\mathcal{H}$ we find that equation (3.5) is equivalent to

$$\underline{x} = \underline{\Phi}(\underline{x}) := (1 + 2\mathcal{H}\beta\mathcal{K}\hat{\mathcal{D}})^{-1}\mathcal{H}[\underline{e} + \mathcal{N}(\underline{x}) - \beta^2\hat{\mathcal{K}}\underline{x}]. \tag{3.39}$$

We claim that equation (3.39) has a solution $(\underline{x}, \hat{\mathcal{D}}\underline{x}) \in \mathcal{X}^{4/3-2\delta,4/3-2\delta} \times \mathcal{Y}^{4/3,4/3}$ with δ from Definition 3.1. In order to prove this, we have to recall some mapping properties from [1].

PROPOSITION 3.20. *We have the estimate*

$$\|\mathcal{N}(\underline{x}) - \mathcal{N}(\underline{y})\|_{\mathcal{Y}^{4/3+5/4,4/3+5/4}} \lesssim \|\underline{x} - \underline{y}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}}$$

for all $\underline{x}$, $\underline{y}$ in the unit ball in $\mathcal{X}^{4/3-2\delta,4/3-2\delta}$.

Proof. This follows by inspection of the proofs of the corresponding results in [1], in particular, Lemmas 4.4–4.10. Note also that the loss of $\tau^{1/4}$ discussed in Remark 4.8 in [1] does not occur in our case since $\kappa(\tau)^{-2}$ is bounded for all $\tau \ge \tau_0$ (unlike the corresponding $\tilde{\lambda}(\tau)^{-2}$ in [1]). In fact, as in [1], we prove that the nonlinearity maps the space X to Y, and for the time decay, we gain at least $\tau^{-(4/3-2\delta)}$ (from the quadratic part $\mathcal{N}_2$, all other contributions are even better; the linear part $\mathcal{N}_1$ yields a gain of τ^{-2}; cf. Lemma 4.5 in [1]). Since $\delta > 0$ is assumed to be small, the stated bound follows. □

Another result from [1] we require concerns the mapping properties of $\mathcal{K}$ and $\hat{\mathcal{K}}$.

PROPOSITION 3.21. *The operators $\mathcal{K}$ and $\hat{\mathcal{K}}$ satisfy the bounds*

$$\begin{aligned}
\|\mathcal{K}(a,f)\|_{\mathbb{C}\times Y} &\lesssim \|(a,f)\|_{\mathbb{C}\times Y},\\
\|\mathcal{K}(a,f)\|_{\mathbb{C}\times X} &\lesssim \|(a,f)\|_{\mathbb{C}\times X},\\
\|\hat{\mathcal{K}}(a,f)\|_{\mathbb{C}\times Y} &\lesssim \|(a,f)\|_{\mathbb{C}\times X}.
\end{aligned}$$

Proof. This follows from Corollaries 5.7 and 5.10 in [1]. □

Now we can prove the existence of a solution to equation (3.39).

THEOREM 3.22. *The function $\underline{\Phi}$ as defined in equation (3.39) maps the closed unit ball of $\mathcal{X}^{4/3-2\delta,4/3-2\delta}$ to itself and is contractive. As a consequence, there exists a unique solution $\underline{x}$ of equation (3.39) in the closed unit ball of $\mathcal{X}^{4/3-2\delta,4/3-2\delta}$.*

Proof. From Lemma 3.4 and Proposition 3.20 we infer

$$\|\mathcal{H}\mathcal{N}(\underline{x}) - \mathcal{H}\mathcal{N}(\underline{y})\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}} \lesssim \tau_0^{-\epsilon} \|\underline{x} - \underline{y}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}}$$

for all $\underline{x}$, $\underline{y}$ in the unit ball in $\mathcal{X}^{4/3-2\delta,4/3-2\delta}$ and some $\epsilon > 0$. Thus, in view of Lemma 3.19 and the contraction mapping principle, it suffices to prove smallness of the remaining terms in the appropriate spaces. In the discussion following Lemma 3.6, we have already noted that

$$\|\mathcal{H}\underline{e}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}} \lesssim \nu^{2\delta}\tau_0^{-\epsilon}$$

for some $\epsilon > 0$. Furthermore, we have

$$\|\mathcal{H}\beta^2\hat{\mathcal{K}}\underline{x}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}} \lesssim \nu^{2\delta}\tau_0^{-1+2\delta}\|\underline{x}\|_{\mathcal{X}^{4/3-2\delta,4/3-2\delta}}$$

by Lemma 3.4 and Proposition 3.21. □

Finally, we consider the derivative $\hat{\mathcal{D}}\underline{x}$.

COROLLARY 3.23. *Let $\underline{x}$ be the solution from Theorem 3.22. Then $\hat{\mathcal{D}}\underline{x}$ belongs to the closed unit ball of $\mathcal{Y}^{4/3,4/3}$.*

Proof. The claim follows by noting that equation (3.38) implies

$$\hat{\mathcal{D}}\underline{x} = (1 + 2\hat{\mathcal{D}}\mathcal{H}\beta\mathcal{K})^{-1}\hat{\mathcal{D}}\mathcal{H}[\underline{e} + \mathcal{N}(\underline{x}) - \beta^2\hat{\mathcal{K}}\underline{x}],$$

and by Propositions 3.20 and 3.21 and Lemmas 3.6, 3.4, and 3.17 we see that $\hat{\mathcal{D}}\underline{x}$ belongs to the unit ball of $\mathcal{Y}^{4/3,4/3}$. □

In light of Lemma 4.3 in [1], we infer that

$$\varepsilon(\tau, R) := R^{-1}\left[x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi\right]$$

satisfies $(\varepsilon(\tau,\cdot), \varepsilon_\tau(\tau,\cdot)) \in H^{5/4} \times H^{1/4}$, with norm vanishing as $\tau \to +\infty$. We have thus proved the following result.

THEOREM 3.24. *Let $\lambda(t)$ be as in (1.5). Then equation (2.1) admits a solution $u(t,r)$ of the form*

$$u(t,r) = \lambda^{1/2}(t)W(\lambda(t)r) + \varepsilon(t,r), \quad t \in (0, t_0),$$

with $(\varepsilon(t,\cdot), \varepsilon_t(t,\cdot)) \in H^{5/4} \times H^{1/4}$. Given $\delta > 0$, we may arrange

$$\|(\varepsilon, \varepsilon_t)\|_{(\dot{H}^1 \times L^2)(r \geq t)} < \delta.$$

Proof. This follows from Theorem 3.22 as in [11] by exploiting energy conservation and smallness of energy outside the light cone. □

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