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Weighted Multilinear Square Function Bounds

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ABSTRACT. We study the boundedness of Littlewood–Paley–Stein square functions associated to multilinear operators. We prove weighted Lebesgue space bounds for square functions under relaxed regularity and cancellation conditions that are independent of weights, which is a new result even in the linear case. For a class of multilinear convolution operators, we prove necessary and sufficient conditions for weighted Lebesgue space bounds. Using extrapolation theory, we extend weighted bounds in the multilinear setting for Lebesgue spaces with index smaller than one.

1. Introduction

Given a function $\psi : \mathbb{R}^n \to \mathbb{C}$, define $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ and the associated Littlewood–Paley–Stein-type square function

$$g_\psi(f) = \left(\int_0^\infty |\psi_t * f|^2 \frac{dt}{t}\right)^{1/2}. \tag{1.1}$$

These convolution-type square functions were introduced by Stein in the 1960s, see for example [32] or [33], and have been studied extensively since then, including classical works by Stein [32], Kurtz [24], Duoandikoetxea and Rubio de Francia [11], and more recent works by Duoandikoetxea and Seijo [12], Cheng [4], Sato [30], Duoandikoetxea [10], Wilson [35], Lerner [25], and Cruz-Uribe, Martell, and Perez [8]. Of particular relevance to this work are [24; 12; 30; 35; 8] and [25], which prove bounds for g_ψ on weighted Lebesgue spaces under various conditions on ψ. Nonconvolution variants of (1.1) were studied by Carleson [3], David, Journé, and Semmes [9], Christ and Journé [5], Semmes [31], Hofmann [22; 21], and Auscher [2], where they replaced the convolution $\psi_t * f(x)$ with

$$\Theta_t f(x) = \int_{\mathbb{R}^n} \theta_t(x,y) f(y)\, dy.$$

In [9] and [31], the authors proved L^p bounds for Littlewood–Paley–Stein square functions associated to Θ_t when $\Theta_t(b) = 0$ for some para-accretive function b. In [22; 21], this type of mean zero assumption is replaced by a local cancellation testing condition on dyadic cubes. In [3; 5] and [2], the authors replace the mean zero assumption with a Carleson measure condition for θ_t to prove L^2 bounds for the square function. The work of Carleson [3] was phrased as a characterization

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of *BMO* in terms of Carleson measures, but nonconvolution-type square function bounds are implicit in his work.

In all of the works studying g_ψ cited above, the authors assume that ψ has mean zero. In fact, if g_ψ is bounded on L^2, then ψ must have mean zero, but in the nonconvolution setting, the mean zero condition is no longer a strictly necessary one, as demonstrated in [3; 22; 21], and [2]. This phenomenon persists in the multilinear square function setting, and in this work, we explore subtle cancellation conditions for multilinear convolution and nonconvolution-type square functions and their connection with weighted Lebesgue space estimates.

The nonconvolution form of the kernel $\theta_t(x, y)$ allows for a natural extension to the multilinear setting. Define, for appropriate $\theta_t : \mathbb{R}^{(m+1)n} \to \mathbb{C}$,

$$S(f_1, \ldots, f_m)(x) = \left(\int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x)|^2 \frac{dt}{t}\right)^{1/2}, \quad \text{where} \tag{1.2}$$

$$\Theta_t(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} \theta_t(x, y_1, \ldots, y_m) \prod_{i=1}^{m} f_i(y_i)\, d\mathbf{y}, \tag{1.3}$$

and we use the notation $d\mathbf{y} = dy_1 \cdots dy_m$. When $m = 1$, that is, in the linear setting, this is the operator Θ_t mentioned above, so we use the same notation for it. We wish to find cancellation conditions on θ_t that imply boundedness properties for S, given that θ_t also satisfies some size and regularity estimates. In particular, we assume that θ_t satisfies

$$|\theta_t(x, y_1, \ldots, y_m)| \lesssim \prod_{i=1}^{m} \frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^N}, \tag{1.4}$$

$$|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x, y_1, \ldots, y_i', \ldots, y_m)| \lesssim t^{-mn}(t^{-1}|y_i - y_i'|)^\gamma \tag{1.5}$$

for all $x, y_1, \ldots, y_m, y_1', \ldots, y_m' \in \mathbb{R}^n$ and $i = 1, \ldots, m$ and some $N > n$ and $0 < \gamma \le 1$. Note that we do not require any regularity for $\theta_t(x, y_1, \ldots, y_m)$ in the x variable. Square functions associated to this type of operators have been studied in a number of recent works. In Maldonado [26] and Maldonado and Naibo [27], the authors introduce the operators (1.3), and make a natural extension of Semmes's point of view in [31] to prove bounds for a Besov-type relative of the square function S in (1.2), which they define by

$$(f_1, \ldots, f_m) \mapsto \left(\int_0^\infty \|\Theta_t(f_1, \ldots, f_m)\|_{L^p}^2 \frac{dt}{t}\right)^{1/2}.$$

When $p = 2$, this Besov-type square function agrees with the square function in (1.2). Hart [19; 20], Grafakos and Oliveira [17], and Grafakos, Liu, Maldonado, and Yang [15] proved boundedness results for discretized versions of the square function S in Lebesgue spaces under various cancellation and regularity conditions on θ_t. Strictly speaking, the discrete- and continuous-parameter square functions are different operators, but typically their boundedness properties and proof techniques are similar. That is, in each of these works, the authors proved bounds of the form $\|S(f_1,\ldots,f_m)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}}$ for minor modifications of S in various ranges of indices $p, p_1, \ldots, p_m$. The first goal of this work includes proving a weighted version of these results,

$$\|S(f_1,\ldots,f_m)\|_{L^p(w^p)} \lesssim \prod_{i=1}^{m} \|f\|_{L^{p_i}(w_i^{p_i})} \tag{1.6}$$

for appropriate $1 < p_1, \ldots, p_m < \infty$, $w_i^{p_i} \in A_{p_i}$, and $w = w_1 \cdots w_m$. We use L^p to denote $L^p(\mathbb{R}^n, dx)$ and $L^p(w) = L^p(\mathbb{R}^n, w(x)\,dx)$, where dx is the Lebesgue measure on $\mathbb{R}^n$, and $w \geq 0$ is a locally integrable function. Our main result is the following theorem.

THEOREM 1.1. *Assume that θ_t satisfies* (1.4) *and* (1.5). *Then the following cancellation conditions are equivalent*:

i. *Θ_t satisfies the strong Carleson condition*,
ii. *Θ_t satisfies the Carleson and two-cube testing conditions*.

Furthermore, if the equivalent conditions (i) *and* (ii) *hold, then S satisfies* (1.6) *for all $w_i^{p_i} \in A_{p_i}$ where $w = w_1 \cdots w_m$, $1 < p_1, \ldots, p_m < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_m$, and $f_i \in L^{p_i}(w_i^{p_i})$.*

We denote by A_p the class of Muckenhoupt weights, which will be precisely defined in the next section. For the definitions of the Carleson, strong Carleson, and two-cube testing conditions, see Section 3. For now, we only note that these conditions quantify some cancellation of θ_t and that $\Theta_t(1,\ldots,1) = 0$ for all $t > 0$ implies all three of these conditions. It is of interest to note that there is no mention of weighted estimates in the hypotheses of Theorem 1.1, but we conclude the boundedness of S in weighted Lebesgue spaces. Also this is the first result for multilinear square functions of this type where S is bounded for $1/m < p < 2$ and $\Theta_t(1,\ldots,1)$ is not necessarily zero for all t.

An approach that has been used to prove bounds for S with $1/m < p \leq 1$ is to view $\{\Theta_t\}_{t>0}$ as a Calderón–Zygmund operator taking values in $L^2(\mathbb{R}_+, \frac{dt}{t})$, reproduce the classical Calderón–Zygmund theory to prove a weak endpoint bound, and interpolate with bounds for $p > 1$. But in order for $\{\Theta_t\}_{t>0}$ to be a Calderón–Zygmund operator, one must require a regularity condition in the first variable of θ_t. In this paper, we prove estimates for $1/m < p \leq 1$ without assuming any regularity for θ_t in the x variable. We use almost orthogonality estimates and Carleson-type bounds adapted to a weighted setting and extend bounds to indices $p < 1$ by the weight extrapolation of Grafakos and Martell [16].

We also prove a stronger result for square functions associated to a certain class of multiconvolution operators. We prove necessary and sufficient cancellation conditions for boundedness properties of S when Θ_t is given by convolution for each $t > 0$. We state these results precisely in the following theorem.

THEOREM 1.2. *Suppose that $\theta_t(x, y_1, \ldots, y_m) = t^{-mn}\Psi^t(t^{-1}(x - y_1), \ldots, t^{-1}(x - y_1))$ satisfies* (1.4) *and* (1.5) *for some collection of functions $\Psi^t : \mathbb{R}^{mn} \to \mathbb{C}$ depending on $t > 0$. Then the following are equivalent*:

i. Θ_t *satisfies the Carleson condition.*
ii. S *satisfies the unweighted version of* (1.6) *for some* $1 < p_1, \dots, p_m < \infty$ *and* $2 \le p < \infty$ *that satisfy* $1/p = 1/p_1 + \cdots + 1/p_m$, *that is*, (1.6) *with* $w_1 = \cdots = w_m = w = 1$.
iii. S *satisfies* (1.6) *for all* $1 < p_1, \dots, p_m < \infty$ *that satisfy* $1/p = 1/p_1 + \cdots + 1/p_m$, $w_i^{p_i} \in A_{p_i}$, *where* $w = w_1 \cdots w_m$, *and* $f_i \in L^{p_i}(w_i^{p_i})$.
iv. Θ_t *satisfies the strong Carleson condition.*

Furthermore, if $\Psi^t = \Psi$ *is constant in* t, *then conditions* (i)–(iv) *are equivalent to* $\Theta_t(1, \dots, 1) = 0$ *as well.*

It should be noted that parts of Theorem 1.1 are already known. It was proved by Carleson [3] (with minor modifications to adapt to the multilinear setting) that if Θ_t satisfies the Carleson condition, then S satisfies the unweighted version of (1.6) with $p = 2$, where $w_1 = \cdots = w_m = w = 1$. If regularity in the x variable is assumed as well, then square function estimates for $p \ge 2$ can be obtained, see Corollary 4.2 of [18]. We do not assume this regularity in x, so the interpolation result from [18] cannot be applied here. We obtain the same estimates and more without assuming regularity in the x variable. Prior to this work, there do not seem to be any bounds for square functions when $p \ne 2$, and the kernel θ_t does not satisfy regularity estimates in x.

If $\theta_t(x, y_1, \dots, y_m) = \Psi_t(x - y_1, \dots, x - y_m)$, then some of the estimates in Theorem 1.2 are known as well. Note that in this convolution situation, (1.5) implies a regularity estimate in the x variable as well and $\Theta_t(1, \dots, 1) = 0$. With these conditions satisfied, all linear results have been shown in [32; 33; 24; 11; 12; 4; 30; 35; 25; 8; 10], and the multilinear unweighted estimates in (1.6) were shown in [27; 19; 20; 17; 15]. The contribution of Theorem 1.2 is largely in the multilinear weighted setting and when $\theta_t(x, y_1, \dots, y_m) = \Psi^t(x - y_1, \dots, x - y_m)$. Theorem 1.2 also provides evidence that the strong Carleson condition is not too restrictive since when Θ_t is a multiconvolution operator, the strong Carleson condition is equivalent to the Carleson condition.

We organize the article in the following way. In Section 2, we prove some convergence and boundedness results for S when $\Theta_t(1, \dots, 1) = 0$. In Section 3, we prove various properties relating the Carleson, strong Carleson, and two-cube testing conditions to each other and some bounds for S. Finally in Section 4, we prove Theorems 1.1 and 1.2.

2. A Reduced T1 Theorem for Square Functions on Weighted Spaces

It is well known that (1.4) implies that $|\Theta_t(f_1, \dots, f_m)(x)| \lesssim Mf_1(x) \cdots Mf_m(x)$, where M is the Hardy–Littlewood maximal function, and hence

$$\sup_{t>0} \|\Theta_t(f_1, \dots, f_m)\|_{L^p} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}$$

when $1 < p_1, \dots, p_m < \infty$ satisfy the Hölder-type relationship

$$\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}. \tag{2.1}$$

So it is natural to expect that $p_1, \dots, p_m$ satisfy this relationship for square function bounds of the form (1.6). For the remainder of this work, we assume that $1 < p_1, \dots, p_m < \infty$ and p is defined by (2.1).

When we are in the linear setting, with a convolution operator $\theta_t(x, y) = \psi_t(x - y) = t^{-n}\psi(t^{-1}(x - y))$, we use the g_ψ notation from (1.1) to avoid confusion with the square function S and to emphasize that we are using the known Littlewood–Paley–Stein theory.

DEFINITION 2.1. Let w be a nonnegative locally integrable function. For $p > 1$, we say that w is an $A_p = A_p(\mathbb{R}^n)$ weight, written $w \in A_p$, if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Also define the Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \quad \text{for } \xi \in \mathbb{R}^n.$$

The following lemma says that approximation to the identity operators have essentially the same convergence properties in weighted L^p spaces as in unweighted spaces. This result is well known (an explicit proof is available, for example, in the work of Wilson [35]), but for the reader's convenience, we state the result precisely and give a short proof.

LEMMA 2.2. *Let $P_t f = \phi_t * f$ where $|\phi(x)| \lesssim 1/(1 + |x|)^N$ for some $N > n$ with $\widehat{\phi}(0) = 1$ and $w \in A_p$ for some $1 < p < \infty$.*

i. *If $f \in L^p(w)$, then $P_t f \to f$ in $L^p(w)$ as $t \to 0$.*
ii. *If $f \in L^p(w)$ and there exists a $1 \le q < \infty$ such that $f \in L^q$, then $P_t f \to 0$ in $L^p(w)$ as $t \to \infty$.*

Proof. We first prove (i) by estimating

$$\|P_t f - f\|_{L^p(w)} \le \int_{\mathbb{R}^n} |\phi(y)| \|f(\cdot - ty) - f(\cdot)\|_{L^p(w)}\, dy.$$

The integrand $|\phi(y)| \|f(\cdot - ty) - f(\cdot)\|_{L^p(w)}$ is controlled by $2\|f\|_{L^p(w)} |\phi(y)|$, which is an integrable function. So, by dominated convergence,

$$\lim_{t\to 0} \|P_t f - f\|_{L^p(w)} \le \int_{\mathbb{R}^n} |\phi(y)| \lim_{t\to 0} \|f(\cdot - ty) - f(\cdot)\|_{L^p(w)}\, dy = 0.$$

Therefore (i) holds. Now for (ii), suppose that $f \in L^p(w) \cap L^q(\mathbb{R}^n)$ for some $1 \le q < \infty$. Then it follows that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |P_t f(x)| &\le \|\phi_t\|_{L^{q'}} \|f\|_{L^q} \\ &\lesssim t^{-n/q} \left(\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{Nq'}} \right)^{1/q'} \|f\|_{L^q} \\ &\lesssim t^{-n/q} \|f\|_{L^q}, \end{aligned}$$

which tends to 0 as $t \to \infty$. So $P_t f \to 0$ a.e. in $\mathbb{R}^n$. Furthermore, $|P_t f(x)| \lesssim Mf(x)$, where M is the Hardy–Littlewood maximal operator, and $Mf \in L^p(w)$ since $f \in L^p(w)$ and $1 < p < \infty$. Then by dominated convergence we have

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} |P_t f(x)|^p w(x)\, dx = \int_{\mathbb{R}^n} \lim_{t\to\infty} |P_t f(x)|^p w(x)\, dx = 0.$$

So it follows that $P_t f \to 0$ in $L^p(w)$ as $t \to \infty$. □

LEMMA 2.3. *Suppose that* θ_t *satisfies* (1.4), $P_t f = \phi_t * f$ *where* $\phi \in C_0^\infty$ *with* $\widehat{\phi}(0) = 1$, *and* $w_i^{p_i} \in A_{p_i}$ *for* $1 < p, p_1, \dots, p_m < \infty$ *satisfying* (2.1). *Define* $w = w_1 \cdots w_m$. *Then, for* $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$,

$$\Theta_t(f_1, \dots, f_m) = \sum_{j=1}^m \int_0^\infty \Theta_t \Pi_{j,s}(f_1, \dots, f_m) \frac{ds}{s}, \tag{2.2}$$

where the convergence holds in $L^p(w^p)$, *and for* $j = 1, \dots, m$, $\Pi_{j,s}$ *is defined by*

$$\Pi_{j,t}(f_1, \dots, f_m) = P_t^2 f_1 \otimes \cdots \otimes P_t^2 f_{j-1} \otimes Q_t f_j \otimes P_t^2 f_{j+1} \otimes \cdots \otimes P_t^2 f_m,$$

$Q_t f = \psi_t * f$, *and* $\psi_t = -t \frac{d}{dt}(\phi_t * \phi_t)$. *Furthermore, there exist* $Q_t^{i,k} f = \psi_t^{i,k} * f$ *where* $\psi^{i,k} \in C_0^\infty$ *have mean zero for* $i = 1, 2$ *and* $k = 1, \dots, n$ *and*

$$Q_t = \sum_{k=1}^n Q_t^{1,k} Q_t^{2,k}.$$

Proof. We note that since $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$, by Lemma 2.2, $P_t^2 f_i \to f_i$ as $t \to 0$ and $P_t^2 f_i \to 0$ as $t \to \infty$ in $L^{p_i}(w_i^{p_i})$. Then it follows that

$$\begin{aligned} &\left\| \Theta_t(f_1, \dots, f_m) - \sum_{j=1}^m \int_\varepsilon^{1/\varepsilon} \Theta_t \Pi_{j,s}(f_1, \dots, f_m) \frac{ds}{s} \right\|_{L^p(w^p)} \\ &\quad = \left\| \Theta_t(f_1, \dots, f_m) + \int_\varepsilon^{1/\varepsilon} s \frac{d}{ds} \Theta_t(P_s^2 f_1, \dots, P_s^2 f_m) \frac{ds}{s} \right\|_{L^p(w^p)} \\ &\quad \le \|\Theta_t(f_1, \dots, f_m) - \Theta_t(P_\varepsilon^2 f_1, \dots, P_\varepsilon^2 f_m)\|_{L^p(w^p)} \\ &\qquad + \|\Theta_t(P_{1/\varepsilon}^2 f_1, \dots, P_{1/\varepsilon}^2 f_m)\|_{L^p(w^p)} \\ &\quad \le \sum_{j=1}^m \|\Theta_t(P_\varepsilon^2 f_1, \dots, P_\varepsilon^2 f_{j-1}, f_j - P_\varepsilon^2 f_j, f_{j+1}, \dots, f_m)\|_{L^p(w^p)} \end{aligned}$$

$$+ \|\Theta_t(P^2_{1/\varepsilon} f_1, \dots, P^2_{1/\varepsilon} f_m)\|_{L^p(w^p)}$$
$$\lesssim \sum_{j=1}^m \|Mf_1 \cdots Mf_{j-1} M(f_j - P_\varepsilon f_j) f_{j+1} \cdots f_m\|_{L^p(w^p)}$$
$$+ \|MP^2_{1/\varepsilon} f_1 \cdots MP^2_{1/\varepsilon} f_m\|_{L^p(w^p)}$$
$$\lesssim \sum_{j=1}^m \|f_j - P_\varepsilon f_j\|_{L^{p_j}(w_j^{p_j})} \prod_{i \neq j} \|f_i\|_{L^{p_i}(w_i^{p_i})} + \prod_{i=1}^m \|P^2_{1/\varepsilon} f_i\|_{L^{p_i}(w_i^{p_i})}.$$

As $\varepsilon \to 0$, the above expression tends to zero. Therefore, (2.2) holds, where the convergence is in the topology of $L^p(w^p)$. One can verify that $\psi^{1,k}(x) = -2\partial_{x_k}\phi(x)$ and $\psi^{2,k}(x) = x_k\phi(x)$ satisfy the conditions given above. For details, this decomposition of Q_t was done in the linear one-dimensional case by Coifman and Meyer [6]; the n-dimensional version can be found, for example, in Grafakos [14]. □

Lemma 2.4. *Let P_t, Q_t, $Q_t^{i,j}$, and $\Pi_{j,s}$ be as in Lemma 2.3. If θ_t satisfies (1.4)–(1.5) and $\Theta_t(1,1) = 0$ for all $t > 0$, then for all $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$, $s > 0$, $j = 1, \dots, m$, and $x \in \mathbb{R}^n$,*

$$|\Theta_t \Pi_{j,s}(f_1, \dots, f_m)(x)| \lesssim \left(\frac{s}{t} \wedge \frac{t}{s}\right)^{\gamma'} \sum_{k=1}^n MQ_s^{2,k} f_j(x) \prod_{i \neq j} Mf_i(x)$$

for some $0 < \gamma' \leq \gamma$, where $u \wedge v = \min(u, v)$ for $u, v > 0$.

This lemma is a pointwise result that was proved in the discrete bilinear setting in [19]. We make the appropriate modifications here to prove this multilinear continuous version.

Proof of Lemma 2.4. For this proof, we define, for $M, t > 0$ and $x \in \mathbb{R}^n$,

$$\Phi_t^M(x) = \frac{t^{-n}}{(1 + t^{-1}|x|)^M}. \tag{2.3}$$

It follows immediately that $\Phi_t^{M+d} \leq \Phi_t^M$ for any $d \geq 0$, and there is a well-known almost orthogonality result, for any $M, L > n$ and $s, t > 0$,

$$\int_{\mathbb{R}^n} \Phi_t^M(x-u)\Phi_s^L(u-y)\,du \lesssim \Phi_s^{M\wedge L}(x-y) + \Phi_t^{M\wedge L}(x-y). \tag{2.4}$$

It is not entirely clear who first formulated this estimate as stated here, but a proof can be a found in the appendix of [14]. Note also that if we take $\eta = \frac{N-n}{2(N+\gamma)}$, $\gamma' = \eta\gamma$, and $N' = (1-\eta)N - \gamma'$, then using a geometric mean with weights $1-\eta$ and η of estimates (1.4) and (1.5), it follows that

$$|\theta_t(x, y_1, \dots, y_m) - \theta_t(x, y_1', y_2, \dots, y_m)|$$
$$\lesssim t^{-\eta mn}(t^{-1}|y_1 - y_1'|)^{\eta\gamma} \left(\prod_{j=2}^m \Phi_t^N(x - y_j)\right)^{1-\eta}$$

$$
\begin{aligned}
&\times (\Phi_t^N(x-y_1)+\Phi_t^N(x-y_1'))^{1-\eta}\\
&\le (t^{-1}|y_1-y_1'|)^{\gamma'}(\Phi_t^{N'+\gamma'}(x-y_1)+\Phi_t^{N'+\gamma'}(x-y_1'))\\
&\times \prod_{j=2}^m \Phi_t^{N'+\gamma'}(x-y_j).
\end{aligned}
$$

It is a direct computation to show that $0<\gamma'=\gamma\frac{N-n}{2(N+\gamma)}<\gamma$ and $n<N'=\frac{N+n}{2}\le N-\gamma'$. We will first look at the kernel of $\Theta_t(Q_s^{1,k}\cdot, P_s\cdot,\dots,P_s\cdot)$ for $k=1,\dots,m$, which is

$$\sum_{k=1}^n \int_{\mathbb{R}^{mn}} \theta_t(x,u_1,\dots,u_m)\psi_s^{1,k}(u_1-y_1)\prod_{i=2}^m \phi_s(u_i-y_i)\,d\mathbf{u}.$$

Our goal here is to bound this kernel by a product of $\Phi_s^{N'}(x-y_j)+\Phi_t^{N'}(x-y_j)$. So in the following computations, whenever possible, we pull out terms of this form. There will also appear terms of the form $\Phi_t^{N'}(x-u_j)$ and $\Phi_s^{N'}(u-y_j)$, for which we will use (2.4) and bound by appropriate functions Φ depending on s, t, N', and $x-y_j$. We estimate the kernel for a fixed $k=1,\dots,m$ and simplify the notation; define

$$\lambda_s(y_1,\dots,y_m)=\psi_s^{1,k}(y_1)\prod_{i=2}^m \phi_s(y_i).$$

Then for $s<t$, using that $\lambda_s(y_1,\dots,y_m)$ has mean zero in y_1 (since $\psi_s^{1,k}$ has mean zero), $\psi^{1,k},\phi\in C_0^\infty$, and θ_t satisfies (1.4) and (1.5), it follows that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^{mn}} \theta_t(x,u_1,\dots,u_m)\lambda_s(u_1-y_1,\dots,u_m-y_m)\,d\mathbf{u}\right|\\
&\quad\lesssim \int_{\mathbb{R}^{mn}} |\theta_t(x,u_1,\dots,u_m)-\theta_t(x,y_1,u_2,\dots,u_m)|\\
&\quad\quad\times\left(\prod_{j=1}^m \Phi_s^{N'+\gamma'}(u_j-y_j)\right)d\mathbf{u}\\
&\quad\lesssim \int_{\mathbb{R}^{mn}} (t^{-1}|u_1-y_1|)^{\gamma'}\Phi_t^{N'+\gamma'}(x-y_1)\Phi_s^{N'+\gamma'}(u_1-y_1)\\
&\quad\quad\times\prod_{j=2}^m(\Phi_t^{N'+\gamma'}(x-u_j)\Phi_s^{N'+\gamma'}(u_j-y_j))\,d\mathbf{u}\\
&\quad\quad+\int_{\mathbb{R}^{mn}}(t^{-1}|u_1-y_1|)^{\gamma'}\prod_{j=1}^m(\Phi_t^{N'+\gamma'}(x-u_j)\Phi_s^{N'+\gamma'}(u_j-y_j))\,d\mathbf{u}\\
&\quad\le \frac{s^{\gamma'}}{t^{\gamma'}}\Phi_t^{N'+\gamma'}(x-y_1)
\end{aligned}
$$

$$\times \int_{\mathbb{R}^{mn}} \Phi_s^{N'}(u_1 - y_1) \prod_{j=2}^{m} (\Phi_t^{N'+\gamma'}(x - u_j)\Phi_s^{N'+\gamma'}(u_j - y_j))\, d\mathbf{u}$$

$$+ \frac{s^{\gamma'}}{t^{\gamma'}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} (\Phi_t^{N'+\gamma'}(x - u_j)\Phi_s^{N'}(u_j - y_j))\, d\mathbf{u}$$

$$\lesssim \frac{s^{\gamma'}}{t^{\gamma'}} \prod_{j=1}^{m} (\Phi_s^{N'}(x - y_j) + \Phi_t^{N'}(x - y_j)). \tag{2.5}$$

We use that $(t^{-1}|u_1 - y_1|)^{\gamma'} \Phi_s^{N'+\gamma'}(u_1 - y_1) \le (s^{\gamma'}/t^{\gamma'})\Phi_s^{N'}(u_1 - y_1)$. Now for $s > t$, we use the assumption $\Theta_t(1, \dots, 1) = 0$ and that θ_t satisfies (1.4) for the following estimate:

$$\left| \int_{\mathbb{R}^{mn}} \theta_t(x, u_1, \dots, u_m)\lambda_s(u_1 - y_1, \dots, u_m - y_m)\, d\mathbf{u} \right|$$

$$\lesssim \int_{\mathbb{R}^{mn}} \prod_{j=1}^{m} \Phi_t^{N'+\gamma'}(x - u_j)$$

$$\times |\lambda_s(u_1 - y_1, \dots, u_m - y_m) - \lambda_s(x - y_1, \dots, x - y_m)|\, d\mathbf{u}. \tag{2.6}$$

Next, we work to control the second term in the integrand on the right-hand side of (2.6). Adding and subtracting successive terms, we get

$$|\lambda_s(u_1 - y_1, \dots, u_m - y_m) - \lambda_s(x - y_1, \dots, x - y_m)|$$

$$\le \sum_{\ell=1}^{m} |\lambda_s(u_1 - y_1, \dots, u_{\ell-1} - y_{\ell-1}, x - y_\ell, \dots, x - y_m)$$

$$- \lambda_s(u_1 - y_1, \dots, u_\ell - y_\ell, x - y_{\ell+1}, \dots, x - y_m)|$$

$$\lesssim \sum_{\ell=1}^{m} (s^{-1}|x - u_\ell|)^{\gamma'} \left(\prod_{r=1}^{\ell-1} \Phi_s^{N'+\gamma'}(u_r - y_r) \right)$$

$$\times (\Phi_s^{N'+\gamma'}(u_\ell - y_\ell) + \Phi_s^{N'+\gamma'}(x - y_\ell))$$

$$\times \left(\prod_{r=\ell+1}^{m} \Phi_s^{N'+\gamma'}(x - y_r) \right).$$

Here we use the convention that $\prod_{j=1}^{0} A_j = \prod_{j=m+1}^{m} A_j = 1$ to simplify the notation. Then (2.6) is bounded by a constant times

$$\sum_{\ell=1}^{m} \int_{\mathbb{R}^{mn}} \left(\prod_{j=1}^{m} \Phi_t^{N'+\gamma'}(x - u_j) \right) (s^{-1}|x - u_\ell|)^{\gamma'} \left(\prod_{r=1}^{\ell-1} \Phi_s^{N'+\gamma'}(u_r - y_r) \right)$$

$$\times (\Phi_s^{N'+\gamma'}(u_\ell - y_\ell) + \Phi_s^{N'+\gamma'}(x - y_\ell)) \left(\prod_{r=\ell+1}^{m} \Phi_s^{N'+\gamma'}(x - y_r) \right) d\mathbf{u}$$

$$\leq \frac{t^{\gamma'}}{s^{\gamma'}} \sum_{\ell=1}^{m} \int_{\mathbb{R}^{mn}} \Big(\prod_{j=1}^{m} \Phi_t^{N'}(x-u_j) \Big) \Big(\prod_{r=1}^{\ell-1} \Phi_s^{N'+\gamma'}(u_r - y_r) \Big)$$
$$\times (\Phi_s^{N'+\gamma'}(u_\ell - y_\ell) + \Phi_s^{N'+\gamma'}(x-y_\ell)) \Big(\prod_{r=\ell+1}^{m} \Phi_s^{N'+\gamma'}(x-y_r) \Big) d\mathbf{u}$$
$$\leq \frac{t^{\gamma'}}{s^{\gamma'}} \sum_{\ell=1}^{m} \Big(\prod_{r=1}^{\ell-1} \int_{\mathbb{R}^n} \Phi_t^{N'}(x-u_r) \Phi_s^{N'+\gamma'}(u_r - y_r)\, du_r \Big)$$
$$\times \Big(\int_{\mathbb{R}^n} \Phi_t^{N'}(x-u_\ell)(\Phi_s^{N'+\gamma'}(u_\ell - y_\ell) + \Phi_s^{N'+\gamma'}(x-y_\ell))\, du_\ell \Big)$$
$$\times \Big(\prod_{r=\ell+1}^{m} \int_{\mathbb{R}^n} \Phi_t^{N'}(x-u_r) \Phi_s^{N'+\gamma'}(x-y_r)\, du_r \Big)$$
$$\lesssim \frac{t^{\gamma'}}{s^{\gamma'}} \prod_{r=1}^{m} (\Phi_s^{N'}(x-y_r) + \Phi_t^{N'}(x-y_r)). \tag{2.7}$$

The following estimate easily follows from (2.5) and (2.7):

$$\Big| \int_{\mathbb{R}^{mn}} \theta_t(x, u_1, \ldots, u_m) \psi_s^{1,k}(u_1 - y_1) \prod_{i=2}^{m} \phi_s(u_i - y_i)\, d\mathbf{u} \Big|$$
$$\lesssim \Big(\frac{s}{t} \wedge \frac{t}{s} \Big)^{\gamma'} \prod_{j=1}^{m} (\Phi_s^{N'}(x-y_j) + \Phi_t^{N'}(x-y_j)).$$

Since $|\Phi_t^{N'} * f(x)| \lesssim Mf(x)$ uniformly in t and $\Theta_t \Pi_{s,1} = \sum_{k=1}^{n} \Theta(Q_s^{1,k} Q_s^{2,k}, P_s^2, \ldots, P_s^2)$, it follows that

$$|\Theta_t \Pi_{s,1}(f_1, \ldots, f_m)(x)| \lesssim \Big(\frac{s}{t} \wedge \frac{t}{s} \Big)^{\gamma'} \sum_{k=1}^{n} M Q_s^{2,k} f_1(x) \prod_{j=2}^{m} Mf_j(x).$$

By symmetry, this completes the proof. □

Next, we work to set the square function results of [19; 20; 17] and [15] in weighted Lebesgue spaces. This is a type of reduced T1 theorem for $L^2(\mathbb{R}_+, \frac{dt}{t})$-valued singular integral operators, where we assume that $\Theta_t(1, \ldots, 1) = 0$ for all $t > 0$. We refer to Theorem 2.5 as a reduced T1 theorem since it applies to operators that satisfy the relatively strong cancellation condition $\Theta_t(1, \ldots, 1) = 0$ for $t > 0$. Also, to prove the more general Theorem 1.1, we reduce the boundedness of operators with Carleson-type cancellation to those with $\Theta_t(1, \ldots, 1) = 0$ cancellation, as in Theorem 2.5 We now state and prove a reduced T(1) theorem for square functions on weighted spaces.

THEOREM 2.5. *Let* Θ_t *and* S *be defined as in* (1.3) *and* (1.2), *where* θ_t *satisfies* (1.4) *and* (1.5). *If* $\Theta_t(1, \ldots, 1) = 0$ *for all* $t > 0$, *then* S *satisfies* (1.6) *for all* $w_i^{p_i} \in A_{p_i}$, $1 < p, p_1, \ldots, p_m < \infty$ *satisfying* (2.1), *where* $w = \prod_{i=1}^{m} w_i$, *and*

$f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$. *Furthermore, the constant for this bound is at most a constant independent of* $w_1, \ldots, w_m$ *times*

$$\prod_{i=1}^{m}(1 + [w_i^{p_i}]_{A_{p_i}}^{\max(1,p_i'/p_i)+\max(1/2,p_i'/p_i)}).$$

Proof. Let P_t, Q_t, et cetera be defined as in Lemma 2.3, $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$ for $i = 1, \ldots, m$ and $h_t \in L^{p'}$ for all $t > 0$ such that

$$\left\| \left(\int_0^\infty |h_t|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^{p'}(w^p)} \le 1.$$

Recall that the dual of $L^p(w^p)$ can be realized as $L^{p'}(w^p)$ if we take the measure space $(\mathbb{R}^n, w(x)^p\,dx)$. We estimate (1.6) by duality, making use of Lemmas 2.3 and 2.4:

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \int_0^\infty \Theta_t(f_1, \ldots, f_m)(x) h_t(x) \frac{dt}{t} w(x)^p\,dx \right| \\
&\qquad \le \int_{\mathbb{R}^n} \int_0^\infty \sum_{j=1}^m \int_0^\infty |\Theta_t \Pi_{j,s}(f_1, \ldots, f_m)(x)| \\
&\qquad\quad \times w(x)|h_t(x)|w(x)^{p/p'} \frac{ds}{s} \frac{dt}{t}\,dx \\
&\qquad \le \sum_{j=1}^m \left\| \left(\int_{(0,\infty)^2} \left(\frac{s}{t} \wedge \frac{t}{s} \right)^{-\gamma'} |\Theta_t \Pi_{j,s}(f_1, \ldots, f_m)(x)|^2 \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w^p)} \\
&\qquad\quad \times \left\| \left(\int_{(0,\infty)^2} \left(\frac{s}{t} \wedge \frac{t}{s} \right)^{\gamma'} |h_t|^2 \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \right\|_{L^{p'}(w^p)} \\
&\qquad \lesssim \sum_{j=1}^m \sum_{k=1}^n \left\| \left(\int_{[0,\infty)^2} \left(\frac{s}{t} \wedge \frac{t}{s} \right)^{\gamma'} \left(M Q_s^{2,k} f_j \prod_{i \ne j} M f_i \right)^2 \frac{dt}{t} \frac{ds}{s} \right)^{1/2} \right\|_{L^p(w^p)} \\
&\qquad \lesssim \sum_{j=1}^m \sum_{k=1}^n \left\| \left(\int_0^\infty (M Q_s^{2,k} f_j)^2 \frac{ds}{s} \right)^{1/2} \prod_{i \ne j} M f_i \right\|_{L^p(w^p)} \\
&\qquad \lesssim \sum_{j=1}^m \sum_{k=1}^n [w_j^{p_j}]_{A_{p_j}}^{\max(1/2,p_j'/p_j)} \|g_{\psi^{2,k}}(f_j)\|_{L^{p_j}(w_j^{p_j})} \prod_{i \ne j} \|M f_i\|_{L^{p_i}(w_i^{p_i})} \\
&\qquad \lesssim \sum_{j=1}^m [w_j^{p_j}]_{A_{p_j}}^{\max(1,p_j'/p_j)+\max(1/2,p_j'/p_j)} \\
&\qquad\quad \times \|f_j\|_{L^{p_j}(w_j^{p_j})} \prod_{i \ne j} [w_i^{p_i}]_{A_{p_i}}^{p_i'/p_i} \|f_i\|_{L^{p_i}(w_i^{p_i})} \\
&\qquad \lesssim \prod_{i=1}^m (1 + [w_j^{p_j}]_{A_{p_j}}^{\max(1,p_j'/p_j)+\max(1/2,p_j'/p_j)}) \|f_i\|_{L^{p_i}(w_i^{p_i})}.
\end{aligned}$$

Here we have used the weighted bound for the Hardy–Littlewood maximal function, the Fefferman–Stein vector-valued maximal function bound proved originally by Andersen–John [1] and proved with the sharp dependence on the weight constant by Cruz-Uribe, Martell, and Perez [8]. We also used the weighted square function estimate for $g_{\psi^{2,k}}$ for $k = 1, \dots, m$ originally proved by Kurtz [24] and proved with sharp dependence on the weight constant by Lerner [25]. □

Although we use sharp estimates to track the weight constant dependence, we do not claim that this bound for S is sharp. In the above argument, once we have bounded the dual pairing by products of maximal functions and g_ψ functions, the estimates may be sharp, but there is no evidence provided here that the estimates up to that point are sharp. We track the constant so that we can explicitly apply the extrapolation theorem of Grafakos and Martell [16].

3. Carleson and Strong Carleson Measures

This section is dedicated to defining the cancellation conditions that we will use for θ_t and proving some properties about them. We start with a discussion to motivate these definitions and describe the role that they will play in this work.

As discussed in the introduction, in the linear convolution operator setting with convolutions kernel ψ_t, if g_ψ is bounded, then necessarily $\psi_t * 1 = 0$ for all $t > 0$. So when working with the square function g_ψ with $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, it is not useful to consider Carleson measure type cancellation conditions like (i) from Theorem 1.1. But if one does not require the convolution kernels ψ_t to be the dilations of a single function ψ or allows for the nonconvolution operators, then mean zero is not a necessary condition for square function bounds. From the classical theory of Carleson measures [3] we know that, in the linear setting, S is bounded on L^2 if and only if $|\Theta_t(1)(x)|^2 \frac{dt\,dx}{t}$ is a Carleson measure, although this may not in general be sufficient for S to be bounded for all $1 < p < \infty$. We will define the strong Carleson condition for Θ_t and prove that it does imply bounds for all $1 < p < \infty$.

There is a stronger notion of Carleson measure defined in terms of A_2 weights by Journé [23] that is related to some of the Carleson conditions in this work. For more information, see Chapter 6, Section II, in [23]. We will discuss this in a little more depth in Section 4.

DEFINITION 3.1. A nonnegative measure $d\mu(x,t)$ on $\mathbb{R}^{n+1}_+ = \{(x,t)\colon\ x \in \mathbb{R}^n, t > 0\}$ is a *Carleson* measure if

$$\|d\mu\|_{\mathcal{C}} = \sup_Q \frac{1}{|Q|} d\mu(T(Q)) < \infty, \tag{3.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, $|Q|$ denotes the Lebesgue measure of a cube Q, $T(Q) = Q \times (0, \ell(Q)]$ denotes the *Carleson box* over Q, and $\ell(Q)$ is the side length of Q.

Suppose that $d\mu$ is a nonnegative measure on $\mathbb{R}^{n+1}_+$ defined by

$$d\mu(x,t) = F(x,t)\,d\tau(t)\,dx \tag{3.2}$$

for some $F \in L^1_{\text{loc}}(\mathbb{R}^{n+1}_+, d\tau(t)\,dx)$. We say that $d\mu$ is a *strong Carleson* measure if

$$\|d\mu\|_{\mathcal{SC}} = \sup_Q \sup_{x\in Q} \int_0^{\ell(Q)} F(x,t)\,d\tau(t) < \infty. \tag{3.3}$$

Given an operator Θ_t with kernel satisfying (1.4), we say that Θ_t satisfies the Carleson condition, respectively strong Carleson condition, if $|\Theta_t(1,\dots,1)(x)|^2 \frac{dt}{t}\,dx$ is a Carleson measure, respectively a strong Carleson measure.

There are a few related notions of Carleson measures that appear to be very similar to the strong Carleson condition defined here, but there are subtle differences between them. For example, Fefferman and Stein [13] verify weighted estimates for measures $d\mu$ and weights w that satisfy

$$\sup_{Q\ni x} \frac{d\mu(T(Q))}{|Q|} \le C w(x).$$

This estimate has a weight on the right-hand side of the inequality, but no weight on the left-hand side. On the other hand, our estimate (3.3) involves no weights at all. These two conditions are related somehow, but they differ in the way that they interact with weight functions and weighted estimates. The measures studied by Fefferman and Stein [13] are generalized to a sort of A_p weight condition for measures by Ruiz [28] and Ruiz and Torrea [29], although they differ from our Carleson measures in the same way that the measures of Fefferman and Stein [13] do.

We use these Carleson conditions for Θ_t to quantify weaker cancellation conditions on the kernels θ_t. The situation $\Theta_t(1,\dots,1) = 0$ for $t > 0$ is, in a way, "perfect" cancellation for Θ_t since the integral of $\theta_t(x, y_1, \dots, y_m)$ in the $d\mathbf{y}$ vanishes. These Carleson conditions relax this cancellation condition by requiring that $\Theta_t(1,\dots,1)$ is small, rather than 0, in the sense that $|\Theta_t(1,\dots,1)(x)| \frac{dt\,dx}{t}$ defines a Carleson or strong Carleson measure. Using Carleson measure estimates for $|\Theta_t(1,\dots,1)(x)|^2 \frac{dt\,dx}{t}$ to derive boundedness properties for Θ_t and S is a very common technique. In the language Christ and Journé [5] and Auscher [2], a Carleson function is a function $G : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ such that $|G(x,t)|^2 \frac{dt}{t}\,dx$ is a Carleson measure. So our definition of the Carleson condition for Θ_t is exactly that $G(x,t) = \Theta_t(1,\dots,1)(x)$ is a Carleson function.

We define strong Carleson measures with a general measure $d\tau(t)$ instead of just $\frac{dt}{t}$ because this allows us to apply results in Section 4 to the discrete relative of Θ_t and S by letting $d\tau(t) = \sum_{k\in\mathbb{Z}} \delta_{2^{-k}}(t)$, like the ones in [11; 27; 19] and [15], among many others.

It is trivial to see that if a nonnegative measure $d\mu(x,t) = F(x,t)\,d\tau(t)\,dx$ is a strong Carleson measure, then it is a Carleson measure, and $\|\mu\|_{\mathcal{C}} \le \|\mu\|_{\mathcal{SC}}$, but we can also prove a partial converse to this for nonnegative measures of the form

$|\Theta_t(1,\dots,1)|^2 \frac{dt\,dx}{t}$ for θ_t satisfying (1.4) and (1.5). In Propositions 3.4 and 3.5, we prove that Θ_t satisfies what we call the two-cube and Carleson conditions if and only if it satisfies the strong Carleson condition. We first define the two-cube testing condition.

DEFINITION 3.2. Let θ_t satisfy (1.4) and Θ_t be defined as in (1.3). We say that Θ_t satisfies the *two-cube testing condition* if

$$\sup_{R\subset Q} \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c})(x) - \Theta_t(\chi_{(2Q)^c},\dots,\chi_{(2Q)^c})(x)|^2 \frac{dt}{t}\,dx < \infty, \tag{3.4}$$

where the supremum is taken over all cubes R and Q with $R \subset Q$.

In the linear case, the two-cube condition for Θ_t becomes

$$\sup_{R\subset Q} \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{2Q\backslash 2R})(x)|^2 \frac{dt}{t}\,dx < \infty.$$

The two-cube testing condition is a technical condition that arises in a number of estimates for $\Theta_t(1,\dots,1)$; however, it is analogous to certain cancellation conditions that appear in singular integral operator theory. See Remark 3.9 and the discussion preceding it for more details of this analogy. Before we verify the equivalence between these conditions in Theorem 1.1, we prove a lemma.

LEMMA 3.3. *Suppose that θ_t satisfies* (1.4). *Then we have the following*:

i. *If $E_1,\dots,E_m \subset \mathbb{R}^n$, then*

$$\sup_{x\in\mathbb{R}^n} |\Theta_t(\chi_{E_1},\dots,\chi_{E_m})(x)| \lesssim t^{-n}\min(|E_1|,\dots,|E_m|). \tag{3.5}$$

ii. *If $E_1,\dots,E_m \subset \mathbb{R}^n$ and $2Q \subset \mathbb{R}^n\backslash E_i$ for some i and cube Q* (*here $2Q$ is the double of Q with the same center*), *then*

$$\sup_{x\in Q} |\Theta_t(\chi_{E_1},\dots,\chi_{E_m})(x)| \lesssim t^{N-n}\ell(Q)^{-(N-n)}. \tag{3.6}$$

Proof. For $E_1,\dots,E_m \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, using (1.4), we have

$$|\Theta_t(\chi_{E_1},\dots,\chi_{E_m})(x)| \lesssim \prod_{j=1}^m \int_{\mathbb{R}^n} \frac{t^{-n}}{(1+t^{-1}|x-y_j|)^N}\chi_{E_j}(y_j)\,dy_j \lesssim t^{-n}|E_i|$$

for each $i = 1,\dots,m$. For (ii), for $x \in Q \subset 2Q \subset \mathbb{R}^n\backslash E_i$, it follows that $|x-y_i| > \ell(Q)$ for all $y_i \in E_i$. Then, using (1.4), it follows that

$$\begin{aligned}|\Theta_t(\chi_{E_1},\dots,\chi_{E_m})(x)| &\lesssim \prod_{j=1}^m \int_{\mathbb{R}^n} \frac{t^{-n}}{(1+t^{-1}|x-y_j|)^N}\chi_{E_j}(y_j)\,dy_j \\ &\lesssim \int_{E_i} \frac{t^{-n}}{(t^{-1}|x-y_i|)^N}\,dy_i\end{aligned}$$

$$\lesssim t^{N-n} \int_{|x-y_i|>\ell(Q)} \frac{1}{|x-y_i|^N}\,dy_i$$
$$\lesssim t^{N-n} \ell(Q)^{-(N-n)}. \qquad \square$$

PROPOSITION 3.4. *Suppose that θ_t satisfies* (1.4) *and* (1.5). *If $\Theta_t(x)$ satisfies the Carleson and two-cube testing conditions, then Θ_t satisfies the strong Carleson condition.*

Proof. We first prove a multilinear analog of the result of Carleson [3] and Christ and Journé [5] mentioned above; if Θ_t satisfies the Carleson condition, then S satisfies the unweighted bound (1.6) for $p = 2$. That is, if $d\mu(x,t) = |\Theta_t(1,\dots,1)(x)|^2 \frac{dt\,dx}{t}$ is a Carleson measure, then S is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^2 for all $1 < p_1, \dots, p_m < \infty$ satisfying (2.1) with $p = 2$. To prove this, we adapt a familiar technique of Coifman and Meyer, see for example [6] or [7]. Decompose $\Theta_t = (\Theta_t - M_{\Theta_t(1,\dots,1)}\mathbb{P}_t) - M_{\Theta_t(1,\dots,1)}\mathbb{P}_t = R_t + U_t$, where

$$\mathbb{P}_t(f_1,\dots,f_m) = \prod_{i=1}^m P_t f_i, \tag{3.7}$$

and P_t is a smooth compactly supported approximation to the identity. The operator R_t satisfies the conditions of Theorem 2.5, and hence the square function associated to R_t is bounded on the appropriate spaces. The second term is bounded as well, using the following Carleson measure bound:

$$\left\| \left(\int_0^\infty |U_t(f_1,\dots,f_m)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2} \le \prod_{i=1}^m \left(\int_{\mathbb{R}^{n+1}_+} |P_t f_i(x)|^{p_i}\, d\mu(x,t) \right)^{1/p_i}$$
$$\lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

We use a bound proved by Carleson [3] which is that $\{P_t\}_{t>0}$ defines a bounded operator from $L^q(\mathbb{R}^n)$ into $L^q(\mathbb{R}^{n+1}_+, d\mu(x,t))$ for all $1 < q < \infty$ whenever $d\mu(x,t)$ is a Carleson measure. We now move on to estimate (3.3), so take a cube $Q \subset \mathbb{R}^n$ and define

$$G_Q(x) = \chi_Q(x) \int_0^{\ell(Q)} d\mu(x,t).$$

To prove that μ is a strong Carleson measure, it is sufficient to show that $\|G_Q\|_{L^\infty} \lesssim 1$ where the constant is independent of $Q \subset \mathbb{R}^n$. Since $d\mu$ is locally integrable in $\mathbb{R}^{n+1}_+$ and $d\mu$ is a Carleson measure, it follows that $G_Q \in L^1(\mathbb{R}^n)$. Then $G_Q(x) \le MG_Q(x)$ for almost every $x \in \mathbb{R}^n$. So we estimate $\|MG_Q\|_{L^\infty}$:

$$MG_Q(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(1,\dots,1)(y)|^2 \chi_Q(y) \frac{dt}{t}\,dy$$
$$= \sup_{R \ni x:\, R \subset Q} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(1,\dots,1)(y)|^2 \frac{dt}{t}\,dy$$

$$\begin{aligned}
&\leq \sup_{R\ni x:\, R\subset Q} \frac{1}{|R|}\int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{2R},\dots,\chi_{2R})(y)|^2 \frac{dt}{t}\,dy \\
&\quad + \sup_{R\ni x:\, R\subset Q} \sum_{\mathbf{F}\in\Lambda} \frac{1}{|R|}\int_R \int_0^{\ell(R)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(y)|^2 \frac{dt}{t}\,dy \\
&\quad + \sup_{R\ni x:\, R\subset Q} \sum_{\mathbf{F}\in\Lambda} \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(y)|^2 \frac{dt}{t}\,dy \\
&= I + II + III,
\end{aligned}$$

where

$$\Lambda = \{\mathbf{F} = (F_1,\dots,F_m):\ \ F_i = 2R \text{ or } F_i = (2R)^c\}\backslash\{(2R,\dots,2R)\}.$$

Note that we may make the reduction to cubes $R \subset Q$ since $\operatorname{supp}(G_Q) \subset Q$ and $G_Q \geq 0$. For each cube $R \subset Q \subset \mathbb{R}^n$, the boundedness of S gives

$$\begin{aligned}
&\frac{1}{|R|}\int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{2R},\dots,\chi_{2R})(y)|^2 \chi_R(y) \frac{dt}{t}\,dy \\
&\qquad \leq \frac{1}{|R|}\int_{\mathbb{R}^n} \int_0^{\infty} |\Theta_t(\chi_{2R},\dots,\chi_{2R})(y)|^2 \frac{dt}{t}\,dy \\
&\qquad \lesssim \frac{1}{|R|}\prod_{i=1}^m \|\chi_{2R}\|_{L^{p_i}}^2 \lesssim 1.
\end{aligned}$$

Therefore, I is bounded independent of x and Q. In each of the terms in the sum defining II, there is at least one F_i such that $F_i = (2R)^c$. Then using (3.6) from Lemma 3.3, it follows that

$$\begin{aligned}
\frac{1}{|R|}\int_R \int_0^{\ell(R)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(y)|^2 \frac{dt}{t}\,dy &\lesssim \frac{1}{|R|}\int_R \int_0^{\ell(R)} \frac{t^{2(N-n)}}{\ell(R)^{2(N-n)}} \frac{dt}{t}\,dy \\
&\lesssim 1.
\end{aligned}$$

Since $|\Lambda| = 2^m - 1$, this is sufficient to bound II. Now for the third term III, we first take $\mathbf{F} \in \Lambda$ such that at least one component $F_i = 2R$. Then it follows from (3.5) in Lemma 3.3 that

$$\begin{aligned}
\frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(y)|^2 \frac{dt}{t}\,dy &\lesssim \frac{1}{|R|}\int_R \int_{\ell(R)}^{\infty} t^{-2n}|2R|^2 \frac{dt}{t}\,dy \\
&\lesssim 1.
\end{aligned}$$

This bounds all but one term for III. It remains to bound the term where $\mathbf{F} = ((2R)^c,\dots,(2R)^c)$. We do so using (3.6) from Lemma 3.3 and the two-cube condition (3.4):

$$\begin{aligned}
&\frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c})(y)|^2 \frac{dt}{t}\,dy \\
&\qquad \leq \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{(2Q)^c},\dots,\chi_{(2Q)^c})(y)|^2 \frac{dt}{t}\,dy
\end{aligned}$$

$$+\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(\chi_{(2Q)^c},\dots,\chi_{(2Q)^c})(y)$$
$$-\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c})(y)|^2\frac{dt}{t}\,dy$$
$$\lesssim\frac{1}{|R|}\int_R\int_0^{\ell(Q)}t^{2(N-n)}\ell(Q)^{-2(N-n)}\frac{dt}{t}\,dy+1\lesssim 1.$$

Therefore, $\|MG_Q\|_{L^\infty}\le I+II+III\lesssim 1$ for all $Q\subset\mathbb{R}^n$, where the constant is independent of Q. Now we can easily verify that $d\mu$ satisfies the strong Carleson condition:

$$\sup_{Q\subset\mathbb{R}^n}\sup_{x\in Q}\int_0^{\ell(Q)}|\Theta_t(1,\dots,1)(x)|^2\frac{dt}{t}\le\sup_{Q\subset\mathbb{R}^n}\|G_Q\|_{L^\infty}\le\sup_{Q\subset\mathbb{R}^n}\|MG_Q\|_{L^\infty}$$
$$\lesssim 1.$$

This completes the proof. □

PROPOSITION 3.5. *If θ_t satisfies (1.4), (1.5) and Θ_t satisfies the strong Carleson condition, then Θ_t satisfies the two-cube condition (3.4).*

Proof. We estimate (3.4) for $R\subset Q\subset\mathbb{R}^n$:

$$\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c})(x)-\Theta_t(\chi_{(2Q)^c},\dots,\chi_{(2Q)^c})(x)|^2\frac{dt}{t}\,dx$$
$$\le\sum_{j=1}^{m}\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c}$$
$$-\chi_{(2Q)^c},\dots,\chi_{(2Q)^c})(x)|^2\frac{dt}{t}\,dx$$
$$\le\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2Q\setminus 2R})(x)|^2\frac{dt}{t}\,dx$$
$$+\sum_{j=1}^{m-1}\frac{1}{|R|}\int_R\int_0^{\ell(Q)}|\Theta_t(\chi_{(2R)^c},\dots,\chi_{2Q\setminus 2R},\dots,\chi_{(2Q)^c})(x)|^2\frac{dt}{t}\,dx$$
$$\le\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(1,\dots,1)(x)$$
$$-\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2Q\setminus 2R})(x)|^2\frac{dt}{t}\,dx$$
$$+\frac{1}{|R|}\int_R\int_{\ell(R)}^{\ell(Q)}|\Theta_t(1,\dots,1)(x)|^2\frac{dt}{t}\,dx$$
$$+\sum_{j=1}^{m-1}\frac{1}{|R|}\int_R\int_0^{\ell(Q)}t^{2(N-n)}\ell(Q)^{-2(N-n)}\frac{dt}{t}\,dx$$

$$\lesssim \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(1,\dots,1)(x) - \Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2Q\setminus 2R})(x)|^2 \frac{dt}{t}\,dx + 1.$$

Here the middle term is bounded by the assumption that $|\Theta_t(1,\dots,1)(x)|^2 \frac{dt}{t}\,dx$ is a strong Carleson measure and the third by direction computation. Now we bound the reaming term in the following way:

$$\begin{aligned}
&|\Theta_t(1,\dots,1)(x) - \Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2Q\setminus 2R})(x)| \\
&\quad\le \sum_{j=1}^{m-1} |\Theta_t(\chi_{2R},\dots,\chi_{2R},1,\dots,1)(x)| \\
&\qquad + |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},1-\chi_{2Q\setminus 2R})(x)| \\
&\quad\lesssim \sum_{j=1}^{m-1} t^{-n}|R| + |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},1-\chi_{2Q\setminus 2R})(x)| \\
&\quad\lesssim t^{-n}|R| + |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{(2Q)^c})(x)| \\
&\qquad + |\Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2R})(x)| \\
&\quad\lesssim t^{-n}|R| + t^{N-n}\ell(Q)^{-(N-n)}.
\end{aligned}$$

In the second-to-last line, we bound the last term by $t^{-n}|R|$ and absorb it into the first term of the last line. Therefore, we have that

$$\begin{aligned}
&\frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(1,\dots,1)(x) - \Theta_t(\chi_{(2R)^c},\dots,\chi_{(2R)^c},\chi_{2Q\setminus 2R})(x)|^2 \frac{dt}{t}\,dx \\
&\quad\lesssim \frac{1}{|R|}\int_R \int_{\ell(R)}^{\infty} t^{-2n}|R|^2 \frac{dt}{t}\,dx + \frac{1}{|R|}\int_R \int_0^{\ell(Q)} t^{2(N-n)}\ell(Q)^{-2(N-n)} \frac{dt}{t}\,dx \\
&\quad\lesssim 1,
\end{aligned}$$

and hence Θ_t satisfies the two-cube condition (3.4). □

We also prove that if S is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some $1 < p_1,\dots,p_m < \infty$ and $2 \le p < \infty$ satisfying (2.1), then Θ_t satisfies the Carleson condition. A partial converse to this was proved within the proof of Proposition 3.4: if Θ_t satisfies the Carleson condition, then S is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^2 for all $1 < p_1,\dots,p_m < \infty$.

PROPOSITION 3.6. *Assume that θ_t satisfies (1.4) and S is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p for some $1 < p_1,\dots,p_m < \infty$ and $2 \le p < \infty$ satisfying (2.1). Then Θ_t satisfies the Carleson condition.*

Proof. We fix a cube $Q \subset \mathbb{R}^n$ and estimate

$$\frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Theta_t(1,\dots,1)(x)|^2 \frac{dt}{t}\,dx$$

$$\le \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Theta_t(\chi_{2Q},\dots,\chi_{2Q})(x)|^2 \frac{dt}{t}\,dx$$
$$+ \sum_{\mathbf{F}\in\Lambda} \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(x)|^2 \frac{dt}{t}\,dx$$
$$= I + II, \tag{3.8}$$

where again we define

$$\Lambda = \{\mathbf{F} = (F_1,\dots,F_m):\ F_i = 2Q \text{ or } F_i = (2Q)^c\}\backslash\{(2Q,\dots,2Q)\}.$$

For each cube $Q \subset \mathbb{R}^n$, we estimate I:

$$\frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Theta_t(\chi_{2Q},\dots,\chi_{2Q})(x)|^2 \frac{dt}{t}\,dx$$
$$\le \frac{1}{|Q|}\int_Q S(\chi_{2Q},\dots,\chi_{2Q})(x)^2\,dx$$
$$\le \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} S(\chi_{2Q},\dots,\chi_{2Q})(x)^p\,dx\right)^{2/p}$$
$$\lesssim |Q|^{-2/p}\prod_{i=1}^m \|\chi_{2Q}\|_{L^{p_i}}^2 \lesssim 1.$$

Now for the second term II, we fix $\mathbf{F} \in \Lambda$, which has at least one component $F_i = (2Q)^c$. Then by (3.6) from Lemma 3.3 we have

$$\frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Theta_t(\chi_{F_1},\dots,\chi_{F_m})(x)|^2 \frac{dt}{t}\,dx$$
$$\lesssim \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} t^{2(N-n)}\ell(Q)^{-2(N-n)} \frac{dt}{t}\,dx \lesssim 1.$$

Then $II \lesssim 1$ as well, and Θ_t satisfies the Carleson condition. □

In fact, this proves that if θ_t satisfies (1.4)–(1.5) and Θ_t satisfies the Carleson condition, then Θ_t satisfies the strong Carleson condition if and only if Θ_t satisfies the two-cube testing condition (3.4). We conclude this section with a few examples of Carleson measures obtained from operators Θ_t and a discussion of the two-cube condition.

In Example 3.7, we define operators that give rise to strong Carleson measures, and in Example 3.8, we define operators that give rise to Carleson measures, but not to strong Carleson measures. For the examples, let P_t be a smooth compactly supported approximation to the identity, and $\mathbb{P}_t$ be as defined in (3.7).

EXAMPLE 3.7. Suppose that $\psi \in L^1$ with integral zero satisfying $|\psi(x)| \lesssim 1/(1+|x|)^N$ for some $N > n$ and

$$\sup_{\xi\ne 0}\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} < \infty. \tag{3.9}$$

Define $Q_t f = \psi_t * f$. Let $b \in L^q$ for some $1 \le q < \infty$ with $|b(x) - b(x')| \le L|x - x'|^\alpha$ where $0 < \alpha < N - n$ and $\beta \in L^\infty(\mathbb{R}^{n+1}_+)$, and define

$$\Theta_t(f_1, \dots, f_m)(x) = \beta(x,t) Q_t b(x) \mathbb{P}_t(f_1, \dots, f_m)(x).$$

It follows that the kernels of Θ_t, which are

$$\theta_t(x, y_1, \dots, y_m) = \beta(x,t) Q_t b(x) \prod_{i=1}^m \phi_t(x - y_i)$$

for $t > 0$, satisfy (1.4) and (1.5). We also have that $\Theta_t(1, \dots, 1) = \beta(x,t) Q_t b$, so we estimate

$$\begin{aligned} |Q_t b(x)| &= \left| \int_{\mathbb{R}^n} \psi_t(x-y)(b(y) - b(x))\, dy \right| \le L \int_{\mathbb{R}^n} |\psi_t(x-y)| |x-y|^\alpha \, dy \\ &\lesssim t^\alpha \int_{\mathbb{R}^n} \frac{t^{-n}}{(1 + t^{-1}|x-y|)^{N-\alpha}}\, dy \lesssim t^\alpha. \end{aligned}$$

Also, we have that

$$|Q_t b(x)| \le \|\psi_t\|_{L^{q'}} \|b\|_{L^q} \lesssim t^{-n/q}.$$

Then it follows that

$$\begin{aligned} &\int_0^{\ell(Q)} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt}{t} \\ &\quad \lesssim \|\beta\|^2_{L^\infty(\mathbb{R}^{n+1}_+)} \int_0^1 t^{2\alpha} \frac{dt}{t} + \|\beta\|^2_{L^\infty(\mathbb{R}^{n+1}_+)} \int_1^\infty t^{-2n/q} \frac{dt}{t} \lesssim 1. \end{aligned}$$

Therefore, with this selection of b and β, it follows that Θ_t satisfies the strong Carleson condition. By Theorem 1.1 it follows that

$$\left\| \left(\int_0^\infty |\Theta_t(f_1, \dots, f_m)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w^p)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}$$

for all $1 < p_1, \dots, p_m < \infty$ and $w_i^{p_i} \in A_{p_i}$, where $w = w_1 \cdots w_m$, and p is defined by (2.1), which allows for $1/m < p < \infty$. Note that with an appropriate selection of β_t, the kernels $\theta_t(x, y)$ are not be smooth in the x variable. The previous results can be applied to this operator to prove (1.6) when $w_1 = \cdots = w_m = 1$ and $p \ge 2$, although we can now apply Theorem 1.1 to prove (1.6) for all $w_i^{p_i} \in A_{p_i}$ and $1 < p_1, \dots, p_m < \infty$ satisfying (2.1). This is an operator to which one could not apply the previous results. Even in the linear case, this provides new results for Littlewood–Paley–Stein square functions whose kernels lack regularity in x.

EXAMPLE 3.8. The purpose of this example is to construct an operator Θ_t satisfying (1.4) and (1.5) such that Θ_t satisfies the Carleson condition, but not the strong Carleson condition. Define $\psi(x) = \chi_{(0,1)}(x) - \chi_{(-1,0)}(x)$, $Q_t f = \psi_t * f$, $b(x) = \chi_{(0,1)}(x)$, and like above, $\Theta_t(f_1, \dots, f_m)(x) = Q_t b(x) \mathbb{P}_t(f_1, \dots, f_m)(x)$. As above, we have that $\Theta_t(1, \dots, 1) = Q_t b$. It is a quick computation to show that

$$\widehat{\psi}(\xi) = 2 \frac{1 - \cos(\xi)}{i\xi}$$

with the appropriate modification when $\xi = 0$. It follows that $|\widehat{\psi}(\xi)| \lesssim \min(|\xi|, |\xi|^{-1})$ and that

$$|\Theta_t(1,\ldots,1)(x)|^2 \frac{dt}{t} dx = |\psi_t * b(x)|^2 \frac{dt}{t} dx$$

is a Carleson measure since $b \in L^\infty \subset BMO$. Now we show that Θ_t does not satisfy the strong Carleson condition. Let $Q = [-1, 0]$, $x \in [-1, 0) \subset Q$, and we estimate (3.3) with the following computation:

$$\begin{aligned}
\int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dt}{t} &= \int_0^1 \left| \int_{\mathbb{R}} \psi_t(y) \chi_{(0,1)}(x-y)\, dy \right|^2 \frac{dt}{t} \\
&\geq \int_{-x}^1 \left| \int_{-t}^x \psi_t(y)\, dy \right|^2 \frac{dt}{t} \\
&= \int_{-x}^1 \frac{(x+t)^2}{t^2} \frac{dt}{t} \\
&= x^2 \int_{-x}^1 \frac{dt}{t^3} + 2x \int_{-x}^1 \frac{dt}{t^2} + \int_{-x}^1 \frac{dt}{t} \\
&\geq x^2 \int_0^1 dt - 2x - 2 - \log(-x) \\
&\geq -\log(-x) - 2.
\end{aligned}$$

Therefore,

$$\sup_{x \in [-1,0]} \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dt}{t} \geq \sup_{x \in [-1,0)} -\log(-x) - 2 = \infty,$$

and hence Θ_t satisfies the Carleson condition, but not the strong Carleson condition.

The two-cube condition in (3.4) can be viewed as a cancellation condition similar to the $T1$-type cancellation conditions defined for singular integral operators. For example, $T1$-type conditions for a linear operator T with kernel K can be expressed as estimates for

$$\left| \int_{a<|x-y|<b} K(x, y)\, dy \right|$$

that are uniform in $0 < a < b < \infty$. For a precise formulation of this type of condition, see [34], where Stein proves that such an estimate is necessary and sufficient for certain boundedness properties. In the following remark, we construct an estimate of this type for θ_t that implies the two-cube condition for Θ_t. We only work in the linear setting here to demonstrate the parallel with cancellation conditions in singular integral operator theory. One can formulate a cancellation condition for multilinear operators as well, but the notation becomes cumbersome.

REMARK 3.9. Assume that $\theta_t(x,y)$ satisfies (1.4) with $m=1$ and that there exists $\gamma > 0$ such that, for all $0 < a < b < \infty$ and $a \le t \le b$,

$$\left|\int_{a\le|x-y|\le b} \theta_t(x,y)\,dy\right| \le \omega_{a,b}(t), \quad \text{where } A = \sup_{0<a<b<\infty} \int_a^b \omega_{a,b}(t)^2 \,\frac{dt}{t} < \infty.$$

Then Θ_t satisfies the two-cube condition.

Proof. Let $R \subset Q$ be cubes, and let $x \in R$. Define $B_R = B(x, \ell(R))$ and $B_Q = B(x, 4\sqrt{n}\ell(Q))$. Note that $2Q \subset B_Q$ and $B_R \subset B_Q$. Then

$$\begin{aligned}|\Theta_t(\chi_{2Q\setminus 2R})(x)| &\le |\Theta_t(\chi_{B_Q\setminus B_R})(x)| + |\Theta_t(\chi_{B_Q\setminus 2Q})(x)| \\ &\quad + |\Theta_t(\chi_{B_R})(x)| + |\Theta_t(\chi_{2R})(x)| \\ &\lesssim |\Theta_t(\chi_{B_Q\setminus B_R})(x)| + \left(\frac{t}{\ell(Q)}\right)^{N-n} + \left(\frac{\ell(R)}{t}\right)^n.\end{aligned}$$

We use Lemma 3.3 to bound the last three terms above. Therefore,

$$\begin{aligned}&\frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{2Q\setminus 2R})(x)|^2 \,\frac{dt}{t}\,dx \\ &\quad\lesssim \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} |\Theta_t(\chi_{B_Q\setminus B_R})(x)|^2 \,\frac{dt}{t}\,dx \\ &\qquad + \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} \left(\frac{t}{\ell(Q)}\right)^{2(N-n)} \frac{dt}{t}\,dx \\ &\qquad + \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} \left(\frac{\ell(R)}{t}\right)^{2n} \frac{dt}{t}\,dx \\ &\quad\lesssim \frac{1}{|R|}\int_R \int_{\ell(R)}^{\ell(Q)} \left|\int_{\ell(R)\le|x-y|\le 4\sqrt{n}\ell(Q)} \theta_t(x,y)\,dy\right|^2 \frac{dt}{t}\,dx + 1 \\ &\quad\le \frac{1}{|R|}\int_R \int_{\ell(R)}^{4\sqrt{n}\ell(Q)} \omega_{\ell(R),4\sqrt{n}\ell(Q)}(t)^2 \,\frac{dt}{t}\,dx + 1 \le A+1.\end{aligned}$$

In the last line of this string of inequalities, we apply the integral estimate above with $a = \ell(R)$ and $b = 4\sqrt{n}\ell(Q)$. Hence, the two-cube condition is verified. □

4. A Full Weighted T1 Theorem for Square Functions on L^2

In this section, we develop some classical Carleson measure results in a weighted setting with strong Carleson measures. With these new tools, we can apply some familiar arguments to complete the proofs of Theorems 1.1 and 1.2. More precisely, Lemmas 4.1 and 4.2 and Proposition 4.3 are weighted versions of results proved by Carleson [3], where we use assume strong Carleson in place of Carleson conditions.

LEMMA 4.1. *If $d\mu$ is a strong Carleson measure, then for any locally integrable function $w \geq 0$ and $E \subset \mathbb{R}^n$,*

$$d\mu_w(\widehat{E}) \leq \|d\mu\|_{\mathcal{SC}} w(E), \tag{4.1}$$

where $d\mu_w(x,t) = w(x)\,d\mu(x,t)$ and $\widehat{E} = \{(x,t) \in \mathbb{R}^{n+1}_+ :\ B(x,t) \subset E\}$.

In [23], Journé says that $d\mu_w$ is a Carleson measure with respect to $w \in A_2$ if there is a constant $C > 0$ such that $d\mu_w(T(Q)) \leq Cw(Q)$ for all cubes Q. He uses this definition to prove that measures satisfying this estimate also verify weighted analogs of Carleson measure bounds. By Lemma 4.1, if $d\mu$ is a strong Carleson measure, then $d\mu$ is a Carleson measure with respect to w for all $w \in A_2$. It is not clear if the converse of this statement is true, but this may be an interesting question for further exploration.

Proof of Lemma 4.1. Let Q_j be the Calderón–Zygmund decomposition of χ_E at height $\frac{1}{2}$. This means that Q_j is a collection of disjoint dyadic cubes such that

$$|\chi_E(x)| \leq \frac{1}{2} \quad \text{for a.e. } x \notin \bigcup_j Q_j,$$

$$\Big|\bigcup_j Q_j\Big| \leq 2\|\chi_E\|_{L^1} = |E|,$$

and

$$\frac{1}{2} < \frac{1}{|Q_j|}\int_{Q_j} \chi_E(x)\,dx \leq 2^{n-1}.$$

The Calderón–Zygmund decomposition is a well-known result in the literature, see for example [14] for the construction. Then it follows that

$$E \subset \bigcup_j Q_j \quad \text{and} \quad |E| \leq \sum_j |Q_j| \leq 2|E|.$$

Let Q_j^* be the dyadic cube with double the side length of Q_j containing Q_j and take $(x,t) \in \widehat{E}$. Since $B(x,t) \subset E$ and $Q_j^* \not\subset E$, it follows that $B(x,t) \subset B(x, 2\sqrt{n}\ell(Q_j))$. Then

$$\widehat{E} \subset \bigcup_j Q_j \times \big(0, 2\sqrt{n}\ell(Q_j)\big].$$

Now $d\mu(x,t) = F(x,t)\,d\tau(t)\,dx$ for some nonnegative $F \in L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Using that $d\mu$ is a strong Carleson measure, it follows that

$$\begin{aligned} d\mu_w(\widehat{E}) &\leq \sum_j d\mu_w\big((E \cap Q_j) \times \big(0, 2\sqrt{n}\ell(Q_j)\big]\big) \\ &= \sum_j \int_{E\cap Q_j} \int_0^{2\sqrt{n}\ell(Q_j)} F(x,t)\,d\tau(t) w(x) \chi_{Q_j}(x)\,dx \end{aligned}$$

$$\leq \|d\mu\|_{\mathcal{SC}} \sum_j \int_{E\cap Q_j} w(x)\,dx$$
$$\leq \|d\mu\|_{\mathcal{SC}} w(E).$$

In the last line, we use that $E \cap Q_j$ are disjoint since Q_j are disjoint. □

LEMMA 4.2. *Suppose that $d\mu(x,t) = F(x,t)\,d\tau(t)\,dx$ is a strong Carleson measure and $|\phi(x)| \lesssim 1/(1+|x|)^N$ for some $N > n$. Then for all $w \in A_p$ for $1 < p < \infty$,*

$$\left(\int_{\mathbb{R}^{n+1}_+} |\phi_t * f(x)|^p w(x)\,d\mu(x,t)\right)^{1/p} \lesssim \|\mu\|_{\mathcal{SC}}^{1/p} [w]_{A_p}^{1/(p-1)} \|f\|_{L^p(w)}. \tag{4.2}$$

It is worth noting that this lemma has additional interest outside the scope of our application since it reproduces a part of the classical characterization of a Carleson measure in [3]. The characterization says a nonnegative measure $d\mu(x,t)$ is a Carleson measure if and only if the map $f \mapsto \phi_t * f$ defines a bounded operator from L^p into $L^p(\mathbb{R}^{n+1}_+, d\mu(x,t))$. It would be interesting to explore if the converse of Lemma 4.2 as well, but for the purposes of this work, Lemma 4.2 suffices; so we leave it at that.

Proof of Lemma 4.2. Define the nontangential maximal function

$$M_\phi f(x) = \sup_{t>0} \sup_{|x-y|<t} |\phi_t * f(t)|.$$

For $\lambda > 0$, define

$$E_\lambda = \{x \in \mathbb{R}^n : \ M_\phi f(x) > \lambda\} \quad \text{and} \quad \widehat{E}_\lambda = \{(x,t) \in \mathbb{R}^{n+1}_+ : \ B(x,t) \subset E_\lambda\}.$$

It follows from Lemma 4.1 that $\mu_w(\widehat{E}_\lambda) \leq \|\mu\|_{\mathcal{SC}} w(E_\lambda)$, where again $d\mu_w(x, t) = w(x)\,d\mu(x,t)$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} |\phi_t * f(x)|^p w(x)\,d\mu(x,t)\\
&\quad = p\int_0^\infty \lambda^p \mu_w(\{(x,t) \in \mathbb{R}^{n+1}_+ : \ |\phi_t * f(x)| > \lambda\})\,\frac{d\lambda}{\lambda}\\
&\quad \leq p\int_0^\infty \lambda^p \mu_w(\widehat{E}_\lambda)\,\frac{d\lambda}{\lambda}\\
&\quad \leq p\|\mu\|_{\mathcal{SC}} \int_0^\infty \lambda^p w(E_\lambda)\,\frac{d\lambda}{\lambda}\\
&\quad = \|\mu\|_{\mathcal{SC}} \int_{\mathbb{R}^n} M_\phi f(x)^p w(x)\,dx\\
&\quad \lesssim \|\mu\|_{\mathcal{SC}} [w]_{A_p}^{p/(p-1)} \|f\|_{L^p(w)}^p.
\end{aligned}$$

Here we use as before that $|\phi_t * f(x)| \lesssim Mf(x)$ and $\|Mf\|_{L^p(w)} \lesssim [w]_{A_p}^{1/(p-1)} \times \|f\|_{L^p(w)}$. □

PROPOSITION 4.3. *Suppose that θ_t satisfies (1.4) and (1.5). If Θ_t satisfies the strong Carleson condition, then S satisfies (1.6) for all $w_i^{p_i} \in A_{p_i}$ and $1 < p_1, \dots, p_m < \infty$ satisfying (2.1) with $p = 2$, where $w = w_1 \cdots w_m$. Furthermore, the constant for this bound is at most a constant independent of $w_1, \dots, w_m$ times*

$$C_{m,w_1,\dots,w_m,p_1,\dots,p_m} = \prod_{i=1}^{m}(1 + [w_i^{p_i}]_{A_{p_i}}^{\max(1,p_i'/p_i)+\max(1/2,p_i'/p_i)}) + \|d\mu\|_{\mathcal{SC}}^{m/2} \prod_{i=1}^{m} [w_i^{p_i}]_{A_{p_i}}^{p_i'/p_i}. \tag{4.3}$$

Proof. Define $R_t = \Theta_t - M_{\Theta_t(1,\dots,1)}\mathbb{P}_t$ and $U_t = M_{\Theta_t(1,\dots,1)}\mathbb{P}_t$. Then R_t satisfies (1.4), (1.5), and in addition $R_t(1, \dots, 1) = 0$ for all $t > 0$. Then by Theorem 2.5 it follows that

$$\left\| \left(\int_0^\infty |R_t(f_1, \dots, f_m)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w^p)} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

Now we turn to the U_t term. Define $d\mu(x,t) = |\Theta_t(1, \dots, 1)|^2 \frac{dt\,dx}{t}$, and let $w_i^{p_i} \in A_{p_i}$ with $1 < p_1, \dots, p_m < \infty$ satisfying (2.1) and $p = 2$, Then it follows that

$$\begin{aligned}
&\left\| \left(\int_0^\infty |U_t(f_1, \dots, f_m)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2(w^2)}^2 \\
&= \int_{\mathbb{R}^{n+1}_+} \left(\prod_{i=1}^{m} |P_t f_i(x)| w_i(x) \right)^2 d\mu(x,t) \\
&\le \prod_{i=1}^{m} \left(\int_{\mathbb{R}^{n+1}_+} |P_t f_i(x)|^{p_i} w_i(x)^{p_i}\, d\mu(x,t) \right)^{2/p_i} \\
&\lesssim \|d\mu\|_{\mathcal{SC}}^{m} \prod_{i=1}^{m} [w_i^{p_i}]_{A_{p_i}}^{2/(p_i-1)} \|f_i\|_{L^{p_i}(w_i^{p_i})}^2.
\end{aligned}$$

The final inequality holds by Lemma 4.2. The first term in the constant (4.3) is from the bound of R_t by Theorem 2.5, and the second term is from the bound of U_t above. □

These results almost complete the proof of Theorem 1.1, except for a minor issue with $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$ and applying weight extrapolation. Propositions 3.4 and 3.5 verify the equivalence of (i) and (ii) from Theorem 1.1. By Proposition 3.4, (i) implies that S satisfies (1.6) for all $w_i^{p_i} \in A_{p_i}$ with $1 < p_1, \dots, p_m$ and $p = 2$ for $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$. In order to conclude boundedness for all $L^{p_i}(w_i^{p_i})$, we make a short density argument and apply the extrapolation theorem of Grafakos and Martell [16] to complete the proof of Theorem 1.1. We will use a lemma to prove this.

LEMMA 4.4. *If* $w \in A_p$ *and* $1 < p < \infty$, *then* $1/(d + |x_0 - \cdot|)^n \in L^p(w)$ *for any* $x_0 \in \mathbb{R}^n$ *and* $d > 0$.

Proof. We start by noting that, for any $x \in \mathbb{R}^n$,

$$M\chi_{B(x_0,d)}(x) \geq \frac{1}{|B(x, |x - x_0| + d)|} \int_{B(x,|x-x_0|+d)} \chi_{B(0,d)}(x)\, dx = \frac{|\chi_{B(x_0,d)}(x)|}{|B(x, |x - x_0| + d)|} = \frac{d^n}{(d + |x - x_0|)^n}.$$

Then it follows that

$$\left(\int_{\mathbb{R}^n} \frac{1}{(d + |x - x_0|)^{np}} w(x)\, dx\right)^{1/p} \leq d^{-n} \|M\chi_{B(x_0,d)}\|_{L^p(w)} \lesssim \|\chi_{B(x_0,d)}\|_{L^p(w)} < \infty.$$

Here we use the Hardy–Littlewood maximal operator bound on $L^p(w)$ and that $w \in L^1_{\text{loc}}$. □

Proof of Theorem 1.1. First, we restrict to the case $p = 2$ and take $f_i \in L^{p_i}(w_i^{p_i})$ and $f_{i,k} \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$ with $f_{i,k} \to f_i$ in $L^{p_i}(w_i^{p_i})$ as $k \to \infty$. It follows that $f_{1,k} \otimes \cdots \otimes f_{m,k} \to f_1 \otimes \cdots \otimes f_m$ as $k \to \infty$ in the weighted product Lebesgue space $L^{p_1}(w_1^{p_1}) \cdots L^{p_m}(w_m^{p_m})$. For all $x \in \mathbb{R}^n$,

$$\begin{aligned} &|\Theta_t(f_1, \ldots, f_m)(x) - \Theta_t(f_{1,k}, \ldots, f_{m,k})(x)| \\ &\quad \leq \int_{\mathbb{R}^{mn}} |\theta_t(x, y_1, \ldots, y_m)||f_1(y_1) \cdots f_m(y_m) - f_{1,k}(y_1) \cdots f_{m,k}(y_m)|\, d\mathbf{y} \\ &\quad \leq \prod_{i=1}^m t^{N-n} \left(\int_{\mathbb{R}^n} \frac{w_i(y_i)^{-p_i'}\, dy_i}{(t + |x - y_i|)^{p_i' N}}\right)^{1/p_i'} \\ &\qquad \times \|f_1 \otimes \cdots \otimes f_m - f_{1,k} \otimes \cdots \otimes f_{m,k}\|_{L^{p_1}(w_1^{p_1}) \cdots L^{p_m}(w_1^{p_m})}, \end{aligned}$$

which tends to zero as $k \to \infty$ almost everywhere since $w_i^{p_i} \in A_{p_i}$ implies that $w_i^{-p_i'} \in A_{p_i'}$ and so the first term is finite almost everywhere by Lemma 4.4. Therefore, $\Theta_t(f_{1,k}, \ldots, f_{m,k}) \to \Theta_t(f_1, \ldots, f_m)$ pointwise as $k \to \infty$ for a.e. $x \in \mathbb{R}^n$ and $t > 0$. Then by Fatou's lemma

$$\begin{aligned} \|S(f_1, \ldots, f_m)\|_{L^2(w^2)}^2 &= \int_{\mathbb{R}^n} \int_0^\infty \lim_{k \to \infty} |\Theta_t(f_{1,k}, \ldots, f_{m,k})(x)|^2 \frac{dt}{t} w(x)^2\, dx \\ &\leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \int_0^\infty |\Theta_t(f_{1,k}, \ldots, f_{m,k})(x)|^2 \frac{dt}{t} w(x)^2\, dx \\ &\leq C_{n,m,w_1,\ldots,w_m,p_1,\ldots,p_m} \liminf_{k \to \infty} \prod_{i=1}^m \|f_{i,k}\|_{L^{p_i}(w_i^{p_i})}^2 \\ &= C_{n,m,w_1,\ldots,w_m,p_1,\ldots,p_m} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}^2. \end{aligned}$$

Therefore, S satisfies (1.6) for all $1 < p_1, \ldots, p_m < \infty$ satisfying (2.1) with $p = 2$, for all $w_i^{p_i} \in A_{p_i}$, and for all $f_i \in L^{p_i}(w_i^{p_i})$. We complete the proof by applying the multilinear extrapolation theorem of Grafakos and Martell [16], which we state now.

THEOREM 4.5 (Grafakos–Martell [16]). *Let $1 \le q_1, \ldots, q_m < \infty$ and $1/m \le q < \infty$ be fixed indices that satisfy (2.1), and T be an operator defined on $L^{q_1}(w_1^{q_1}) \times \cdots \times L^{q_m}(w_m^{q_m})$ for all tuples of weights $w_i^{q_i} \in A_{q_i}$. We suppose that for all $B > 1$, there is a constant $C_0 = C_0(B) > 0$ such that for all tuples of weights $w_i^{q_i} \in A_{q_i}$ with $[w_i^{q_i}]_{A_{q_i}} \le B$ and all functions $f_i \in L^{q_i}(w_i^{q_i})$, T satisfies*

$$\|T(f_1, \ldots, f_m)\|_{L^q(w^q)} \le C_0 \prod_{i=1}^m \|f_i\|_{L^{q_i}(w_i^{q_i})}.$$

Then for all indices $1 < p_1, \ldots, p_m < \infty$ and $1/m < p < \infty$ that satisfy (2.1), all $B > 1$, and all weights $w_i^{p_i} \in A_{p_i}$ with $[w_i^{p_i}]_{A_{p_i}} < B$, there is a constant $C = C(B)$ such that for all $f_i \in L^{p_i}(w_i^{p_i})$,

$$\|T(f_1, \ldots, f_m)\|_{L^p(w^p)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}.$$

Fix $q_1 = \cdots = q_m = 2m$ and $q = 2$. Then we have just proved that for all $B > 1$ and $w_i^{q_i} \in A_{q_i}$ with $[w_i^{q_i}]_{A_{q_i}} \le B$,

$$\|S(f_1, \ldots, f_m)\|_{L^2(w^2)} \le C_{n,m,q_1,\ldots,q_m} C_{m,n,p_1,\ldots,p_m,w_1,\ldots,w_m} \prod_{i=1}^m \|f_i\|_{L^{q_i}(w_i^{q_i})},$$

where $C_{m,n,w_1,\ldots,w_m,q_1,\ldots,q_m}$ is defined in (4.3). Since $C_{m,n,w_1,\ldots,w_m,q_1,\ldots,q_m}$ is an increasing sum of power functions of $[w_i^{q_i}]_{A_{q_i}}$, one can define $C_0(B)$ by replacing the weight constants with B in (4.3) times a constant independent of the weights:

$$C_0(B) = C_{n,m,q_1,\ldots,q_m}\Bigg[\prod_{i=1}^m 2B^{\max(1,1/(q_i-1))+\max(1/2,1/(q_i-1))} + \|\mu\|_{SC}^{m/2} \prod_{i=1}^m B^{1/(q_i-1)}\Bigg],$$

which verifies the hypotheses of Theorem 4.5 for S. Therefore, for all $B > 1$, there exists C depending on $B, n, m, q_1, \ldots, q_m$ such that

$$\|S(f_1, \ldots, f_m)\|_{L^p(w^p)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i^{p_i})}$$

for all $1 < p_1, \ldots, p_m < \infty$, $w_i^{p_i} \in A_{w_i}$ with $[w_i^{p_i}]_{A_{p_i}} \le B$, and $f_i \in L^{p_i}(w_i^{p_i})$. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) have already been proved in a more general context. So it is sufficient to show that (i) $\Rightarrow$ (iv). Since $\theta_t(x, y_1, \dots, y_m) = t^{-mn}\Psi^t(t^{-1}(x - y_1), \dots, t^{-1}(x - y_m))$, it follows that $\Theta_t(1, \dots, 1)(x)$ is constant in x: for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\Theta_t(1, \dots, 1)(x) &= \int_{\mathbb{R}^{mn}} t^{-mn}\Psi^t(t^{-1}(x - y_1), \dots, t^{-1}(x - y_m))\, d\mathbf{y} \\ &= \int_{\mathbb{R}^{mn}} \Psi^t(y_1, \dots, y_m)\, d\mathbf{y} = F(t),\end{aligned}$$

where we take the last line here as the definition of F. But we have assumed that Θ_t satisfies the Carleson condition, and hence $|F(t)|^2 \frac{dt}{t}\, dx$ is a Carleson measure. The strong Carleson condition follows: for all cubes $Q \subset \mathbb{R}^n$,

$$\int_0^{\ell(Q)} |\Theta_t(1, \dots, 1)(x)|^2 \frac{dt}{t} = \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |F(t)|^2 \frac{dt}{t}\, dx \lesssim 1.$$

If we assume also that $\Psi^t = \Psi$ is constant in t, then it follows that $F(t) = c_0$ is a constant function. But then $|c_0|^2 \frac{dt}{t}\, dx$ is a Carleson measure and hence integrable on $Q \times (0, \ell(Q)]$ for all cubes $Q \subset \mathbb{R}^n$. This forces $c_0 = 0$ when Ψ^t is constant in t, which completes the proof. □

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