Michigan Math. J. 63 (2014), 293–332

Common Boundary Values of Holomorphic Functions for Two-Sided Complex Structures

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Abstract. Let Ω_1, Ω_2 be two disjoint open sets in $\mathbf{R}^{2n}$ whose boundaries share a smooth real hypersurface M as a relatively open subset. Assume that Ω_i is equipped with a complex structure J^i that is smooth up to M. Suppose that at each point $x \in M$ there is a vector $v \in T_x M$ such that $J_x^1 v$ and $J_x^2 v$ are in the same connected component of $T_x\mathbf{R}^{2n} \setminus T_x M$. If f is holomorphic with respect to both structures in the open sets and continuous on $\Omega_1 \cup M \cup \Omega_2$, then f must be smooth on the union $\Omega_1 \cup M$. Although the result, as stated, is far more meaningful for integrable structures, our methods make it much more natural to deal with the general almost complex structures without the integrability condition. The result is therefore proved in the framework of almost complex structures.

1. Introduction

In this paper, we study the regularity of boundary values of two functions that are holomorphic with respect to two complex structures defined on two domains separated by a real hypersurface. We are interested in the situation where the two functions have the same continuous boundary values on the hypersurface. Notice that the regularity becomes an interior property when the two structures are the restriction of the same complex structure, which is well established by the Newlander–Nirenberg theorem; our results are concerned with a pair of distinct structures. To highlight the relevance of our problem to yet another classical regularity problem, we recall the edge-of-the-wedge theorem, which deals with two holomorphic functions on two wedges in $\mathbf{C}^n$ that have the same appropriate boundary values on the edge. Under suitable assumptions on the wedges, the theorem concludes that both functions are actually the restriction of the same holomorphic function defined on a domain containing the edge. The edge-of-the-wedge theorem, originally due to Bogolyubov, has been extended in great generality by many authors. For instance, see Rudin [20] and references therein, Morimoto [15], Pinčuk [18], Bedford [2], Straube [23], Rosay [19], Forstnerič [8], and Eastwood and Graham [7]. Despite some similarity between the classical edge-of-the-wedge theorem and our results, the study of the regularity in this paper apparently gives rise to a new

Received April 10, 2013. Revision received February 7, 2014.
Research of the first author supported in part by Austrian Science Fund FWF grant M1461-N25.
Research of the second author supported in part by NSF grant DMS-0705426.

type of boundary regularity problem even in one complex variable. Our methods are effective to study the case of not necessarily integrable almost complex structures, and they allow us to deal with a pair of systems of nonhomogeneous equations.

Our main result is the following theorem.

THEOREM 1.1. *Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{R}^{2n}$ such that their boundaries $\partial\Omega^1$, $\partial\Omega^2$ share a $\mathcal{C}^\infty$ smooth real hypersurface M. Suppose that M is relatively open in each $\partial\Omega^\ell$. For $\ell = 1, 2$, let J^ℓ be an almost complex structure of class $\mathcal{C}^\infty$ on $\Omega_\ell \cup M$. Suppose that at each point $p \in M$, there is a vector v tangent to M at p such that $J^1_p v$ and $J^2_p v$ belong to the same connected component of $T_p\mathbf{R}^{2n} \setminus T_pM$. Let f be a continuous function on $\Omega_1 \cup M \cup \Omega_2$. Suppose that $(\partial_{x_j} + iJ^\ell\partial_{x_j})f$ and $(\partial_{y_j} + iJ^\ell\partial_{y_j})f$, originally defined on Ω_ℓ, extend to functions of class $\mathcal{C}^\infty$ on $\Omega_\ell \cup M$ for $\ell = 1, 2$ and $1 \le j \le n$. Then f is of class $\mathcal{C}^\infty$ on $\Omega_1 \cup M$.*

We will actually prove a more precise version of Theorem 1.1 under finite smoothness assumptions made on the hypersurface, the structures, and the set of the derivatives of f in the theorem. We emphasize that we make no convexity assumption on M with respect to either of the almost complex structures. Therefore, it is not clear if the smoothness of the function restricted to the hypersurface can be achieved via classical one-sided techniques such as of Bishop discs, and, although it leads to a loss of regularity, the use of the Fourier transform appears to be essential in our approach to the boundary value problem.

As mentioned earlier, the interior regularity of f for integrable almost complex structures is ensured by the well-known Newlander–Nirenberg theorem [16] (see also Nijenhuis and Woolf [17] and Webster [26]). There are results on the Newlander–Nirenberg theorem for pseudoconvex domains with boundary by Catlin [5] and Hanges and Jacobowitz [10]. See earlier work of Hill [11] on failure of a Newlander–Nirenberg-type theorem with boundary.

Let us observe how the common boundary values arise from the Cauchy–Green operator for $\overline{\partial}$ in $\mathbf{C}$. Let $X = \partial_{\overline{z}} + a(z)\partial_z$ with $|a(z)| < 1$ and a being a $\mathcal{C}^\infty$ function on the closure of a bounded domain Ω with smooth boundary in $\mathbf{C}$. To seek new coordinates $z + f(z)$ to transform $\partial_{\overline{z}} + a\partial_z$ into a multiple of $\partial_{\overline{z}}$, we consider the equation

$$\partial_{\overline{z}} f + a(z)\partial_z f + b(z) = 0, \quad z \in \Omega, \tag{1.1}$$

where b is $\mathcal{C}^\infty$ on the closure of Ω. To solve it, one considers the integro-differential equation

$$f(z) + T(a\partial_z f)(z) + Tb(z) = 0, \quad z \in \Omega. \tag{1.2}$$

Here $T = T_\Omega$ is the Cauchy–Green operator

$$Tf(z) = \frac{1}{\pi}\int_\Omega \frac{f(\zeta)}{z-\zeta}\, d\xi\, d\eta$$

with $\xi = \operatorname{Re} \zeta$ and $\eta = \operatorname{Im} \zeta$. Equation (1.2) is equivalent to (1.1) and an extra equation

$$\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0, \quad z \in \Omega. \tag{1.3}$$

When $f \in \mathcal{C}(\partial\Omega)$, the jump formula implies that (1.3) is equivalent to f being the boundary values of a function that is holomorphic on $\Omega' = \mathbf{C} \setminus \overline{\Omega}$, continuous on $\overline{\Omega'}$, and vanishing at ∞. See Lemmas 6.3 and 6.6 for details. To find a solution f that is $\mathcal{C}^\infty$ on $\overline{\Omega}$ for equation (1.1), we would like to invert the operator $\mathrm{I} + Ta\partial_z$ in $\mathcal{C}^k$ space for each finite k. In complex dimension one, we also obtain a sharp version of Theorem 1.1 under finite smoothness assumptions; see Theorem 6.2. As an application of Theorem 6.2, we will prove the following.

THEOREM 1.2. *Let $0 < \alpha < 1$, and let $\Omega \subset \mathbf{C}$ be a bounded domain with $\mathcal{C}^{1+\alpha}$ boundary. Let $a, b \in \mathcal{C}^\alpha(\overline{\Omega})$. Assume that $|a|_0 < 1$. Then (1.2) admits a unique solution $f \in \mathcal{C}^{1+\alpha}(\overline{\Omega})$. Assume further that $a, b \in \mathcal{C}^{k+\alpha}(\overline{\Omega})$ and $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$ for an integer $k \geq 0$. Then $f \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega})$. Moreover, the linear map $\mathrm{I} + Ta\partial_z$ from $\mathcal{C}^{k+1+\alpha}(\overline{\Omega})$ into itself has a bounded inverse.*

Theorem 1.2 yields a method to solve equation (1.2) for the boundary regularity of the solutions. We will prove a version of the above theorem when equation (1.2) depends smoothly on a parameter. As a consequence, we will obtain a $\mathcal{C}^\infty$ version of the Riemann mapping theorem for complex structures and simply connected bounded domains with smooth boundaries in the complex plane that depend ($\mathcal{C}^\infty$) smoothly on a parameter.

Finally, we would like to explain the condition in Theorem 1.1 that $J_x^1 v$, $J_x^2 v$ be in the same connected component of $T_x\mathbf{R}^{2n} \setminus T_x M$. In one complex variable, this condition is equivalent to J^1, J^2 defining the same orientation for $T_x\mathbf{R}^2$ when $x \in M$. This condition is also necessary as illustrated by the following counterexample. Let J^1 be the standard complex structure defined by $\partial_{\overline{z}_1}$ on the upper half-plane Ω^+. Let J^2 be the complex structure defined by ∂_{z_1} on the lower half-plane Ω^-. Let M be defined by $\operatorname{Im} z_1 = 0$, which is the boundary of both Ω^+ and Ω^-. Let f be a holomorphic function on $\operatorname{Im} z_1 > 0$ that extends continuously to $\Omega^+ \cup M$. On Ω^-, let $f(z_1) = f(\overline{z}_1)$. We can find f that does not extend as a $\mathcal{C}^1$ function on $\Omega^+ \cup M$. However, for any integer $n > 1$, the validity of the assertion on the smoothness of f has no obvious connection with the orientations defined by J^1, J^2. When $n = 2$, it is easy to extend the above counterexample to both cases where J^1, J^2 define the same orientation or opposite orientations by adding ∂_{z_2} or $\partial_{\overline{z}_2}$ to the one-dimensional structures.

The paper is organized as follows.

In Section 2 we recall some basic facts about the solution operator T for $\overline{\partial}$ equations on bounded domains in the complex plane. We will briefly address the invertibility of $\mathrm{I} + Ta\partial$ when a has compact support in a domain Ω. The invertibility for this special case is used in Section 3 to study the interior regularity of J-holomorphic curves depending on a parameter.

In Section 3 we give a detailed proof of the existence and regularity of J-holomorphic curves that depend on a parameter. The result is due to Nijenhuis and Woolf [17] for the finite smoothness case. We take the opportunity to modify their proof to treat the $\mathcal{C}^\infty$ case in the z-variable, which is not in [17]. However, the authors do not know if the regularity result holds for $\mathcal{C}^\infty$ class in the parameter variable, which is another case left open in [17].

Section 4 contains some elementary estimates for Cauchy integrals for domains depending on a parameter.

In Section 5 we prove our main theorem by establishing a more precise finite smoothness version of Theorem 1.1. The main step of the proof is to establish the smoothness of f on the common boundary M. It is for this step that we need the assumptions on both structures. Our basic technique is the Fourier transform applied on families of lines, or straightened curves, in M. We will show that the Fourier transform of f on these lines will decay uniformly within the families as if f has the desired regularity. To apply the two almost complex structures, we will use the flexibility that we can attach two families of approximate J-holomorphic curves, one for each almost complex structure, to the same family of the lines in M. After we establish the regularity of f on M, we will obtain the regularity of f from one side of M by using families of genuine J-holomorphic curves. The regularity in all variables will be obtained after we establish uniform bounds on the derivatives of f on families of J-holomorphic curves attached to M. We should mention that the methods of establishing the smoothness of a function via uniform boundedness of its derivatives on families of curves have appeared in other works (for instance, see Tumanov [24], Coupet, Gaussier, and Sukhov [6]).

In Section 6 we treat the one complex variable case by establishing some sharp regularity results. We will conclude the paper with some open problems.

2. Inverting $\mathrm{I} + TA\overline{\partial_z}$

In this section, we recall estimates on the Cauchy–Green operator T and $\partial_z T$. We will discuss the inversion of $\mathrm{I} + TA\partial_z$ in spaces of higher-order derivatives when A has a small $\mathcal{C}^\alpha$ norm. When A has compact support, we can invert $\mathrm{I} + TA\partial_z$ and $\mathrm{I} + TA\overline{\partial_z}$ by direct estimates. This is obtained in this section. In Section 6 we will finish the proof that $\mathrm{I} + TA\partial_z$ is indeed invertible when A is a suitable scalar function, that is, Theorem 1.2 is valid. In fact, we will prove a parameter version for derivatives of any order by using our main theorem and a method from [17].

We will systematically use the functions that depend on parameters as follows. To prove Theorem 1.1, we need to build up smoothness of a function via its smoothness on a web of curves and on uniform bounds of its derivatives on these curves. We will need two families of curves; one consists of real curves in M, and another consists of J-holomorphic curves intersecting M transversally. We will need to study the Cauchy–Green operator T on domains in J-holomorphic curves depending on parameters. The regularity of J-holomorphic curves depending on parameters is given in Proposition 3.7. The estimates on operators T on domains depending on parameters are in Lemma 4.1.

Throughout the paper, when a parameter set P is involved in $\Omega \times P$, Ω is a bounded open set in a Euclidean space, and P is the closure of a bounded open set in a Euclidean space. We assume that two points a, b in the interior of $\overline{\Omega} \times P$ can be connected by a smooth curve in the interior of length at most $C|b-a|$.

We now recall spaces of functions with parameter defined in [17]. Let $\mathcal{C}^k(\overline{\Omega})$ denote the set of functions f such that all partial derivatives of order k are continuous functions on Ω that extend continuously to $\overline{\Omega}$. The usual norm on $\mathcal{C}^{k+\alpha}(\overline{\Omega})$ is denoted by $|\cdot|_{k+\alpha}$. For integers $k, j \geq 0$ and $0 \leq \alpha < 1$, we define $\hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$ to be the set of functions f defined on $\overline{\Omega} \times P$ such that for all integer $0 \leq \ell \leq j$, the map $t \mapsto \partial_t^\ell f(\cdot, t)$ is continuous from P into $\mathcal{C}^k(\overline{\Omega})$ and such that

$$\|f\|_{k+\alpha,j} := \max_{0\leq \ell \leq j} \sup_{t\in P} |\partial_t^\ell f(\cdot,t)|_{k+\alpha} < \infty.$$

Here ∂_t^ℓ denotes the partial derivatives of order ℓ in parameter variables t. Define

$$\hat{\mathcal{C}}^{\infty,j}(\overline{\Omega}, P) := \bigcap_{k=1}^{\infty} \hat{\mathcal{C}}^{k,j}(\overline{\Omega}, P), \qquad \mathcal{C}^{\infty,j}(\overline{\Omega}, P) := \hat{\mathcal{C}}^{\infty,j}(\overline{\Omega}, P).$$

To simplify notation, the parameter set P will not be indicated sometimes.

Let Ω be a bounded domain in $\mathbf{C}$. The $\overline{\partial}$ solution operator T and $S = \partial_z T$ are

$$Tf(z) := \frac{1}{\pi}\int_\Omega \frac{f(\zeta)}{z-\zeta}\,d\xi\,d\eta, \qquad Sf(z) := -\frac{1}{\pi} p.v. \int_\Omega \frac{f(\zeta)}{(z-\zeta)^2}\,d\xi\,d\eta. \tag{2.1}$$

It is well known that $\partial_{\overline{z}} T$ is the identity on $L^p(\Omega)$ when $p > 2$. Assume now that $0 < \alpha < 1$. When $f \in \mathcal{C}^\alpha(\overline{\Omega})$ and $\partial\Omega \in \mathcal{C}^{1+\alpha}$, one has

$$Sf(z) = -\frac{1}{\pi}\int_\Omega \frac{f(\zeta)-f(z)}{(z-\zeta)^2}\,d\xi\,d\eta - \frac{f(z)}{2\pi i}\int_{\partial\Omega} \frac{d\overline{\zeta}}{\zeta - z}. \tag{2.2}$$

If f has compact support in Ω, or if $f \in \mathcal{C}^{k+\alpha}(\overline{\Omega})$ and $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$, then

$$|Tf|_{k+1+\alpha} \leq C_{k+1+\alpha}|f|_{k+\alpha}, \qquad |Sf|_{k+\alpha} \leq C_{k+1+\alpha}|f|_{k+\alpha}. \tag{2.3}$$

See Bers [3] and Vekua [25, p. 56]. The above estimates for domains with parameter will be derived in Section 4. Recall that

$$\partial_z Sf = S\partial_z f, \qquad \partial_{\overline{z}} Sf = \partial_z f, \tag{2.4}$$

where the first identity needs f to have compact support in Ω.

For $f \in \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$, define $Tf(z,t)$ and $Sf(z,t)$ by (2.1) and (2.2) by fixing the parameter t.

Lemma 2.1. *Let $k, j \geq 0$ be two integers, and let $0 < \alpha < 1$. Let $\Omega \subset \mathbf{C}$ be a bounded domain with $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$. Then*

$$\begin{aligned} T &: \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P) \to \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P), \qquad \|Tf\|_{k+1+\alpha,j} \leq C_{k+1+\alpha}\|f\|_{k+\alpha,j}, \\ S &: \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P) \to \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P), \qquad \|Sf\|_{k+\alpha,j} \leq C_{k+1+\alpha}\|f\|_{k+\alpha,j} \end{aligned} \tag{2.5}$$

for some positive constant $C_{k+1+\alpha}$.

Proof. By (2.2) we get $S(\hat{\mathcal{C}}^{k+\alpha,j}) \subset \hat{\mathcal{C}}^{k,j}$. We can verify that $\partial_t S = S\partial_t$ on $\hat{\mathcal{C}}^{\alpha,j}$ for $j \geq 1$. Thus, $S(\hat{\mathcal{C}}^{k+\alpha,j}) \subset \hat{\mathcal{C}}^{k+\alpha,j}$ by (2.3).

The Cauchy kernel is integrable. So $T(\hat{\mathcal{C}}^{0,j}(\overline{\Omega}, P)) \subset \hat{\mathcal{C}}^{0,j}(\overline{\Omega}, P)$. Also, $\partial_t T = T\partial_t$ on $\hat{\mathcal{C}}^{0,j}$ for $j \geq 1$. The rest of assertions follows from $\partial_z T = S$ and $\partial_{\overline{z}} T = \mathrm{I}$. □

By an abuse of notation, we define $\overline{\partial_z} f = \overline{\partial_z \overline{f}}$.

LEMMA 2.2. *Let $k, j \geq 0$ be two integers, and let $0 < \alpha < 1$. Let Ω be a bounded domain in* $\mathbf{C}$. *Let $A \in \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$ be an $m \times m$ matrix. There exists $\varepsilon_\alpha > 0$, depending only on α, such that the following hold.*

(i) *If $\partial\Omega \in \mathcal{C}^{1+\alpha}$ and $|A|_{\alpha,0} < \varepsilon_\alpha$, then*

$$\mathrm{I} + TA\partial_z, \mathrm{I} + TA\overline{\partial_z} : [\hat{\mathcal{C}}^{1+\alpha,j}(\overline{\Omega}, P)]^m \to [\hat{\mathcal{C}}^{1+\alpha,j}(\overline{\Omega}, P)]^m$$

have bounded inverses.

(ii) *If $A(\cdot, t)$ have compact support in Ω for all $t \in P$ and $|A|_{\alpha,0} < \varepsilon_\alpha$, then*

$$\mathrm{I} + TA\partial_z, \mathrm{I} + TA\overline{\partial_z} : [\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P)]^m \to [\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P)]^m$$

have bounded inverses.

Proof. We want to show that the inverse of $\mathrm{I} + TA\partial_z$ is given by

$$L = \mathrm{I} + \sum_{\ell=1}^{\infty} (-1)^\ell (TA\partial_z)^\ell.$$

Since $(TA\partial_z)^\ell = TA(SA)^{\ell-1}\partial_z$, we need to show that the norms of $(SA)^{\ell-1}$ for various derivatives tend to zero sufficiently fast. When S operates on functions with compact support, it commutes with $\partial_t, \partial_z, \partial_{\overline{z}}$ somewhat. However, differentiating the operator product $(SA)^\ell$ requires counting terms efficiently as ℓ tends to ∞.

(i) Fix $0 < \theta < 1/2$. Note that

$$\|fg\|_{k+\alpha,j} \leq C_{k,j} \|f\|_{k+\alpha,j} \|g\|_{k+\alpha,j}.$$

By (2.1) we have $\|SA\|_{\alpha,0} \leq C'_\alpha \|A\|_{\alpha,0}$. Thus,

$$\|(SA)^\ell\|_{\alpha,0} \leq (C_\alpha \|A\|_{\alpha,0})^\ell \leq \theta^\ell$$

if $\|A\|_{\alpha,0}$ is sufficiently small. Then

$$\|TA(SA)^{\ell-1}\partial_z f\|_{1+\alpha,0} \leq C_\alpha \theta^{\ell-1} \|A\|_{\alpha,0} \|f\|_{1+\alpha,0}.$$

This shows that for $f \in \hat{\mathcal{C}}^{1+\alpha,0}$, the series $\sum_{\ell=0}^{\infty} (-1)^\ell TA(SA)^{\ell-1}\partial_z f$ converges to $Lf \in \hat{\mathcal{C}}^{1+\alpha,0}$. Moreover, $\|Lf\|_{1+\alpha,0} \leq C\|f\|_{1+\alpha,0}$. It is straightforward that $L(\mathrm{I} + TA\partial_z)$ and $(\mathrm{I} + TA\partial_z)L$ are the identity on $\hat{\mathcal{C}}^{1+\alpha,0}$. This verifies (i) for $j = 0$. The case of $j > 0$ will follow from the argument in (ii) by using $\partial_t T = T\partial_t$ and $\partial_t S = S\partial_t$.

(ii) We need to show that $\sum \|(SA)^\ell\|_{k+\alpha,j}$ converges when A has compact support in Ω. Denote by $C_{k+\alpha,j}$ a positive constant depending only on k, j, and $\|A\|_{k+\alpha,j}$. By (2.4) and $\partial_t S = S\partial_t$, we can write

$$\partial SA = \tilde{S}\tilde{\partial}A,$$

where $\tilde{S}$ is either S or I, and ∂, $\tilde{\partial}$ are of form ∂_z, $\partial_{\overline{z}}$, or ∂_t. Denote by ∂^K a derivative in z and $\overline{z}$ of order $|K|$. Then $\partial(SA)^\ell$ equals a sum of terms of the form

$$S_{m_1}(\partial^{K_1}A)\cdots S_{m_\ell}(\partial^{K_\ell}A)\partial^{K_{\ell+1}}, \quad |K_1|+\cdots+|K_{\ell+1}|=1.$$

Here S_{m_i} is either S or I; in particular, $\|S_{m_i}\|_{k+\alpha,j} \leq C_{k+\alpha,j}$ for all m_i. The sum has at most $\ell+1$ terms. Thus, $\partial^K\partial_t^I(SA)^\ell$ is a sum of at most $(\ell+1)^{|K|+|I|}$ terms of the form

$$S_{m_1}(\partial^{K_1}\partial_t^{I_1}A)\cdots S_{m_\ell}(\partial^{K_\ell}\partial_t^{I_\ell}A)S_{m_\ell}\partial^{K_{\ell+1}}\partial_t^{I_{\ell+1}}. \tag{2.6}$$

Assume that $|K|\leq k$, $|I|\leq j$, and $\ell > k+j$. With $C_\alpha \geq 1$,

$$\begin{aligned}\|\partial^K\partial_t^I((SA)^\ell f)\|_{\alpha,0} &\leq (\ell+1)^{k+j}C_\alpha^\ell(1+\|A\|_{k+\alpha,j})^{k+j}\|A\|_{\alpha,0}^{\ell-k-j}\|f\|_{k+\alpha,j}\\ &\leq (\ell+1)^{k+j}C_{k+\alpha,j}\|f\|_{k+\alpha,j}\theta^{\ell-k-j}.\end{aligned}$$

This shows that $\|(SA)^\ell\|_{k+\alpha,j} \leq C'_{k+\alpha,j}(\ell+1)^{k+j}\theta^{\ell-k-j}$. Hence,

$$\|TA(SA)^\ell\partial_z\|_{k+1+\alpha,j} \leq C_{k+\alpha,j}(\ell+1)^{k+j}\theta^{n-k-j}.$$

We conclude that $\|(\mathrm{I}+TA\partial_z)^{-1}\|_{k+1+\alpha,j} < \infty$.

The proof for $\mathrm{I}+TA\overline{\partial_z}$ is obtained by minor changes. Indeed, denoting by $Cf = \overline{f}$ the complex conjugation, we write $(TA\overline{\partial_z})^\ell = TAC(SAC)^{\ell-1}\partial_z$. We have $\partial_z C = C\partial_{\overline{z}}$ and $\partial_{\overline{z}}C = C\partial_z$, and since we may assume that t are real variables, we also have $\partial_t C = C\partial_t$. Thus, $\partial^K\partial_t^I(SAC)^n$ is a sum of at most $(\ell+1)^{|K|+|I|}$ terms of the form

$$S_{m_1}(\partial^{K_1}\partial_t^{I_1}A)C\cdots S_{m_\ell}(\partial^{K_\ell}\partial_t^{I_\ell}A)C\partial^{K_{\ell+1}}\partial_t^{I_{\ell+1}}.$$

Substitute the above for (2.6). The remaining argument follows easily. □

We need a simple version of Borel theorem with parameter.

LEMMA 2.3. *Let N be a positive integer or ∞. Let $0\leq j<\infty$ be an integer, and $0\leq\alpha<1$. Let ε_k be positive numbers for $1\leq k<N$, and let $\varepsilon_N=0$ for $N<\infty$. Let $f_I \in \hat{\mathcal{C}}^{N-1-|I|+\alpha,j}(\mathbf{R}^n,P)$ for $0\leq|I|<N$ with $I=(i_1,\ldots,i_m)$. Assume that all $f_I(\cdot,t)$ have support contained in a compact subset K of the unit ball B^n. There exists $Ef\in\hat{\mathcal{C}}^{N-1+\alpha,j}(\mathbf{R}^n\times\mathbf{R}^m,P)$ such that $\partial_y^I Ef(x,0,t)=f_I(x,t)$. Moreover, $Ef(\cdot,t)$ have compact support in the unit ball of $\mathbf{R}^n\times\mathbf{R}^m$ and*

$$\|Ef\|_{k+\alpha,j} \leq \varepsilon_{k+1} + C_{N,k,K}\sum_{|I|\leq k}\|f_I\|_{k-|I|+\alpha,j}, \quad 0\leq k<N. \tag{2.7}$$

Here $C_{N,k,K}$ depends also on $\varepsilon_1,\ldots,\varepsilon_k$ and the upper bound of $\|f_I\|_{k-|I|+\alpha,j}$ for $|I|\leq k$.

Proof. We extend f one dimension at a time. Start with $m = 1$. We consider an extension of the form

$$Ef(x, y, t) = \sum_{\ell < N} \frac{y^\ell}{\ell!} g_\ell(x, t) g(\delta_\ell^{-1} y), \tag{2.8}$$

where g is a smooth function of compact support in $(-1, 1)$, and $g(y) = 1$ for $|y| < 1/2$. We will also choose δ_ℓ that decreases to 0 so rapidly that Ef is in $\hat{\mathcal{C}}^{N-1+\alpha,j}$ and (2.7) holds. Here $y^\ell g_\ell(x, y, t)$ is a substitute of $y^\ell f_\ell(x, t)$ such that $y^\ell g_\ell(x, y, t)$ is in $\hat{\mathcal{C}}^{N-1+\alpha,j}$. We also need the correct ℓth y-derivative of $y^\ell g_\ell(x, y, t)$ due to the presence of $y^{\ell'} g_{\ell'}(x, y, t)$ for $\ell' < \ell$.

Denote by B_δ^n the open ball in $\mathbf{R}^n$ centered at the origin with radius $0 < \delta < \frac{1}{2}\operatorname{dist}(K, \partial B^n)$. Let ϕ be a smooth function on $\mathbf{R}^n$ with support in B_δ^n satisfying $\int_{\mathbf{R}^n} \phi(y)\, dy = 1$. With $a_\ell \in \hat{\mathcal{C}}^{N-1-\ell,j}$ to be determined, consider

$$g_\ell(x, y, t) = \int_{\mathbf{R}^n} a_\ell(x - yz, t)\phi(z)\, dz.$$

Fix t. We first consider $\ell \le k < N$. For $|I| = k$ and $y \ne 0$, we have

$$\partial^I (y^\ell g_\ell(x, y, t)) = \sum_{i_1 + |I_2| = k} C_{i_1 I_2} \partial_y^{i_1} y^\ell \partial^{I_2} \int a_\ell(x - yz, t)\phi(z)\, dz \tag{2.9}$$

with $i_1 \le \ell$. Write $I_2 = I_3 + I_4$ with $|I_3| = k - \ell$ and $|I_4| = \ell - i_1$. We have

$$\partial^{I_3} g_\ell(x, y, t) = \sum_{|L| = |I_3|} \int \partial^L a_\ell(x - yz, t)\phi_{I_3 L}(z)\, dz$$

for some $\phi_{I_3 L}$ with support in B_δ^n. When $y \ne 0$, changing variables and taking derivative ∂^{I_4}, we get

$$\partial^{I_2} g_\ell(x, y, t) = \sum_{|L| = |I_3|} \int \frac{1}{y^{|I_4| + n}} \partial^L a_\ell(z, t) \tilde{\phi}_{I_3 I_4 L}\left(\frac{x - z}{y}\right) dz.$$

Changing variables again, we get

$$y^{|I_4|} \partial^{I_2} g_\ell(x, y, t) = \sum_{|L| = |I_3|} \int \partial^L a_\ell(x - yz, t) \tilde{\phi}_{I_3 I_4 L}(z)\, dz.$$

The right-hand side and its derivatives in t of order at most j are clearly continuous functions. Then, by (2.9), $b_\ell(x, y, t) := y^\ell g_\ell(x, y, t)$ are of class $\hat{\mathcal{C}}^{k+\alpha,j}$ for $l \le k$. This shows that $b_\ell \in \hat{\mathcal{C}}^{N-1+\alpha,j}$ for all ℓ. By the product rule, at $y = 0$,

$$\partial_y^\ell b_\ell(x, y, t) = \ell!\, a_\ell(x, t), \qquad \partial_y^{\ell'} b_{\ell'}(x, y, t) = 0, \quad \ell' < \ell.$$

Starting with $a_0 = f_0$, we find $a_\ell \in \hat{\mathcal{C}}^{N-1-\ell+\alpha,j}$ inductively such that $\partial_y^\ell \sum_{\ell' \le \ell} b_{\ell'}(x, y, t)$ equals $\ell!\, f_\ell(x, t)$ at $y = 0$. We also have

$$\|b_\ell\|_{\ell+\alpha,j} \le C_{\ell,K} \|a_\ell\|_{\alpha,j}.$$

Without loss of generality, we may assume that ε_ℓ decreases. We choose small δ_ℓ with $0 < \delta_\ell < 1/m$ such that $\tilde{b}_\ell(x, y, t) := y^\ell g_\ell(x, t) g(\delta_\ell^{-1} y)$ satisfies

$$\|\tilde{b}_\ell\|_{\ell+\alpha, j} \leq 2^{-l} \varepsilon_\ell.$$

Let Ef be defined by (2.8). Estimate (2.7) is then immediate, and Ef is in $\hat{\mathcal{C}}^{N-1+\alpha, j}$ and satisfies $\partial_y^\ell Ef(x, y, t) = f_\ell(x, t)$ for $\ell < N$.

For $m > 1$, set $y = (y', y_m)$ and $y' = (y_1, \ldots, y_{m-1})$. Suppose that we have found extensions $\tilde{f}_\ell \in \hat{\mathcal{C}}^{N-1-\ell+\alpha, j}(\mathbf{R}^n \times \mathbf{R}^{m-1}, P)$ such that $\partial_{y'}^{I'} \tilde{f}_\ell = f_{I'\ell}$ at $y' = 0$ for all $|I'| < N - \ell$ and

$$\|\tilde{f}_\ell\|_{k-\ell+\alpha, j} \leq e'_{k+1,K} + C_{k,K} \sum_{|I'| \leq k-\ell} \|f_{I'\ell}\|_{k-|I'|-\ell+\alpha, j}, \quad k < N, \tag{2.10}$$

where $e'_{k,K} > 0$ will be determined later. Moreover, assume that $\tilde{f}_\ell(\cdot, t)$ have support in a compact subset K' of the unit ball of $\mathbf{R}^{n+m-1}$, where K' depends only on K. Using the one-dimensional result again, we get $Ef \in \hat{\mathcal{C}}^{N-1+\alpha, j}(\mathbf{R}^n \times \mathbf{R}^m, P)$ with compact support in the unit ball of $\mathbf{R}^{n+m}$. Furthermore, $\partial_{y_n}^\ell Ef = \tilde{f}_\ell$ at $y_n = 0$, and

$$\|Ef\|_{k+\alpha, j} \leq e'_{k+1,K} + C'_{k,K} \sum_{0 \leq \ell \leq k} \|\tilde{f}_\ell\|_{k-\ell+\alpha, j}, \quad k < N.$$

Choose $e'_{k+1,K} > 0$ small enough to ensure that combining the above inequality with (2.10) yields (2.7). $\square$

The above proof for the nonparameter case is in [12, pp. 16 and 18]. When f is defined on $y_n \leq 0$ with $\partial_{x_n}^k f = f_k$ on $y_n = 0$, the above extension Ef can be replaced by f on $y_n \leq 0$. The same conclusions on Ef hold. Seeley [22] has a linear extension $E : \mathcal{C}^\infty(\overline{\mathbf{R}}_+^n) \to \mathcal{C}^\infty(\mathbf{R}^n)$ such that $E : \mathcal{C}^k(\overline{\mathbf{R}}_+^n) \to \mathcal{C}^k(\mathbf{R}^n)$ have bounds depending only on k.

3. J-Holomorphic Curves and Derivatives on Curves

This section is mainly devoted to the study of J-holomorphic curves with parameter. The result is essentially in the work of Nijenhuis and Woolf [17]. See also Ivashkovich and Rosay [14] for another regularity proof and the existence of J-holomorphic curves with prescribed jets. The proof below relies only on some basic facts about the Cauchy–Green operator and the inversion of $\mathrm{I} + TA\overline{\partial_\zeta}$ discussed in Section 2. We also study how to obtain the smoothness of a function from its smoothness on a web of curves and uniform bounds of its derivatives on the curves (see Proposition 3.9). This result will be one of main ingredients in the proof of Theorem 1.1. Our results are local. Throughout the paper, a real hypersurface M will be a relatively open subset of the boundary of a domain $\Omega \subset \mathbf{R}^{2n}$ or a closed subset without boundary in the domain.

Definition 3.1. Let $k \geq 0$ be an integer, and let $0 \leq \alpha < 1$. An almost complex structure J of class $\mathcal{C}^{k+\alpha}$ on Ω is a $\mathcal{C}^{k+\alpha}$ mapping from $T\Omega$ onto itself such that each J_p is an $\mathbf{R}$-linear mapping from $T_p\mathbf{R}^{2n}$ onto itself with $J_p^2 = -\,\mathrm{id}$. An almost complex structure J of class $\mathcal{C}^{k+\alpha}$ on $\Omega \cup M$ is the restriction to $\Omega \cup M$ of some $\mathcal{C}^{k+\alpha}$ almost complex structure defined on some open neighborhood of $\Omega \cup M$ in $\mathbf{R}^{2n}$.

Note that for each $p \in \Omega$ (resp. $\Omega \cup M$), there are $\mathcal{C}^{k+\alpha}$ real vector fields $v_1, \dots, v_n$ defined near p such that

$$X_1 = v_1 + iJv_1, \quad \dots, \quad X_n = v_n + iJv_n \tag{3.1}$$

and their complex conjugates are pointwise $\mathbf{C}$-linearly independent. Conversely, if $X_1, \dots, X_n$ are $\mathcal{C}^{k+\alpha}$ vector fields such that the n vector fields and their complex conjugates are pointwise $\mathbf{C}$-linearly independent, there exists a unique $\mathcal{C}^{k+\alpha}$ almost complex structure J satisfying (3.1). The *operator norm* $\|A\|$ of a linear map $A : T_p\Omega \to T_p\Omega$ is defined as $\max\{|Av| \colon \ |v| = 1\}$ with $|a\partial_x + b\partial_y| = (\sum_{\ell=1}^n |a_\ell|^2 + |b_\ell|^2)^{1/2}$ being the Euclidean norm on $T_p\Omega$.

DEFINITION 3.2. We say that a diffeomorphism φ transforms $X_1, \dots, X_n$ into $\tilde{X}_1, \dots, \tilde{X}_n$ if $d\varphi(X_\ell)$ are locally in the span of $\tilde{X}_1, \dots, \tilde{X}_n$.

A linear complex structure J on $\mathbf{R}^{2n}$ is given by

$$X_\ell = \sum_{1 \le s \le n} (b_{\ell s}\partial_{\overline{z}_s} + a_{\ell s}\partial_{z_s}), \quad \ell = 1, \dots, n, \tag{3.2}$$

where constant matrices $A := (a_{\ell s})$ and $B := (b_{\ell s})$ satisfy

$$\begin{vmatrix} B & A \\ \overline{A} & \overline{B} \end{vmatrix} \ne 0.$$

The map $z = w\overline{B} + \overline{w}A$ transforms $\partial_{\overline{w}_1}, \dots, \partial_{\overline{w}_n}$ into $X_1, \dots, X_n$, and hence J_{st} into J given by

$$J = KJ_{st}K^{-1}, \quad J_{st} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \mathrm{Re}(B+A) & \mathrm{Im}(A-B) \\ \mathrm{Im}(A+B) & \mathrm{Re}(B-A) \end{pmatrix}. \tag{3.3}$$

Thus, under a local change of coordinates by shrinking Ω or $\Omega \cup M$, an almost complex structure J is locally given by

$$X_\ell = \partial_{\overline{z}_s} + \sum_{1 \le s \le n} a_{\ell s}(z)\partial_{z_s}, \quad \ell = 1, \dots, n, \tag{3.4}$$

where the operator norm of $A = (a_{\ell s})$ satisfies $\|A(z)\| < 1$ on Ω (resp. $\Omega \cup M$). Conversely, notice that the condition $\|A\| < 1$ ensures that n vector fields of the form (3.4) define an almost complex structure.

In the next lemma, we give a quantitative condition and a geometric condition, each ensuring the assumption made on the two almost complex structures in Theorem 1.1.

LEMMA 3.3. *Let J^1, J^2 be two linear complex structures on $\mathbf{R}^{2n}$. Let M be a hyperplane in $\mathbf{R}^{2n}$. If $\|J^2 - J^1\| < 2$ or if $T_0M \cap J^1T_0M \ne T_0M \cap J^2T_0M$, then* *there exists $v \in T_0M$ such that J^1v, J^2v are in the same connected component of $T_0\mathbf{R}^{2n} \setminus T_0M$.*

Proof. Denote by $T_0(M, J^\ell) := T_0M \cap J_0^\ell T_0M$ the J^ℓ-holomorphic tangent space at the origin for $\ell = 1, 2$. Let ω_1, ω_2 be the two connected components of $T_0\mathbf{R}^{2n} \setminus T_0M$. Note that J^ℓ sends one of two connected components of $T_0M \setminus T_0(M, J^\ell)$ into ω_1 and the other into ω_2. Thus, the assertion is trivial if $T_0(M, J^1) \neq T_0(M, J^2)$. Assume that they are identical. By choosing an orthonormal basis for $T_0\mathbf{R}^{2n}$ we may assume that $T_0(M, J^1) = \{x_n = y_n = 0\}$. Since M contains $x_n = y_n = 0$, then M is defined by $ax_n + by_n = 0$ with $a^2 + b^2 = 1$. After a change of orthonormal coordinates, M and $T_0(M, J^1)$ are defined by $y_n = 0$ and $x_n = y_n = 0$, respectively. We still have $\|J^2 - J^1\| < 2$ since orthogonal transformations preserve the operator norm. Now write

$$J^\ell \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} = B_\ell \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix}, \qquad J^\ell \begin{pmatrix} \partial_{x_n} \\ \partial_{y_n} \end{pmatrix} = C_\ell \begin{pmatrix} \partial_{x'} \\ \partial_{y'} \end{pmatrix} + D_\ell \begin{pmatrix} \partial_{x_n} \\ \partial_{y_n} \end{pmatrix},$$

where B_ℓ, C_ℓ, D_ℓ are real matrices, $x' = (x_1, \dots, x_{n-1})$ and $y' = (y_1, \dots, y_{n-1})$. In particular, $D_\ell^2 = -\mathrm{I}$. We want to show that the coefficients of ∂_{y_n} in $J^\ell \partial_{x_n}$ have the same sign. Otherwise, we can write

$$D_1 = \begin{pmatrix} a_1 & b_1 \\ -\frac{1+a_1^2}{b_1} & -a_1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} a_2 & -b_2 \\ \frac{1+a_2^2}{b_2} & -a_2 \end{pmatrix}, \quad b_1 > 0, b_2 > 0.$$

Since $\|D_2 - D_1\| < 2$, we have $b_1 + b_2 < 2$ and $b_1^{-1} + b_2^{-1} < 2$, which is a contradiction. □

It is easy to see that when $n = 1$, the condition that J^1v and J^2v are on the same side of M for some $v \in T_0M$ is equivalent to that J^1, J^2 define the same orientation of $\mathbf{R}^2$. In higher dimensions, whether J^1, J^2 define the same orientation or not is not related to the validity of assertions in Theorem 1.1.

EXAMPLE 3.4. Lemma 3.3 and Theorem 1.1 fail easily for the triplet

$$\{J_{st}, -J_{st}, \{y_1 = 0\}\}.$$

Note that J_{st} and $-J_{st}$ define the same orientation if and only if n is even. On the other hand, Theorem 1.1 is valid for

$$\{J_{st}, J_{st}^2 \times (-J_{st}^{2n-2}), \{y_1 = 0\}\}.$$

Here J_{st}^{2k} denotes the standard complex structure on $\mathbf{R}^{2k}$. A less trivial example is given by the following. Let $0 \le t \le \pi$, and let J^t be defined by

$$X_1^t = (\cos t\,\partial_{x_1} + \sin t\,\partial_{x_2}) + i\partial_{y_1}, \qquad X_2^t = (-\sin t\,\partial_{x_1} + \cos t\,\partial_{x_2}) + i\partial_{y_2}.$$

The conclusions in Lemma 3.3 and Theorem 1.1 fail for $\{J^0, J^\pi, \{y_2 = 0\}\}$ with $\|J^0 - J^\pi\| = 2$. Under new orthonormal coordinates $w_1 = (x_2 + iy_1)/\sqrt{2}$, $w_2 = (-x_1 + iy_2)/\sqrt{2}$, J^t is given by

$$\begin{gathered}(1 + \sin t)\partial_{\overline{w}_1} - \cos t\,\partial_{\overline{w}_2} - (1 - \sin t)\partial_{w_1} - \cos t\,\partial_{w_2}, \\ \cos t\,\partial_{\overline{w}_1} + (1 + \sin t)\partial_{\overline{w}_2} + \cos t\,\partial_{w_1} - (1 - \sin t)\partial_{w_2}.\end{gathered}$$

The above can be put into the form (3.4) with

$$A^t = \begin{pmatrix} 0 & -\frac{\cos t}{1+\sin t} \\ \frac{\cos t}{1+\sin t} & 0 \end{pmatrix}.$$

Note that $\|A^t\| \le 1$. However, the authors do not know if Theorem 1.1 or Lemma 3.3 holds for two structures J^1, J^2 of the form (3.4) with corresponding $\|A^\ell\| < 1$ for $\ell = 1, 2$. Note that such an almost complex structure preserves the standard orientation of $\mathbf{C}^n$.

DEFINITION 3.5. Let J be an almost complex structure defined by vector fields $X_1, \dots, X_n$ of class $\mathcal{C}^{k+\alpha}$ on Ω with $k \ge 0$ and $0 < \alpha < 1$.

(i) A $\mathcal{C}^1$ map $u : \mathbb{D} \to \Omega$ is called a *J-holomorphic curve* if $du(\partial_{\bar\zeta})$ is in the span of $X_1, \dots, X_n$, namely if

$$du(\partial_{\bar\zeta}) = V(\zeta) \cdot X(u(\zeta)) = V_1(\zeta) \cdot X_1(u(\zeta)) + \cdots + V_n(\zeta) \cdot X_n(u(\zeta)).$$

(ii) Let $k \ge 1$. A $\mathcal{C}^1$ map $u : \overline{\mathbb{D}}^+ \to \Omega$ is called an *approximate J-holomorphic curve of order k* attached to the curve $u(x, 0)$ if

$$du(\partial_{\bar\zeta}) = V(\zeta) \cdot X(u(\zeta)) + F(\zeta) \cdot \overline{X(u(\zeta))}, \quad |F(\zeta)| = o(|\operatorname{Im}\zeta|^{k-1}). \tag{3.5}$$

As emphasized earlier, we may assume that J is locally given by vector fields of the form (3.4). Therefore locally, a J-holomorphic curve $u : \mathbb{D} \to \Omega$ satisfies the following equations:

$$\partial_{\bar\zeta} u_\ell = \sum_{1 \le s \le n} a_{\ell s}(u) \overline{\partial_\zeta u_s}, \quad l = 1, \dots, n,$$

which can be written as a row vector

$$\partial_{\bar\zeta} u = \overline{\partial_\zeta u} A(u). \tag{3.6}$$

Note that one cannot prescribe the boundary for a J-holomorphic disc. However, we can prescribe the boundary of the approximate J-holomorphic disc defined above. This fact will be used in the proof of Theorem 1.1. Also, notice that if f is a function on Ω and u is an approximate J-holomorphic curve, the above equation (3.5) implies that

$$\partial_{\bar\zeta}(f(u(\zeta))) = V(\zeta) \cdot (Xf)(u(\zeta)) + F(\zeta) \cdot (\overline{X} f)(u(\zeta)).$$

When u is J-holomorphic, this identity becomes

$$\partial_{\bar\zeta}(f(u(\zeta))) = (Xf)(u(\zeta)) \cdot \overline{\partial_\zeta u}.$$

The next two results deal with the existence of the two types of curves in Definition 3.5.

LEMMA 3.6. *Let $k \ge 0$ be an integer or $k = \infty$, and let $0 \le \alpha < 1$. Let J be an almost complex structure of class $\mathcal{C}^{k+\alpha}(\Omega)$ defined by vector fields*

$$X_\ell = \sum_{1 \le s \le n} b_{\ell s} \partial_{\bar z_s} + \sum_{1 \le s \le n} a_{\ell s} \partial_{z_s}, \quad \ell = 1, \dots, n.$$

Set $A := (a_{\ell s})$ and $B := (b_{\ell s})$. Let $k' \geq 0$ and $0 \leq j < \infty$ be such that $j + k' \leq k$, and let $0 < r < 1$. Let K be a compact subset of Ω and assume that $u_0 : (-1, 1) \times P \to K$ is a map of class $\hat{\mathcal{C}}^{k'+1+\alpha,j}((-1, 1), P)$. Then there exists a map $u : [-r, r] \times [-\delta, \delta] \times P \to \Omega$ of class $\hat{\mathcal{C}}^{k'+1+\alpha,j}([-r, r] \times [-\delta, \delta], P)$ satisfying the following:

(i) *We have $u(x, 0, t) = u_0(x, t)$ and*

$$du(\partial_{\overline{\zeta}}) = V(\zeta, t) \cdot X(u(\zeta, t)) + F(\zeta, t) \cdot \overline{X(u(\zeta, t))}, \tag{3.7}$$

$$|F(\zeta, t)| = o(|y|^{k'}), \quad \alpha = 0; \qquad |F(\zeta, t)| = O(|y|^{k'+\alpha}), \quad 0 < \alpha < 1, \tag{3.8}$$

where $\zeta = x + iy$.

(ii) *Let $e_{j'+1} > 0$. On $[-r, r] \times [-\delta, \delta] \times P$ the norms of u, V, and F satisfy*

$$\begin{aligned} &\|u\|_{j'+1+\alpha,j} + \|(V, F)\|_{j'+\alpha,j} \\ &\quad \leq C^*_{j'+j} \max(\|u_0\|_{j'+1+\alpha,j}, \|u_0\|^{j'+j+2}_{j'+1+\alpha,j}) + e_{j'+1}, \quad j' < k' + 1. \end{aligned}$$

Moreover, $C^*_0(1 + \|u_0\|_{1,0})\delta > 1$ and $C^*_{j'+j}$ depends on K, Ω, $|J|_{j'+j+\alpha}$ and $\inf_\Omega \left| \begin{smallmatrix} B & A \\ \overline{A} & \overline{B} \end{smallmatrix} \right|$.

Proof. We suppress the parameter t in all expressions. We first determine a unique set of coefficients $a_{j'}(x)$ for $1 \leq j' < k' + 2$ such that, as a power series in y, $u(x, y) = u_0(x) + \sum_{j' \geq 1} a_{j'}(x) y^{j'}$ satisfies (3.7)–(3.8). It is convenient to consider u as the real map $(x, y) \mapsto (\operatorname{Re} u, \operatorname{Im} u)$ still denoted by u and to rewrite the equations as

$$du(\partial_y) = J(u)(du(\partial_x)) + F_1(x, y) \cdot \partial_*, \quad F_1(x, y) = o(|y|^{k'}). \tag{3.9}$$

Here $\partial_* = (\partial_{u_1}, \dots, \partial_{u_{2n}})$ is evaluated at $u(\zeta, t)$. In the matrix form, let J be the matrix defined by (3.3). Then we need to solve

$$\partial_y u = \partial_x u J(u) + F_2(x, y), \quad |F_2(x, y)| = o(|y|^{k'}).$$

We solve the equation formally, which determines $a_{j'}(x)$ uniquely for $1 \leq j' < k' + 2$, and then apply Lemma 2.3. This gives a map $u : ([-r, r] \times [-1, 1]) \times P \to \mathbf{R}^n$ of class $\hat{\mathcal{C}}^{k'+1+\alpha,j}$ satisfying the stated norm estimate in (ii). By $|u(x, y) - u(x, 0)| \leq C^*_0(\|u_0\|_{1,0} + e_1)|y|$ and the compactness of K, we find $\delta > 0$ such that u maps $[-r, r] \times [-\delta, \delta] \times P$ into Ω. We have obtained (3.9). Thus,

$$2\, du(\partial_{\overline{\zeta}}) = du(\partial_x) + i\, du(\partial_y) = du(\partial_x) + i J(u)(du(\partial_x)) + i F_1(x, y) \cdot \partial_*.$$

Note that $du(\partial_x) + i J(u)(du(\partial_x)) = V_1(z) \cdot X(u(z))$. Write ∂_* in terms of X_ℓ and $\overline{X}_\ell$ by using the inverse of $\left(\begin{smallmatrix} B & A \\ \overline{A} & \overline{B} \end{smallmatrix} \right)$ where A and B are given by (3.2). We get (3.7) and (3.8). Finally, we can estimate the norms of V and F via V_1, F_1, F_2 and the inverse of $\left(\begin{smallmatrix} B & A \\ \overline{A} & \overline{B} \end{smallmatrix} \right)$. □

In Section 2, we have defined $\hat{\mathcal{C}}^{k+\alpha,j}$ and $\| \cdot \|_{k+\alpha,j}$. Following [17], we define, for $j \leq k < \infty$ and $0 \leq \alpha < 1$,

$$\mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P) := \bigcap_{0 \leq \ell \leq j} \hat{\mathcal{C}}^{k-\ell+\alpha,\ell}(\overline{\Omega}, P), \qquad |u|_{k+\alpha,j} := \max_{0 \leq \ell \leq j} \|u\|_{k-\ell+\alpha,\ell}.$$

We also define

$$\mathcal{C}^{\infty,j}(\overline{\Omega}, P) := \bigcap_{k=1}^{\infty} \mathcal{C}^{k,j}(\overline{\Omega}, P).$$

One can see that $(\mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P), |\cdot|_{k+\alpha,j})$ is a Banach space. By the assumptions on Ω and P we have

$$\mathcal{C}^{k+\alpha,k}(\overline{\Omega}, P) \supset \mathcal{C}^{k+\alpha}(\overline{\Omega} \times P).$$

In particular, if $f \in \mathcal{C}^{k+\alpha,j} \cap \mathcal{C}^1$ and $u \in \mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P)$, then $f \circ u \in \mathcal{C}^{k+\alpha,j}(\overline{\Omega}, P)$ whenever the composition is well defined. In general, for $\varphi(x,t) = (\tilde{\varphi}(x,t), t)$ with $\tilde{\varphi}$ being a map from $\Omega \times P$ into Ω' of class $\mathcal{C}^{k+\alpha,j} \cap \mathcal{C}^1$, we have

$$|v \circ \varphi|_{k+\alpha,j} \le C(1 + |\tilde{\varphi}|_{1,0} + |\tilde{\varphi}|_{k+\alpha,j})^{1+k+j} |v|_{k+\alpha,j}.$$

Let $\mathbb{D}$ be the unit disc in $\mathbf{C}$, $\mathbb{D}_r$ the disc of radius $r > 0$, and $\mathbb{D}_r^+ = \mathbb{D}_r \cap \{\operatorname{Im} z > 0\}$. The following result gives coordinate maps in J-holomorphic curves. To estimate the Cauchy–Green operator T on domains in J-holomorphic curves, we will also need to reparameterize the J-holomorphic half-discs, which are obtained by cutting a J-holomorphic disc by the real hypersurface M. The reparameterization, which is not necessarily J-holomorphic, is given by a mapping $R(\cdot,t)$ that sends $\overline{\mathbb{D}}_r^+$ onto the half-discs. The existence and regularity of J-holomorphic curves for finite smoothness case is proved by Nijenhuis and Woolf [17, pp. 459–460]. They also stated a version of J-holomorphic curves with parameter passing through the same point [17, p. 461]. Since the following precise result is needed for our main results, we prove it in detail. We will also deal with $\mathcal{C}^\infty$ structures.

PROPOSITION 3.7. *Let $k \ge 1$ be an integer or $k = \infty$, and let $0 < \alpha < 1$. Let $j \le k$ be an integer. Let J be an almost complex structure of class $\mathcal{C}^{k+\alpha}$ defined on Ω by vector fields $X_1, \dots, X_n$. Let $M \subset \Omega$ be a $\mathcal{C}^{k+1+\alpha}$ real hypersurface containing the origin 0. Let $e : M \to \mathbf{C}^n$ be a $\mathcal{C}^k$ map such that $e \cdot X = e_1X_1 + \dots + e_nX_n$ is not tangent to M at each point of M. Then there exist two $\mathcal{C}^j$ diffeomorphisms u and R from $\mathbb{D}_r^n$ onto their images in Ω satisfying the following*:

(i) *For each $t \in \mathbb{D}_r^{n-1}$, $u(\cdot,t)$ is J-holomorphic and embeds $\mathbb{D}_r$ onto $D(t)$.*
(ii) *$u(0,t) \in M$, and $D(t)$ intersects M transversally along a curve $\gamma(t)$. Also, $u(0) = 0$ and $du(0,t)(\partial_{\overline{\zeta}}) = (e \cdot X)(u(0,t))$.*
(iii) *$R(\cdot,t)$, with $R(0) = 0$, sends $\mathbb{D}_r^+$, $(-r,r)$, $\mathbb{D}_r$ into $\Omega^+ \cap D(t)$, $M \cap D(t)$, $D(t)$, respectively.*
(iv) *Moreover, u and R are in $\mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_r, \mathbb{D}_r^{n-1})$.*

Proof. Since the result is purely local, we may assume that

$$X_\ell = \partial_{\overline{z}_\ell} + \sum_{1 \le s \le n} a_{\ell s}(z)\partial_{z_s}, \quad \ell = 1, \dots, n,$$

where $A := (a_{\ell s})$ satisfies $A(0) = 0$ and $\|A(z)\| < 1$ on Ω. Applying a unitary change of coordinates, we may assume that $T_0M = \{y_n = 0\}$. By a $\mathcal{C}^{k+1+\alpha}$ change of coordinates which is tangent to the identity, we may assume that M is in $y_n = 0$.

By dilation, we may assume that Ω contains $\mathbb{D}_2^n$ and that on it we have $\|A(z)\| < 1/4$ and $|A|_{j'+1+\alpha} < 1/C_*$. Here C_* will be determined later, and

$$j' := \max\{0, j-1\} < k.$$

Finally, by a dilation in the unit disc $\mathbb{D}$ we achieve $\|e\|_{\mathcal{C}^j(M)} < 1/4$.

Step 1. Existence of a solution u in class $\mathcal{C}^{j'+1+\alpha,j'}$.

At the origin, $X_\ell(0) = \partial_{\overline{z}_\ell}$. Let $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ be the standard basis of $\mathbf{C}^{n-1} \times \{0\}$ and set $P := \overline{\mathbb{D}}^{n-1}_{1/(2n)}$. It follows from the assumption made on $e \cdot X$ that $\tilde{e}_1, \ldots, \tilde{e}_{n-1}, e(t,0)$ are $\mathbf{C}$-linearly independent.

Recall that for $u : \mathbb{D} \times P \to \Omega$, $u(\cdot, t)$ is a J holomorphic disc for all $t \in P$ if and only if u satisfies the equation

$$\partial_{\overline{\zeta}} u = \overline{\partial_\zeta u} A(u). \tag{3.10}$$

Denote by $\mathcal{B}_1$ the closed unit ball of the Banach space $\mathcal{B} := [\mathcal{C}^{j'+1+\alpha,j'}(\overline{\mathbb{D}}, P)]^n$ equipped with the norm $|\cdot|_{j'+1+\alpha,j'}$. By an abuse of notation we will drop the exponent n in the rest of the proof. Consider, for $u \in \mathcal{B}_1$,

$$\Psi(u)(\zeta,t) := \frac{1}{4}(t,0) + \zeta \overline{e(t,0)} + [\Phi(u) - P_1\Phi(u)](\zeta,t), \tag{3.11}$$

where $\Phi(u) := T_{\mathbb{D}}(\overline{\partial_\zeta u} A(u))$ and $P_1\Phi(u)(\zeta,t) := \Phi(u)(0,t) + \zeta\,\partial_\zeta \Phi(u)(0,t)$. One can verify that $\Psi(u)(0,t) = \frac{1}{4}(t,0)$ and $\partial_\zeta \Psi(u)(0,t) = \overline{e(t,0)}$. Therefore, if $u = \Psi(u)$, then $u(\cdot,t)$ is a J-holomorphic disc satisfying $u(0,t) \in M$ and $du(0,t)(\partial_{\overline{\zeta}}) = (e \cdot X)(t,0)$. According to Lemma 2.1, Φ maps $\mathcal{B}_1$ into $\mathcal{B}$. Moreover, $P_1\Phi(u)$ is of class $\mathcal{C}^{j'}$ in t and is a polynomial in ζ. In particular, $P_1\Phi(u)$ and $\Psi(u)$ are in $\mathcal{B}$. Moreover, since $|A|_{j'+1+\alpha} < 1/C_*$ on $\mathbb{D}_2 \times P$, the map Ψ is a contraction from $\mathcal{B}_1$ into itself. Indeed, we have

$$\begin{aligned} A(u_2) - A(u_1) = \int_0^1 &\{(u_2 - u_1) \cdot (\partial_u A)(u_1 + s(u_2 - u_1)) \\ &+ (\overline{u}_2 - \overline{u}_1) \cdot \partial_{\overline{u}} A(u_1 + s(u_2 - u_1))\}\, ds, \end{aligned}$$

$$\begin{aligned} |\Phi(u_2) - \Phi(u_1)|_{j'+1+\alpha,j'} &\le C|\overline{\partial_\zeta u_2} A(u_2) - \overline{\partial_\zeta u_1} A(u_1)|_{j'+\alpha,j'} \\ &\le C'(1 + |(u_1,u_2)|^{j+2}_{j'+1+\alpha,j'}) \\ &\quad \times |A|_{j'+1+\alpha} |u_2 - u_1|_{j'+1+\alpha,j'}. \end{aligned} \tag{3.12}$$

Notice that here we need A to be in $\mathcal{C}^{j'+1+\alpha}$ instead of $\mathcal{C}^{j'+\alpha}$ and we have $j'+1 = k$ if $j' = k-1$. We also have

$$|P_1\Phi(u_2) - P_1\Phi(u_1)|_{j'+1+\alpha,j'} \le C|\Phi(u_2) - \Phi(u_1)|_{j'+1+\alpha,j'}.$$

Recall that after dilation, $|A|_{j'+1+\alpha} < 1/C_*$. It follows that $\Phi(u)$ and $P_1(\Phi(u))$ have small norms on $\mathcal{B}_1$. Thus, Ψ is a contraction map on $\mathcal{B}_1$ into itself, and therefore it has a fixed point $u \in \mathcal{B}_1$.

Step 2. Regularity of u in class $\hat{\mathcal{C}}^{1+\alpha,k}$ for finite k.

This step is the most subtle step of the proof and certainly less classical than the others. We follow the arguments in [17]. This type of arguments has also been used in the Dirichlet and Neumann problems on planar domains with parameter

[4], including the $\mathcal{C}^\infty$ case. Here we need to restrict to spaces of finite derivatives as in [17].

In this step, we assume that k is finite and $j = k = j' + 1$. We have proved that $u \in \mathcal{B}_1 \subset \mathcal{C}^{k+\alpha,k-1}(\overline{\mathbb{D}}, P)$. In order to prove that $u \in \mathcal{C}^{k+1+\alpha,k}$, we first need to show that $\partial_t^k u \in \hat{\mathcal{C}}^{1+\alpha,0}(\overline{\mathbb{D}}, P)$, namely that u is k times differentiable in the t-variable, that $t \mapsto \partial_t^k u$ is a continuous map from P to $\mathcal{C}^1(\overline{\mathbb{D}})$, and that $|\partial_t^k u(\cdot,t)|_{1+\alpha}$ is uniformly bounded. We will then show that $u \in \hat{\mathcal{C}}^{2+\alpha,k-1}$, which will be achieved in the next step by shrinking $\mathbb{D}$.

Recall that u satisfies $\Psi(u) = u$ where Ψ is defined by (3.11). Rewrite $u = \Psi(u)$ as

$$u = h + \Phi(u) - P_1\Phi(u),$$
$$\Phi(u) = T(\overline{\partial_\zeta u}A(u)), \quad P_1\Phi(u)(\zeta,t) = \Phi(u)(0,t) + \zeta\partial_\zeta\Phi(u)(0,t),$$

where $T = T_{\mathbb{D}}$, h is holomorphic in ζ and of class $\mathcal{C}^k$ in t, and hence $h \in \mathcal{C}^{k+1+\alpha,k}$. By differentiating $k-1$ times the previous equation in the t-variable, we obtain

$$v = TA_1(u)v - P_1TA_1(u)v + B_{k-1}(u). \tag{3.13}$$

Here $v := \partial_t^{k-1}u \in \hat{\mathcal{C}}^{1+\alpha,0}(\overline{\mathbb{D}}, P)$ and

$$A_1(u)v := \overline{\partial_\zeta v}A(u) \quad \text{when } k = 1;$$
$$A_1(u)v := \overline{\partial_\zeta v}A(u) + \overline{\partial_\zeta u}(\partial_u A(u)v + \partial_{\overline{u}}A(u)\overline{v}) \quad \text{when } k \geq 2;$$
$$P_1TA_1(u)v := (TA_1(u)v)(0,t) + \zeta\partial_\zeta(TA_1(u)v)(0,\cdot),$$
$$E_{k-1}(u) := \sum_{1\leq\ell\leq k-2}\binom{k-1}{\ell}\overline{\partial_\zeta\partial_t^\ell u}\,\partial_t^{k-1-\ell}(A(u)),$$
$$B_{k-1}(u) := \partial_t^{k-1}h + T(E_{k-1}(u)) - P_1T(E_{k-1}(u))(0,\cdot).$$

Notice that $B_{k-1}(u)$ does not depend on v and $B_{k-1}(u) \in \hat{\mathcal{C}}^{1+\alpha,1}$; $E_{k-1} = 0$ when $k = 1, 2$. Also, $A_1(u)$ is a differential operator whose coefficients are in $\hat{\mathcal{C}}^{\alpha,1}$ when $k > 1$ and in $\hat{\mathcal{C}}^{1+\alpha,0}$ when $k = 1$. Fix t in the interior of P and $V \in \mathbf{R}^{2n-2} \setminus \{0\}$. Consider $t' = t + \lambda V \in P$ where $\lambda \in \mathbf{R}^*$. We first need to prove that $(v(\cdot,t') - v(\cdot,t))/\lambda$ converges as $\lambda \to 0$. We claim that $|(v(\cdot,t') - v(\cdot,t))/\lambda|_{1+\alpha}$ is bounded for λ in a neighborhood of 0. Let

$$I(t',t) := TA_1(u(\cdot,t'))v(\cdot,t') - TA_1(u(\cdot,t))v(\cdot,t).$$

Let us first consider the case $k = 1$. In this case $v = u$. We decompose

$$I(t',t) = T\{(\overline{\partial_\zeta u(\cdot,t')} - \overline{\partial_\zeta u(\cdot,t)})A(u(\cdot,t'))$$
$$+ \overline{\partial_\zeta u(\cdot,t)}(A(u(\cdot,t')) - A(u(\cdot,t)))\}.$$

Using $A(u(\cdot,t')) - A(u(\cdot,t)) = \int_{s=0}^1 \frac{d}{ds}\{A(su(\cdot,t') + (1-s)u(\cdot,t))\}\,ds$, we obtain

$$|A(u(\cdot,t')) - A(u(\cdot,t))|_\alpha \leq C|A|_{1+\alpha}|u(\cdot,t') - u(\cdot,t)|_\alpha.$$

Thus,

$$|\lambda^{-1}I(t',t)|_{1+\alpha} \leq c|A|_{1+\alpha}|\lambda^{-1}(u(\cdot,t') - u(\cdot,t))|_{1+\alpha} \tag{3.14}$$

for some positive constant c. Here, we have used the fact that $|u(\cdot,t)|_{1+\alpha}$ is bounded in t due to $u \in \mathcal{C}^{1+\alpha,0}$. From the definition of $P_1TA_1(u)v$ we obtain from (3.14)

$$|\lambda^{-1}\{P_1TA_1(u)u(\cdot,t') - P_1TA_1(u)u(\cdot,t)\}|_{1+\alpha} \le c'|\lambda^{-1}I(t',t)|_{1+\alpha}.$$

We also have $|\lambda^{-1}\{B_1(\cdot,t') - B_1(\cdot,t)\}|_{1+\alpha} \le C|h|_{2+\alpha}$. Since $|A|_{1+\alpha}$ can be chosen arbitrarily small, it follows that $|(u(\cdot,t')-u(\cdot,t))/\lambda|_{1+\alpha}$ is bounded for λ in a neighborhood of 0. Thus, there exist a sequence $\lambda_\nu \to 0$ and a map $w(\cdot,t)$ of class $\mathcal{C}^{1+\alpha}$ such that $|w(\cdot,t)|_{1+\alpha}$ is uniformly bounded and such that for $t_\nu := t+\lambda_\nu V$, we have $|(u(\cdot,t_\nu)-u(\cdot,t))/\lambda_\nu - w(\cdot,t)|_{L^\infty} \to 0$, $|(\partial_\zeta u(\cdot,t_\nu) - \partial_\zeta u(\cdot,t))/\lambda_\nu - \partial_\zeta w(\cdot,t)|_{L^\infty} \to 0$, and $|(\partial_{\overline{\zeta}} u(\cdot,t_\nu) - \partial_{\overline{\zeta}} u(\cdot,t))/\lambda_\nu - \partial_{\overline{\zeta}} w(\cdot,t)|_{L^\infty} \to 0$ (see 7.1e in [17]). Since $T : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D})$ is bounded, it follows that $w(\cdot,t)$ satisfies the following equation:

$$w = (\mathrm{I}-P_1)T\{\overline{\partial_\zeta w}A(u) + \overline{\partial_\zeta u}(\partial_u A(u)w + \partial_{\overline{u}}A(u)\overline{w})\} + D_V h. \qquad (3.15)$$

Here D_V denotes the derivative in the direction V in the parameter space. Now, if $w'(\cdot,t) \in \mathcal{C}^{1+\alpha}$ is another solution of (3.15), then

$$|w(\cdot,t) - w'(\cdot,t)|_{1+\alpha} \le C|A|_{1+\alpha}(|u|_{1,0}^{\alpha} + |u|_{1+\alpha,0})|w(\cdot,t) - w'(\cdot,t)|_{1+\alpha}.$$

Recall that $|u|_{1+\alpha,1} \le 1$. Thus, $w = w'$. Therefore, the solution w is unique, which proves that $(u(\cdot,t')-u(\cdot,t))/\lambda$ converges to $w(\cdot,t) = D_V u(\cdot,t)$ in $|\cdot|_1$ as $\lambda \to 0$. Next, we want to show that the map $t \mapsto w(\cdot,t)$ is continuous from P into $\mathcal{C}^1$. Let t_ν be a sequence in P converging to t. Since $w(\cdot,t_\nu)$ is a sequence of bounded maps in $\mathcal{C}^{1+\alpha}(\overline{\mathbb{D}})$, it admits a subsequence, still denoted by $w(\cdot,t_\nu)$, that converges in $|\cdot|_{\mathcal{C}^1}$ to a map $\tilde{w} \in \mathcal{C}^{1+\alpha}$. It follows that $\tilde{w}$ satisfies equation (3.15) with w replaced by $\tilde{w}$. By the uniqueness of the solution of (3.15) we obtain $w(\cdot,t) = \tilde{w}$, and thus $|w(\cdot,t_\nu) - w(\cdot,t)|_1 \to 0$ as $|t_\nu - t| \to 0$. This proves that $w = D_V u \in \hat{\mathcal{C}}^{1+\alpha,0}(\overline{\mathbb{D}},P)$.

Consider now the case $k \ge 2$, which produces some extra terms. The above arguments can be repeated easily. Now $(D_V)^{k-1}u$ is different from u. Set $w_1 := \overline{\partial_\zeta u}\partial_u A(u)$, $w_2 := \overline{\partial_\zeta u}\partial_{\overline{u}}A(u)$. We decompose $I(t',t)$ in the following way:

$$\begin{aligned} I(t',t) = T\{&(\overline{\partial_\zeta v(\cdot,t')} - \overline{\partial_\zeta v(\cdot,t)})A(u(\cdot,t')) \\ &+ \overline{\partial_\zeta v(\cdot,t)}(A(u(\cdot,t')) - A(u(\cdot,t)))\} \\ &+ T\{(w_1(\cdot,t') - w_1(\cdot,t))v(\cdot,t') + (v(\cdot,t') - v(\cdot,t))w_1(\cdot,t')\} \\ &+ T\{(w_2(\cdot,t') - w_2(\cdot,t))\overline{v}(\cdot,t') + (\overline{v}(\cdot,t') - \overline{v}(\cdot,t))w_2(\cdot,t')\}. \end{aligned}$$

Since $k \ge 2$, we have $|w_\ell(\cdot,t)|_\alpha \le C|A|_{2+\alpha}|u|_{1+\alpha,0}$. Recall that $|u|_{2+\alpha,1} \le 1$. We can get

$$|w_\ell(\cdot,t') - w_\ell(\cdot,t)|_\alpha \le C|A|_{2+\alpha}|u(\cdot,t') - u(\cdot,t)|_{1+\alpha} \le C|A|_{2+\alpha}|\lambda||u|_{2+\alpha,1}.$$

We obtain

$$|\lambda^{-1}I(t',t)|_{1+\alpha} \le C|A|_{2+\alpha}|\lambda^{-1}(v(\cdot,t') - v(\cdot,t))|_{1+\alpha} + C|u|_{2+\alpha,1}. \qquad (3.16)$$

Again, from the definition of $P_1TA_1(u)v$ we obtain from (3.16)

$$|\lambda^{-1}\{P_1TA_1(u)v(\cdot,t') - P_1TA_1(u)v(\cdot,t)\}|_{1+\alpha} \le c'|\lambda^{-1}I(t',t)|_{1+\alpha}.$$

We also have $|\lambda^{-1}\{B_{k-1}(\cdot,t') - B_{k-1}(\cdot,t)\}|_{1+\alpha} \le C(|h|_{k+1+\alpha,k} + |u|_{k+\alpha,k-1})$. Since $|A|_{2+\alpha}$ can be chosen arbitrarily small, it follows that as before there exist a sequence $\lambda_\nu \to 0$ and a map $w(\cdot,t)$ of class $\mathcal{C}^{1+\alpha}$ such that $|w(\cdot,t)|_{1+\alpha}$ is uniformly bounded and such that for $t_\nu := t + \lambda_\nu V$, the sequence $(v(\cdot,t_\nu) - v(\cdot,t))/\lambda_\nu - w(\cdot,t)$, together its first-order derivatives, converges to zero as $\lambda_\nu \to 0$. Also,

$$w = (\mathrm{I} - P_1)T\{\overline{\partial_\zeta w}A(u) + \overline{\partial_\zeta v}\partial_t A(u) + \partial_t(\overline{\partial_\zeta u \partial_u A(u)})v + \overline{\partial_\zeta u \partial_u A(u)}w\}$$
$$+ (\mathrm{I} - P_1)T\{\partial_t(\overline{\partial_\zeta u}\partial_{\overline{u}} A(u))\overline{v} + \overline{\partial_\zeta u}(\partial_{\overline{u}} A(u))\overline{w}\} + \partial_t B_{k-1}(u).$$

This equation is similar to (3.15) since $|A(u)|_\alpha \le C|A|_\alpha$ and

$$|\overline{\partial_\zeta u \partial_u A(u)}|_\alpha \le C|u|_{1+\alpha,0}|A|_{1+\alpha}|u|_1^\alpha \le C|A|_{1+\alpha|}$$

can be chosen arbitrarily small. By repeating the rest of arguments from the case $k=1$ we can verify that $(D_V)^k u \in \hat{\mathcal{C}}^{1+\alpha,0}(\overline{\mathbb{D}}, P)$.

Step 3. Regularity of u in $\mathcal{C}^{k+1+\alpha,k}$ for finite k or in $\mathcal{C}^{\infty,j}$.

Recall by Step 2 that we need to show that $u \in \hat{\mathcal{C}}^{2+\alpha,k-1}$ when $j = k$ is finite. We have also proved that $u \in \mathcal{C}^{j'+1+\alpha,j'}(\overline{\mathbb{D}}, P) \cap \hat{\mathcal{C}}^{1+\alpha,k}(\overline{\mathbb{D}}, P)$. We will only be able to improve the regularity in interior of $\mathbb{D}$. To achieve Step 3 and to demonstrate the differences between Steps 1 and 2, we will show a stronger result. Assume that the almost complex structure is of class $\mathcal{C}^{k+\alpha}$ with $k > j$ (this includes the case $j = k$ treated in Step 2 for finite k). Assume that for all $\ell \le j$, $\partial_t^\ell u$ is continuous on $\mathbb{D} \times P$ and that distributional derivatives $\partial_\zeta \partial_t^\ell u(\cdot,t)$ have bounded $L^p(\mathbb{D})$ norms on P for $p > 2$. Moreover, suppose that $u(\cdot,t)$ is J-holomorphic on $\mathbb{D}$. Then $u \in \mathcal{C}^{k+1+\beta,j}(\mathbb{D}_r, P)$ for $r < 1$ and $\beta = \min(\alpha, 1-2/p)$. The proof is achieved by induction on the order of the derivative in the ζ-variable.

According to (3.10), the first-order derivatives of $\partial_t^\ell u(\cdot,t)$ have bounded $L^p(\mathbb{D})$ norms on P. By Morrey's inequalities, $u \in \hat{\mathcal{C}}^{\beta,j}(\mathbb{D}_r, P)$ for any $r < 1$ (see Lemma 7.16 and Theorem 7.17 in [9, pp. 162–163]). Fix $\zeta_0 \in \mathbb{D}$ and set $A_0 := A(u(\zeta_0,t))$, $u = \tilde{u} + \overline{\tilde{u}}A_0$, and $u_*(\zeta,t) := \tilde{u}(\zeta_*,t)$ with $\zeta_* := \zeta_0 + \mu\zeta$. Here $0 < \mu < \frac{1}{2}(1 - |\zeta_0|)$ will be determined later. According to (3.10), we get on $\mathbb{D}$

$$\partial_{\overline{\zeta}} u_* = \overline{\partial_\zeta u_*}A_*(\zeta,t), \quad A_*(0,t) = 0, \tag{3.17}$$

$$A_*(\zeta,t) = [A(u(\zeta_0 + \mu\zeta,t)) - A_0][\mathrm{I} - \overline{A}_0 A]^{-1}. \tag{3.18}$$

We emphasize that $A_*(\zeta,t)$ is considered as a matrix function in ζ and t, but not in u_*. Let χ be a smooth real function with compact support in $\mathbb{D}_{1/4}$. Let $v := \chi u_*$. Multiply (3.17) by χ and rewrite it as

$$\partial_{\overline{\zeta}} v - \overline{\partial_\zeta v}A_*(\zeta,t) = \partial_{\overline{\zeta}}\chi u_* - \overline{\partial_\zeta \chi u_*}A_*(\zeta,t). \tag{3.19}$$

Let $\tilde{\chi}$ be a smooth real function with compact support in $\mathbb{D}$ such that $\tilde{\chi} = 1$ on $\mathbb{D}_{1/4}$ and $|\tilde{\chi}|_1 < 5$. Replacing A_* by $\tilde{\chi}A_*$, we may assume that $A_*(\cdot,t)$ has compact support in $\mathbb{D}$. Using (3.18), we get, for $\zeta, \zeta' \in \mathbb{D}$,

$$|A_*(\zeta,t)| \le C|A(u(\cdot,t))|_\beta \mu^\beta,$$
$$|A_*(\zeta',t) - A_*(\zeta,t)| \le C|A(u(\cdot,t))|_\beta \mu^\beta |\zeta' - \zeta|^\beta.$$

Therefore,

$$\|A_*\|_{\beta,0} \leq C|A \circ u|_{\beta,0}\mu^\beta < \varepsilon_\beta.$$

Here ε_β is the constant in Lemma 2.2, and μ is sufficiently small. Apply $T = T_{\mathbb{D}}$ to (3.19). Since v has compact support, we have

$$v - T(\overline{\partial_\zeta v} A_*) = T(\partial_{\overline{\zeta}}\chi u_* - \overline{\partial_\zeta \chi u_*} A_*) =: w. \tag{3.20}$$

The transpose of the solution v is equal to $(\mathrm{I} - TA_*^t\overline{\partial_\zeta})^{-1}w^t$. Since u_* is in $\hat{\mathcal{C}}^{\beta,j}(\overline{\mathbb{D}}, P)$, A_* and w are in $\hat{\mathcal{C}}^{1+\beta,j}(\overline{\mathbb{D}}, P)$. By Lemma 2.2, v is in $\hat{\mathcal{C}}^{1+\beta,j}(\overline{\mathbb{D}}, P)$. Hence, $u \in \hat{\mathcal{C}}^{1+\beta,j}(\overline{\mathbb{D}}_r, P)$ for any $0 < r < 1$. Assume that we have achieved $u \in \hat{\mathcal{C}}^{\ell+\beta,k}(\overline{\mathbb{D}}_r, P)$ for any $0 < r < 1$ and $\ell < k + 1 - j$. Since A_* has compact support in $\mathbb{D}$, $A_* \in \hat{\mathcal{C}}^{\ell+\beta,k}(\overline{\mathbb{D}}, P)$. By $|A_*|_{\alpha,0} < \varepsilon_\alpha$, (3.20), and Lemma 2.2 we get $v \in \hat{\mathcal{C}}^{\ell+1+\beta,j}(\overline{\mathbb{D}}, P)$. This shows that $u \in \hat{\mathcal{C}}^{k+1+\beta,j}(\overline{\mathbb{D}}_r, P)$ for any $r < 1$.

Step 4. Construction of R.

We assume that Ω^+ and M are subsets defined by $y_n > 0$ and $y_n = 0$, respectively. Let $\overline{e(t,0)} = (a, b' + ib'')$. Since $e(t,0) \cdot \partial_{\overline{z}}$ is not tangent to M, $b' + ib'' \neq 0$. Recall that $u = \Psi(u)$. By (3.11), $D(t) \cap M$ is defined by

$$\begin{aligned} &b''\xi + b'\eta = F(\xi,\eta,t), \\ &\quad F(\xi,\eta,t) = \mathrm{Im}\{P_1\Phi(u(\xi+i\eta,t)) - \Phi(u(\xi+i\eta,t))\}. \end{aligned} \tag{3.21}$$

Without loss of generality, we may assume that $b' \geq |b''|$. We already know that $F \in \mathcal{C}^{k+1+\alpha,j}(\mathbb{D}_r, P)$. We may also achieve $|\partial_\eta F| < b'/2$ by assuming $|A|_{j+\alpha} < 1/C_*$. By the implicit function theorem, (3.21) has a solution $\eta = h(\xi,t)$ for $|\xi| < r/c$ and $t \in P$. Now

$$(\partial_\xi h, \partial_t h) = (b' - \partial_\eta F(\xi,\eta,t))^{-1}(\partial_\xi F - b'', \partial_t F)$$

implies that $\partial_t^\ell h \in \mathcal{C}^{k+1+\alpha-\ell}$ for all $\ell \leq j$. On $\mathbb{D}_{r/c} \times P$, define

$$R(\zeta,t) := u(\xi + i(\eta + h(\xi,t)), t).$$

It follows that $R(\cdot,t)$ sends $\mathbb{D}^+_{r/c}$ into $D^+(t)$. Replace $R(\zeta,t)$ by $R(\zeta/c,t)$. The remaining assertions can be verified easily. □

We point out that Proposition 3.7 fails if the almost complex structure is merely Hölderian. Indeed, Ivashkovich, Pinchuk, and Rosay [13] define an almost complex structure of class $\mathcal{C}^{1/2}$ on $\Omega = \mathbb{D}_2 \times \mathbb{D}_{1/10} \subset \mathbf{R}^4$ together with a family of pseudoholomorphic discs $u(\cdot,t) : \mathbb{D} \to \Omega$ of class $\mathcal{C}^{1+1/2}$ such that $u(\zeta,0) = (2\zeta,0)$ and such that for $t \neq 0$, $u(0,t) = (0,t)$ and $|\partial_\zeta u(0,t)| \leq A$ for some $A < 2$. In particular, the map $t \mapsto u(\cdot,t) \in \mathcal{C}^1$ is not continuous at 0, and so $u \notin \mathcal{C}^{1+1/2,0}$.

It is well known that via the Fourier transform, the boundedness of derivatives of a function on all lines parallel to coordinates axes yields some smoothness of the function in all variables (see Rudin [21, p. 203]). To limit the loss of derivatives, we will use the Fourier transform only on curves. This requires us to bound derivatives of a function on a larger family of curves.

Let γ be a curve of class $\mathcal{C}^k$ in $\mathbf{R}^n$, and let f be a function of class $\mathcal{C}^k$ on $\mathbf{R}^n$. We have

$$\begin{aligned}\partial_t^k f(\gamma(t)) &= ((\gamma'(t)\cdot\partial)^k f)(\gamma(t)) \\ &\quad + \sum_{1\le|I|<k} Q_{k,I}(\partial_t^{(k+1-|I|)}\gamma)(\partial^I f)(\gamma(t)),\end{aligned} \tag{3.22}$$

where $Q_{k,I}$ are polynomials, $\partial^{(k)}$ denotes derivatives of order $\le k$, and

$$v\cdot\partial = v_1\partial_{x_1} + \cdots + v_n\partial_{x_n}.$$

For the convenience of the reader, we prove the following elementary result.

LEMMA 3.8. *Let k be a positive integer, and let $\varepsilon > 0$.*

(i) *There exist N vectors $v_j = (1, v_j') \in \mathbf{R}^n$ such that $|v_j'| < \varepsilon$ and*

$$\partial^I = c_{I,1}(v_1\cdot\partial)^k + \cdots + c_{I,N}(v_N\cdot\partial)^k, \quad |I| = k. \tag{3.23}$$

(ii) *If $v_1, \dots, v_N$ satisfy (3.23), there exists $\delta > 0$ such that if $|u - v| < \delta$, then*

$$\partial^I = Q_{I,1}(u)(u_1\cdot\partial)^k + \cdots + Q_{I,N}(u)(u_N\cdot\partial)^k, \quad |I| = k. \tag{3.24}$$

Here $Q_{I,j}$ are rational functions with $Q_{I,j}(v) = c_{I,j}$. Moreover, N depends only on k, n.

Proof. (i) Equivalently, we need to verify (3.23)–(3.24) when ∂ is replaced by $\xi \in \mathbf{R}^n$. It holds for $n = 1$. Assume that it holds when n is replaced by $n-1$. For ξ_n^k, we take distinct nonzero constants $\lambda_1, \dots, \lambda_k$. Then ξ_n^k is in the linear span of ξ_1^k, $(\xi_1 + \lambda_1\xi_n)^k, \dots, (\xi_1 + \lambda_k\xi_n)^k$. Let $\xi_n^j P(\xi_1, \dots, \xi_{n-1})$ be a monomial of degree $k > j$. Then by the induction assumption

$$\xi_n^j P(\xi_1, \dots, \xi_{n-1}) = \xi_n^j [c_1(v_1\cdot\xi)^{k-j} + \cdots + c_\ell(v_\ell\cdot\xi)^{k-j}],$$

where $v_j = (1, v_j'', 0)$ with $|v_j''| < \varepsilon/2$. Then $\xi_n^i(v_\ell\cdot\xi)^{k-i}$ are in the linear span of $(v_\ell\cdot\xi)^k$, $(v_\ell\cdot\xi + \lambda_1\xi_n)^k, \dots, (v_\ell\cdot\xi + \lambda_k\xi_n)^k$. Note that λ_j can be arbitrarily small. Thus, (i) is verified.

(ii) For $|I| = k$, we have the following expansions:

$$\xi^I = \sum_{1\le j\le N} c_{I,j}(v_j\cdot\xi)^k, \qquad \xi^I = \sum_{1\le j\le N} c_{I,j}(u_j\cdot\xi)^k + \sum_{|I'|=k} \widetilde{Q}_{I,I'}(v-u)\xi^I.$$

Clearly, $\widetilde{Q}_{I,I'}(0) = 0$. Moving the last sum to the left-hand side and inverting $\mathrm{I} - \widetilde{Q}_{I,I'}$ yield (3.24). □

It is elementary that the smoothness of a function on all lines does not yield the smoothness of the function. However, if the norms of derivatives on lines are uniformly bounded, we can achieve the smoothness of the function. For the proof of Theorem 1.1, we need to use derivatives of functions on families of curves. This is the content of the following proposition (see [24] for a similar statement). Set $t' = (t_2, \dots, t_n)$ and $t = (t_1, t')$.

PROPOSITION 3.9. *Let k, N be positive integers. For $j = 1, \dots, N$, let R_j be $\mathcal{C}^1$ diffeomorphisms from $\Omega_j \subset \mathbf{R}^n$ onto an open subset $\Omega \subset \mathbf{R}^n$. Assume that $R_j(\cdot, t') \in \mathcal{C}^k$ and that $R_j(0) = 0$. Suppose that, at $0 \in \Omega$,*

$$\partial^I = \sum_{1 \le j \le N} c_{I,j}(\partial_{t_1} R_j(0) \cdot \partial)^{|I|}, \quad 1 \le |I| \le k. \tag{3.25}$$

Let $f \in \mathcal{C}^0(\Omega)$. Then the following hold:

(i) *Let f be of class $\mathcal{C}^k$ near $0 \in \Omega$. Then, for $x = R_j(t^j)$ near 0,*

$$\begin{aligned}\partial^I f(x) = \sum_{1 \le \ell \le m} \sum_{1 \le j \le N} & Q_{I,\ell,j}(\partial_{t_1^1}^{(m-\ell+1)} R_1(t^1), \dots, \partial_{t_1^N}^{(m-\ell+1)} R_N(t^N)) \\ & \times \partial_{t_1^j}^{\ell} f(R_j(t^j)),\end{aligned} \tag{3.26}$$

where $Q_{I,\ell,j}$ are rational functions without pole at $(\partial_{t_1^1}^{(m-\ell+1)} R_1(0), \dots, \partial_{t_1^N}^{(m-\ell+1)} R_N(0))$, and $1 \le m = |I| \le k$. Moreover, (3.26) holds on a domain ω if $f \in \mathcal{C}^k(\omega)$ and $0 \in \partial\omega \subset B_\varepsilon^n$ where ε is sufficiently small.

(ii) *Suppose that R_j are affine, that is, $R_j(t) - R_j(y) = R_j(t - y)$ wherever they are defined. Suppose that the $L_{t_1}^\infty$ norms of one-dimensional distributions $\partial_{t_1}^m (f \circ R_j)(\cdot, t')$ are bounded in t' for all $m \le k$. Then, near 0, $\partial^I f$ are Lipschitz functions for all $|I| < k$.*

(iii) *Let R_j be of class $\mathcal{C}^{k+1}$ near $0 \in \mathbf{R}^n$, and let $n < p < \infty$. Suppose that the $L_{t_1}^p$ norms of one-dimensional distributions $\partial_{t_1}^m (f \circ R_j)(\cdot, t')$ are bounded in t' for all $m \le k$. Then, near 0, f is of class $\mathcal{C}^{k-n/p}$.*

Proof. (i) follows from (3.22) and (3.24), by hypothesis (3.25).

(ii) Applying dilation and replacing f by χf, we may assume that f has compact support in Δ^n. Set $\chi_\varepsilon(x) := \varepsilon^{-n} \chi(\varepsilon^{-1} x)$ for a smooth function χ with support in Δ^n and $\int \chi \, dx = 1$. Set $f_\varepsilon(x) := \int f(y) \chi_\varepsilon(x - y) \, dy$ and $f_{\varepsilon,j} := f_\varepsilon \circ R_j$. Changing variables via R_j, we get

$$f_{\varepsilon,j}(t) = \int f(R_j(t) - R_j(y)) \chi_\varepsilon(R_j(y)) \det R_j'(y) \, dy.$$

Using $R_j(t) - R_j(y) = R_j(t - y)$, we get $|f_{\varepsilon,j}(\cdot, t')|_k < C$ for C independent of ε and t'. In (3.26), we substitute f_ε for f. Therefore, $\partial^I f_\varepsilon$ are bounded near 0. We can find a sequence f_{ε_j} such that as ε_j tends to 0, $\partial^I f_{\varepsilon_j}$ converges uniformly for $|I| < k$, and the Lipschitz norms of $\partial^I f_{\varepsilon_j}$ are bounded. Since f_ε converges uniformly to f as $\varepsilon \to 0^+$, it follows that $\partial^{k-1} f \in Lip_{\mathrm{loc}}$.

(iii) For such a function f, we define a distribution $T_j f$ by

$$T_j f(\phi) := (-1)^k \int_{\mathbf{R}^n} f \circ R_j(t) \partial_{t_1}^k \phi(R_j(t)) \, dt,$$

where ϕ is a test function supported in Δ_ε^n with ε small. It is clear that defined near 0, $T_j f$ is a distribution of order $(\le)$ k. Integrating by parts in the t_1-variable yields

$$|T_j f(\phi)| \le C \int_{\mathbf{R}^{n-1}} \|\partial_{t_1}^k [f \circ R_j](\cdot, t')\|_{L^p_{t_1}} \|\phi \circ R_j(\cdot, t')\|_{L^q_{t_1}} \, dt'$$
$$\le C_1 \int_{\mathbf{R}^{n-1}} \|\phi \circ R_j(\cdot, t')\|_{L^q_{t_1}} \, dt' \le C_2 \|\phi \circ R_j\|_{L^q} \le C_3 \|\phi\|_{L^q}.$$

Here the first and the second last inequalities are obtained from the Hölder inequality and from $\operatorname{supp}\phi \subset \Delta^n_\varepsilon$. Hence, near 0, $T_j f \in L^p$ for $p > 1$. Next, we find a differential operator $P_{j,k}(\partial)$ of order k such that $P_{j,k}(\partial) f = T_j f$. To this end, we use a smooth function g to obtain

$$T_j g(\phi) = \int \partial_{t_1}^k [g \circ R_j(t)] \phi(R_j(t)) \, dt = \int (\phi \tilde{P}_{j,k}(\partial) g) \circ R_j(t) \, dt$$
$$= \int [\det((R_j^{-1})') \tilde{P}_{j,k}(\partial) g] \phi \, dx =: (P_{j,k}(\partial) g)(\phi).$$

Since $R_j \in \mathcal{C}^{k+1}$, it is easy to see that

$$P_{j,k}(\partial) = \det((R_j^{-1})') \tilde{P}_{j,k}(\partial) = \sum_{|I| \le k} a_{j,k,I} \partial^I, \quad a_{j,k,I} \in \mathcal{C}^{|I|}.$$

The last assertion implies that $P_{j,k}(\partial)$ has order k. The definition of $T_j f$ and the identity $P_{j,k}(\partial) g = T_j g$ imply that as distributions defined near 0, we have $P_{j,k}(\partial) f = T_j f$.

Note that

$$\sum_{|I|=k} a_{j,k,I}(x) \partial_x^I = \sum_{|I|=k} C_I \det((R_j^{-1})'(x)) (\partial_{t_1^j} R_j(t^j))^I \partial_x^I, \quad C_I \ne 0.$$

Here $t^j = R_j^{-1}(x)$. Combining with (3.25), we get, for $g \in \mathcal{C}^k$ and $1 \le |I| \le k$,

$$\partial^I g = \sum_{1 \le \ell \le m} \sum_{1 \le j \le N} b_{I,j,\ell} P_{j,\ell}(\partial) g, \quad b_{I,j,\ell} \in \mathcal{C}^\ell.$$

The last assertion, combined with $\operatorname{ord} P_{j,\ell}(\partial) \le \ell$, $a_{j,k,I} \in \mathcal{C}^{|I|}$, and $P_{j,\ell}(\partial) f \in L^p$, implies that near 0, $\partial^I f$ are in L^p for $1 \le |I| \le k$. Therefore, $f \in W^{k,p}_{\text{loc}}$, and $f \in \mathcal{C}^{k-n/p}$ by the Sobolev embedding theorem (see [12, p. 123]). □

4. Cauchy–Green Operator on Domains with Parameter

The following result is certainly classical; see [25, Section 8.1, pp. 56–61]. For the convenience of the reader, we present details for a parameter version. Recall that P is the closure of a bounded open set in a Euclidean space and that two points a, b in the interior of P can be connected by a smooth curve in the interior of length at most $C|b-a|$.

LEMMA 4.1. *Let τ be a complex-valued function on $\overline{\mathbb{D}}^+ \times P$ of class $\mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}^+, P)$. Suppose that for $z, z' \in \mathbb{D}^+$ and $t \in P$,*

$$|\tau(z', t) - \tau(z, t)| \ge |z' - z|/C. \tag{4.1}$$

(i) *Let f be a continuous function on $[-1,1]\times P$. For $z\in\mathbb{D}^+$, define*

$$C_0f(z,t):=\frac{1}{2\pi i}\int_{-1}^{1}\frac{f(s,t)}{\tau(s,t)-\tau(z,t)}\,ds.$$

Then $|\partial_z^k C_0 f(z,t)|\leq C_k|f|_{L^\infty}/|\operatorname{Im} z|^{k+1}$, where C_k depends only on $|\tau|_{k,0}$.

(ii) *Let f be a function of class $\mathcal{C}^{k+\alpha,0}([-1,1],P)$. Then C_0f extends continuously to $(\mathbb{D}^+\cup(-1,1))\times P$. Moreover, $C_0f\in\mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}_r^+,P)$ for any $r<1$ and satisfies $|C_0f|_{k+\alpha,0}\leq C|f|_{k+\alpha,0}$.*

(iii) *Let f be a function of class $\mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}^+,P)$. For $z\in\mathbb{D}^+$, define*

$$S_0f(z,t):=-\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{\zeta\in\mathbb{D}^+:\ |\tau(\zeta,t)-\tau(z,t)|>\varepsilon\}}\frac{f(\zeta,t)}{(\tau(\zeta,t)-\tau(z,t))^2}\,d\xi\,d\eta,$$

$$T_0f(z,t):=-\frac{1}{\pi}\int_{\mathbb{D}^+}\frac{f(\zeta,t)}{\tau(\zeta,t)-\tau(z,t)}\,d\xi\,d\eta.$$

Then $S_0f\in\mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}_r^+,P)$ and $T_0f\in\mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}_r^+,P)$ for any $r<1$. Moreover, $|S_0f|_{k+\alpha,0}+|T_0f|_{k+1+\alpha,0}\leq C|f|_{k+\alpha,0}$.

Proof. (i) Note that (4.1) implies that $|\tau(z,t)-\tau(s,t)|\geq \operatorname{Im} z/C$ for $-1\leq s\leq 1$ and $z\in\mathbb{D}^+$. The proof is straightforward by taking the derivatives in $z,\overline{z}$ directly onto the kernel.

(ii) Let $z=x+iy$. Let χ be a smooth function with compact support in $(-1,1)$. Replacing $f(x,t)$ with $\chi(x)f(x,t)/\partial_x\tau(x,t)$, it suffices to get the norm estimate on $\mathbb{D}_r^+\times P$ for

$$\begin{aligned}C_0f(z,t)&=\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f(\zeta,t)}{\tau(\zeta,t)-\tau(z,t)}\,d\tau(\zeta,t)\\&=\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f(\zeta,t)-f(x,t)}{\tau(\zeta,t)-\tau(z,t)}\,d\tau(\zeta,t)+\varepsilon f(x,t).\end{aligned}\tag{4.2}$$

Here the differentiation and integration are in ζ, and $\varepsilon=1$ if $\tau(\cdot,t)$ preserves the orientation of $\mathbb{D}^+$; otherwise, $\varepsilon=-1$. From (4.1) and $\tau\in\mathcal{C}^{1,0}(\mathbb{D}^+,P)$ we know that ε is independent of t. Let C_1f denote the second integral in (4.2). Let ∂^j denote the jth derivative in x, y. In what follows, the norms $|\cdot|_{j+\alpha,0}$ for f, τ are on $\mathbb{D}^+$, and norms $|\cdot|_{j+\alpha,0}$ for C_0f are on $\mathbb{D}_r^+$ with $r<1$. These norms will be denoted by the same notation $|\cdot|_{j+\alpha}$. Since t is fixed, we suppress it in all expressions. All constants are independent of t.

That C_0f extends continuously to $\overline{\mathbb{D}}^+\times P$ follows from the continuity of f and

$$\left|\frac{f(s)-f(x)}{\tau(s)-\tau(z)}\right|\leq C|f|_\alpha|x-s|^{\alpha-1}.$$

Take (4.2) as the definition of C_0. Differentiating it gives

$$\partial C_0f(z)=\frac{\partial\tau(z)}{2\pi i}\int_{\partial\mathbb{D}^+}\frac{f(\zeta)-f(x)}{(\tau(\zeta)-\tau(z))^2}\,d\tau(\zeta).\tag{4.3}$$

Using $|\tau(s)-\tau(z)| \ge (|s-x|+|y|)/C$, we get

$$|\partial C_0 f(z)| \le |\tau|_1 \int_{\partial\mathbb{D}^+} \frac{C|f|_{\alpha,0}|s-x|^\alpha}{|y|^2+|s-x|^2}\, ds \le C'_\alpha |f|_\alpha |y|^{\alpha-1}.$$

By a Hardy–Littlewood-type lemma we obtain $|C_0 f|_{\alpha,0} \le C|f|_{\alpha,0}$. For higher derivatives of $C_0 f$, we differentiate (4.2) in the z-variable and transport derivatives to f via integration by parts. We get, for $|I| = k$,

$$\partial^I C_0 f(z) = \sum_{|K|=|I|} \partial^{K_1}\tau(z)\cdots\partial^{K_\ell}\tau(z) \int_{\partial\mathbb{D}^+} \frac{f_{IK}(\zeta)}{\tau(\zeta)-\tau(z)}\, d\tau(\zeta). \tag{4.4}$$

Here $f_{IK}(s)$ are polynomials in $(\partial_s\tau(s))^{-1}$, $\partial_s^\ell f(s)$, $\partial_s^{\ell+1}\tau(s)$ with $\ell \le k$. As before, we have the continuity of

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \frac{f_{IK}(\zeta)}{\tau(\zeta)-\tau(z)}\, d\tau(\zeta) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}^+} \frac{f_{IK}(\zeta)-f_{IK}(x)}{\tau(\zeta)-\tau(z)}\, d\tau(\zeta) + \varepsilon f_{IK}(x).$$

By differentiating the integral in (4.4) one more time, we get a formula analogous to (4.3). As in the case $k=0$, we can verify that the $\mathcal{C}^\alpha$ norms of $\partial^k C_0 f(\cdot,t)$ on $\overline{\mathbb{D}}_r^+$ are bounded.

(iii) We first show that $S_0 f \in \mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}^+, P)$. Let $-2i\, d\xi \wedge d\eta = A(\zeta,t)\, d\tau(\zeta,t) \wedge d\overline{\tau(\zeta,t)}$. Let χ be a smooth function with compact support in $\mathbb{D}_{r'}^+ \cup (-r', r')$, where $0 < r < r' < 1$. By replacing $f(\zeta,t)$ by $\chi(\zeta)f(\zeta,t)A(\zeta,t)$ we may reduce to the case where $f(\cdot,t)$ is supported in $\overline{\mathbb{D}}_{r'}^+$ with $r < r' < 1$. We may also replace the domain of integration by a smooth domain D with $\mathbb{D}_r^+ \subset \overline{D} \subset \overline{\mathbb{D}}_{r'}^+$. Again, we suppress the parameter t in all expressions and write

$$\begin{aligned} S_0 f(z) = {} & \frac{1}{2\pi i}\int_D \frac{(f(\zeta)-f(z))\, d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{(\tau(\zeta)-\tau(z))^2} \\ & - \frac{f(z)}{2\pi i}\int_{\partial D} \frac{d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z)}. \end{aligned} \tag{4.5}$$

On ∂D, write $d\overline{\tau(\zeta)} = a_0(\zeta)\, d\tau(\zeta)$. By (ii) we know that the last integral in (4.5) is in $\mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}_r^+, P)$. Denote the first integral in (4.5) by $\tilde g(z)/(2\pi i)$. That $\tilde g$ extends continuously follows from the continuity of f and $|f(\zeta)-f(z)|/|\tau(\zeta)-\tau(z)|^2 \le C|\zeta - z|^{\alpha-2}$. Write

$$\begin{aligned} \tilde g(z_2) - \tilde g(z_1) = {} & \int_D \frac{(f(z_1)-f(z_2))\, d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{(\tau(\zeta)-\tau(z_2))(\tau(\zeta)-\tau(z_1))} \\ & + \int_D \frac{(f(\zeta)-f(z_2))(\tau(z_2)-\tau(z_1))}{(\tau(\zeta)-\tau(z_2))^2(\tau(\zeta)-\tau(z_1))}\, d\tau(\zeta)\wedge d\overline{\tau(\zeta)} \\ & + \int_D \frac{(f(\zeta)-f(z_1))(\tau(z_2)-\tau(z_1))}{(\tau(\zeta)-\tau(z_1))^2(\tau(\zeta)-\tau(z_2))}\, d\tau(\zeta)\wedge d\overline{\tau(\zeta)}. \end{aligned}$$

The last two integrals can be estimated by a standard argument for Hölder estimates, bounded in absolute value by $C_\alpha \|f\|_{\alpha,0}|z_2 - z_1|^\alpha$. The first integral can be rewritten as the product of $f(z_1)-f(z_2)$ and $\mathcal{I}$, where

$$\begin{aligned}\mathcal{I} &:= \frac{1}{\tau(z_2)-\tau(z_1)}\int_D\left\{\frac{d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z_2)}-\frac{d\tau(\zeta)\wedge d\overline{\tau(\zeta)}}{\tau(\zeta)-\tau(z_1)}\right\}\\ &=2\pi i\varepsilon\frac{\overline{\tau(z_2)-\tau(z_1)}}{\tau(z_1)-\tau(z_2)}+\frac{1}{\tau(z_1)-\tau(z_2)}\int_{\partial D}\left\{\frac{\overline{\tau(\zeta)}\,d\tau(\zeta)}{\tau(\zeta)-\tau(z_2)}-\frac{\overline{\tau(\zeta)}\,d\tau(\zeta)}{\tau(\zeta)-\tau(z_1)}\right\}.\end{aligned}$$

A derivative of $\int_{\partial D}\overline{\tau(\zeta)}\,d\tau(\zeta)/(\tau(\zeta)-\tau(z))$ is $\partial_z\tau(z)\int_{\partial D}d\overline{\tau(\zeta)}/(\tau(\zeta)-\tau(z))$, which is bounded by (ii). By the mean-value theorem, the last term in $\mathcal{I}$ is bounded. This shows that $S_0f\in\mathcal{C}^{\alpha,0}(\overline{\mathbb{D}}_r^+,P)$.

For higher-order derivatives, we transport derivatives to f. Define $f_*(\tau(z)):=f(z)$ and $\omega(t):=\tau(\cdot,t)(D)$. Let $C_*:=C_{\partial\omega(t)}$, $T_*:=T_{\omega(t)}$, and $S_*:=S_{\omega(t)}$. Rewrite (4.5) as $g_*(\tau)=\varepsilon S_*f_*$. By integrating by parts we obtain

$$g_*(\tau)=\frac{1}{2\pi i}\int_{\omega(t)}\frac{\partial_\sigma f_*(\sigma)}{\sigma-\tau}\,d\sigma\wedge d\overline{\sigma}-\frac{1}{2\pi i}\int_{\partial\omega(t)}\frac{f_*(\sigma)}{\sigma-\tau}\,d\overline{\sigma}.$$

On $\partial\omega(t)$, we write $d\overline{\tau}=a(\tau,t)\,d\tau$ with $a\in\mathcal{C}^{k+\alpha,0}(\partial D,P)$. Taking derivatives, we get

$$\partial_{\overline{\tau}}S_*f_*=\partial_\tau f_*,\qquad \partial_\tau S_*f_*=S_*\partial_\tau f_*-\partial_\tau C_*af_*.$$

Using the last formula k times, we get

$$(\partial_\tau)^kS_*f_*=S_*\partial_\tau^kf_*-\sum_{0\le j<k}\partial_\tau^{k-j}C_*a\partial_\tau^jf_*.$$

We return to the z-variable. Let $\tilde a(z,t):=a(\tau(z,t),t)$. Let ∂_z^K be a derivative in $z,\overline{z}$ of order $|K|=k$. Let $\partial^{(j)}$ denote derivatives of orders $\le j$. Then

$$\begin{aligned}\partial_z^KS_0f(z)&=p^1(\partial_z^{(k)}\tau)\cdot(\partial_\tau^{(k)}S_*f_*,\partial_\tau^{(k)}f_*)\circ\tau\\ &=\sum_{0\le j\le k}p_j^2(\partial_z^{(k)}\tau)\cdot(S_*\partial_\tau^{(k)}f_*,\partial_\tau^{(k-j)}C_*(a\partial_\tau^{(j)}f_*),\partial_\tau^{(k)}f_*)\circ\tau\\ &=\sum_{0\le j\le k}q_{j,k}^1\cdot(S_0q_k^2\partial_z^{(k)}f,\partial_\tau^{(k-j)}C_0(\tilde aq_j^3\partial_\tau^{(j)}f),\partial_\tau^{(k)}f).\end{aligned}\tag{4.6}$$

Here the integral operator S_0 is over the domain D, and C_0 is over ∂D. Furthermore, p_j^i are vectors of polynomials, and $q_{j,k}^1$, q_k^2, q_j^3 are matrices of polynomials in $(\det\tau')^{-1}$ and $\partial_z^{(j)}\tau$. It follows from the assertion for $k=0$ and (ii) that $S_0f\in\mathcal{C}^{k+\alpha,0}(\overline{\mathbb{D}}^+,P)$.

Note that $\partial_\tau T_*=S_*$ and $\partial_{\overline{\tau}}T_*=\mathrm{I}$. Thus, it follows from (ii), the product rule, and the chain rule as used in (4.6) that $T_0f\in\mathcal{C}^{k+1+\alpha,0}(\overline{\mathbb{D}}_r^+,P)$. □

5. Proof of the Main Theorem

Let $\Delta:=[-1,1]$ and $\Delta_r:=[-r,r]$. Let Δ_r^n, Δ_r^{2n-1}, and Δ_r^{2n} be the corresponding cubes in the x-subspace, hyperplane $y_n=0$, and $\mathbf{R}^{2n}$, respectively.

In this section, we state and prove a more precise version of Theorem 1.1.

THEOREM 5.1. *Let $k \geq 3$ be an integer. Let Ω_1, Ω_2, M, α be as in Theorem 1.1 with $M \in \mathcal{C}^{k+1+\alpha}$. For $\ell = 1, 2$, let J^ℓ be an almost complex structure of class $\mathcal{C}^{k+\alpha}(\Omega_\ell \cup M)$ on $\Omega_\ell \cup M$. Suppose that at each point $p \in M$, there is a tangent vector $v_p \in T_pM$ such that $J^1_p v_p$, $J^2_p v_p$ are in the same connected component of $T_p\mathbf{R}^{2n} \setminus T_pM$. Let f be a continuous function on $\Omega_1 \cup M \cup \Omega_2$ such that $(\partial_{x_j} + iJ^\ell\partial_{x_j})f$ and $(\partial_{y_j} + iJ^\ell\partial_{y_j})f$, defined on Ω_ℓ, extend to functions in $\mathcal{C}^k(\Omega_\ell \cup M)$ for $\ell = 1, 2$ and $1 \leq j \leq n$. Then $f \in \mathcal{C}^{k-2}(\Omega_1 \cup M) \cap \mathcal{C}^{k-2+\beta}(\Omega_1)$ for any $\beta < 1$.*

Notice that no integrability condition is assumed. A byproduct of our proof is an interior regularity of f with $f \in \mathcal{C}^\beta(\Omega_1)$ for any $\beta < 1$ when $k = 2$; of course, the assumptions on f, J^2, M, and Ω_2 are not needed in order to obtain the regularity of f on Ω_1. The result might not be sharp. Indeed, when the structures are integrable and $k \geq 1$, the Newlander–Nirenberg theorem [26] yields $f \in \mathcal{C}^{k+\beta}(\Omega_1)$ for any $\beta < 1$.

Proof of Theorem 5.1. We first describe the main ingredients of the proof.

Step 1. Let $\Omega^+ = \Omega_1$ and $\Omega^- = \Omega_2$. We first assume the interior regularity that f is $\mathcal{C}^1$ on $\Omega^+ \cup \Omega^-$. We will show that the Fourier transforms of f on lines L in M decay in the ξ-variable. To use the differential equations for f, lines L need to be transversal to the complex tangent vectors of M of both structures. Two almost complex structures yield decay of the Fourier transform at opposite rays. This is the only place we need both structures. According to Proposition 3.9, this gives the smoothness of f on M.

Step 2. In order to obtain the smoothness of f on each side of M (up to the boundary) via the one-sided almost complex structure, we attach a family of pseudoholomorphic discs to M by using Proposition 3.7. Such a disc will have regularity as good as the structure provides. This is achieved by extending the structure to a neighborhood of M. Using Lemma 4.1, we prove that the regularity of f on M yields uniform bounds of pointwise derivatives of f along the discs up to their boundaries in M.

Step 3. After obtaining the smoothness of f on families of discs in $\Omega^+ \cup M$, we obtain the interior regularity of f, including the $\mathcal{C}^1$ regularity, by Proposition 3.9. Using Proposition 3.9 again, we conclude the smoothness of f on $\Omega^+ \cup M$.

We now carry out details. We need a preparation for Step 1.

Step 0. Match approximate J-holomorphic half-discs in M.

We fix a finite integer $k \geq 3$. We may assume that $M = \Delta^{2n-1} \times 0$, $\Omega^+ = \mathbb{D}^n \cap \{y_n > 0\}$, and $\Omega^- = \mathbb{D}^n \cap \{y_n < 0\}$. By assumptions, there is a vector $v_0 \in T_0M$ such that the vectors $J^1_0 v_0$, $J^2_0 v_0$ are transversal to T_0M and are in Ω^+. Thus, the line segments $tJ^1_0 v_0$, $tJ^2_0 v_0$ $(0 < t \leq 1)$ are transversal to M and are contained in Ω^+, by shrinking v_0 if necessary. Here we have identified $\mathbf{R}^{2n}$ with $T_p\mathbf{R}^{2n}$ by sending v to the tangent vector of $p + tv$; consequently, J^ℓ_p acts on $\mathbf{R}^{2n}$ linearly.

Let $\varepsilon > 0$ be sufficiently small such that if $p \in M$ and $v \in T_pM$ satisfy $|p| < \varepsilon$ and $|v - v_0| < \varepsilon$ then $J^1_p v$ and $J^2_p v$ are still in the same component of $T_p\mathbf{R}^{2n} \setminus T_pM$. By transversality, $p + tJ^1_p v$ and $p + tJ^2_p v$ are in Ω^+ for $0 < t \leq 1$. Define the line segment

$$L = L(v, p) := \{p + tv\colon\ -2 < t < 2\} \subset M.$$

Let $e_1, \dots, e_{2n-1}$ be the standard basis of $\mathbf{R}^{2n-1}$. We find an affine coordinate map ϕ on $\mathbf{R}^{2n}$ such that $\phi(p) = 0$, $\phi(p+v) = e_n$, and $\phi(p + v_j) = e_j$. We may also assume that the sup norms of the derivatives of ϕ and ϕ^{-1} have an upper bound independent of p and v. In what follows, all constants are independent of p and v. Proposition 3.9 (ii) will be used for this family of ϕ (with $p = 0$) depending on the parameter v with v to be chosen for various cases.

We want to apply Lemma 3.6 to $L(v, p)$. Here v and p are considered as parameters, and we suppress them in all expressions. For the above $L(p, v)$, we attach an approximate J-holomorphic curve u^1 of class $\mathcal{C}^{k+1+\alpha}$ such that

$$\begin{aligned} du^1(\partial_{\overline{z}}) &= V^1(z) \cdot X^1(u^1(z)) + F^1(z) \cdot \overline{X^1(u^1(z))}, \\ |F^1(z)| &\leq C|y|^{k+\alpha}, \quad (x, y) \in Q := (-1, 1) \times (0, \varepsilon). \end{aligned} \tag{5.1}$$

We have an analogous u^2 and $\Omega^- \cup M$. We also require

$$u^1(x, 0) = p + xv = u^2(x, 0) \quad \text{on } [-1, 1].$$

We know that $u^1(x, 0)$ is contained in $M \subset \overline{\Omega}^+ \cap \overline{\Omega}^-$ for $|x| < 1$. When $p = 0$ and $v = v_0$, we have $du^1(0)(\partial_x) = v_0$. Moreover, $du^1(0)(\partial_y) = J_0^1\, du^1(0)(\partial_x) = J_0^1 v_0$ is contained in Ω^+, $-J_0^2 v_0$ is contained in Ω^-, and both are transversal to M. Thus,

$$u^1(x, y) \in \Omega^+, \quad (x, y) \in Q; \qquad u^2(x, y) \in \Omega^-, \quad (x, y) \in -Q. \tag{5.2}$$

The above holds for $v = v_0$ and $p = 0$. Since the derivatives of u are continuous in p and v, the above holds for $|p| < \varepsilon$ and $|v - v_0| < \varepsilon$. And for a constant $C > 1$ independent of p and v, we have

$$\operatorname{dist}(u^\ell(x, y), M) \geq |y|/C, \quad (x, y) \in (-1)^{\ell-1} Q. \tag{5.3}$$

Step 1. Uniform bound of Fourier transform of f on transversal lines L in M. In Steps 1 and 2 we will assume that f is $\mathcal{C}^1$ on $\Omega^+ \cup \Omega^-$. We will verify this interior regularity in the final step (Step 4). Of course, the final step does not rely on the assumption that $f \in \mathcal{C}^1$ as we will emphasize later.

Fix k. Recall from Step 0 that M is contained in $\mathbf{R}^{2n-1}$. Let v_0, ε be as in Step 0. By Lemma 3.8 there exist N vectors v_j in $\mathbf{R}^{2n-1}$ such that

$$(\partial_x, \partial_{y'})^I = \sum_{1 \leq j \leq N} c_{I,j} (v_j \cdot (\partial_x, \partial_{y'}))^{|I|}, \quad 1 \leq |I| \leq k. \tag{5.4}$$

Here $|v_j - v_0| < \varepsilon$. Recall that the line segment L is $\{p + tv_j\colon\ -1 \leq t \leq 1\}$ with $p \in M$ such that $|p| < \varepsilon$. Fix such a segment L and denote its tangent vector v_j by v.

Note that when ε is sufficiently small, L has length $> |v_0|/2$. Let χ_0 be a cutoff function on M with compact support in $\Delta^{2n}_{|v_0|/(4n)} \cap M$. Then $\chi_0|_L$ has compact support. We will show that the Fourier transform of $\chi_0 f$ on L satisfies

$$(1 + |\xi|)^{k-1+\alpha-\beta} |\widehat{\chi_0 f|_L}(\xi)| < C_\beta \tag{5.5}$$

for any $\beta > 0$, where C_β will be independent of $p, v_1, \dots, v_N$. We will verify (5.5) for $\xi = -|\xi|v$, using $X_j^1 f = g_j^1$ on Ω^+ with $g_j^1 \in \mathcal{C}^k(\Omega^+ \cup M)$. For $\xi = |\xi|v$, we use $X_j^2 f = g_j^2$ on Ω^- with $g_j^2 \in \mathcal{C}^k(\Omega^- \cup M)$.

We now use approximate J-holomorphic curves u^1, u^2 defined in Step 0. We drop the superscript in u^1, g_j^1, a_{jk}^1, etc. Applying Lemma 2.3, we extend $\chi_0 \circ u(x,0)$ to $\chi \in \mathcal{C}^\infty(Q)$ that has compact support in each $(-1,1)\times\{y\}$. Moreover, $|\partial_{\overline{z}}\chi(x,y)| \le C|y|^{k+\alpha}$. For brevity, denote $f\circ u$ and $g_j \circ u$ by f and g_j. Combining with (5.1), we get on Q

$$\begin{aligned} \partial_y f(x,y) &= i\partial_x f(x,y) - 2iV(x,y)\cdot g(u(x,y)) - 2iF(x,y)\cdot \overline{X(u)}f, \\ \partial_y \chi(x,y) &= i\partial_x \chi(x,y) + E(x,y). \end{aligned} \tag{5.6}$$

Moreover, $(|E|+|F|)(x,y) \le C|y|^{k+\alpha}$, and V, E, F are in $\mathcal{C}^{k+\alpha}(\overline{Q})$. Also, g is in $\mathcal{C}^k(\overline{Q})$.

In what follows, as required by (5.5), the constants do not depend on L, p, v_j. By (5.2), $u(x,y)$ is in Ω^+ for $|x|<1$, $0<y<\varepsilon$. Define, for $y \ge 0$,

$$\lambda(\xi,y) := \int_{\mathbf{R}} (\chi f)(x,y) e^{-i(x+iy)\xi}\,dx.$$

Notice that

$$\widehat{\chi f|_L}(\xi) \equiv \lambda(\xi,0) = \lambda(\xi,\eta) - \int_0^\eta \partial_y \lambda(\xi,y)\,dy.$$

By (5.6) we obtain

$$\begin{aligned} \partial_y\lambda(\xi,y) = &\int_{\mathbf{R}} i\partial_x[(\chi f)(x,y)e^{-i(x+iy)\xi}]\,dx \\ &- 2i\int_{\mathbf{R}} (g(u)\cdot V\chi)(x,y)e^{-i(x+iy)\xi}\,dx \\ &+ \int_{\mathbf{R}} (f(u)E)(x,y)e^{-i(x+iy)\xi}\,dx \\ &- 2i\int_{\mathbf{R}} \chi F(x,y)\cdot(\overline{X}f)(u(x,y))e^{-i(x+iy)\xi}\,dx. \end{aligned}$$

By integration by parts the first integral is zero. Since $g(u(x,y)), V(x,y) \in \mathcal{C}^k$, and $y\xi \le 0$, the second one, via using integration by parts k times, is bounded by $C(1+|\xi|)^{-k}$. The third one is bounded by $C|E(x,y)| \le Cy^{k+\alpha}$.

We now estimate the last integral. This amounts to controlling the blow-up of derivatives of f at $u(x,y)$, for which we apply Proposition 3.7 to a domain of fixed size. By (5.3), Ω^+ contains $\mathbb{D}^n_{y/C}(u(x,y))$. Let $\hat{w} = \psi(w) := u(x,y) + yw/C$ with $w \in \mathbf{C}^n$. So ψ^{-1} transforms J, X_ℓ into $\hat{J}$, $\hat{X}_\ell = C^{-1}y\,d\psi^{-1}X_\ell$. On $\mathbb{D}^n$, we have

$$\hat{X}_\ell = \sum_{1\le s\le n} (b_{\ell s}\circ\psi\,\partial_{\overline{\hat{w}_s}} + a_{\ell s}\circ\psi\,\partial_{\hat{w}_s}).$$

Let $A' := (a_{\ell s}\circ\psi)$ and $B' =: (b_{\ell s}\circ\psi)$. It is easy to see that on $\mathbb{D}^n$, $\inf\left|\begin{smallmatrix} B' & A' \\ \overline{A'} & \overline{B'}\end{smallmatrix}\right| \ge 1/C$ and $|(A',B')|_{k+\alpha} \le C$ for some constant independent of v and p. Fix $1 \le m \le 2n$ and let $(\hat{w}_1,\dots,\hat{w}_{2n})$ be the standard coordinates of $\mathbf{R}^{2n}$. Applying

Proposition 3.7 to $\{\hat{X}_j\}$, we get a $\hat{J}$-holomorphic curve $\hat{u} : \mathbb{D}_r \to \mathbb{D}^n$ satisfying $\hat{u}(0) = 0$ and $d\hat{u}(0)(\partial_\zeta) = \partial_{\hat{w}'_m} - i\hat{J}_0\partial_{\hat{w}'_m}$. Here $r > 0$ is a constant independent of y, and $|\partial^j \hat{u}| < C$ for $j \le k+1$. Then the disc $\tilde{u}(\zeta) := \psi \circ \hat{u}(C\zeta/y)$ is J-holomorphic. We have $\tilde{u} : \mathbb{D}_{y/c} \to \Omega_1^+ := \psi(\Omega^+)$, $\tilde{u}(0) = u(x, y)$, and

$$d\tilde{u}(0)(\partial_\zeta) = \partial_{w'_m} - iJ_{\tilde{u}(0)}\partial_{w'_m}.$$

So $d\tilde{u}(0)(\partial_{\overline{\zeta}}) = \partial_{w'_m} + iJ_{\tilde{u}(0)}\partial_{w'_m}$. A direct computation shows that the first- and second-order derivatives of $\tilde{u}$ are bounded by C and C/y, respectively. It follows that $d\tilde{u}(\partial_{\overline{\zeta}}) = \tilde{V}(\zeta) \cdot X(\tilde{u}(\zeta))$ and the first-order derivative of $\tilde{V}(\xi, \eta)$ is bounded by C/η. Let $g_j := X_j f$. We obtain

$$\partial_{\overline{\zeta}}(f(\tilde{u}(\zeta))) = g(\tilde{u}(\zeta)) \cdot \tilde{V}(\zeta).$$

By the Cauchy–Green identity we have

$$f(\tilde{u}(\zeta)) = \frac{1}{2\pi i}\int_{|\zeta_*|=y/c} \frac{f(\tilde{u}(\zeta_*))}{\zeta_* - \zeta}\, d\zeta_* + \frac{1}{\pi}\int_{|\zeta_*|<y/c} \frac{g(\tilde{u}(\zeta_*)) \cdot \tilde{V}(\zeta_*)}{\zeta_* - \zeta}\, d\xi_*\, d\eta_*.$$

At $\zeta = 0$, the first-order derivatives of the first integral are bounded by C/y. Write $g(\tilde{u}(\zeta)) \cdot \tilde{V}(\zeta)$ as $h_1(\zeta) + h_2(\zeta)$. Here $\mathcal{C}^1$ norms of h_1, h_2 are bounded by C/y, $h_1(\zeta) = 0$ on $|\zeta| < y/(4c)$, and $h_0(\zeta) = 0$ on $|\zeta| > y/(2c)$. The first-order derivatives of the integral involving h_1 are bounded by C at $\zeta = 0$. After applying a translation $\zeta' = \zeta_* - \zeta$, the integral involving h_0 has bounded derivatives at $z = 0$ too. We obtain

$$2|(\partial_{w'_m} f)(\tilde{u}(0))| = |\partial_{\overline{z}}(f(\tilde{u}))(0) + \partial_z(f(\tilde{u}))(0)| \le C/y.$$

Thus,

$$|\partial_y \lambda(\xi, y)| \le C(y^{k-1+\alpha} + (1 + |\xi|)^{-k}).$$

We also have, for $\eta\xi \le 0$,

$$|\lambda(\xi, \eta)| \le Ce^{\eta\xi} \le C_L|\eta\xi|^{-L}.$$

We may assume that $\xi \le -1$. For $0 < \alpha' < \alpha$, choose $\eta = 1/(C|\xi|^{1-\varepsilon})$ with $\varepsilon > 0$ sufficiently small. Finally, $\lambda(\xi, 0) = \lambda(\xi, \eta) - \int_0^\eta \partial_y \lambda(\xi, y)\, dy$ satisfies

$$\begin{aligned} |\lambda(\xi, 0)| &\le C_L(C^{-1}|\xi|^\varepsilon)^{-L} + \frac{C}{k+\alpha}\eta^{k+\alpha} + C(1 + |\xi|)^{-k}\frac{1}{C|\xi|^{1-\varepsilon}} \\ &\le C_{\alpha'}(1 + |\xi|)^{-(k+\alpha')} \end{aligned} \tag{5.7}$$

for $\xi \le 0$. Reasoning with $X_j^2 f = g_j^2$ for $y_n \le 0$ and u^2, we get (5.7) for $\xi \ge 0$ and hence for $-\infty < \xi < \infty$.

By the Fourier inversion formula,

$$\begin{aligned} \chi f(p + xv) &= \frac{1}{2\pi}\int_{\mathbf{R}} \lambda(\xi, 0)e^{i\xi x}\, d\xi, \\ \partial_x^{k-1}(\chi f(p + xv)) &= \frac{1}{2\pi}\int_{\mathbf{R}} \lambda(\xi, 0)(i\xi)^{k-1}e^{i\xi x}\, d\xi. \end{aligned} \tag{5.8}$$

Using (5.7), we obtain $\chi f(p+xv) \in \mathcal{C}^{k-1}$. Let $0 < \alpha'' < \alpha' < \alpha$. Note that $|e^{ix_2} - e^{ix_1}| \le 2|x_2 - x_1|^{\alpha''}$ for all real numbers x_1, x_2. By (5.7) and (5.8) again, we obtain

$$|\partial_x^{k-1}(\chi f)(p+x_2 v) - \partial_x^{k-1}(\chi f)(p+x_1 v)| \le C_{\alpha'} \int_{\mathbf{R}} \frac{|x_2 - x_1|^{\alpha''}|\xi|^{\alpha''}}{(1+|\xi|)^{1+\alpha'}}\, d\xi.$$

We have $|(\chi f(p+\cdot v))|_{k-1+\alpha''} < C_{\alpha''}$. Therefore,

$$|\chi_0 f|_L|_{k-1+\alpha''} < C_{\alpha''},$$

where L is any line which is tangent to one of $v_1, \dots, v_N$ and which passes through $p \in M$ near the origin. For such a line L, we can find an affine diffeomorphism R with $R(0) = 0 \in M$, sending Δ^{2n-1} into M, such that $R(\cdot, t)$ are lines parallel to L for $t \in \mathbb{D}^{n-1}$. By Proposition 3.9 (ii) and hypothesis (5.4) we get $\partial^{k-2}(\chi_0 f) \in Lip(M)$.

Step 2. Uniform bound of derivatives of f on transversal J-holomorphic curves.

By Lemma 3.8 there exist N vectors $v_j \in \mathbf{R}^{2n}$ with $|v_j| < 1$ such that

$$(\partial_x, \partial_y)^I = \sum_{1 \le j \le N} c_{I,j}(v_j \cdot (\partial_x, \partial_y))^{|I|}, \quad 1 \le |I| \le k. \tag{5.9}$$

By perturbing v_j we may assume that $J_p(v_j \cdot (\partial_x, \partial_y))$ are not tangent to M at $p = 0$ and hence in a neighborhood of 0 in M.

We are given n vector fields $X_1, \dots, X_n$ defined on $\Omega^+ \cup M$. Recall that M is contained in $y_n = 0$ and Ω^+ is contained in $y_n > 0$. Applying the Borel theorem (Lemma 2.3) via restriction and then extension, we may assume that $X_1, \dots, X_n$ define an almost complex structure on a neighborhood of M. We emphasize that even if we start with an integrable almost complex structure, the resulting almost complex structure may not be integrable. We now apply Proposition 3.7 with the parameter set $P = \overline{\mathbb{D}}_{r_0}^{n-1}$. We find diffeomorphisms u_j, R_j of class $\mathcal{C}^{k+1+\alpha,k}$, which map $\mathbb{D}_{r_0} \times \mathbb{D}_{r_0}^{n-1}$ into Ω, such that

$$du_j(0,t)(\partial_\xi) = v_j \cdot (\partial_x, \partial_y). \tag{5.10}$$

Moreover, $D_{j,r}(t) := u_j(\mathbb{D}_r, t) = R_j(\omega_{j,r}(t), t)$ satisfies

$$\mathbb{D}_{r/c_1} \subset \omega_{j,r}(t) \subset \mathbb{D}_{c_1 r}.$$

Also, $R_j(0) = 0$, $u_j(0) = 0$, and

$$-2i\, du_j(\cdot, t)(\partial_{\overline{\zeta}}) = J_0(v_j \cdot (\partial_x, \partial_y)) - i(v_j \cdot (\partial_x, \partial_y)).$$

We choose $r < r_0$ sufficiently small so that various compositions in u_j, R_j are well defined. Also, $\omega_{j,r}^+(t) := \omega_{j,r}(t) \cap \{y > 0\}$ satisfies $R_j(\omega_{j,r}^+(t), t) = D_{j,r}^+(t) := D_{j,r}(t) \cap \Omega^+$. Write $(\tilde{D}_{j,r}^+(t), t) = u_j^{-1}(D_{j,r}^+(t))$. According to the Cauchy–Green formula, we have

$$f(u_j(\tilde{z}, t)) = \frac{1}{2\pi i} \int_{\partial \tilde{D}_{j,r}^+(t)} \frac{f(u_j(\tilde{\zeta}, t))}{\tilde{\zeta} - \tilde{z}}\, d\tilde{\zeta} - \frac{1}{2\pi i} \int_{\tilde{D}_{j,r}^+(t)} \frac{\partial_{\overline{\tilde{\zeta}}} f(u_j(\tilde{\zeta}, t))}{\tilde{\zeta} - \tilde{z}}\, d\overline{\tilde{\zeta}} \wedge d\tilde{\zeta}.$$

Set $(\tilde{z}, t) = u_j^{-1} \circ R_j(z,t) = (\tau_j(z,t), t)$. By $du_j(\partial_{\overline{\zeta}}) = V(\zeta) \cdot X(u_j(\zeta))$ and $X_j f = g_j$ we get $\partial_{\overline{\zeta}} f(u_j) = V \cdot g(u_j)$ and

$$\begin{aligned}\varepsilon f(R_j(z,t)) = \frac{1}{2\pi i}\int_{\zeta\in\partial\omega^+_{j,r}(t)} \frac{f(R_j(\zeta,t))}{\tau_j(\zeta,t)-\tau_j(z,t)}\, d\tau_j(\zeta,t)\\ - \frac{1}{2\pi i}\int_{\omega^+_{j,r}(t)} \frac{g(R_j(\zeta,t))\cdot V(\tau_j(\zeta,t))}{\tau_j(\zeta,t)-\tau_j(z,t)}\, d\overline{\tau_j(\zeta,t)} \wedge d\tau_j(\zeta,t).\end{aligned}$$

Here $\varepsilon = 1$ if $\tau_j(\cdot,t)$ is orientation-preserving and otherwise $\varepsilon = -1$. Notice that ε is independent of t. Recall that $R_j(\cdot,t)$ sends $[-r/c, r/c]$ into $\partial D^+_{j,r}(t) \cap M$. Since $u \in \mathcal{C}^{k+1+\alpha,k}$, it is easy to see that $V(\tau_j(\zeta,t))$ are in $\mathcal{C}^{k+\alpha,0}$. Applying Lemma 4.1 (ii), $\partial^{k-2}_{\xi,\eta}(f(R_j))$ are continuous on $\Delta_{r/c_*} \times [0, r/c_*) \times \mathbb{D}^{n-1}_{r/c_*}$, and on it, $|f(R_j)|_{k-2+\beta,0} < C_\beta$ for any $\beta < 1$, where $\mathbb{D}^{n-1}_{r/c_*}$ is the parameter space.

Step 3. Smoothness of f via families of J-holomorphic curves.

In this step, we will first remove the assumption stated at the beginning of Step 1 that f is $\mathcal{C}^1$ on Ω_ℓ. We also address the comment made after Theorem 5.1 on the interior regularity of f. Let $J = J^\ell \in \mathcal{C}^{k+\alpha}$ and $\Omega = \Omega_\ell$.

Let v_j satisfy (5.9). According to Proposition 3.7, we find a $\mathcal{C}^{k+1+\alpha,k}$ diffeomorphism u_j defined in neighborhood of $0 \in \Omega$ such that $\zeta \mapsto u_j(\zeta,t)$ is J-holomorphic for fixed $t \in \mathbb{D}^{n-1}_\varepsilon$ and $u_j(0) = 0$, $du_j(0)(\partial_\xi) = v_j$. Drop the subscript j in u_j. Then u^{-1} defines a $\mathcal{C}^k$ coordinate system in a neighborhood of the origin, and u^{-1} transforms J into $\hat{J}$. It follows that $\mathbb{D}_\varepsilon \times \{t\}$ are $\hat{J}$-holomorphic curves for $|t| < \varepsilon$ and ε small enough. Thus, we can take $\hat{X}_1 = a(\zeta,t)\partial_{\overline{\zeta}} + b(\zeta,t)\partial_\zeta$ with $a, b \in \mathcal{C}^{k+\alpha,k}$ and t being the parameter. Now $\hat{f} := f \circ u$ satisfies $\hat{X}_1 \hat{f} = \hat{g}_1 \in \mathcal{C}^k$. Here $\hat{X}_1 \hat{f} = \hat{g}_1$ holds in the sense of distributions. We want to show that when restricted to $\mathbb{D}_\varepsilon \times \{t\}$, $\hat{X}_1 \hat{f} = \hat{g}_1$ still holds as distributions. To verify it, we fix a test function ϕ on $\mathbb{D}_\varepsilon$ and take a sequence of test functions ϕ_ν in $\mathbf{C}^{n-1}$ such that $\int_{\mathbf{C}^{n-1}} \phi_\nu = 1$ and $\operatorname{supp}\phi_\nu \subset \{t\} + \mathbb{D}^{n-1}_{1/\nu}$. Note that the formal adjoint $\hat{X}_1^*$ does not contain derivatives in the t-variable and satisfies

$$\int_{\mathbf{C}^n} \hat{g}_1 \phi\phi_\nu = \int_{\mathbf{C}^n} \hat{f}\hat{X}_1^*(\phi\phi_\nu) = \int_{\mathbf{C}^n} \hat{f}\phi_\nu \hat{X}_1^*(\phi).$$

Since all functions in the integrands are continuous, as ν tends to ∞, we obtain

$$\int_{\mathbf{C}} \hat{g}_1(\cdot,t)\phi = \int_{\mathbf{C}} \hat{f}(\cdot,t)\hat{X}_1^*(\phi).$$

Thus, we have proved that in the sense of distributions $\hat{X}_1 \hat{f} = \hat{g}_1 \in \mathcal{C}^k \subset \mathcal{C}^{k-1+\beta}$ for any $\beta < 1$. The coefficients of $\hat{X}_1$ are in $\mathcal{C}^{k+\alpha,k}$. Reasoning as at the end of Step 2, by using Lemma 4.1 (i) and (iii), we obtain $f \circ u_j \in \mathcal{C}^{k+\beta,0}$ for any $\beta < 1$. (*Note that this part of Step 2 does not use the assumption that $f \in \mathcal{C}^1$.*) Now $f \circ u_j \in \mathcal{C}^{k-1,0}$ with $k - 1 \geq 1$, $u_j \in \mathcal{C}^k$, and Proposition 3.9 (iii) implies that $f \in \mathcal{C}^{k-2+\beta}$ for all $\beta < 1$. In particular, we get $f \in \mathcal{C}^1$ on $\Omega_1 \cup \Omega_2$ for $k \geq 3$, which is used in Steps 1 and 2.

We now finish the proof of the theorem. By the end of Step 2, we know that when r is sufficiently small, $\partial^{k-2}_{\xi,\eta}(f \circ R_j(\xi + i\eta, t))$ are continuous on $\mathbb{D}^n_r \cap \{\eta \geq 0\}$. Assume that $k > 2$. We also know that the $\mathcal{C}^k$ diffeomorphisms R_j, u_j satisfy $u_j(0) = R_j(0) = 0 \in M$, $R_j(\omega_{j,r}(t), t) = u_j(\mathbb{D}_r, t)$, and $R_j(\omega_{j,r}(t) \cap \{\eta \geq 0\}, t) = u_j(\mathbb{D}_r, t) \cap \overline{\Omega}^+$. Thus, $(\partial^{k-2}_{\xi}(f \circ u_j)) \circ u_j^{-1}$ are continuous on $\mathbb{D}^n_r \cap \overline{\Omega}^+$. Since $du_j(0,t)(\partial_\xi) = v_j \cdot (\partial_x, \partial_y)$, in view of (5.9), we can apply Proposition 3.9 to the family of diffeomorphisms $u_j(\xi + i\eta, t)$ by treating (η, t) as parameters. Since $f \in \mathcal{C}^{k-2}(\Omega^+)$, using Proposition 3.9 (i), we express $\partial^I f$ on $\Omega^+ \cap \Delta^{2n}_r$ via functions $(\partial^\ell_\xi(f \circ u_j)) \circ u_j^{-1}$ for $1 \leq |I| \leq k-2$. The latter are continuous on $\overline{\Omega}^+ \cap \mathbb{D}^n_r$. We have proved that $\partial^I f(x)$ extends continuously to $\overline{\Omega}^+ \cap \Delta^{2n}_r$ for all $|I| \leq k-2$ and $k > 2$. Therefore, $f \in \mathcal{C}^{k-2}(\Omega^+ \cup M)$ by $M \in \mathcal{C}^{k-2}$. The proof of Theorem 5.1 is complete. $\square$

6. One-Dimensional Riemann Mapping for Structures with Parameter

In this section, we will study the regularity of inhomogeneous Beltrami equation on a planar domain for complex structures that depend on a parameter. The regularity of lower-order derivatives of the solutions of such equations have been studied by Ahlfors and Bers [1]. We will emphasize the regularity of solutions in $\hat{\mathcal{C}}^{k+1+\alpha,j}$ spaces up to the boundary. Throughout this section, Ω is a bounded open set in $\mathbf{C}$. We start with the existence of isothermal coordinates with parameter. Recall that a diffeomorphism φ is said to transform a vector field X into $\tilde{X}$ if locally $d\varphi(X) = \mu\tilde{X}$. Denote by $\hat{\mathcal{C}}^{k+\alpha,j}(\Omega \cup \gamma, P)$ the set of functions that are in $\hat{\mathcal{C}}^{k+\alpha,j}(K, P)$ for any compact subset K of $\Omega \cup \gamma$ where γ is an embedded curve. Set $\mathcal{C}^{k+\alpha,j}(\Omega \cup \gamma, P) := \bigcap_{0 \leq \ell \leq j} \hat{\mathcal{C}}^{k-\ell+\alpha,\ell}(\Omega \cup \gamma, P)$ for $k \geq j$. Recall that $\mathcal{C}^{\infty,j}(\overline{\Omega}, P) = \bigcap_{k=1}^{\infty} \mathcal{C}^{k,j}(\overline{\Omega}, P)$ and set $\mathcal{C}^{\infty,\infty}(\Omega, P) := \bigcap_{j=1}^{\infty} \mathcal{C}^{\infty,j}(\overline{\Omega}, P)$.

PROPOSITION 6.1. *Let k, j be integers or ∞ such that $j \leq k$, and let $0 < \alpha < 1$. Let Ω be a domain in $\mathbf{C}$, and P be an open set in an Euclidean space. Let $a \in \hat{\mathcal{C}}^{k+\alpha,j}(\Omega, P)$ (resp. $\mathcal{C}^{k+\alpha,j}(\Omega, P)$) satisfying $|a|_{L^\infty} < 1$. If $x \in \Omega$, then there exist a neighborhood U of x and a map $\varphi \in \hat{\mathcal{C}}^{k+1+\alpha,j}(U, P)$ (resp. $\mathcal{C}^{k+1+\alpha,j}(U, P)$) such that $\varphi(\cdot, t)$ are diffeomorphisms that map U onto their images and $\partial_{\overline{z}} + a(z,t)\partial_z$ into $\partial_{\overline{z}}$.*

Proof. Fix $x = 0 \in \Omega$. Let $\varphi(z,t) = z - a(0,t)\overline{z}$. Then $z \mapsto \varphi(z,t)$ is invertible and transforms $\partial_{\overline{z}} + a(z,t)\partial_z$ into $\partial_{\overline{z}} + \tilde{a}(z,t)\partial_z$ with $\tilde{a}(0,t) = 0$. The problem is local. By dilation, we may assume that Ω contains $\overline{\mathbb{D}}$. Let χ be a smooth function on $\mathbb{D}$ that has compact support and equals 1 on $\mathbb{D}_{1/2}$. Applying a dilation and replacing $\tilde{a}$ by $\chi\tilde{a} = b$, we achieve $|b|_{\alpha,0} < \varepsilon_\alpha$ on $\mathbb{D}$ for the ε_α in Lemma 2.2. Set $f := -(\mathrm{I} + Tb\partial_z)^{-1}Tb$. On $\mathbb{D}$, we have $f \in \hat{\mathcal{C}}^{k+1+\alpha,j}$ and $|f|_{1,0} \leq C_\alpha |(\mathrm{I} + Tb\partial_z)^{-1}|_{1+\alpha,0} |b|_{\alpha,0}$. With the dilation for $\tilde{a}$, $|f|_{1,0}$ can be arbitrarily small. Therefore, $z \mapsto z + f(z,t)$ are indeed diffeomorphisms. Since $z + f(z,t)$ is annihilated by $\partial_{\overline{z}} + \tilde{a}\partial_z$, it transforms $\partial_{\overline{z}} + \tilde{a}(z,t)\partial_z$ into $\partial_{\overline{z}}$. $\square$

It is important that the above classical result (for the nonparameter case) allows us to interpret $\partial_{\overline{z}} f + a\partial_z f = g$ when a is merely $\mathcal{C}^\alpha$. Let $w = \varphi(z)$ be a local $\mathcal{C}^{1+\alpha}$ diffeomorphism such that $d\varphi(\partial_{\overline{z}} + a\partial_z) = \mu(w)\partial_{\overline{w}}$. Then $\partial_{\overline{z}} f + a\partial_z f = g$ in the w-coordinates if $\partial_{\overline{w}}(f \circ \varphi^{-1}) = g \circ \varphi^{-1}(w)/\mu(w)$ in the sense of distributions. Recall that in the proof of Theorem 1.1, there is a loss of derivatives. We now turn to a sharp version for planar domains. In fact, we will prove a parameter version.

THEOREM 6.2. *Let k, k', j be integers or ∞ such that $j \leq k'$, and let $0 < \alpha < 1$. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$ (resp. $\mathcal{C}^{k'+1+\alpha}$). Let Ω_1, Ω_2 be disjoint open subsets of $\mathbf{C}$ such that both $\partial\Omega_1$, $\partial\Omega_2$ contain γ as a relatively open subset. Assume that $a_\ell \in \hat{\mathcal{C}}^{k+\alpha,j}(\Omega_\ell \cup \gamma, P)$ (resp. $\mathcal{C}^{k'+\alpha,j}(\Omega_\ell \cup \gamma, P)$) and satisfies $|a_\ell|_{L^\infty} < 1$ on $(\Omega_\ell \cup \gamma) \times P$. Let $f \in \hat{\mathcal{C}}^{0,j}(\Omega_1 \cup \gamma \cup \Omega_2, P)$ and $b_\ell \in \hat{\mathcal{C}}^{k+\alpha,j}(\Omega_\ell \cup \gamma, P)$ (resp. $\mathcal{C}^{k'+\alpha,j}(\Omega_\ell \cup \gamma, P)$) be such that*

$$\partial_{\overline{z}} f + a_\ell \partial_z f = b_\ell \quad \text{on } \Omega_\ell, \ell = 1, 2.$$

Then $f \in \hat{\mathcal{C}}^{k+1+\alpha,j}(\Omega_\ell \cup \gamma, P)$ (resp. $\mathcal{C}^{k'+1+\alpha,j}(\Omega_\ell \cup \gamma, P)$).

Proof. As in the proof of Theorem 1.1 in Section 5, we may assume that γ is the x-axis and $\Omega_1 = \mathbb{D}_r^+$, $\Omega_2 = \mathbb{D}_r^-$. In the following, all functions a_ℓ, b_ℓ, etc. are defined on Ω_ℓ for some $r > 0$, and we will take smaller values for r for a few times. Applying Lemma 2.3, we first find a function $\phi_\ell \in \hat{\mathcal{C}}^{k+1+\alpha,j}$ with $\phi_\ell(\cdot, t) \in \mathcal{C}^2(\Omega_\ell \cup \gamma)$ such that $\phi_\ell(x, 0, t) = x$ and $\partial_{\overline{z}}\phi_\ell + a_\ell\partial_z\phi_\ell = O(|y|^{k+\alpha})$. Then ϕ_ℓ sends $\partial_{\overline{z}} + a_\ell\partial_z$ into $\mu_\ell(\partial_{\overline{z}} + \tilde{a}_\ell\partial_z)$. Replace f, a_ℓ by $f \circ \phi_\ell^{-1}, \tilde{a}_\ell$ on $\overline{\Omega}_\ell$. Therefore, we may assume that $a_\ell(z, t) = O(|y|^{k+\alpha})$. Now, define $a := a_\ell$ on $\overline{\Omega}_\ell$. It follows that a is of class $\hat{\mathcal{C}}^{k+\alpha,j}(\Omega_1 \cup \gamma \cup \Omega_2, P)$. Set $X := \partial_{\overline{z}} + a(z, t)\partial_z$.

Next, we find $g_\ell \in \hat{\mathcal{C}}^{k+1+\alpha,j}$ on $\overline{\Omega}_\ell$ such that

$$Xg_\ell - b_\ell = O(|y|^{k+\alpha}), \quad g_\ell(x, 0) = 0.$$

Replace f by $f - g_\ell$ on $\overline{\Omega}_\ell$. Therefore, we may assume that $b_\ell(z, t) = O(|y|^{k+\alpha})$. Define $b := b_\ell$ on $\overline{\Omega}_i$. Then b is of class $\hat{\mathcal{C}}^{k+\alpha,j}(\Omega_1 \cup \gamma \cup \Omega_2, P)$.

By Proposition 6.1 there are diffeomorphisms $\psi(\cdot, t)$ with $\psi \in \hat{\mathcal{C}}^{k+1+\alpha,j}(\mathbb{D}_r, P)$, which send X into $\mu\partial_{\overline{z}}$ with $\mu \in \hat{\mathcal{C}}^{k+\alpha,j}(\mathbb{D}_r, P)$. Then $\partial_{\overline{z}}(f \circ \psi^{-1}) = b \circ \psi^{-1}/\mu$. Let $h := T_{\mathbb{D}_r}(b \circ \psi^{-1}/\mu)$ where r is sufficiently small. Then $h \in \hat{\mathcal{C}}^{k+1+\alpha,j}$. Now $f \circ \psi^{-1} - h$ is holomorphic away from $\psi(\gamma)$, continuous up to the $\mathcal{C}^1$ curve $\psi(\gamma)$. Take a small disc D_r, independent of t and centered at $p \in \psi(\cdot, t_0)(\gamma)$. By the Cauchy formula, we express $f(\cdot, t)$ on D_r via the Cauchy transform on ∂D_r when t is in a small neighborhood of t_0. From $f \in \hat{\mathcal{C}}^{0,j}$ and the compactness of P we conclude that $f \in \hat{\mathcal{C}}^{k+1+\alpha,j}(\mathbb{D}_{r/2}, P)$. Recall that f is replaced by $f \circ \phi_\ell^{-1}$. Therefore, the original f is in $\hat{\mathcal{C}}^{k+1+\alpha,j}(\Omega_\ell \cup \gamma, P)$. □

LEMMA 6.3. *Let $\Omega \subset \mathbf{C}$ be a bounded domain with $\partial\Omega \in \mathcal{C}^1$. Suppose that $v \in \mathcal{C}^1(\Omega)$ and b are continuous functions on $\overline{\Omega}$. Then v satisfies*

$$v + Tb = 0 \tag{6.1}$$

if and only if it satisfies

$$\partial_{\bar z} v + b = 0, \tag{6.2}$$

$$\mathcal{C}v = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{v(\zeta)}{\zeta - z}\,d\zeta = 0. \tag{6.3}$$

Here three identities are on Ω. *Moreover,* (6.3) *holds on* Ω *if and only if* v *is the boundary value of a function that is holomorphic on* $\mathbf{C}\setminus\overline{\Omega}$, *continuous on* $\mathbf{C}\setminus\Omega$, *and vanishing at* ∞.

Proof. Applying $\partial_{\bar z}$ to (6.1) gives us (6.2). On Ω, $\mathcal{C}v = v - T\partial_{\bar z}v$. Applying $T\partial_{\bar z}$ to (6.1) and using (6.1) again, we get $v - T\partial_{\bar z}v = 0$. Conversely, if v satisfies (6.2), then $v + Tb = v - T\partial_{\bar z}v = \mathcal{C}v$. The latter is zero by (6.3). Thus, v satisfies (6.1).

It is a standard fact that when Ω is a bounded domain with $\mathcal{C}^1$ boundary and v is continuous on $\partial\Omega$, $\mathcal{C}v(z - tn(z)) - \mathcal{C}v(z + tn(z))$ converges to $v(z)$ uniformly on $\partial\Omega$ as $t \to 0^+$. Here n is the unit outer normal vector of $\partial\Omega$. Then (6.3) implies that $\mathcal{C}v$ is continuous on $\mathbf{C}\setminus\Omega$ and agrees with v on $\partial\Omega$. That $\mathcal{C}v$ vanishes at ∞ is trivial. The converse follows from the Cauchy formula. □

Now, we prove a version of Theorem 1.2 with parameter.

THEOREM 6.4. *Let* k, j *be integers, and let* $0 < \alpha < 1$. *Let* Ω *be a bounded domain in* $\mathbf{C}$ *with* $\partial\Omega \in \mathcal{C}^{k+1+\alpha}$. *Let* $a \in \hat{\mathcal{C}}^{k+\alpha,j}(\overline{\Omega}, P)$, $b \in \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P)$ *be (scalar) functions satisfying* $\|a\|_{0,0} < 1$. *Then*

$$v(\cdot,t) + T_\Omega(a(\cdot,t)\partial_z v(\cdot,t)) = b(\cdot,t) \tag{6.4}$$

has a unique solution $v(\cdot,t)$ *with* $v \in \hat{\mathcal{C}}^{k+\alpha+1,j}(\overline{\Omega}, P)$. *Moreover,*

$$\mathrm{I} + Ta\partial_z : \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P) \to \hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega}, P)$$

has a bounded inverse.

Proof. Let us first show the existence of solution $v \in \hat{\mathcal{C}}^{\beta,0}$ for some $0 < \beta < 1$. We want to differentiate (6.4) in z in order to transform the equation into a corresponding equation by using $\partial_z T = S$. Recall that $\partial_{\bar z}T$ is the identity on $L^p(\Omega)$ when $p > 2$ and Ω is bounded. Therefore, the set of solutions v to (6.4) with $w(z,t) = \partial_z v(\cdot,t) \in L^p(\Omega)$ is completely determined by the set of solutions w to

$$w(z,t) + S(a(\cdot,t)w(\cdot,t))(z) = \partial_z b(z,t), \qquad w(\cdot,t) \in L^p(\Omega), \quad p > 2.$$

In fact, if $w(z,t)$ is a solution to the above equation, then

$$v(z,t) = b(z,t) - T(a(\cdot,t)w(\cdot,t))(z) \tag{6.5}$$

is a solution to (6.4). We also recall that

$$\|Sf\|_{L^p(\Omega)} \le \varepsilon_p \|f\|_{L^p(\Omega)}, \tag{6.6}$$

where the best constant ε_p depends on Ω only, and ε_p tends to 1 as $p \to 2$ (see [25, p. 81]). We also recall that for $2 < p < \infty$, there exists a positive constant C_p

such that

$$\|Tf\|_{\beta} \leq C_p \|f\|_{L^p(\Omega)}, \tag{6.7}$$

where $\beta = 1 - 2/p$ (see [25, Theorem 1.19]). We fix $2 < p < \infty$ such that

$$\|Sa\|_{L^p(\Omega)} \leq 1/2, \tag{6.8}$$

and we set $\beta = 1 - 2/p$. By the contraction mapping theorem, for each t, there exists a unique $w(\cdot, t)$ in $L^p(\Omega)$ such that on Ω,

$$w(z,t) + S(a(\cdot,t)w(\cdot,t))(z) = \tilde{b}(z,t). \tag{6.9}$$

Moreover, the unique solution satisfies

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leq 2\|\tilde{b}(\cdot,t)\|_{L^p(\Omega)}. \tag{6.10}$$

By (6.5) and (6.7) we have

$$\|v(\cdot,t)\|_{\beta} \leq C\|b\|_{1,0}. \tag{6.11}$$

Let us also show that $v(z,t)$ is continuous. We set $\tilde{w}(z,t',t) = w(z,t') - w(z,t)$, $\tilde{a}(z,t',t) = a(z,t') - a(z,t)$, $\tilde{b}(z,t',t) = b(z,t') - b(z,t)$, and $\tilde{v}(z,t',t) = v(z,t') - v(z,t)$. Then we have

$$\tilde{w}(z,t',t) + S(a(\cdot,t')\tilde{w}(\cdot,t',t))(z) = \tilde{b}(z,t',t) - S(\tilde{a}(\cdot,t',t)w(\cdot,t))(z).$$

According to (6.6) and (6.8), we obtain

$$\|\tilde{w}(\cdot,t',t)\|_{L^p(\Omega)} \leq C_p(\|\tilde{a}(\cdot,t',t)\|_0 + \|\tilde{b}(\cdot,t',t)\|_0). \tag{6.12}$$

Using (6.7), we get

$$\begin{aligned}\|\tilde{v}(\cdot,t',t)\|_0 \leq{} & C_p'\|\tilde{a}(\cdot,t',t)\|_0\|\partial_z b(\cdot,t')\|_{L^p(\Omega)} \\ & + C_p'\|a\|_{0,0}\|\tilde{w}(\cdot,t',t)\|_{L^p(\Omega)} + \|\tilde{b}(\cdot,t',t)\|_0.\end{aligned}$$

This shows that $v \in \hat{\mathcal{C}}^{\beta,0}$. By Theorem 6.2 we obtain $v \in \hat{\mathcal{C}}^{k+1+\alpha,0}$.

Since the solution to (6.9) is unique, as mentioned earlier, the solution to (6.4) is also unique. Therefore, $\mathrm{I} + Ta\partial_z : \hat{\mathcal{C}}^{k+1+\alpha,0}(\overline{\Omega}, P) \to \hat{\mathcal{C}}^{k+1+\alpha,0}(\overline{\Omega}, P)$ is continuous, injective, and onto. The last assertion in the proposition for $j = 0$ follows from the open mapping theorem applied to the Banach space $\hat{\mathcal{C}}^{k+1+\alpha,0}(\overline{\Omega}, P)$.

We now apply induction on j. Here we need to use a method from [17], which we have used in Step 2 of the proof of Proposition 3.7. In fact, the proof is valid without any essential changes. We briefly point out the arguments. Let D_V denote the derivative in direction V in the parameter space. For $j = 1$, the difference quotient $\delta_\lambda v(\cdot,t) = \frac{v(\cdot,t+\lambda V)-v(\cdot,t)}{\lambda}$ satisfies

$$\delta_\lambda v(\cdot,t) + T(a(\cdot,t)\partial_z\delta_\lambda v(\cdot,t)) = \delta_\lambda b(\cdot,t) - T(\delta_\lambda a(\cdot,t)\partial_z v(\cdot,t')).$$

Here t, t' are in the interior of P, and $t' = t + \lambda V$. As in (6.11), we can verify that

$$\|\delta_\lambda v(\cdot,t)\|_{\beta} \leq C(\|\delta_\lambda b(\cdot,t)\|_{1,0} + \|T(\delta_\lambda a(\cdot,t)\partial_z v(\cdot,t))\|_{1,0}).$$

For fixed t and small λ, by the intermediate value theorem we obtain

$$\|\delta_\lambda v(\cdot,t)\|_{\beta} \leq C(\|b\|_{1,1} + \|a\|_{1,1}\|v\|_{0,0}).$$

Then we can show that $\delta_\lambda v(\cdot,t)$ converges to $\omega(\cdot,t)$ and that $\omega(\cdot,t)$ is the unique solution to

$$\omega(\cdot,t)+T(a(\cdot,t)\partial_z\omega(\cdot,t))=D_Vb(\cdot,t)-T((D_Va(\cdot,t))\partial_z v(\cdot,t)).$$

Here $D_Vb(\cdot,t)$ denotes the directional derivative in vector V for the t variables. As before, we can verity that $w(z,t)$ is continuous in z, t. Thus, $w=D_Vv\in\hat{\mathcal{C}}^{k+1+\alpha,0}$ by Theorem 6.2. We now have

$$D_Vv(z,t)+T(a(\cdot,t)\partial_z D_Vv(\cdot,t))(z)=D_Vb(\cdot,t)-T((D_Va(\cdot,t))\partial_z v(\cdot,t))(z),$$

and we can verify that $(D_V)^2v(z,t)\in\hat{\mathcal{C}}^{k+1+\alpha,0}$. Inductively, this shows that v is in $\hat{\mathcal{C}}^{k+1+\alpha,j}(\overline{\Omega},P)$. By the open mapping theorem again, the last assertion of the theorem is true. $\square$

In [4], the Riemann mapping with parameter is proved for simply connected domains in $\mathbf{C}$. We now extend the result to complex structures. The reader is referred to [4] for some elementary properties of $\mathcal{C}^{k+1+\alpha,j}(\overline{\Omega},[0,1])$, where $\mathcal{C}^{k+1+\alpha,j}(\overline{\Omega},[0,1])$ is denoted by $\mathcal{B}^{k+1+\alpha,j}(\overline{\Omega},[0,1])$.

THEOREM 6.5. *Let k, j be integers or infinity such that $0\le j\le k$, and let $0<\alpha<1$. Let $P=[0,1]$. Let $\Gamma(\cdot,t):\overline{\mathbb{D}}\to\overline{\Omega}^t$ be embeddings for all $t\in P$ that satisfy $\Gamma\in\mathcal{C}^{k+1+\alpha,j}(\overline{\mathbb{D}},P)$. Let J^t be a family of complex structures on $\overline{\Omega}^t$ such that the pull-back of J^t by $\Gamma(\cdot,t)$ defines a family of complex structures on $\overline{\mathbb{D}}$ of class $\mathcal{C}^{k+\alpha,j}(\overline{\mathbb{D}},P)$. There exists a family of mappings $R(\cdot,t)$ from Ω^t onto $\mathbb{D}$ such that $(z,t)\mapsto R(\Gamma(z,t),t)$ is a mapping of class $\mathcal{C}^{k+1+\alpha,j}(\overline{\mathbb{D}},P)$ and the push-forward of J^t via $R(\cdot,t)$ is the standard complex structure on $\mathbb{D}$ for each $t\in P$.*

Proof. To localize the problem in the parameter space, we mark three points a^t, b^t, c^t on $\partial\Omega^t$ such that $t\mapsto(a^t,b^t,b^t)$ is of class $\mathcal{C}^j$. Let $R(\cdot,t)$ be the Riemann mapping sending a^t, b^t, c^t to a, b, c on the unit circle. We want to show that $R(\cdot,t)$ has the desired regularity for t near a given point $t_0\in P$.

We first want to find a family of mappings $S(\cdot,t)$ from $\overline{\Omega}^t$ onto $\overline{D}^t$ such that the push-forward of J^t agrees with the standard complex structure. To see this, we extend J^t to a larger simply connected domain $\tilde{\Omega}^t$ that contains the closure of Ω^t. By the uniformization theorem there exists a diffeomorphism $S(\cdot,t_0)$ of class $\mathcal{C}^{k+1+\alpha}$ that maps $\tilde{\Omega}^{t_0}$ onto the unit disc or $\mathbf{C}$ such that $S(\cdot,t_0)_*J^{t_0}$ is the standard complex structure. By Kellogg's theorem we can find the Riemann mapping sending $S(\cdot,t_0)(\Omega^{t_0})$ onto the unit disk and match three marked points. For t close to t_0, we can find a family of diffeomorphisms of class $\mathcal{C}^{k+1+\alpha,j}$, defined near $\overline{\mathbb{D}}$, that fixes $S(\cdot,t_0)(\Omega^{t_0})$ pointwise and maps $S(\cdot,t_0)(\Omega^t)$ onto the unit disc. Therefore, for regularity near t_0, we have simplified the problem to the case where J^{t_0} is J_{st} and Ω^t are the unit disk. Then J^t are defined by vector fields

$$\partial_{\overline{z}}+a(z,t)\partial_z$$

with $a(\cdot,t_0)=0$. Let $f(\cdot,t):=u$ be the solution to

$$u+T_{\mathbb{D}}(a(\cdot,t)\partial_z u)=-T_{\mathbb{D}}a(\cdot,t).$$

Since a is of class $\mathcal{C}^{k+\alpha,j}$, $|a|_{0,0}$ can be made small enough on $\mathbb{D}\times\tilde{P}$ by choosing a small neighborhood $\tilde{P}$ of t_0. In particular, $|a|_{0,0}<1$, and Theorem 6.4 shows that f is of class $\mathcal{C}^{k+1+\alpha,j}$. Moreover, using

$$|a|_{\alpha/2,0}\leq c|a|_{0,0}^{1/2}|a|_{\alpha,0}^{1/2}$$

for some positive constant c, we obtain $|f|_{1+\alpha/2,0}\leq C|a|_{\alpha/2,0}<1$. Therefore, $F(\cdot,t): z\mapsto z+f(z,t)$ is a family of diffeomorphisms from $\overline{\mathbb{D}}$ onto $\overline{D}^t$ and such that $F(\cdot,t)_*J^t$ are the standard complex structure. By the Riemann mapping with parameter [4] we conclude that there exists a family of Riemann mappings $R_0(\cdot,t)$ from D^t onto the unit disc such that $(z,t)\mapsto R_0(F(z,t),t)$ is of class $\mathcal{C}^{k+1+\alpha,j}(\overline{\mathbb{D}},\tilde{P})$. Let $M(\cdot,t)$ be the linear fractional transformation sending

$$(\tilde{a}^t,\tilde{b}^t,\tilde{c}^t)=(R_0(F(a^t,t),t),R_0(F(b^t,t),t),R_0(F(c^t,t),t))$$

to (a,b,c). Since the mapping $t\mapsto(\tilde{a}^t,\tilde{b}^t,\tilde{c}^t)$ is of class $\mathcal{C}^j$, it is easy to verify that M is of class $\mathcal{C}^{k+1+\alpha,j}$ on $\overline{\mathbb{D}}\times\tilde{P}$. Then $R(\cdot,t)=M(R_0(\cdot,t),t)$ is of class $\mathcal{C}^{k+1+\alpha,j}$ on $\overline{\mathbb{D}}\times\tilde{P}$. □

From the proof of Lemma 6.3 we also have the following for vector-valued functions.

LEMMA 6.6. *Let $\Omega\subset\mathbf{C}$ be a bounded domain with $\partial\Omega\in\mathcal{C}^1$. Let v and b be vectors of n continuous functions on $\overline{\Omega}$. Suppose that $v\in\mathcal{C}^1(\Omega)$.*

(i) *Let A be an $n\times n$ matrix of continuous functions defined on $\overline{\Omega}$. Then v satisfies*

$$v+T(b+A\partial_z v)=0 \tag{6.13}$$

if and only if v satisfies (6.3) *and*

$$\partial_{\overline{z}}v+b+A\partial_z v=0. \tag{6.14}$$

(ii) *Assume further that $v(\overline{\Omega})$ is contained in an open subset D of $\mathbf{C}^n$. Let $A\in\mathcal{C}^1(D)$ be an $n\times n$ matrix. Then v satisfies*

$$v+T(b+A(v)\overline{\partial}_z v)=0 \tag{6.15}$$

if and only if v satisfies (6.3) *and*

$$\partial_{\overline{z}}v+b+A(v)\overline{\partial_z v}=0. \tag{6.16}$$

Here, equations (6.3) *and* (6.13)–(6.16) *hold on Ω.*

We now use the proof of Theorem 6.2 to study a problem in different directions.

PROPOSITION 6.7. *Let k be an integer, and let $0<\alpha<1$. Let γ be an embedded curve in $\mathbf{C}$ of class $\mathcal{C}^{k+1+\alpha}$. Let Ω_1,Ω_2 be disjoint two open subsets of $\mathbf{C}$ such that both $\partial\Omega_1,\partial\Omega_2$ contain γ as a relatively open subset. Assume that $a_\ell\in\mathcal{C}^{k+\alpha}(\Omega_\ell\cup\gamma)$ satisfies $|a_\ell|_{L^\infty}<1$ on $\Omega_\ell\cup\gamma$. Let E be an embedded $\mathcal{C}^1$ curve in $\mathbb{D}$ such that $\mathbb{D}\setminus E$ is open in $\mathbf{C}$ and has exactly two connected components ω_1,ω_2. Assume that u is a continuous map from $\mathbb{D}$ into $\Omega_1\cup\gamma\cup\Omega_2$ such that $u:\omega_\ell\to\Omega_\ell$ are J-holomorphic with respect to $\partial_{\overline{z}}+a_\ell\partial_z$. Then E is a curve of class $\mathcal{C}^{k+1+\alpha}$.*

Proof. The proof is a slight modification of the proof of Theorem 6.2. The problem is local. Fix $z_0 \in E$ and let $p = u(z_0)$. We may assume that near p, γ is contained in the real axis and Ω_1, Ω_2 are contained in the lower and upper half planes. Applying a local change of coordinates φ_ℓ, which is of class $\mathcal{C}^{k+1+\alpha}$ on $\Omega_\ell \cup \gamma$ and fixes γ pointwise, we may assume that $a_\ell = O(|y|^{k+\alpha})$. Define $a := a_\ell$ on $\Omega_\ell \cup \gamma$. Then $X := \partial_{\overline{z}} + a\partial_z$ is of class $\mathcal{C}^{k+\alpha}$ on $\Omega_1 \cup \gamma \cup \Omega_2$. Near $p \in \gamma$, we apply a diffeomorphism ϕ of class $\mathcal{C}^{k+1+\alpha}$ that transforms X into $\partial_{\overline{z}}$. Let $g = \phi \circ \varphi_\ell \circ u$ on $\omega_\ell \cup E$, which is holomorphic away from E. Since g is continuous and E is an embedded $\mathcal{C}^1$ curve, then g is holomorphic at z_0. It is easy to verify that g is biholomorphic near z_0. Consequently, E is of class $\mathcal{C}^{k+1+\alpha}$ near z_0. □

However, the above result fails in higher dimensions.

EXAMPLE 6.8. Let E be an embedded $\mathcal{C}^1$ curve connecting i, $-i$ and dividing $\mathbb{D}$ into two components ω_1, ω_2. Let λ be a $\mathcal{C}^\infty$ function on $\mathbb{D}$ that is positive on ω_1 and negative on ω_2. The existence of such a function is trivial, by taking it vanishing to infinity order along E. We use the standard complex structure on $\mathbb{D} \times \mathbf{C} = \{\operatorname{Im} w < \lambda(z)\} \cup \{\operatorname{Im} w = \lambda(z)\} \cup \{\operatorname{Im} w > \lambda(z)\}$. Let $u(z) := (z, 0)$. Then $u : \omega_\ell \to \Omega_\ell$ are holomorphic, $\gamma = \{\operatorname{Im} w = \lambda(z)\}$ is $\mathcal{C}^\infty$, $u(\mathbb{D}) = \mathbb{D} \times \{0\}$, but E needs not to be $\mathcal{C}^\infty$.

We would like to mention that our main result fails for harmonic functions. For instance, we take a continuous function f on $[-1, 1]$ and then extend continuously to $\partial\mathbb{D}$. Extend f harmonically by solving two Dirichlet problems on $\mathbb{D}^+$ and $\mathbb{D}^-$. Then f is not $\mathcal{C}^\infty$ on $\overline{\mathbb{D}}^+$ in general. One sees a similar result for the Neumann problem. Let Ω be a bounded domain in $\mathbf{C}$ with $\partial\Omega \in \mathcal{C}^\infty$. Suppose that f is continuous on $\partial\Omega$ and dt is the arc-length element on $\partial\Omega$. Then $W_f(z) = \frac{1}{\pi}\int_{\partial\Omega} f(t)\log|\gamma(t) - z|\,dt$ is harmonic on $\mathbf{C} \setminus \partial\Omega$ and continuous on $\mathbf{C}$. However,

$$\partial_{n(s)} W_f = f(s) + \frac{1}{\pi}\int_{\partial\Omega} f(t)\partial_s \arg(\gamma(s) - \gamma(t))\,dt,$$
$$\partial_{-n(s)} W_f = f(s) - \frac{1}{\pi}\int_{\partial\Omega} f(t)\partial_s \arg(\gamma(s) - \gamma(t))\,dt.$$

Here $n(s)$ is the unit outer normal vector of $\partial\Omega$. In particular, if f is not smooth, then W_f cannot be smooth simultaneously on $\overline{\Omega}$ and $\mathbf{C} \setminus \Omega$. It is interesting that if $W_f \in \mathcal{C}^1(\mathbf{C})$, then f and W_f must be zero.

We conclude the paper by mentioning two open problems. Recall that Theorem 6.2 is essential to the proof of Theorem 1.2. Both theorems deal with the case where f is a function. If f is vector-valued, we have the following open problem.

PROBLEM A. *Let $m \geq 2$ be an integer, and let $0 < \alpha < 1$. Let Ω be a bounded domain in $\mathbf{C}$ with $\mathcal{C}^\infty$ boundary. Let $a \in \mathcal{C}^\infty(\overline{\Omega})$ be an $m \times m$ matrix with a sufficiently small $\mathcal{C}^\alpha$ norm on $\overline{\Omega}$. Does $\mathrm{I} + T_\Omega a\partial_z : [\mathcal{C}^{k+\alpha}(\overline{\Omega})]^m \to [\mathcal{C}^{k+\alpha}(\overline{\Omega})]^m$ have a bounded inverse for all positive integers k?*

The following boundary regularity problem arises from the proof of Proposition 3.7 on J-holomorphic curves.

PROBLEM B. *Let Ω be a bounded domain in $\mathbf{C}$ with $\mathcal{C}^\infty$ boundary. Let D be a domain in $\mathbf{C}^n$ with $n \geq 1$. Let A be an $n \times n$ matrix of $\mathcal{C}^\infty$ functions on D. Suppose that the operator norm $\|A(w)\|$ is less than 1 for each $w \in D$. Let $u : \overline{\Omega} \to D$ be a $\mathcal{C}^1$ map such that*

$$u + T_\Omega((\overline{\partial_z u})A(u)) \in \mathcal{C}^\infty(\overline{\Omega}). \tag{6.17}$$

Is $u \in \mathcal{C}^\infty(\overline{\Omega})$?

ACKNOWLEDGMENT. The authors are grateful to the referee for pointing out an additional argument to relax our original condition $|a|_\alpha < \varepsilon$ in Theorems 1.2 and 6.4 to the optimal condition $|a|_0 < 1$.

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