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Lower Bound for the Geometric Type from a Generalized Estimate in the $\bar\partial$-Neumann Problem – a New Approach by Peak Functions

Tran Vu Khanh

1. Introduction

In a series of seminal papers in the Annals of Mathematics [Cat83; Cat87], Catlin proved the equivalence of the finite type of a boundary (cf. [D'A82]) with the existence of a subelliptic estimate for the $\bar\partial$-Neumann problem by triangulating through the t^ε-property (see below)

(i) finite type $m \Rightarrow t^\varepsilon$-property with $\varepsilon = m^{-n^2 m^{n^2}}$;
(ii) t^ε-property $\Rightarrow \varepsilon$-subelliptic estimate;
(iii) ε-subelliptic estimate $\Rightarrow$ finite type m for $m \le \frac{1}{\varepsilon}$.

Here, the t^ε-property of a boundary $b\Omega$ is a special case of a more general "f-property" defined as follows. For a smooth strictly increasing function $f : [1+\infty) \to [1, +\infty)$ with $f(t) \le t^{1/2}$, the f-property at z_o means the existence of a neighborhood U of z_o, of constants C_1, C_2, and of a family of functions $\{\phi_\delta\}$ such that

1) ϕ_δ are plurisubharmonic and C^2 on U, and $-1 \le \phi_\delta \le 0$;
2) $\partial\bar\partial\phi_\delta \ge C_1 f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \le C_2\delta^{-1}$ for any $z \in U \cap \{z \in \Omega\colon\ -\delta < r(z) < 0\}$, where r is a defining function of Ω.

The results in steps (ii) and (iii) were generalized in [KZ10; KZ12]. In particular, in [KZ10] it was shown that the f-property implies an f-estimate for any f, and in [KZ12] that an f-estimate with $\frac{f}{\log} \to \infty$ at ∞ implies that the type along a complex analytic variety has a lower bound with the rate G with

$$G(\delta) = \left(\left(\frac{f}{\log}\right)^*(\delta^{-1})\right)^{-1}, \tag{1.1}$$

where the superscript * denotes the inverse function. Combining the above results, we obtain the following:

Theorem 1.1 (Catlin [Cat83; Cat87]; Khanh and Zampieri [KZ10; KZ12]). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^∞-smooth boundary $b\Omega$, and z_o be a boundary point. Assume that the f-property holds at z_o with $\frac{f}{\log} \nearrow \infty$ as $t \to \infty$.*

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Then, if $b\Omega$ has type $\le F$ along a one-dimensional complex analytic variety Z at z_o, that is,

$$|r(z)| \le F(|z - z_o|), \quad z \in Z, z \to z_o, \tag{1.2}$$

then $F(\delta) \ge \alpha G(\delta)$ for a suitable constant $\alpha > 0$ and for any δ small.

The purpose of this note is to give a short proof of Theorem 1.1, which has also the advantage of requiring only a minimal smoothness of $b\Omega$ if a slightly stronger assumption on f is given. More precisely, we prove the following:

THEOREM 1.2. *Let Ω be a pseudoconvex domain of $\mathbb{C}^n$ with C^2-smooth boundary $b\Omega$, and z_o be a boundary point. Assume that the f-property holds at z_o with f satisfying $(g(t))^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$ for some $t \ge 1$, and set $G(\delta) = (g^*(\delta^{-1}))^{-1}$. Then, if $b\Omega$ has type $\le F$ along a one-dimensional complex analytic variety Z at z_o, then $F(\delta) \ge \alpha G(\beta\delta)$ for suitable constants $\alpha, \beta > 0$ and for any δ small.*

Some remarks are in order. First, the C^∞-smoothness of the boundary in the results of Catlin and Khanh–Zampieri (Theorem 1.1) is required since they applied the regularity of the $\bar\partial$-Neumann problem. In Theorem 1.2, the condition of smoothness is reduced because of the use of a plurisubharmonic peak function. However, in the construction of the family of the plurisubharmonic peak functions, we need a slightly stronger hypothesis on f (e.g., $f(t) = \log t \cdot \log^\varepsilon(\log t)$ with $0 < \varepsilon \le 1$), which fulfills the hypothesis in Theorem 1.1 but does not in Theorem 1.2. Finally, the statements of the two theorems are equivalent in the cases $f(t) = \log^\beta t$ for $\beta > 1$ or $f(t) = t^\varepsilon$ for any $0 < \varepsilon \le \frac{1}{2}$.

2. Proof of Theorem 1.2

The proof of Theorem 1.2 follows immediately from Theorems 2.1 and 2.2. In [Kha13], we showed that there exists a family of plurisubharmonic functions with good estimates.

THEOREM 2.1. *Under the assumptions of Theorem 1.2, for a fixed constant $0 < \eta \le 1$, there are a neighborhood V of z_o and positive constants c_1, c_2, c_3 such that the following holds. For any $w \in V \cap b\Omega$, there is a plurisubharmonic function ψ_w on $V \cap \Omega$ verifying*

(1) $|\psi_w(z) - \psi_w(z')| \le c_1|z - z'|^\eta$,
(2) $\psi_w(z) \le -G^\eta(c_2|z - w|)$, *and*
(3) $\psi_{\pi(z)}(z) \ge -c_3\delta_{b\Omega}(z)^\eta$

for any z and z' in $V \cap \bar\Omega$ (where $\delta_{b\Omega}(z)$ and $\pi(z)$ denote the distance and projection of z to the boundary, respectively).

Using Theorem 2.1 for $w = z_0$, we get the following:

THEOREM 2.2. *Let Ω be a C^2-smoothly pseudoconvex domain in $\mathbb{C}^n$, and z_o be a boundary point. Assume that there are a neighborhood V of z_o and a plurisubharmonic function ψ on $V \cap \Omega$ such that*

$$-c_1|z - z_o|^\eta \le \psi(z) \le -G^\eta(c_2|z - z_o|), \quad z \in V \cap \Omega, \tag{2.1}$$

for suitable $c_1, c_2 > 0$ and $\eta \in (0, 1]$. If $b\Omega$ has type $\le F$ along a one-dimensional complex analytic variety Z, then $F(\delta) \ge \alpha G(\beta\delta)$ for some $\alpha, \beta > 0$ and for any small δ.

Proof. Let Ω be a domain in $\mathbb{C}^n$ and assume that there is a function F and an one-dimensional complex analytic variety Z passing through z_o such that (1.2) is satisfied for $z \in Z$. Then, in any neighborhood U of z_o, there are constants $c_3, c_4 > 0$ and a family $\{Z_\delta\}$ of one-dimensional complex manifolds $Z_\delta \subset U$ defined by $h_\delta : \overline{\Delta} \to U$ with $h_\delta(0) = z_o$ such that

$$\delta = \sup_{t \in \overline{\Delta}} |h_\delta(t) - z_o| \ge |h'_\delta(0)| \ge c_3\delta \tag{2.2}$$

and

$$\sup_{t \in \overline{\Delta}} |\delta_{b\Omega}(h_\delta(t))| < c_4 F(\delta), \tag{2.3}$$

where Δ denotes the unit disc in $\mathbb{C}$.

Let ν be the outward normal vector to $b\Omega$ at z_o. From (2.3) we have $h_\delta(t) - c_4 F(\delta)\nu \in \Omega \cap U$ for any $t \in \overline{\Delta}$. Applying the submean value inequality to the subharmonic function $\psi(h_\delta(t) - c_4 F(\delta)\nu)$ on $\overline{\Delta}$, we get

$$\psi(z_o - c_4 F(\delta)\nu) \le \frac{1}{2\pi} \int_0^{2\pi} \psi(h_\delta(e^{i\theta}) - c_4 F(\delta)\nu)\, d\theta. \tag{2.4}$$

Now, we use the first inequality in (2.1) for the left-hand side term of (2.4):

$$-\psi(z_o - c_4 F(\delta)\nu) \le c_1 c_4^\eta F^\eta(\delta)^\eta.$$

For the right-hand side term of (2.4), we use the second inequality of (2.1):

$$\begin{aligned} -\frac{1}{2\pi} &\int_0^{2\pi} \psi(h_\delta(e^{i\theta}) - c_4 F(\delta)\nu)\, d\theta \\ &\ge \frac{1}{2\pi} \int_0^{2\pi} G^\eta(c_2|h_\delta(e^{i\theta}) - c_4 F(\delta)\nu - z_o|)\, d\theta. \end{aligned} \tag{2.5}$$

Using (2.2) and the Jensen inequality for the increasing, convex function G^η, we get

$$\begin{aligned} G^\eta(c_2 c_3 \delta) &\le G^\eta(c_2|h'_\delta(0)|) \\ &\le G^\eta\left(\frac{1}{2\pi} \int_0^{2\pi} c_2|h_\delta(e^{i\theta}) - CF(\delta)\nu - z_o|\, d\theta\right) \\ &\le \frac{1}{2\pi} \int_0^{2\pi} G^\eta(c_2|h_\delta(e^{i\theta}) - CF(\delta)\nu - z_o|)\, d\theta. \end{aligned} \tag{2.6}$$

Combining (2.4), (2.5), and (2.6), we obtain

$$F(\delta) \geq \alpha G(\beta\delta)$$

with $\alpha = (c_1^{1/\eta} c_4)^{-1}$ and $\beta = c_2 c_3$. The proof of Theorem 1.2 is completed. □

REMARK 2.3. In the case $G(t) = t^m$, the result was obtained by Fornaess and Sibony [FS89].

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Tan Tao University
Tan Tao avenue, Tan Duc e-city
Long An province
Vietnam

khanh.tran@ttu.edu.vn