

Classification of Involutions on Enriques Surfaces

HIROKI ITO & HISANORI OHASHI

ABSTRACT. We present the classification of involutions on Enriques surfaces. We classify them into 18 types with the help of lattice theory due to Nikulin. We also give geometric realizations to all types.

1. Introduction

An Enriques surface Y is a compact complex surface satisfying the following conditions:

- (1) the geometric genus and the irregularity vanish,
- (2) the bi-canonical divisor on Y is linearly equivalent to 0.

Every Enriques surface Y is the quotient of a $K3$ surface X by a fixed point free involution ε . In this work, we give the classification of involutions on Enriques surfaces.

Given an involution ι on Y , we get two lifted involutions g and τ on X , which together with ε form an action by the Klein four-group $K_4 \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Here g is the so-called symplectic or Nikulin involution, namely which acts on the space $H^0(X, \Omega^2)$ trivially. The other two act nonsymplectically. Conversely, if an action by K_4 on X contains a fixed point free involution ε , then the group $K_4/\langle \varepsilon \rangle$ determines an involution on $Y = X/\varepsilon$. Therefore our problem is equivalent to the classification of such K_4 -actions.

By the Torelli theorem [PS], group actions on $K3$ surfaces are determined by the representation on the second cohomology group $H^2(X, \mathbb{Z})$, which has the natural structure of a unimodular lattice given by the cup product. To classify K_4 -actions, we use the theory of classification of involutions of a lattice with condition on a sublattice, due to V. V. Nikulin [Nik4] (see Section 3 for a review and notation).

Let S be a fixed lattice and θ be an involution of S . In [Nik4], the determining condition of a triple (L, ϕ, i) with the condition (S, θ) satisfying the following commutative diagram is given.

$$\begin{array}{ccc}
 L & \xrightarrow{\phi} & L \\
 \uparrow i & & \uparrow i \\
 S & \xrightarrow{\theta} & S
 \end{array}$$

Received February 18, 2013. Revision received July 9, 2013.

Supported in part by the JSPS Grant-in-Aid for Scientific Research (S) 22224001 and for Young Scientists (B) 23740010.

Here L is a unimodular lattice, ϕ is an involution of L , and $i : S \rightarrow L$ is a primitive embedding. To investigate (L, ϕ, i) , we use the following invariants: Let $L_{\pm} = \{x \in L \mid \phi(x) = \pm x\}$ and $S_{\pm} = \{x \in S \mid \theta(x) = \pm x\}$. From the primitive embedding $i : S \rightarrow L$, we get primitive embeddings $i_{\pm} : S_{\pm} \rightarrow L_{\pm}$. Hence we have the orthogonal complements $K_{\pm} = (S_{\pm})_{L_{\pm}}^{\perp}$ and images of the projection

$$H_- = p_{S_-} \left(\frac{(L_+ \oplus S_-)_{L_-}^{\wedge}}{L_+ \oplus S_-} \right) \subset A_{S_-},$$

$$\widetilde{H}_- = p_{S_-} \left(\frac{(K_+ \oplus S_-)_{L_-}^{\wedge}}{K_+ \oplus S_-} \right) \subset H_-,$$

where A_{S_-} is the discriminant group of S_- and M_L^{\wedge} denotes the primitive closure of M inside L (see Section 2).

We apply this theory as $L = H^2(X, \mathbb{Z})$, $S = \{x \in H^2(X, \mathbb{Z}) \mid g^*(x) = -x\}$, and $\phi = \varepsilon^*$. The notation k_- will be introduced in Section 3, (3.2). We also determine the geometric invariant (r, l, δ) , the main invariant of the involution τ , and the topology of the fixed curves in Y^l in Section 5. The following tables describe our main result.

THEOREM 1.1. *Involutions of Enriques surfaces are classified into 18 types as shown in Tables 1–3.*

In Table 1, the blank entry in $q_{S_-}|_{\widetilde{H}_-}$ stands for the same as $q_{S_-}|_{H_-}$.

The Enriques surface of type [1] was constructed by Horikawa [Hor] and studied by Dolgachev [Dol] and Barth–Peters [BP]. Type [2] was found by Kondo [Kon] and constructed generally by Mukai [Muk1]. Type [3] was constructed by Lieberman (cf. [MN]). The Enriques surfaces of type [1]–[3] were studied by Mukai–Namikawa [MN] and Mukai [Muk1] in the context of numerically trivial involutions. Moreover, type [5] was studied by Mukai [Muk2] as numerically reflective involutions.

REMARK 1.2. Geometrically, K_- corresponds generically to the transcendental lattice of X , see the proof of Corollary 4.12. It might be noteworthy to point out that in type [2] and [3], type [10] and [11], type [12] and [13], respectively, we have isomorphic K_- . Thus the corresponding $K3$ surfaces are isomorphic. However, note that this isomorphism does not respect the K_4 -action. It seems to be a difficult question whether there exists a model of X on which the two K_4 -actions are both projectively realizable.

In Section 2 we collect some basic definitions and notation of lattice theory. In Section 3 we introduce Nikulin’s theory [Nik4]. Using this, we classify the lattice structure of involutions in Section 4. We determine the lattices S_{\pm} , K_{\pm} and forms $q_{S_-}|_{H_-}$, $q_{S_-}|_{\widetilde{H}_-}$, k_- there. In Section 5 we give the models and other geometric invariants in Table 3, and thus complete Theorem 1.1.

Table 1 Invariants and the model

No.	$S_+(\frac{1}{2})$	$S_-(\frac{1}{2})$	$q_{S_-} _{H_-}$	$q_{S_-} _{\widetilde{H}_-}$	Horikawa model
[1]	{0}	E_8	u^4		
[2]	{0}	E_8	$u^3 \oplus w$		
[3]	{0}	E_8	$u^3 \oplus z$		
[4]	A_1	E_7	$u^3 \oplus w$		
[5]	A_1	E_7	$u^2 \oplus w^2$		
[6]	A_1^2	D_6	$u^2 \oplus w^2$		
[7]	A_1^2	D_6	$u \oplus w^3$		
[8]	A_1^3	$D_4 \oplus A_1$	$u \oplus w^3$		
[9]	A_1^3	$D_4 \oplus A_1$	w^4		
[10]	D_4	D_4	$v \oplus z^2$		
[11]	D_4	D_4	$v \oplus z^2$	$w \oplus z^2$	
[12]	D_4	D_4	$w \oplus z^2$		
[13]	D_4	D_4	$w \oplus z^2$	z^2	(see Subsection 5.2)
[14]	A_1^4	A_1^4	w^4		
[15]	$D_4 \oplus A_1$	A_1^3	w^3		
[16]	D_6	A_1^2	w^2		

Table 1 Continued

No.	$S_+(\frac{1}{2})$	$S_-(\frac{1}{2})$	$q_{S_-} _{H_-}$	$q_{S_-} _{\widetilde{H}_-}$	Horikawa model
[17]	E_7	A_1	w		
[18]	E_8	$\{0\}$	–		

Table 2 Further invariants

No.	k_-	K_-	K_+
[1]	u	$U \oplus U(2)$	$U(2) \oplus E_8(2)$
[2]	u^2	$U(2) \oplus U(2)$	$U(2) \oplus E_8(2)$
[3]	u^2	$U(2) \oplus U(2)$	$U(2) \oplus E_8(2)$
[4]	$u \oplus \langle \frac{-1}{4} \rangle$	$U \oplus U(2) \oplus A_1(2)$	$U(2) \oplus E_7(2)$
[5]	$u^2 \oplus \langle \frac{-1}{4} \rangle$	$U(2) \oplus U(2) \oplus A_1(2)$	$U(2) \oplus E_7(2)$
[6]	$u \oplus \langle \frac{-1}{4} \rangle^2$	$U \oplus U(2) \oplus A_1(2)^2$	$U(2) \oplus D_6(2)$
[7]	$u^2 \oplus \langle \frac{-1}{4} \rangle^2$	$U(2) \oplus U(2) \oplus A_1(2)^2$	$U(2) \oplus D_6(2)$
[8]	$u \oplus \langle \frac{-1}{4} \rangle^3$	$U \oplus U(2) \oplus A_1(2)^3$	$U(2) \oplus D_4(2) \oplus A_1(2)$
[9]	$u^2 \oplus \langle \frac{-1}{4} \rangle^3$	$U(2) \oplus U(2) \oplus A_1(2)^3$	$U(2) \oplus D_4(2) \oplus A_1(2)$
[10]	$u \oplus v \oplus F_4$	$U \oplus U(2) \oplus D_4(2)$	$U(2) \oplus D_4(2)$
[11]	$u \oplus v \oplus F_4$	$U \oplus U(2) \oplus D_4(2)$	$U(2) \oplus D_4(2)$
[12]	$u^2 \oplus v \oplus F_4$	$U(2) \oplus U(2) \oplus D_4(2)$	$U(2) \oplus D_4(2)$
[13]	$u^2 \oplus v \oplus F_4$	$U(2) \oplus U(2) \oplus D_4(2)$	$U(2) \oplus D_4(2)$
[14]	$u \oplus \langle \frac{1}{4} \rangle^4$	$U \oplus U(2) \oplus A_1(2)^4$	$U(2) \oplus A_1(2)^4$
[15]	$u^2 \oplus \langle \frac{1}{4} \rangle^3$	$U \oplus U(2) \oplus D_4(2) \oplus A_1(2)$	$U(2) \oplus A_1(2)^3$
[16]	$u^3 \oplus \langle \frac{1}{4} \rangle^2$	$U \oplus U(2) \oplus D_6(2)$	$U(2) \oplus A_1(2)^2$
[17]	$u^4 \oplus \langle \frac{1}{4} \rangle$	$U \oplus U(2) \oplus E_7(2)$	$U(2) \oplus A_1(2)$
[18]	u^5	$U \oplus U(2) \oplus E_8(2)$	$U(2)$

The authors wish to express their gratitude to Professor Kondo for suggestions to this problem and many stimulating conversations. We are grateful to Professor Tokunaga for the construction of the curve in Example No. [14].

2. Preliminaries

Our main tool is lattice theory. Here we recall some definitions and notations.

A *lattice* is a pair $(L, (\cdot, \cdot))$, where L is a free \mathbb{Z} -module of finite rank and (\cdot, \cdot) is a nondegenerate integral symmetric bilinear form on L . We abbreviate $(L, (\cdot, \cdot))$ to L . We write $\text{sign } L$ for the signature of L . We denote by $L(m)$ the

Table 3 Geometric invariants

No.	(r, l, δ)	Fixed curves	(K_-)
[1]	(18, 2, 0)	$C^{(1)} + 4\mathbb{P}^1$	$U \oplus U(2)$
[2]	(18, 4, 0)	$4\mathbb{P}^1$	$U(2) \oplus U(2)$
[3]	(18, 4, 0)	$4\mathbb{P}^1$	$U(2) \oplus U(2)$
[4]	(16, 4, 1)	$C^{(1)} + 3\mathbb{P}^1$	$U \oplus U(2) \oplus A_1(2)$
[5]	(16, 6, 1)	$3\mathbb{P}^1$	$U(2) \oplus U(2) \oplus A_1(2)$
[6]	(14, 6, 1)	$C^{(1)} + 2\mathbb{P}^1$	$U \oplus U(2) \oplus A_1(2)^2$
[7]	(14, 8, 1)	$2\mathbb{P}^1$	$U(2) \oplus U(2) \oplus A_1(2)^2$
[8]	(12, 8, 1)	$C^{(1)} + \mathbb{P}^1$	$U \oplus U(2) \oplus A_1(2)^3$
[9]	(12, 10, 1)	\mathbb{P}^1	$U(2) \oplus U(2) \oplus A_1(2)^3$
[10]	(10, 6, 0)	$C^{(2)} + \mathbb{P}^1$	$U \oplus U(2) \oplus D_4(2)$
[11]	(10, 8, 0)	$C_1^{(1)} + C_2^{(1)}$	$U \oplus U(2) \oplus D_4(2)$
[12]	(10, 8, 0)	$C^{(1)}$	$U(2) \oplus U(2) \oplus D_4(2)$
[13]	(10, 10, 0)	\emptyset	$U(2) \oplus U(2) \oplus D_4(2)$
[14]	(10, 10, 1)	$C^{(1)}$	$U \oplus U(2) \oplus A_1(2)^4$
[15]	(8, 8, 1)	$C^{(2)}$	$U \oplus U(2) \oplus D_4(2) \oplus A_1(2)$
[16]	(6, 6, 1)	$C^{(3)}$	$U \oplus U(2) \oplus D_6(2)$
[17]	(4, 4, 1)	$C^{(4)}$	$U \oplus U(2) \oplus E_7(2)$
[18]	(2, 2, 0)	$C^{(5)}$	$U \oplus U(2) \oplus E_8(2)$

lattice $(L, m(\cdot, \cdot))$ for a given lattice $(L, (\cdot, \cdot))$ and $m \in \mathbb{Q}$. L is called *even* if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. For a lattice L , we have an injective homomorphism $\alpha : L \rightarrow L^* = \text{Hom}(L, \mathbb{Z})$ defined by $x \mapsto (x, -)$. L is called *unimodular* if α is bijective. Let U (resp. $\langle n \rangle$) denote the rank 2 (resp. rank 1) lattice given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{resp. } \langle n \rangle).$$

The root lattices A_l, D_m, E_n are considered to be negative definite.

A *finite quadratic form* is a triple (A, b, q) , where A is a finite abelian group, $b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form, and q is a map $q : A \rightarrow \mathbb{Q}/2\mathbb{Z}$ satisfying the following conditions:

- (1) $q(na) = n^2q(a)$ for all $n \in \mathbb{Z}, a \in A$;
- (2) $q(a + a') \equiv q(a) + q(a') + 2b(a, a') \pmod{2\mathbb{Z}}$ for all $a, a' \in A$.

A finite quadratic form is called *nondegenerate* if b is nondegenerate. An element $x \in A$ is called *characteristic* if $b(x, a) \equiv q(a) \pmod{\mathbb{Z}}$ for all $a \in A$. We abbreviate (A, b, q) (resp. $b(a, a'), q(a)$) to (A, q_A) or just q_A (resp. aa', a^2). We denote by w (resp. z) the finite quadratic form on $\mathbb{Z}/2\mathbb{Z}$ whose value is 1 (resp. 0). Note that w and z are degenerate as finite quadratic forms.

The *discriminant (quadratic) form* of an even lattice L is a nondegenerate finite quadratic form (A_L, b_L, q_L) , where $A_L := L^*/L$, $b_L(\bar{x}, \bar{y}) = (x, y) \pmod{\mathbb{Z}}$, and $q_L(\bar{x}) = (x, x) \pmod{2\mathbb{Z}}$. We denote by u (resp. $v, \langle \frac{1}{n} \rangle$) the associated discriminant form of the lattice $U(2)$ (resp. $D_4, \langle n \rangle$). We often use the following discriminant forms:

$$(L, q_L) = \left(A_1(2), \left\langle \frac{-1}{4} \right\rangle \right), \quad (D_4(2), v \oplus F_4),$$

$$\left(D_6(2), u^2 \oplus \left\langle \frac{1}{4} \right\rangle^2 \right), \quad \left(E_7(2), u^3 \oplus \left\langle \frac{1}{4} \right\rangle \right), \quad (E_8(2), u^4),$$

where u^n denotes the direct sum of n copies of u and F_4 denotes

$$\left((\mathbb{Z}/4\mathbb{Z})^2, \left(\begin{array}{cc} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{array} \right) \right).$$

An embedding $i : S \rightarrow L$ of lattices is called *primitive* if $L/i(S)$ is free. Two primitive embeddings $i : S \rightarrow L$ and $i' : S \rightarrow L'$ are called *isomorphic* if there exists $f \in \text{Isom}(L, L')$ such that $f \circ i = i'$. Let S be a sublattice of L . We define the sublattices

$$S^\perp := \{x \in L \mid (x, y) = 0 \ \forall y \in S\},$$

$$S^\wedge := (S \otimes \mathbb{Q}) \cap L$$

of L called the *orthogonal complement* to S and the *primitive closure* of S inside L , respectively. When we emphasize L , we write S_L^\wedge for S^\wedge . Let T be a sublattice of L orthogonal to S . We write

$$\Gamma_{ST} := \frac{(S \oplus T)^\wedge}{S \oplus T}.$$

Let M and N be even lattices, and let $M \rightarrow N$ be an embedding. Then N is called an *overlattice* of M if N/M is a finite abelian group. Let $l(A)$ denote the minimal number of generators of an abelian group A . Note that

$$\text{rank } M \geq l(A_M), \quad l(A_N) \geq l(A_M) - 2l\left(\frac{N}{M}\right) \tag{2.1}$$

for a lattice M and an overlattice N of M .

A lattice M is called *2-elementary* if $A_M = M^*/M$ is a 2-elementary group $(\mathbb{Z}/2\mathbb{Z})^a$.

PROPOSITION 2.1 [Nik2, Theorem 3.6.2]. *The isomorphism class of an even hyperbolic 2-elementary lattice M is determined by the invariants (r, l, δ) , where r is the rank of M , l is the minimal number of generators of A_M , and δ is the parity of q_M , that is,*

$$\delta = \begin{cases} 0 & \text{if } q_M(x) \equiv 0 \pmod{\mathbb{Z}} \text{ for } \forall x \in A_M, \\ 1 & \text{otherwise.} \end{cases}$$

Let L be a lattice and σ be an involution of L . Write

$$L^{(\sigma)} = \{x \in L \mid \sigma(x) = x\},$$

$$L_{(\sigma)} = (L^{(\sigma)})^\perp = \{x \in L \mid \sigma(x) = -x\}.$$

Note that if L is unimodular, then $L^{(\sigma)}$ and $L_{(\sigma)}$ are 2-elementary lattices.

The next proposition is the analogue of Witt’s theorem.

PROPOSITION 2.2 [Nik4, Proposition 1.9.2]. *Let q be a finite quadratic form on a finite 2-elementary group Q whose kernel is zero, that is,*

$$\{x \in Q \mid q(x) \equiv 0 \pmod{2\mathbb{Z}} \text{ and } x \perp Q\} = \{0\},$$

where $x \perp Q$ means $q(x + y) - q(x) - q(y) \equiv 0 \pmod{2\mathbb{Z}}$ for $\forall y \in Q$. Let $f : H_1 \rightarrow H_2$ be an isomorphism of two subgroups of Q that preserves the restrictions $q|_{H_1}$ and $q|_{H_2}$ and that maps the elements of the kernel and the characteristic elements of the bilinear form q into the same sort of elements if they belong to H_1 . Then f extends to an automorphism of q .

3. Involutions of a Lattice with Condition on a Sublattice

In this section we introduce the theory of involutions of a lattice with condition on a sublattice due to Nikulin [Nik4]. See [Nik4, Section 1] for more details.

DEFINITION 3.1 [Nik4, Definition 1.1.1]. By a *condition on an involution* we understand a pair (S, θ) , where S is a nondegenerate lattice and θ is an involution of S .

REMARK 3.2. In [Nik4], a condition on an involution is defined as a triple (S, θ, G) , where S is a (possibly degenerate) lattice, θ is an involution of S , and $G \subset O(S, \theta)$ is a distinguished subgroup of the normalizer of θ in $O(S)$. In this paper, we assume that $G = \{\text{id}_S\}$.

DEFINITION 3.3 [Nik4, Definition 1.1.2]. By a *unimodular involution with the condition* (S, θ) , we understand a triple (L, ϕ, i) , where L is a unimodular lattice, ϕ is an involution of L , and $i : S \rightarrow L$ is a primitive embedding satisfying $\phi \circ i = i \circ \theta$.

Two unimodular involutions (L, ϕ, i) and (L', ϕ', i') with the condition (S, θ) are called *isomorphic* if there exists an isomorphism $f : L \rightarrow L'$ with $\phi' \circ f = f \circ \phi$ and $f \circ i = i'$.

Let $S_\pm = \{x \in S \mid \theta(x) = \pm x\}$. We denote the natural projections $S/(S_+ \oplus S_-) \rightarrow A_{S_\pm}$ by p_{S_\pm} and their images $p_{S_\pm}(S/(S_+ \oplus S_-)) \subset A_{S_\pm}$ by Γ_\pm . Let $\gamma := p_{S_-} \circ p_{S_+}^{-1} : \Gamma_+ \rightarrow \Gamma_-$. We write $\Gamma_\gamma = S/(S_+ \oplus S_-)$ since

$$\frac{S}{S_+ \oplus S_-} = \{(x, y) \in \Gamma_+ \times \Gamma_- \mid y = \gamma(x)\}$$

is the graph of γ .

THEOREM 3.4 [Nik4, Theorem 1.3.1]. *Any unimodular involution with the condition (S, θ) is determined by the 9-tuple (“the list”)*

$$(H_{\pm}, q_r, q, \gamma_r, K_{\pm}, \gamma_{K_{\pm}}), \tag{3.1}$$

where H_{\pm} are subgroups with $\Gamma_{\pm} \subset H_{\pm} \subset (S_{\pm}^* \cap \frac{1}{2}S_{\pm})/S_{\pm}$, q_r is a finite quadratic form on the 2-elementary group $(H_+ \oplus H_-)/\Gamma_{\gamma}$ with $q_r|_{H_{\pm}} = \pm q_{S_{\pm}}|_{H_{\pm}}$, q is the isomorphism class of a nondegenerate 2-elementary finite quadratic form, $\gamma_r : q_r \rightarrow q$ is an embedding of forms, K_{\pm} are even lattices, and $\gamma_{K_{\pm}} : q_{K_{\pm}} \rightarrow k_{\pm}$ are isomorphisms of forms. Here k_{\pm} are defined by

$$k_{\pm} = \frac{((-q_{S_{\pm}} \oplus \pm q)|_{\Gamma_{\gamma_r|H_{\pm}}})}{\Gamma_{\gamma_r|H_{\pm}}}, \tag{3.2}$$

where $\Gamma_{\gamma_r|H_{\pm}}$ are the graphs of the embeddings $H_{\pm} \rightarrow q$ induced by γ_r . Following the original translation of Nikulin’s paper [Nik4], we call this 9-tuple the list hereafter.

Two lists (3.1) and $(H'_{\pm}, q'_r, q', \gamma'_r, K'_{\pm}, \gamma'_{K'_{\pm}})$ determine isomorphic unimodular involutions with the condition (S, θ) if and only if $H_{\pm} = H'_{\pm}$, $q_r = q'_r$, $q = q'$, and there exist isomorphisms $\xi \in O(q)$ and $\psi_{\pm} \in \text{Isom}(K_{\pm}, K'_{\pm})$ such that $\xi \circ \gamma_r = \gamma'_r$ and $(\text{id}, \xi)|_{k_{\pm}} \circ \gamma_{K_{\pm}} = \gamma'_{K'_{\pm}} \circ \overline{\psi_{\pm}}$, where $(\text{id}, \xi)|_{k_{\pm}}$ are isomorphisms between k_{\pm} and k'_{\pm} induced by $\text{id} \in O(q_{S_{\pm}})$ and ξ , and $\overline{\psi_{\pm}}$ are isomorphisms between $q_{K_{\pm}}$ and $q_{K'_{\pm}}$ induced by ψ_{\pm} .

Proof. We divide the proof of this theorem into three parts as follows.

- (1) The construction of the list (3.1) from the unimodular involution with the condition (S, θ) .
- (2) The construction of the unimodular involution with the condition (S, θ) from the list (3.1).
- (3) The equivalence of the lists (3.1) and $(H'_{\pm}, q'_r, q', \gamma'_r, K'_{\pm}, \gamma'_{K'_{\pm}})$.

The construction of the list from the unimodular involution. Let (L, ϕ, i) be a unimodular involution with the condition (S, θ) . We write

$$L_{\pm} = \{x \in L \mid \phi(x) = \pm x\}.$$

Define $q := q_{L_+}$. The primitive embedding $i : S \rightarrow L$ defines primitive embeddings $i_{\pm} : S_{\pm} \rightarrow L_{\pm}$. Hence we define 2-elementary groups

$$H_{\pm} := p_{S_{\pm}}(\Gamma_{L_{\mp}S_{\pm}}) \subset \left(S_{\pm}^* \cap \frac{1}{2}S_{\pm} \right) / S_{\pm}.$$

Note that both projections $p_{S_{\pm}}$ are injective, since the embeddings i_{\pm} are primitive. The group $\Gamma_{L_-S_+}$ (resp. $\Gamma_{L_+S_-}$) is the graph of an injective homomorphism

$$\gamma_{H_+} : H_+ \rightarrow A_{L_-} \quad (\text{resp. } \gamma_{H_-} : H_- \rightarrow A_{L_+}).$$

Note that the notation of $\gamma_{H_{\pm}}$ is slightly different from that of [Nik4]. We define the embedding of forms γ_r and the quadratic form q_r on $(H_+ \oplus H_-)/\Gamma_{\gamma}$ as

$$\gamma_r := (\gamma_{L_+L_-}^{-1} \circ \gamma_{H_+}, \gamma_{H_-}) : \frac{H_+ \oplus H_-}{\Gamma_{\gamma}} \rightarrow q,$$

$$q_r := q \circ \gamma_r,$$

where $\gamma_{L_+L_-}$ is an isomorphism between A_{L_+} and A_{L_-} . The even lattices K_{\pm} are defined by $K_{\pm} := (S_{\pm})_{L_{\pm}}^{\perp}$. The quadratic forms $-k_{\pm}$ in (3.2) are equal to the discriminant forms of $(L_{\mp} \oplus S_{\pm})^{\wedge}$. Hence the sign reversing isometries give $\gamma_{K_{\pm}} : q_{K_{\pm}} \rightarrow k_{\pm}$. Therefore we have the list (3.1).

The construction of the unimodular involution from the list. The invariants $(H_{\pm}, \gamma_r|_{H_{\pm}}, K_{\pm}, \gamma_{K_{\pm}})$, which are part of the list (3.1), determine primitive embeddings $S_{\pm} \rightarrow L_{\pm}$ with orthogonal complements K_{\pm} , respectively, by [Nik2, Proposition 1.15.1]. Here L_{\pm} are the lattices with the discriminant forms $\pm q$, respectively. The diagonal set $\Delta \subset A_{L_+} \oplus A_{L_-}$ is the isotropic subgroup with respect to $q \oplus (-q)$. Hence we have an overlattice L of $L_+ \oplus L_-$. This lattice L is unimodular since Δ is the diagonal set. The embedding $S_+ \oplus S_- \subset L_+ \oplus L_-$ extends to a primitive embedding $i : S \rightarrow L$ since $\gamma_r|_{H_+}$ and $\gamma_r|_{H_-}$ extend to an embedding $\gamma_r : q_r \rightarrow q$. Since L_{\pm} are 2-elementary lattices, there exists an involution $\phi \in O(L)$ such that $\phi|_{L_+} = 1$ and $\phi|_{L_-} = -1$ by [Nik2, Corollary 1.5.2]. We see that $\phi|_S = \theta$. Therefore we have the unimodular involution with the condition (S, θ) .

The equivalence of the lists. This part is omitted in [Nik4]. Let (L, ϕ, i) and (L', ϕ', i') be the unimodular involutions with the condition (S, θ) determined by the lists (3.1) and $(H'_{\pm}, q'_r, q', \gamma'_r, K'_{\pm}, \gamma'_{K'_{\pm}})$, respectively.

Assume that two lists determine isomorphic unimodular involutions. There exists $f \in \text{Isom}(L, L')$ such that $f \circ i = i'$ and $\phi' \circ f = f \circ \phi$. It follows from $\phi' \circ f = f \circ \phi$ that f induces $f_{\pm} := f|_{L_{\pm}} \in \text{Isom}(L_{\pm}, L'_{\pm})$ with $f_{\pm} \circ i_{\pm} = i'_{\pm}$, where $i_{\pm} : S_{\pm} \rightarrow L_{\pm}$ and $i'_{\pm} : S_{\pm} \rightarrow L'_{\pm}$ are primitive embeddings induced by i and i' , respectively. Since f induces $f|_{L_+ \oplus S_-} = (f_+, \text{id}) \in \text{Isom}(L_+, L'_+) \times O(S_-)$, so does an isomorphism between $(L_+ \oplus S_-)^{\wedge}$ and $(L'_+ \oplus S_-)^{\wedge}$. Hence we have

$$H_- = p_{S_-}(\Gamma_{L_+S_-}) = p_{S_-}(\Gamma_{L'_+S_-}) = H'_-$$

and $\overline{f_+} \circ \gamma_r|_{H_-} = \gamma'_r|_{H'_-}$, where $\overline{f_+}$ is an isomorphism between q and q' induced by f_+ .

Similarly, f induces an isomorphism between $(L_- \oplus S_+)^{\wedge}$ and $(L'_- \oplus S_+)^{\wedge}$. Hence we see that $H_+ = H'_+$ and $\overline{f_-} \circ (\gamma_{L_+L_-} \circ \gamma_r|_{H_+}) = \gamma_{L'_+L'_-} \circ \gamma'_r|_{H'_+}$. From $\overline{f_-} \circ \gamma_{L_+L_-} = \gamma_{L'_+L'_-} \circ \overline{f_+}$, we have $\overline{f_+} \circ \gamma_r = \gamma'_r$. Since $(L_+ \oplus S_-)^{\wedge} = (K_-)_{L_-}^{\perp}$ and $(L'_+ \oplus S_-)^{\wedge} = (K'_-)_{L'_-}^{\perp}$, there exists ψ_- with the condition by [Nik2, Corollary 1.5.2]. Similarly, we have ψ_+ with condition. It is clear that $q = q'$ and $q_r = q'_r$.

We show the contrary. Assume that $H_{\pm} = H'_{\pm}$, $q_r = q'_r$, $q = q'$ and there exist $\xi = \xi_{\pm} \in O(q)$ and $\psi_{\pm} \in \text{Isom}(K_{\pm}, K'_{\pm})$ with conditions. Recall that invariants $(H_{\pm}, \gamma_r|_{H_{\pm}}, K_{\pm}, \gamma_{K_{\pm}})$ determine primitive embeddings $i_{\pm} : S_{\pm} \rightarrow L_{\pm}$ with orthogonal complements K_{\pm} , where L_{\pm} are the lattices with discriminant forms $\pm q$, respectively. Let T_1 (resp. T_2) be any lattice which is unique in its genus; and furthermore, $O(T_1) \rightarrow O(q_{T_1})$ (resp. $O(T_2) \rightarrow O(q_{T_2})$) is surjective and $q_{T_1} = q$ (resp. $q_{T_2} = -q$). Note that the existence of such lattices T_1 and T_2 follows from

[Nik2, Theorem 1.14.2] by adding some unimodular lattices if necessary. From $q = q'$ and $K_- \simeq K'_-$ (resp. $K_+ \simeq K'_+$), we see that L_- and L'_- (resp. L_+ and L'_+) are obtained as orthogonal complements of a primitive embedding $T_1 \rightarrow L_1$ (resp. $T_2 \rightarrow L_2$), where L_1 (resp. L_2) is a unimodular lattice with

$$\begin{aligned} \text{sign } L_1 &= \text{sign } L_- + \text{sign } T_1 = \text{sign } L'_- + \text{sign } T_1 \\ (\text{resp. } \text{sign } L_2 &= \text{sign } L_+ + \text{sign } T_2 = \text{sign } L'_+ + \text{sign } T_2). \end{aligned}$$

Moreover, T_1 is obtained as an orthogonal complement of a primitive embedding $T_2 \rightarrow L_3$, where L_3 is a unimodular lattice with

$$\text{sign } L_3 = \text{sign } T_1 + \text{sign } T_2.$$

Hence there exists $\xi_- \in \text{O}(-q)$ such that $\xi_- \circ \gamma_{T_1 T_2} = \gamma_{T_1 T_2} \circ \xi_+$.

Since $\text{O}(T_1) \rightarrow \text{O}(q_{T_1}) = \text{O}(q)$ is surjective, there exists $f_1 \in \text{O}(T_1)$ such that $\overline{f_1} = \xi_+$. By $\xi_+ \circ \gamma_r|_{H_-} = \gamma'_r|_{H'_-}$ and $H_- = H'_-$, it follows that $(f_1, \text{id}) \in \text{O}(T_1) \times \text{O}(S_-)$ extends to an isomorphism

$$\alpha_1 : (T_1 \oplus S_-)^\wedge \rightarrow (T_1 \oplus S_-)^\wedge.$$

Note that the former $(T_1 \oplus S_-)^\wedge$ is equal to $(K_-)^\perp_{L_1}$, and the latter is equal to $(K'_-)^\perp_{L_1}$. From the condition of ψ_- , it follows that (α_1, ψ_-) extends to an automorphism

$$\beta_1 : L_1 \rightarrow L_1.$$

Similarly, there exists an automorphism $\beta_2 : L_2 \rightarrow L_2$ such that $\overline{\beta_2|_{T_2}} \in \text{O}(-q)$, $\beta_2|_{S_+} = \text{id}$ and $\beta_2|_{K_+} = \psi_+$. Therefore we have the following commutative diagram:

$$\begin{array}{ccccccc} A_{L_-} & \longrightarrow & A_{T_1} & \longrightarrow & A_{T_2} & \longrightarrow & A_{L_+} \\ \overline{\beta_1|_{L_-}} \downarrow & & \downarrow \xi_+ & & \xi_- \downarrow & & \downarrow \overline{\beta_2|_{L_+}} \\ A_{L'_-} & \longrightarrow & A_{T_1} & \longrightarrow & A_{T_2} & \longrightarrow & A_{L'_+} \end{array}$$

Hence $(\beta_2|_{L_+}, \beta_1|_{L_-})$ extends to an isomorphism $\beta : L \rightarrow L'$ with $\beta \circ i = i'$ and $\beta \circ \phi = \phi' \circ \beta$, which is the desired isomorphism. □

REMARK 3.5. In the proof of Theorem 3.4, we see that

$$\overline{\beta_2|_{L_+}} = (\overline{\psi_+}, \overline{\text{id}})|\Gamma_{K_+ S_+}^\perp / \Gamma_{K_+ S_+}, \quad \overline{\beta_1|_{L_-}} = (\overline{\psi_-}, \overline{\text{id}})|\Gamma_{K_- S_-}^\perp / \Gamma_{K_- S_-}.$$

Moreover, if both lattices L_\pm are indefinite, then we can take T_1 and T_2 as L_+ and L_- , respectively. Hence we see that

$$\xi_+ = (\overline{\psi_+}, \overline{\text{id}})|\Gamma_{K_+ S_+}^\perp / \Gamma_{K_+ S_+}, \quad \xi_- = (\overline{\psi_-}, \overline{\text{id}})|\Gamma_{K_- S_-}^\perp / \Gamma_{K_- S_-}. \quad (3.3)$$

4. Classification

4.1. Involutions on Enriques Surfaces

Let Y be an Enriques surface and X be its covering $K3$ surface with the covering involution ε . Consider an involution ι of Y . Then ι lifts to two involutions of X . One of them acts on $H^0(X, \Omega^2)$ trivially, which we denote by g . Then another involution is $\tau = g \circ \varepsilon = \varepsilon \circ g$.

The second cohomology group $H^2(X, \mathbb{Z})$ is an even unimodular lattice with the signature $(3, 19)$. Let $S = \{x \in H^2(X, \mathbb{Z}) \mid g^*(x) = -x\}$, where g^* is the involution of $H^2(X, \mathbb{Z})$ induced by g . It is known that S is isomorphic to $E_8(2)$ and this does not depend on g [Mor; Nik1].

LEMMA 4.1. *Let L be a unimodular lattice and S be a 2-elementary lattice. The following are equivalent.*

- (1) *There exists an involution α of L such that $L_{(\alpha)} \simeq S$.*
- (2) *There exists a primitive embedding $S \rightarrow L$.*

Proof. Assume (1). Since $S = (L^{(\alpha)})^\perp$, it follows that the sublattice S is primitive in L .

Assume (2). Let $K = S^\perp$. Since S and K are 2-elementary lattices, there exists an involution $\alpha \in O(L)$ such that $\alpha|_K = 1$ and $\alpha|_S = -1$ by [Nik2, Corollary 1.5.2]. Since S is primitive in L , it follows that $S = L_{(\alpha)}$. □

To classify ι , it suffices to classify the pair of involutions (g^*, ε^*) on the unimodular lattice $H^2(X, \mathbb{Z})$. By Lemma 4.1, giving the pair (g^*, ε^*) is equivalent to giving the involution ε^* on $H^2(X, \mathbb{Z})$ with the condition (S, θ) for some θ . Therefore we will first classify the condition θ on S in Lemma 4.2 and then classify ε^* by using Theorem 3.4. The next subsection is devoted to the realization of these statements.

4.2. Classification of Lattice Structure

We classify the unimodular involutions (L, ϕ, i) with condition by applying Theorem 3.4. Here we identify $L = H^2(X, \mathbb{Z})$, $\phi = \varepsilon^*$, $S = \{x \in L \mid g^*(x) = -x\} \simeq E_8(2)$, and $i : S \rightarrow H^2(X, \mathbb{Z})$. In this case, it is known that

$$L_+ \simeq U(2) \oplus E_8(2), \quad L_- \simeq U \oplus U(2) \oplus E_8(2)$$

and these do not depend on ε (cf. [BP]).

First we classify the action of ε^* on S in Lemma 4.2. This shows all possibilities of the involution θ of the condition (S, θ) . Ultimately, for each condition (S, θ) , we calculate and classify the list (3.1), that is,

$$(H_\pm, q_r, q, \gamma_r, K_\pm, \gamma_{K_\pm}).$$

The invariant q is $q = q_{L_+} = u^5$ in our case. In Lemmas 4.5 and 4.6, we narrow the possibilities of H_\pm down. To investigate γ_r , we prepare \widetilde{H}_\pm in Definition 4.7. Finally, we classify the list (3.1) in Theorem 4.11.

LEMMA 4.2. *Suppose that $S = E_8(2)$ and θ is an involution of S . Then the isomorphism class of (S_+, S_-) is determined by one of the following:*

$$\left(S_+ \left(\frac{1}{2} \right), S_- \left(\frac{1}{2} \right) \right) = (E_8, \{0\}), \quad (E_7, A_1), \quad (D_6, A_1^2), \quad (D_4 \oplus A_1, A_1^3), \\ (D_4, D_4), \quad (A_1^4, A_1^4), \quad (A_1^3, D_4 \oplus A_1), \quad (A_1^2, D_6), \quad (A_1, E_7), \quad (\{0\}, E_8).$$

Proof. It suffices to prove the lemma for $S(\frac{1}{2}) = E_8$. Since θ is an involution, it follows that S_{\pm} are even 2-elementary lattices. By symmetry, we can assume that the rank of S_+ is at most 4. By [Nik2, Theorem 3.6.2], invariants (r, l, δ) of S_+ are one of the following:

$$(0, 0, 0), \quad (1, 1, 1), \quad (2, 2, 1), \quad (3, 3, 1), \quad (4, 4, 1), \quad (4, 2, 0).$$

We see that $\{0\}$, A_1 , A_1^2 , A_1^3 , A_1^4 , and D_4 have the above invariants, respectively, and these lattices have exactly one class in their genus (cf. [Nik2, Remark 1.14.6]). Hence S_+ is one of them. It follows that S_- is obtained as an orthogonal complement to S_+ in S . Interchanging S_+ and S_- , we obtain the claimed list. \square

LEMMA 4.3. *Suppose that S_+ is one of the lattices in Lemma 4.2. Then there exists a unique primitive embedding $S_+ \rightarrow L_+$.*

Proof. Since $S_+(\frac{1}{2})$ is an even negative definite lattice of rank at most 8 and $L_+(\frac{1}{2}) \simeq U \oplus E_8$ is a unimodular lattice of signature $(1, 9)$, the lemma follows from [Nik2, Theorem 1.14.4]. \square

COROLLARY 4.4. *We have $K_+ \simeq U(2) \oplus S_-$ in all cases. In particular, $\Gamma_{S_+S_-} \simeq \Gamma_{K_+S_+}$.*

Proof. Recall that $L_+ \simeq U(2) \oplus E_8(2)$ and $S \simeq E_8(2)$. By Lemma 4.3, a primitive embedding $S_+ \rightarrow L_+$ is unique. Hence $K_+ = (S_+)_{L_+}^{\perp}$ is uniquely determined as $U(2) \oplus S_-$. Therefore we see that

$$\Gamma_{K_+S_+} = \frac{L_+}{K_+ \oplus S_+} \\ \simeq \frac{U(2) \oplus S}{U(2) \oplus S_- \oplus S_+} \simeq \frac{S}{S_+ \oplus S_-} = \Gamma_{S_+S_-}. \quad \square$$

LEMMA 4.5. *Suppose that (S_+, S_-) is one of the pairs in Lemma 4.2. We have the following about H_{\pm} :*

- (1) $\Gamma_+ \subset H_+ = \frac{1}{2}S_+/S_+$, $\Gamma_- \subset H_- \subset \frac{1}{2}S_-/S_-$.
- (2) $q_{S_{\pm}}|_{H_{\pm}} \equiv 0 \pmod{\mathbb{Z}}$.
- (3) $\text{rank } S_- - 1 \leq \text{rank } H_- \leq \text{rank } S_-$.
- (4) $q_{S_+}|_{\Gamma_+}$ (resp. $q_{S_-}|_{\Gamma_-}$) is a direct summand of $q_{S_+}|_{H_+}$ (resp. $q_{S_-}|_{H_-}$).

Proof. Since $S_{\pm}(\frac{1}{2})$ are even lattices, we have $H_{\pm} \subset \frac{1}{2}S_{\pm}/S_{\pm}$. Let $x \in \frac{1}{2}S_+$. From $L_+(\frac{1}{2}) \simeq U \oplus E_8$, we have $x \in L_+^*$. Since L is unimodular, there exists

$y \in L_-^*$ such that $x + y \in L$, which implies $x + y \in (S_+ \oplus L_-)^\wedge$. Therefore $H_+ = \frac{1}{2}S_+/S_+$.

Since $\gamma_r : q_r \rightarrow q$ is an embedding and $q = u^5 \equiv 0 \pmod{\mathbb{Z}}$, q_r also satisfies $q_r \equiv 0 \pmod{\mathbb{Z}}$. Hence $q_{S_\pm}|_{H_\pm} = q_r|_{H_\pm} \equiv 0 \pmod{\mathbb{Z}}$.

By $K_- = (S_-)_{L_-}^\perp$, we see that $\text{rank } K_- = \text{rank } L_- - \text{rank } S_- = 12 - \text{rank } S_-$. From (2.1), we see that

$$l(A_{K_-}) = l(A_{(L_+ \oplus S_-)^\wedge}) \geq l(A_{L_+ \oplus S_-}) - 2l(\Gamma_{L_+ S_-}) = 10 + l(A_{S_-}) - 2l(\Gamma_{L_+ S_-}).$$

Obviously, $l(A_{S_-}) = \text{rank } S_-$. The primitivity of L_+ in $(L_+ \oplus S_-)^\wedge$ gives $l(\Gamma_{L_+ S_-}) = l(H_-) = \text{rank } H_-$. Therefore $\text{rank } K_- \geq l(A_{K_-})$ yields

$$12 - \text{rank } S_- \geq 10 + \text{rank } S_- - 2 \text{rank } H_-.$$

Hence we have (3).

Recall that (S_+, S_-) is one of the pairs in Lemma 4.2. We can write $A_{S_+} = (\mathbb{Z}/2\mathbb{Z})^a \oplus (\mathbb{Z}/4\mathbb{Z})^b$ and $q_{S_+} = q_2 \oplus q_4$, where q_2 (resp. q_4) is a finite quadratic form on $(\mathbb{Z}/2\mathbb{Z})^a$ (resp. $(\mathbb{Z}/4\mathbb{Z})^b$). Since $\Gamma_+ = 2A_{S_+} = \{2x \mid x \in A_{S_+}\}$, we have $q_{S_+}|_{\Gamma_+} = 2q_4$, where $2q_4$ denotes the finite quadratic form whose generators are twice the size of those of q_4 . Since $q_{S_+}|_{(1/2)S_+/S_+} = q_2 \oplus 2q_4$, we see that $q_{S_+}|_{\Gamma_+}$ is a direct summand of $q_{S_+}|_{(1/2)S_+/S_+}$. Hence $q_{S_+}|_{\Gamma_+}$ is also a direct summand of $q_{S_+}|_{H_+}$. The same proof works for $q_{S_-}|_{\Gamma_-}$. \square

LEMMA 4.6. (1) In the cases $S_-(\frac{1}{2}) = E_8, E_7, D_6, D_4 \oplus A_1$, we have $\Gamma_+ = H_+ = \frac{1}{2}S_+/S_+$.

(2) In the cases $S_-(\frac{1}{2}) = A_1^4, A_1^3, A_1^2, A_1, \{0\}$, we have $\Gamma_- = H_- = \frac{1}{2}S_-/S_-$.

Proof. We give the proof only for the case $S_-(\frac{1}{2}) = E_7$; the other cases are left to the reader. In the case $S_-(\frac{1}{2}) = E_7$, we have $S_+(\frac{1}{2}) = A_1$. Hence we see that

$$\begin{aligned} \Gamma_+ &= p_{S_+} \left(\frac{S}{S_+ \oplus S_-} \right) = p_{S_+} \left(\frac{E_8(2)}{A_1(2) \oplus E_7(2)} \right) \\ &\simeq p_{S_+} \left(\frac{E_8}{A_1 \oplus E_7} \right) = A_{A_1} \simeq \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

At the same time, we see that

$$\frac{1}{2}S_+/S_+ = \frac{1}{2}A_1(2)/A_1(2) \simeq \mathbb{Z}/2\mathbb{Z}.$$

The lemma follows from Lemma 4.5 (1). \square

We consider the behavior of $\gamma_{H_\pm} : H_\pm \rightarrow A_{L_\mp}$. Note that

$$\Gamma_{K_\pm S_\pm}^\perp \cap A_{K_\pm} = \frac{\Gamma_{K_\pm S_\pm}^\perp \cap A_{K_\pm}}{\Gamma_{K_\pm S_\pm} \cap A_{K_\pm}} \subset \frac{\Gamma_{K_\pm S_\pm}^\perp}{\Gamma_{K_\pm S_\pm}} \simeq A_{L_\pm}.$$

DEFINITION 4.7. Let $\widetilde{A}_{K_+} := \Gamma_{K_+S_+}^\perp \cap A_{K_+} \subset A_{L_+}$ and $\widetilde{A}_{K_-} := \Gamma_{K_-S_-}^\perp \cap A_{K_-} \subset A_{L_-}$. The subgroups \widetilde{H}_- of H_- and \widetilde{H}_+ of H_+ are defined by

$$\widetilde{H}_- := \gamma_{H_-}^{-1}(\widetilde{A}_{K_+}), \quad \widetilde{H}_+ := \gamma_{H_+}^{-1}(\widetilde{A}_{K_-}).$$

We see that $(\widetilde{H}_-, \gamma_{H_-}|_{\widetilde{H}_-})$ and $(\widetilde{H}_+, \gamma_{H_+}|_{\widetilde{H}_+})$ determine $(K_+ \oplus S_-)^\wedge$ and $(S_+ \oplus K_-)^\wedge$, respectively, since (H_\mp, γ_{H_\mp}) determines $(L_\pm \oplus S_\mp)^\wedge$. It follows from Corollary 4.4 that $\Gamma_+ = p_{S_+}(\Gamma_{K_+S_+})$. Therefore we have

$$\Gamma_- \subset \widetilde{H}_- \subset H_- \tag{4.1}$$

From Theorem 3.4, if two unimodular involutions with the condition (S, θ) determined by the lists (3.1) and $(H'_\pm, q'_r, q', \gamma'_r, K'_\pm, \gamma'_{K'_\pm})$, respectively, are isomorphic, then there exist $\xi_\pm \in O(\pm q)$ and $\psi_\pm \in \text{Isom}(K_\pm, K'_\pm)$ with conditions. As stated in Remark 3.5, we have (3.3). It follows that

$$\widetilde{H}_- = \widetilde{H}'_- \quad (\text{resp. } \widetilde{H}_+ = \widetilde{H}'_+)$$

since $(\overline{\psi}_+, \overline{\text{id}})|_{\Gamma_{K_+S_+}^\perp} / \Gamma_{K_+S_+}$ (resp. $(\overline{\psi}_-, \overline{\text{id}})|_{\Gamma_{K_-S_-}^\perp} / \Gamma_{K_-S_-}$) induces an isomorphism between A_{K_+} and $A_{K'_+}$ (resp. A_{K_-} and $A_{K'_-}$). Hence we define the following equivalence relation:

$$\begin{aligned} \gamma_{H_\mp} &\sim \gamma_{H'_\mp} \\ \iff &\text{there exists } \xi_\pm \in O(\pm q) \text{ such that } \xi_\pm \circ \gamma_{H_\mp} = \gamma_{H'_\mp} \text{ and } \widetilde{H}_\mp = \widetilde{H}'_\mp. \end{aligned}$$

The existence condition of ξ_\pm follows from Proposition 2.2. Thus we have a one-to-one correspondence between $\{\gamma_{H_\mp}\}/\sim$ and $\{\widetilde{H}_\mp\}$.

LEMMA 4.8. *We have an equality*

$$\frac{|H_-|}{|\widetilde{H}_-|} = \frac{|H_+|}{|\widetilde{H}_+|}.$$

Proof. It is easy to check that

$$\frac{|\Gamma_{L_+S_-}|}{|\Gamma_{K_+S_-}|} = \frac{|\Gamma_{S_+L_-}|}{|\Gamma_{S_+K_-}|}.$$

Since L_\pm and K_\pm are primitive in L ,

$$p_{S_-} : \Gamma_{L_+S_-} \rightarrow H_-, \quad p_{S_-} : \Gamma_{K_+S_-} \rightarrow \widetilde{H}_-$$

and

$$p_{S_+} : \Gamma_{S_+L_-} \rightarrow H_+, \quad p_{S_+} : \Gamma_{S_+K_-} \rightarrow \widetilde{H}_+$$

are all bijective, which proves the lemma. □

LEMMA 4.9. *Let $\lambda \in K_+^*$, $\mu \in S_+^*$, $\nu \in S_-^*$. If $\lambda + \mu + \nu \in L$, then $\lambda \in \frac{1}{2}K_+$, $\mu \in \frac{1}{2}S_+$, $\nu \in \frac{1}{2}S_-$.*

Proof. Let T be the primitive sublattice of L spanned by $K_+ \oplus S_-$, that is, $T = (K_+ \oplus S_-)^\wedge$. Since T is also the fixed part of the action of the involution $(g \circ \varepsilon)^*$ on L , it follows that $L/(T \oplus T^\perp)$ is a 2-elementary group. Hence we have $2(\lambda + \nu) + 2\mu \in T \oplus T^\perp$, in particular $2\mu \in T^\perp \subset L$. Since S_+ is the primitive sublattice of L , we see that $2\mu \in S_+^* \cap L \subset (S_+)_L^\wedge = S_+$. We thus get $\mu \in \frac{1}{2}S_+$. The rest of the proof is left to the reader. \square

From this lemma, we see that

$$\gamma_{H_-}(H_-) \subset \left(\frac{1}{2}K_+/K_+ \oplus \frac{1}{2}S_+/S_+ \right) / \Gamma_{K_+S_+}. \tag{4.2}$$

LEMMA 4.10. *We have $\widetilde{H}_\pm = H_\pm$ unless $S_\pm = D_4(2)$.*

Proof. In the cases $S_-(\frac{1}{2}) = A_1^4, A_1^3, A_1^2, A_1, \{0\}$, it follows from (4.1) and Lemma 4.6 that $\widetilde{H}_- = H_-$. In the cases $S_-(\frac{1}{2}) = E_8, E_7, D_6, D_4 \oplus A_1$, we have $\gamma_{H_-}(\Gamma_-) \equiv \frac{1}{2}S_+/S_+ \pmod{\Gamma_{K_+S_+}}$ by Lemma 4.6. From (4.2), we see that $\widetilde{H}_- = H_-$. It follows from Lemma 4.8 that $\widetilde{H}_+ = H_+$. \square

THEOREM 4.11. *The unimodular involutions $(L, \phi, i: S \subset L)$ such that*

- (1) L is the unimodular lattice with the signature $(3, 19)$,
- (2) ϕ is an involution with $L_+ \simeq U(2) \oplus E_8(2)$, $L_- \simeq U \oplus U(2) \oplus E_8(2)$,
- (3) S is isomorphic to $E_8(2)$

are classified into 18 types by the list (3.1) as in Table 1 and Table 2 in Theorem 1.1.

Proof. By Lemmas 4.5 and 4.10, we calculate $(H_-, K_\pm, \gamma_{K_\pm})$ for each (S_+, S_-) except the case $S_\pm = D_4(2)$. In the case $S_\pm = D_4(2)$, we have to calculate $(H_-, \widetilde{H}_-, K_\pm, \gamma_{K_\pm})$. We first calculate H_- .

In the case $S_- = E_8(2)$, we see that $q_{S_-}|_{(1/2)S_-/S_-} = u^4$. By Lemma 4.5 (3), $\text{rank } H_- = 8$ or 7 . For $\text{rank } H_- = 8$, we have $H_- = \frac{1}{2}S_-/S_-$. For $\text{rank } H_- = 7$, we have $q_{S_-}|_{H_-} = u^3 \oplus w$ or $u^3 \oplus z$ by Lemma 4.5 (2).

In the case $S_- = E_7(2)$, we see that $q_{S_-}|_{(1/2)S_-/S_-} = u^3 \oplus w$ and $q_{S_\pm}|_{\Gamma_\pm} = w$. By Lemma 4.5 (3), $\text{rank } H_- = 7$ or 6 . For $\text{rank } H_- = 7$, we have $H_- = \frac{1}{2}S_-/S_-$. For $\text{rank } H_- = 6$, we have $q_{S_-}|_{H_-} = u^2 \oplus w^2$ by Lemma 4.5 (2) and (4) (note that we have $w \oplus z = w^2$). The same proof works for the cases $S_-(\frac{1}{2}) = D_6, D_4 \oplus A_1$. So we omit it.

In the cases $S_-(\frac{1}{2}) = A_1^4, A_1^3, A_1^2, A_1, \{0\}$, we see that $q_{S_-}|_{H_-} = q_{S_-}|_{(1/2)S_-/S_-}$ by Lemma 4.6.

We next deal with the case $S_\pm = D_4(2)$. We see that $q_{S_-}|_{(1/2)S_-/S_-} = v \oplus z^2$ and $q_{S_\pm}|_{\Gamma_\pm} = z^2$. By Lemma 4.5 (3), $\text{rank } H_- = 4$ or 3 . For $\text{rank } H_- = 3$, we have $q_{S_-}|_{H_-} = w \oplus z^2$ by Lemma 4.5 (2) and (4). From (4.1), we have $q_{S_-}|_{\widetilde{H}_-} = w \oplus z^2$ or z^2 . For $\text{rank } H_- = 4$, we have $H_- = \frac{1}{2}S_-/S_-$. From (4.1), a candidate for $q_{S_-}|_{\widetilde{H}_-}$ is one of $v \oplus z^2, w \oplus z^2$, and z^2 . Here we claim that $q_{S_-}|_{\widetilde{H}_-} = z^2$ is impossible.

Suppose that $q_{S_-} |_{\widetilde{H}_-} = z^2$. This yields $\widetilde{H}_- = \Gamma_-$. From Corollary 4.4, we have $K_+ = U(2) \oplus D_4(2)$. Note that $q_{K_+} |_{(1/2)K_+/K_+} = u \oplus v \oplus z^2$. Recall (4.2). Let

$$\begin{aligned} H_- &= G_{H_-} \oplus \widetilde{H}_-, \\ \frac{1}{2}K_+/K_+ &= G_{K_+} \oplus p_{K_+}(\Gamma_{K_+S_+}), \\ \frac{1}{2}S_+/S_+ &= G_{S_+} \oplus p_{S_+}(\Gamma_{K_+S_+}), \end{aligned}$$

where G_{H_-} (resp. G_{K_+} , G_{S_+}) is a subgroup of H_- (resp. $\frac{1}{2}K_+/K_+$, $\frac{1}{2}S_+/S_+$) whose quadratic form is v (resp. $u \oplus v$, v). It follows from $\widetilde{H}_- = \Gamma_-$ and Corollary 4.4 that

$$\Gamma_{K_+S_-} \cong \Gamma_{S_+S_-} \cong \Gamma_{K_+S_+}.$$

Hence we have

$$\gamma_{H_-}(\widetilde{H}_-) = p_{K_+}(\Gamma_{K_+S_-}) = p_{K_+}(\Gamma_{K_+S_+}) \equiv p_{S_+}(\Gamma_{K_+S_+}) \pmod{\Gamma_{K_+S_+}}.$$

It follows from (4.2) that

$$\gamma_{H_-}(G_{H_-}) \subset G_{K_+} \oplus G_{S_+}.$$

Let $0 \neq x \in G_{H_-}$. We see that

$$\gamma_{H_-}(x) \notin G_{K_+}, \quad \gamma_{H_-}(x) \notin G_{S_+},$$

since we have $\widetilde{H}_- = \Gamma_- \subsetneq H_-$. Hence a nonzero element of $\gamma_{H_-}(G_{H_-})$ is a sum of nonzero elements of G_{K_+} and G_{S_+} . This contradicts the fact that the quadratic form of G_{H_-} is v . Now we have Table 1.

We proceed to calculate K_{\pm} . From Corollary 4.4, we see that $K_+ = U(2) \oplus S_-$. By calculating (3.2) we have k_- . From [Nik2, Theorem 1.14.2 and Corollary 1.9.4], K_- is uniquely determined with the signature $(2, 10 - \text{rank } S_-)$ and the discriminant form k_- . Since these K_{\pm} are unique in their genus and $O(K_{\pm}) \rightarrow O(q_{K_{\pm}})$ are surjective, $\gamma_{K_{\pm}}$ are uniquely determined for each K_{\pm} . Therefore we have Table 2. \square

COROLLARY 4.12. *There are 18 types of involutions on Enriques surfaces.*

Proof. We saw that there are at most 18 lattice types by Theorem 4.11. Here we show the converse, namely the existence of geometric involutions on some Enriques surface for each type.

Let us pick a type from Tables 1 and 2, which determines a unimodular involution $(L, \phi, i: S \subset L)$. Using i , we regard S as a sublattice of L . By Lemma 4.1, we get another involution α on L with $L_{\langle \alpha \rangle} = S$. Note that ϕ and α generate a group isomorphic to the Klein group K_4 .

Recall that K_- has the signature $(2, 10 - \text{rank } S_-)$. By an easy computation, we can find an element $\omega \in K_{-, \mathbb{C}}$ (\mathbb{C} denotes the scalar extension) with $\omega^2 = 0$, $\omega\bar{\omega} > 0$ such that for any $l \in K_- - \{0\}$ we have $\omega l \neq 0$. Then by the surjectivity of the period map, there exists a marked K3 surface X with $\gamma: H^2(X, \mathbb{Z}) \simeq L$ and

$H^{2,0}(X) \simeq \mathbb{C}\omega$ by the identification γ . By the construction, $NS(X)$ is mapped bijectively to the primitive closure of $K_+ \oplus S_+ \oplus S_-$ inside L .

Next we look at the ample cone $\mathcal{A}_X \subset NS(X)_{\mathbb{R}}$. It is one of the fundamental domains of the reflection group $W \subset O(NS(X))$ generated by (-2) -elements, acting on the positive cone (one component of $\{x \in NS(X)_{\mathbb{R}} \mid x^2 > 0\}$). Since the orthogonal complement of K_+ inside $\gamma(NS(X))$ is given by $(S_+ \oplus S_-)^{\wedge} = S \simeq E_8(2)$, which contains no (-2) -element, we can find an element $w \in W$ such that $w(\mathcal{A}_X) \cap \gamma^{-1}(K_{+, \mathbb{R}}) \neq \emptyset$. Then the pullback of the K_4 -action on L to $H^2(X, \mathbb{Z})$ by γw preserves the intersection numbers, the Hodge structure and some ample element; hence by the Torelli theorem [PS] we get a K_4 -action on the surface X . By the condition $L_+ \simeq U(2) \oplus E_8(2)$, the pullback of ϕ corresponds to a fixed point free involution ε , and we get the Enriques surface $Y = X/\varepsilon$ with an action by $K_4/\langle \varepsilon \rangle$, which determines an involution on Y as stated.

In the next section, we give projective realizations to these involutions. □

5. Examples

Our Theorem 4.11 and Corollary 4.12 achieve the classification of involutions on Enriques surfaces in the abstract sense. In this section we construct examples of involutions on Enriques surfaces. Additionally, we give the other invariants and complete Theorem 1.1.

We denote by ι an involution on an Enriques surface Y . The $K3$ -cover is denoted by X with the covering transformation ε . The symplectic lift of ι to X is denoted by g and the other nonsymplectic one is $\tau = g \circ \varepsilon = \varepsilon \circ g$.

We first note that the fixed locus of τ ,

$$X^\tau = \{x \in X \mid \tau(x) = x\},$$

can be computed from Theorem 4.11 via the following theorem.

THEOREM 5.1 [Nik3, Theorem 4.2.2]. *Let τ be a nonsymplectic involution of X , and let $T = H^2(X, \mathbb{Z})^{(\tau^*)}$. Since T is 2-elementary, the lattice T is determined by invariants (r, l, δ) by Proposition 2.1. Then the fixed locus X^τ has the following form.*

$$X^\tau = \begin{cases} C^{(g)} \amalg \coprod_{i=1}^k E_i & \text{where } g = \frac{22-r-l}{2} \text{ and } k = \frac{r-l}{2}, \\ C_1^{(1)} \amalg C_2^{(1)} & \text{if } r = 10, l = 8, \delta = 0, \\ \emptyset & \text{if } r = 10, l = 10, \delta = 0. \end{cases}$$

Here we denote by $C^{(g)}$ and E_i a nonsingular curve of genus g and a nonsingular rational curve, respectively.

PROPOSITION 5.2. *The invariant (r, l, δ) for each type in Table 1 is as in Table 3.*

Proof. We see that $T = H^2(X, \mathbb{Z})^{(\tau^*)}$ is exactly the sublattice $(K_+ \oplus S_-)^{\wedge} = ((K_+ \oplus S_-) \otimes \mathbb{Q}) \cap L$ of $L = H^2(X, \mathbb{Z})$. Therefore we get $r = \text{rank } K_+ + \text{rank } S_-$.

Since T is 2-elementary, we have $\det T = 2^l$. It follows from $p_{S_-}(\Gamma_{K_+S_-}) = \widetilde{H_-}$ that

$$|\widetilde{H_-}| = |\Gamma_{K_+S_-}| = \sqrt{\frac{\det(K_+ \oplus S_-)}{\det(K_+ \oplus S_-)^\wedge}} = \sqrt{\frac{\det(K_+ \oplus S_-)}{2^l}}.$$

From this equation we get l .

Next we compute the invariant δ . In cases No. [4], [5], [8], [9], [15]–[17], the invariants (r, l) already determine δ uniquely by the existence condition for the 2-elementary hyperbolic lattices, see [Nik3]. In cases No. [1]–[3], [18], we have that the parity of $K_+ \oplus S_-$ is zero, hence the overlattice T has parity zero, too. In No. [6], we see from Table 1 that the length of $\widetilde{H_-}$ is 6, which equals the rank of S_- . By straightforward computations, we see that the discriminant group of T has elements of noninteger square, that is, we have $\delta = 1$ in this case. In No. [7], we see that T^\perp has rank 8, signature $(2, 6)$, and length 8. Therefore $T^\perp(\frac{1}{2})$ is an integral unimodular lattice, which must be odd because of the signature. We get $T^\perp \simeq A_1(-1)^2 \oplus A_1^6$ and so $\delta = 1$.

The remaining five cases, where $\text{rank } S_+ = \text{rank } S_- = 4$, are established by the next two lemmas.

LEMMA 5.3. *Assume that $S_\pm = A_1(2)^4$ and $(r, l) = (10, 10)$. Then $T = U(2) \oplus A_1^8$ and $\delta = 1$.*

Proof. Let $K_+ = U(2) \oplus A_1(2)^4 = U(2) \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_4 \rangle$, where e_i are generators of $A_1(2)$, respectively. Similarly, let

$$\begin{aligned} S_+ &= A_1(2)^4 = \langle e'_1 \rangle \oplus \cdots \oplus \langle e'_4 \rangle, \\ S_- &= A_1(2)^4 = \langle e''_1 \rangle \oplus \cdots \oplus \langle e''_4 \rangle. \end{aligned}$$

By $p_{S_-}(\Gamma_{S_+S_-}) = \Gamma_- = \langle e''_1/2 \rangle \oplus \cdots \oplus \langle e''_4/2 \rangle$, elements of norm 1 (mod $2\mathbb{Z}$) in Γ_- are of the form either $e''_i/2$ or $(e''_j + e''_k + e''_l)/2$. Hence $\gamma : \Gamma_+ \rightarrow \Gamma_-$ maps $e'_i/2$ to either $e''_j/2$ or $(e''_j + e''_k + e''_l)/2$. In the former case, it contradicts the fact that $S = E_8(2)$ does not contain (-2) -vector. Similarly, the patching $p_{S_+}(\Gamma_{K_+S_+}) \rightarrow p_{K_+}(\Gamma_{K_+S_+})$ maps $e'_i/2$ to $(e_j + e_k + e_l)/2$. Hence $\Gamma_{K_+S_-}$ contains an element of the form

$$\frac{e_i + e_j + e_k + e'_l + e''_m + e''_n}{2}.$$

This element has norm (-6) . The assumption $(r, l) = (10, 10)$ yields that $T(\frac{1}{2}) = U \oplus E_8$ or $U \oplus (-1)^8$. Since $U \oplus E_8$ does not contain (-3) -vector, we conclude $T = U(2) \oplus A_1^8$. □

LEMMA 5.4. *Assume that $S_\pm = D_4(2)$. Then the parity δ of $T = (K_+ \oplus S_-)^\wedge$ is equal to 0.*

Proof. By Corollary 4.4, we see that $K_+ = U(2) \oplus D_4(2)$. Let $q_{K_+} = u \oplus v \oplus F_4 = u \oplus \langle e_1, f_1 \rangle \oplus \langle g_1, h_1 \rangle$, where $\langle e_1, f_1 \rangle$ and $\langle g_1, h_1 \rangle$ are generators of v and

F_4 , respectively. Similarly, let

$$\begin{aligned} q_{S_+} &= v \oplus F_4 = \langle e_2, f_2 \rangle \oplus \langle g_2, h_2 \rangle, \\ q_{S_-} &= v \oplus F_4 = \langle e_3, f_3 \rangle \oplus \langle g_3, h_3 \rangle. \end{aligned}$$

Recall that $L_+ = U(2) \oplus E_8(2)$ and $S = E_8(2)$. We see that $\Gamma_{K_+S_+} = \langle 2g_1 + 2g_2, 2h_1 + 2h_2 \rangle$ and $\Gamma_{S_+S_-} = \langle 2g_2 + 2g_3, 2h_2 + 2h_3 \rangle$. Hence $\Gamma_{K_+S_-}$ contains $\langle 2g_1 + 2g_3, 2h_1 + 2h_3 \rangle$. This shows that T is an overlattice of $U(2) \oplus E_8(2)$. Therefore the parity of T is equal to 0. \square

This completes the proofs for all cases. \square

5.1. Horikawa Constructions

The general construction is as follows.

PROPOSITION 5.5 [BHPV, V. 23]. *Let ψ be an involution on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $\psi : (u, v) \mapsto (-u, -v)$, where u and v are inhomogeneous coordinates of \mathbb{P}^1 , respectively. Let B be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ whose bidegree is $(4, 4)$ with at worst simple singularities and which is preserved under ψ . Assume that B does not pass through any of the fixed points of ψ . Then the minimal resolution X of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along B is a K3 surface. Moreover, ψ lifts to two involutions of X . One of them is a fixed point free involution ε . In particular, $Y = X/\varepsilon$ is an Enriques surface.*

In this construction, the other lift of ψ gives a symplectic involution g on X and induces an involution ι on Y (namely the construction is always associated with an involution on Y). The covering involution τ of $X/\mathbb{P}^1 \times \mathbb{P}^1$ is the same as $\varepsilon \circ g$, which is a nonsymplectic involution of X . In what follows, we exhibit many choices of branch B so that the resulting ι covers all involutions in Theorem 1.1 except for No. [13]. We remark that the condition for B to have the expected number of components, types of singularities, and not to pass through the fixed points of ψ is Zariski open, so that we will always assume that the coefficients (parameters) of the exhibited equation of B are general enough to satisfy these conditions. Further remark is that by choosing very general coefficients, we can assume that the resulting K3 surface X does not contain any accidental divisors. This results in saying that the transcendental lattice T_X is isomorphic to the lattice K_- in Table 3. (In other words, the transcendental lattice is just contained in K_- in not very general cases.) This way of choice is used in No. [11] and No. [12] in order to compute transcendental lattices geometrically.

EXAMPLE No. [1]. This example was constructed by Horikawa [Hor] and studied by Dolgachev [Dol] and Barth–Peters [BP]. Here we give another construction given by Mukai–Namikawa [MN].

Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 1):

$$\begin{aligned} X_{\pm} : u &= \pm 1, & Y_{\pm} : v &= \pm 1, \\ E : u^2v^2 - 1 + a_1(u^2 - 1) + a_2(v^2 - 1) &= 0 \quad (a_i \in \mathbb{C}). \end{aligned}$$

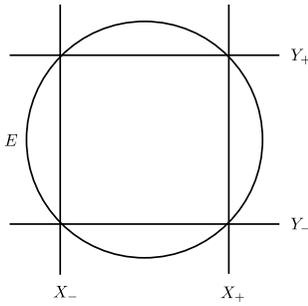


Figure 1

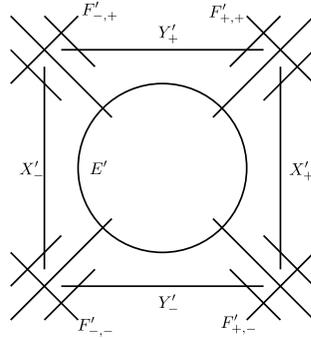


Figure 2

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the four intersection points of X_{\pm}, Y_{\pm} , and E . Let $F_{\pm, \pm}$ be the exceptional curves over $(\pm 1, \pm 1)$, respectively. Blow up again at the 12 intersection points of $F_{\pm, \pm}$ and the strict transforms of X_{\pm}, Y_{\pm} , and E . Let R be the blown-up surface. We denote by $X'_{\pm}, Y'_{\pm}, F'_{\pm, \pm}$, and E' the strict transforms of $X_{\pm}, Y_{\pm}, F_{\pm, \pm}$, and E , respectively. The configuration of curves in R is given in Figure 2. Note that X'_{\pm}, Y'_{\pm} , and $F'_{\pm, \pm}$ are all (-4) -curves, and other rational curves are all (-1) -curves. Let $B' = \sum(X'_{\pm} + Y'_{\pm} + F'_{\pm, \pm}) + E'$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of one elliptic curve and eight rational curves, we see that $(r, l) = (18, 2)$ by Theorem 5.1. This is enough to conclude that this example belongs to No. [1] by Table 2.

EXAMPLE No. [2]. This example was found by Kondo and overlooked in [MN] (cf. [Muk1]).

Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 3):

$$\begin{aligned} X_{\pm}: u = \pm 1, \quad Y_{\pm}: v = \pm 1, \\ C_{\pm}: uv - 1 + a_1(\pm u - 1) + a_2(\pm v - 1) = 0 \quad (a_i \in \mathbb{C}). \end{aligned}$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the ten intersection points of X_{\pm}, Y_{\pm} , and C_{\pm} . Let F_+ and F_- be the exceptional curves over $(1, 1)$ and $(-1, -1)$, respectively. Blow up again at the six intersection points of F_{\pm} and the strict transforms of X_{\pm}, Y_{\pm} , and C_{\pm} . Let R be the blown-up surface. We denote by $X'_{\pm}, Y'_{\pm}, C'_{\pm}$, and F'_{\pm} the strict transforms of $X_{\pm}, Y_{\pm}, C_{\pm}$, and F_{\pm} , respectively. The configuration of curves in R is given in Figure 4. Note that $X'_{\pm}, Y'_{\pm}, C'_{\pm}$, and F'_{\pm} are all (-4) -curves, and the others are all (-1) -curves. Let $B' = \sum(X'_{\pm} + Y'_{\pm} + C'_{\pm} + F'_{\pm})$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of eight rational curves, we see that $(r, l) = (18, 4)$ by Theorem 5.1. Note that the configuration of curves in X is the same as in Figure 4. We notice that the dual graph of the continuous lines in Figure 5 is the Dynkin diagram of type $E_7 \oplus A_1$. Let e_i ($i = 1, \dots, 8$) denote the cohomology classes of these curves. The images

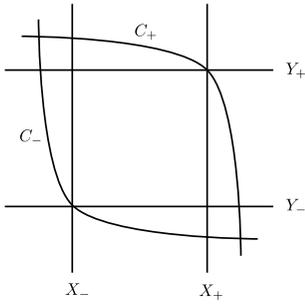


Figure 3

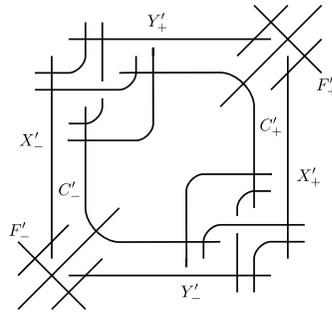


Figure 4

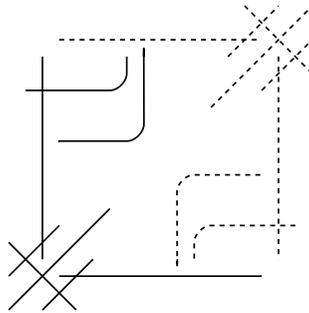


Figure 5

of these curves by ε are given by the dashed lines in Figure 5. Let M be the lattice generated by $e_i - \varepsilon^*(e_i)$ ($i = 1, \dots, 8$). We see that $M \simeq E_7(2) \oplus A_1(2)$ and $M \subset S_-$. For $(e_i - \varepsilon^*(e_i))/2 \in \frac{1}{2}M$, there exists $(e_i + \varepsilon^*(e_i))/2 \in L_+^*$ such that

$$\frac{e_i - \varepsilon^*(e_i)}{2} + \frac{e_i + \varepsilon^*(e_i)}{2} = e_i \in L.$$

It follows that

$$\frac{1}{2}M/S_- \subset H_-.$$

By calculation, we have $q_{E_8(2)}|_{(1/2)(E_7(2) \oplus A_1(2))/E_8(2)} = u^3 \oplus w$. Therefore this is the example of No. [2].

EXAMPLE No. [3]. This example was constructed by Lieberman.

Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 6):

$$\begin{aligned} X_{1\pm}: u = \pm 1, & & Y_{1\pm}: v = \pm 1, \\ X_{2\pm}: u = \pm a_1, & & Y_{2\pm}: v = \pm a_2 \quad (a_i \in \mathbb{C}). \end{aligned}$$

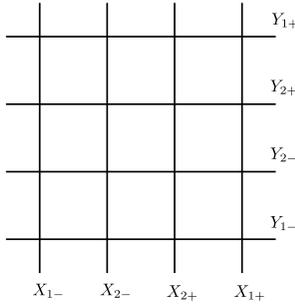


Figure 6

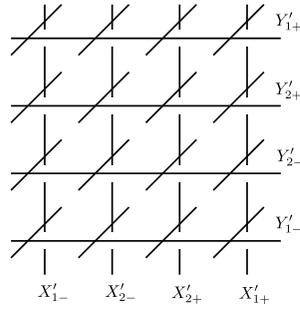


Figure 7

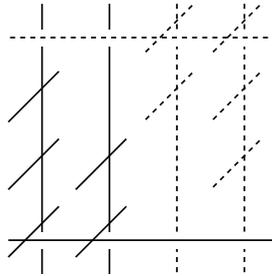


Figure 8

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 intersection points of $X_{1\pm}$, $X_{2\pm}$, $Y_{1\pm}$, and $Y_{2\pm}$. Let R be the blown-up surface. We denote by $X'_{1\pm}$, $X'_{2\pm}$, $Y'_{1\pm}$, and $Y'_{2\pm}$ the strict transforms of $X_{1\pm}$, $X_{2\pm}$, $Y_{1\pm}$, and $Y_{2\pm}$, respectively. The configuration of curves in R is given in Figure 7. Note that $X'_{1\pm}$, $X'_{2\pm}$, $Y'_{1\pm}$, and $Y'_{2\pm}$ are all (-4) -curves, and the others are all (-1) -curves. Let $B' = \sum(X'_{1\pm} + X'_{2\pm} + Y'_{1\pm} + Y'_{2\pm})$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^\tau = B'$ consists of eight rational curves, we see that $(r, l) = (18, 4)$ by Theorem 5.1. Note that the configuration of curves in X is the same as in Figure 7. We notice that the dual graph of the continuous lines in Figure 8 is the Dynkin diagram of type D_8 . Let e_i ($i = 1, \dots, 8$) denote the cohomology classes of these curves. The images of these curves by ε are given by the dashed lines in Figure 8. Let M be the lattice generated by $e_i - \varepsilon^*(e_i)$ ($i = 1, \dots, 8$). We see that $M \simeq D_8(2)$ and $M \subset S_-$. Similarly to Example No. [2], we have $\frac{1}{2}M/S_- \subset H_-$. By calculation, we have $qE_8(2)|_{(1/2)(D_8(2))/E_8(2)} = u^3 \oplus z$. Therefore this is the example of No. [3].

EXAMPLE No. [4]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 9):

$$\begin{aligned}
 X_{\pm}: u &= \pm 1, & Y_{\pm}: v &= \pm 1, \\
 E: u^2v^2 - 1 + a_1(u^2 - 1) + a_2(v^2 - 1) + a_3(uv - 1) &= 0 \quad (a_i \in \mathbb{C}).
 \end{aligned}$$

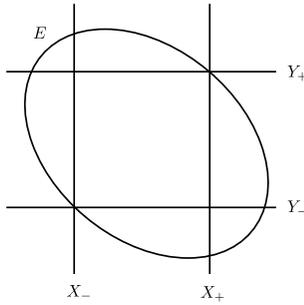


Figure 9

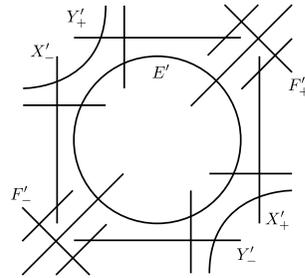


Figure 10

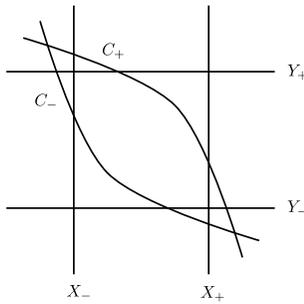


Figure 11

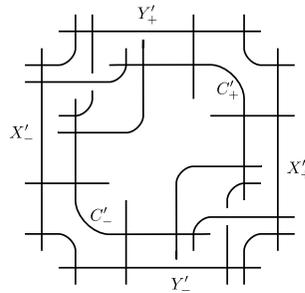


Figure 12

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of X_{\pm} , Y_{\pm} , and E . Let F_+ and F_- be the exceptional curves over $(1, 1)$ and $(-1, -1)$, respectively. Blow up again at the six intersection points of F_{\pm} and the strict transforms of X_{\pm} , Y_{\pm} , and E . Let R be the blown-up surface. We denote by X'_{\pm} , Y'_{\pm} , F'_{\pm} , and E' the strict transforms of X_{\pm} , Y_{\pm} , F_{\pm} , and E , respectively. The configuration of curves in R is given in Figure 10. Note that X'_{\pm} , Y'_{\pm} , and F'_{\pm} are all (-4) -curves, and other rational curves are all (-1) -curves. Let $B' = \sum(X'_{\pm} + Y'_{\pm} + F'_{\pm}) + E'$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of one elliptic curve and six rational curves, we see that $(r, l) = (16, 4)$ by Theorem 5.1. Therefore this is the example of No. [4].

EXAMPLE No. [5]. This example was studied by Mukai [Muk2] as the example of numerically reflective involution.

Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 11):

$$\begin{aligned} X_{\pm}: u &= \pm 1, & Y_{\pm}: v &= \pm 1, \\ C_{\pm}: uv \pm a_1u \pm a_2v + a_3 &= 0 & (a_i \in \mathbb{C}). \end{aligned}$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 14 intersection points of X_{\pm} , Y_{\pm} , and C_{\pm} . Let R be the blown-up surface. We denote by X'_{\pm} , Y'_{\pm} , and C'_{\pm} the strict transforms of X_{\pm} ,

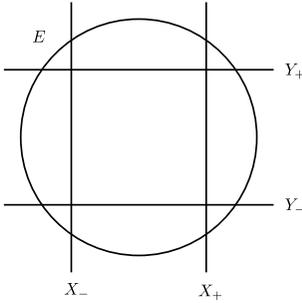


Figure 13

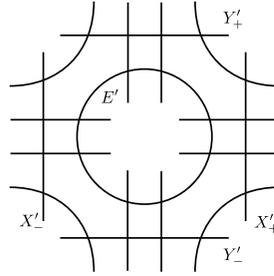


Figure 14

Y_{\pm} , and C_{\pm} , respectively. The configuration of curves in R is given in Figure 12. Note that X'_{\pm} , Y'_{\pm} , and C'_{\pm} are all (-4) -curves and the others are all (-1) -curves. Let $B' = \sum(X'_{\pm} + Y'_{\pm} + C'_{\pm})$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of six rational curves, we see that $(r, l) = (16, 6)$ by Theorem 5.1. Therefore this is the example of No. [5].

EXAMPLE No. [6]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 13):

$$\begin{aligned} X_{\pm}: u &= \pm 1, & Y_{\pm}: v &= \pm 1, \\ E: u^2v^2 + a_1u^2 + a_2v^2 + a_3uv + a_4 &= 0 \quad (a_i \in \mathbb{C}). \end{aligned}$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 12 intersection points of X_{\pm} , Y_{\pm} , and E . Let R be the blown-up surface. We denote by X'_{\pm} , Y'_{\pm} , and E' the strict transforms of X_{\pm} , Y_{\pm} , and E , respectively. The configuration of curves in R is given in Figure 14. Note that X'_{\pm} , Y'_{\pm} are all (-4) -curves and other rational curves are all (-1) -curves. Let $B' = \sum(X'_{\pm} + Y'_{\pm}) + E'$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of one elliptic curve and four rational curves, we see that $(r, l) = (14, 6)$ by Theorem 5.1. Therefore this is the example of No. [6].

EXAMPLE No. [7]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 15):

$$Y_{\pm}: v = \pm 1, \quad C_{\pm}: u^2v \pm uv \pm a_1u^2 + a_2u + a_3v \pm a_4 = 0 \quad (a_i \in \mathbb{C}).$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the 12 intersection points of Y_{\pm} and C_{\pm} . Let R be the blown-up surface. We denote by Y'_{\pm} and C'_{\pm} the strict transforms of Y_{\pm} and C_{\pm} , respectively. The configuration of curves in R is given in Figure 16. Note that Y'_{\pm} and C'_{\pm} are all (-4) -curves and the others are all (-1) -curves. Let $B' = \sum(Y'_{\pm} + C'_{\pm})$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of four rational curves, we see that $(r, l) = (14, 8)$ by Theorem 5.1. Therefore this is the example of No. [7].

EXAMPLE No. [8]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 17):

$$Y_{\pm}: v = \pm 1,$$

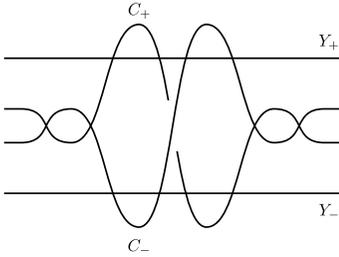


Figure 15

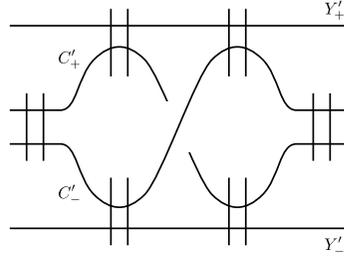


Figure 16

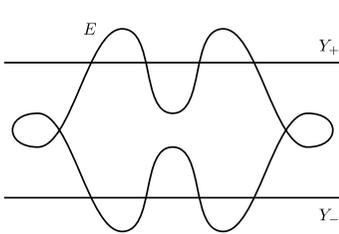


Figure 17

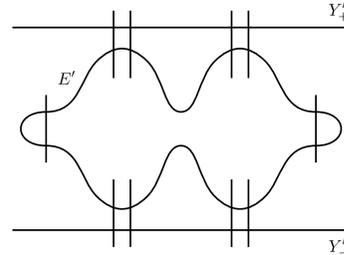


Figure 18

$$E: v^2(u^4 + a_1u^2 + a_2) + 2a_3uv(u^2 - a_4) + a_5(u^2 - a_4)^2 = 0 \quad (a_i \in \mathbb{C}).$$

Note that E has two nodes at $(u, v) = (\pm\sqrt{a_4}, 0)$.

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of Y_{\pm} and E and at two nodes of E . Let R be the blown-up surface. We denote by Y'_{\pm} and E' the strict transforms of Y_{\pm} and E , respectively. The configuration of curves in R is given in Figure 18. Note that Y'_{\pm} are (-4) -curves and other rational curves are all (-1) -curves. Let $B' = Y'_+ + Y'_- + E'$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of one elliptic curve and two rational curves, we see that $(r, l) = (12, 8)$ by Theorem 5.1. Therefore this is the example of No. [8].

EXAMPLE No. [9]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 19):

$$C_{\pm}: v^2(u^2 \pm a_1u + a_2) \pm 2a_3v(u \mp a_4)^2 + a_5(u \mp a_4)^2 = 0 \quad (a_i \in \mathbb{C}).$$

Note that C_+ and C_- have a node at $(u, v) = (a_4, 0)$ and $(-a_4, 0)$, respectively.

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of C_{\pm} and at two nodes of C_{\pm} . Let R be the blown-up surface. We denote by C'_{\pm} the strict transforms of C_{\pm} , respectively. The configuration of curves in R is given in Figure 20. Note that C'_{\pm} are (-4) -curves and the others are all (-1) -curves. Let $B' = C'_+ + C'_-$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} =$

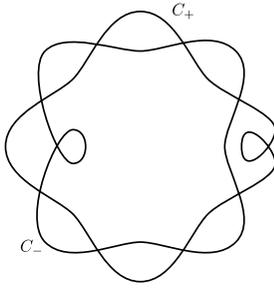


Figure 19

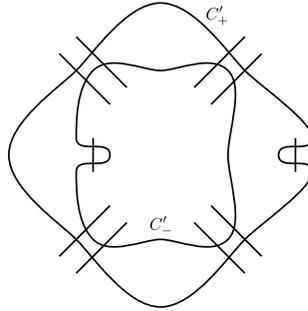


Figure 20

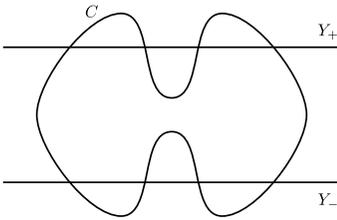


Figure 21

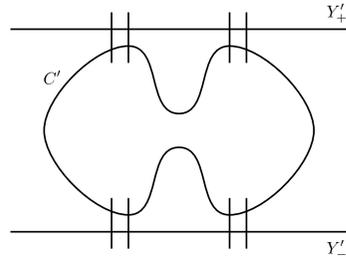


Figure 22

B' consists of two rational curves, we see that $(r, l) = (12, 10)$ by Theorem 5.1. Therefore this is the example of No. [9].

EXAMPLE No. [10]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (Figure 21):

$$Y_{\pm}: v = \pm 1,$$

$$C: v^2(u^4 + u^2 + a_1) + vu(a_2u^2 + a_3) + a_4u^4 + a_5u^2 + a_6 = 0 \quad (a_i \in \mathbb{C}).$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of Y_{\pm} and C . Let R be the blown-up surface. We denote by Y'_{\pm} and C' the strict transforms of Y_{\pm} and C , respectively. The configuration of curves in R is given in Figure 22. Note that Y'_{\pm} are (-4) -curves and other rational curves are all (-1) -curves. Let $B' = Y'_+ + Y'_- + C'$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of a curve of genus 3 and 2 rational curves, we see that $(r, l) = (10, 6)$ by Theorem 5.1. Therefore this is the example of No. [10].

EXAMPLE No. [11]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$E_1: u^2v^2 + u^2 + a_1v^2 + a_2uv + a_3 = 0,$$

$$E_2: u^2v^2 + v^2 + a_4u^2 + a_5uv + a_6 = 0 \quad (a_i \in \mathbb{C}).$$

Then E_i are smooth elliptic curves preserved by ψ (Figure 23).

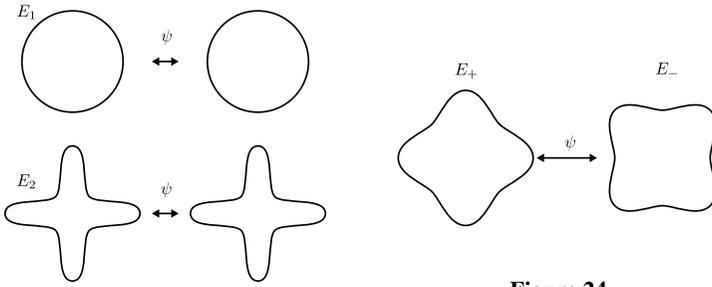


Figure 23

Figure 24

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of E_1 and E_2 . Let R be the blown-up surface. We denote by E'_1 and E'_2 the strict transforms of E_1 and E_2 , respectively. Let $B' = E'_1 + E'_2$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^\tau = B'$ consists of two elliptic curves, we see that $(r, l, \delta) = (10, 8, 0)$ by Theorem 5.1. To see which No. this example belongs to, we discuss as follows.

The involution ψ of $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to the rational elliptic surface R/\mathbb{P}^1 and it acts on the base \mathbb{P}^1 trivially. Hence, by choosing a zero-section, it corresponds to a translation by a 2-torsion section σ . In this case, the Horikawa construction corresponds exactly to the quadratic twist construction discussed in [Kon; HS]: the free involution ε is the lift of the translation automorphism. We remark that generically the elliptic surface R has eight singular fibers $4I_2 + 4I_1$ (Kodaira's notation).

Here we consider a deformation of the $K3$ surface X . Fix the rational elliptic surface R . We move the branch locus $B' = E'_1 + E'_2$, the union of two smooth fibers, to B'_1 , the union of one I_2 fiber plus one smooth fiber. We denote by X_1 the smooth $K3$ surface obtained by the double cover branched along B'_1 and then taking the minimal desingularization. Since only rational double points appear in the construction, X and X_1 are connected by a smooth deformation by the simultaneous resolution. Now X_1 has also an Enriques quotient Y_1 by the quadratic twist construction. By definition of B'_1 , the main invariant of τ_1 on X_1 is $(12, 8, 1)$ and the associated involution on Y_1 has type [8]. We recall that a specialization of $K3$ surfaces $X \rightsquigarrow X_1$ exists if and only if their transcendental lattices satisfy $T_{X_1} \subset T_X$. Recall that when we consider very general cases, the transcendental lattice is isomorphic to K_- as remarked in sentences before Example No. [1]. Since the lattice $K_- \simeq U(2) \oplus U(2) \oplus D_4(2)$ of type [12] does not contain the summand U of type [8], we see that our example belongs to No. [11].

EXAMPLE No. [12]. Consider the following curves on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$E_\pm: v^2(u^2 \pm a_1u + a_2) \pm v(u^2 \pm a_3u + a_4) + (u^2 \pm a_5u + a_6) = 0 \quad (a_i \in \mathbb{C}).$$

Then E_\pm are elliptic curves which are exchanged by ψ (Figure 24).

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight intersection points of E_{\pm} . Let R be the blown-up surface. We denote by E'_{\pm} the strict transforms of E_{\pm} , respectively. Let $B' = E'_+ + E'_-$. The $K3$ surface X is the double cover of R whose branch locus is B' . Since $X^{\tau} = B'$ consists of two elliptic curves, we see that $(r, l, \delta) = (10, 8, 0)$ by Theorem 5.1. To check that they correspond to No. [12] in this case, we discuss as follows.

We remark that case No. [9] is a specialization of our family: it is exactly the case where both E_{\pm} acquire a node. Thus we can regard the $K3$ surface X_0 in No. [9] as a special member of a smooth deformation with general fiber X from our family No. [12]. Here, the two elliptic curves E'_{\pm} deform into the sums of two rational curves $F_{\pm} + F'_{\pm}$, where $(F_{\pm}^2) = ((F'_{\pm})^2) = -2$ and $(F_{\pm}, F'_{\pm}) = 2$ (double sign corresponds).

Moreover, since the formation of ε is preserved in this specialization, our deformation is in fact a family of $K3$ surfaces equipped with free involutions ε on X and ε_0 on X_0 . By the theory of period maps, we have an inclusion $NS(X) \subset NS(X_0)$. The orthogonal complement of this inclusion is generated by the (-4) -vector $F_+ - F_-$, and the overlattice structure of $NS(X_0) \supset NS(X) \oplus \mathbb{Z}\langle F_+ - F_- \rangle$ is given by

$$F_+ = \frac{F_+ + F_-}{2} + \frac{F_+ - F_-}{2} \in NS(X_0).$$

(Here we have used that $(F_+ + F_-) \in H^2(X_0)^{\varepsilon_0} = H^2(X)^{\varepsilon} \subset NS(X)$, the equality in the middle holds since the deformation is with involutions.) Hence, under the condition of being very general as in No. [11], we get $\det NS(X_0) = \det NS(X) \cdot 4/2^2 = \det NS(X)$. Thus, for transcendental lattices, we get $\det T_X = \det T_{X_0} = 2^{10}$ (type [9]). Therefore our example belongs to No. [12].

EXAMPLE No. [14]. We need an irreducible curve on $\mathbb{P}^1 \times \mathbb{P}^1$ which has eight nodes and is stable under ψ , but it seems not easy to construct them in a direct way. The following construction is due to H. Tokunaga.

Let B_0 be a smooth irreducible divisor of bidegree $(2, 2)$ to which the four lines $u = 0, \infty; v = 0, \infty$ are tangent. We remark that, in general, if a divisor is tangent to the branch curve (with local intersection number 2), then by pulling back to the double cover, the divisor acquires a node at the point of tangency. Thus in our case the following construction works: We consider the two self-morphisms $\psi_1 : (u, v) \mapsto (u^2, v)$ and $\psi_2 : (u, v) \mapsto (u, v^2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Then the pullback $C_8 := (\psi_1 \circ \psi_2)^*(B_0)$ has bidegree $(4, 4)$ with eight nodes and is stable under ψ (Figure 25).

For example, we can exhibit the equation for C_8 as follows:

$$(c^2u^4 + 2cbu^2 + b^2)v^4 + (2cau^4 + du^2 + 2b)v^2 + (a^2u^4 + 2au^2 + 1) = 0.$$

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight nodes of C_8 . Let R be the blown-up surface. We denote by C'_8 the strict transforms of C_8 . The $K3$ surface X is the double cover of R whose branch locus is C'_8 . Since $X^{\tau} = C'_8$ is an elliptic curve, we see that $(r, l, \delta) = (10, 10, 1)$ by Theorem 5.1. Therefore this is the example of No. [14].

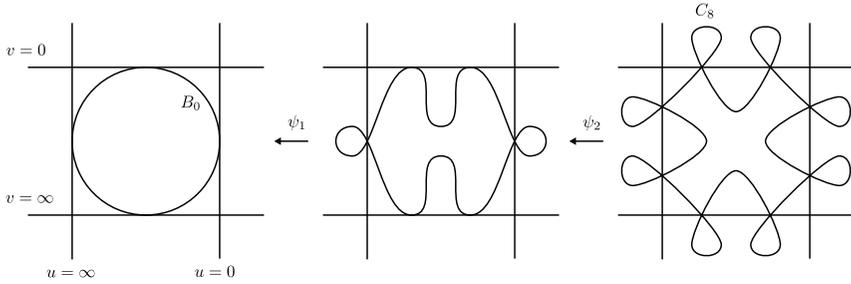


Figure 25

EXAMPLE NOS. [15]–[18]. Let C_{2i} ($i = 0, 1, 2, 3$) be irreducible curves on $\mathbb{P}^1 \times \mathbb{P}^1$ whose bidegree is $(4, 4)$ with $2i$ nodes, respectively.

Blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at $2i$ nodes of C_{2i} . Let R_{2i} be the blown-up surface. We denote by C'_{2i} the strict transforms of C_{2i} . The $K3$ surface X_{2i} is the double cover of R_{2i} whose branch locus is C'_{2i} . Since $X_{2i}^\tau = C'_{2i}$ is a curve of genus $9 - 2i$, we see that $(r, l) = (2i + 2, 2i + 2)$ by Theorem 5.1. Therefore the cases $i = 3, 2, 1$, and 0 are the examples of No. [15], [16], [17], and [18], respectively.

5.2. Enriques' Sextics

The non-normal sextic surface in \mathbb{P}^3 , which is singular along the six edges of a tetrahedron, is a model of an Enriques surface, the one first considered by Enriques himself. In fact its normalization gives a smooth Enriques surface, see [GH]. By setting the tetrahedron as $xyzt = 0$, we have the general equation of such surfaces

$$q(x, y, z, t)xyzt + (x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2) = 0,$$

where q is a quadratic equation. By considering various linear actions on \mathbb{P}^3 , we can get many examples of involutions on Enriques surfaces. The most important for us among them is the following example exhibiting No. [13].

EXAMPLE No. [13]. Let us consider the involution $\iota : (x : y : z : t) \mapsto (y : x : t : z)$ on \mathbb{P}^3 . The general equation of invariant Enriques' sextic \bar{Y} is of the form

$$(a_1(x^2 + y^2) + a_2(z^2 + t^2) + a_3xy + a_4zt + a_5(xz + yt) + a_6(xt + yz))xyzt + (x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2) = 0,$$

where $a_i \in \mathbb{C}$ are general. Then the normalization Y is a smooth Enriques surface with the induced action by ι .

Let us show that they belong to No. [13]. Since in this case τ is also fixed point free, this is equivalent to saying that the fixed locus Y^ι is a finite set. Moreover, since the normalization $Y \rightarrow \bar{Y}$ is a finite morphism, it suffices to show that \bar{Y}^ι is a finite set. However, this set is the intersection of \bar{Y} with the fixed locus in \mathbb{P}^3 ,

$\{x = y, z = t\} \cup \{x + y = 0, z + t = 0\}$. Since the general element does not contain these lines, the intersection is a finite set as desired.

References

- [BHPV] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, 2004.
- [BP] W. Barth and C. Peters, *Automorphisms of Enriques surfaces*, *Invent. Math.* 73 (1983), 383–411.
- [DoI] I. Dolgachev, *On automorphisms of Enriques surfaces*, *Invent. Math.* 76 (1984), 163–177.
- [GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.
- [Hor] E. Horikawa, *On the periods of Enriques surfaces. I–II*, *Math. Ann.* 234 (1978), 73–88; 235 (1978), 217–246.
- [HS] K. Hulek and M. Schütt, *Enriques surfaces and Jacobian elliptic K3 surfaces*, *Math. Z.* 268 (2011), 1025–1056.
- [Kon] S. Kondo, *Enriques surfaces with finite automorphism groups*, *Jpn. J. Math.* 12 (1986), 191–282.
- [Mor] D. R. Morrison, *On K3 surfaces with large Picard number*, *Invent. Math.* 75 (1984), 105–121.
- [Muk1] S. Mukai, *Numerically trivial involutions of Kummer type of an Enriques surface*, *Kyoto J. Math.* 50 (2010), 889–902.
- [Muk2] ———, *Kummer’s quartics and numerically reflective involutions of Enriques surfaces*, *J. Math. Soc. Japan* 64 (2012), 231–246.
- [MN] S. Mukai and Y. Namikawa, *Automorphisms of Enriques surfaces which act trivially on the cohomology groups*, *Invent. Math.* 77 (1984), 383–397.
- [Nik1] V. V. Nikulin, *Finite automorphism groups of Kähler K3 surfaces*, *Trans. Moscow Math. Soc.* 38 (1980), 71–135.
- [Nik2] ———, *Integral symmetric bilinear forms and some of their applications* (English translation), *Math. USSR Izv.* 14 (1980), 103–167.
- [Nik3] ———, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*, *J. Soviet Math.* 22 (1983), 1401–1475.
- [Nik4] ———, *Involutions of integral quadratic forms and their applications to real algebraic geometry*, *Math. USSR Izv.* 22 (1984), 99–172.
- [PS] I. I. Piatetski-Shapiro and I. R. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971), 530–572.

H. Ito
 Graduate School of Mathematics
 Nagoya University
 Furo-cho, Chikusa-ku
 Nagoya 464-8602
 Japan

hiroki.ito@math.nagoya-u.ac.jp

H. Ohashi
 Department of Mathematics
 Faculty of Science and Technology
 Tokyo University of Science
 2641 Yamazaki, Noda
 Chiba 278-8510
 Japan

ohashi@ma.noda.tus.ac.jp