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Euler–Mellin Integrals and A-Hypergeometric Functions

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Abstract. We consider integrals that generalize both Mellin transforms of rational functions of the form $1/f$ and classical Euler integrals. The domains of integration of our so-called Euler–Mellin integrals are naturally related to the coamoeba of f, and the components of the complement of the closure of this coamoeba give rise to a family of these integrals. After performing an explicit meromorphic continuation of Euler–Mellin integrals, we interpret them as A-hypergeometric functions and discuss their linear independence and relation to Mellin–Barnes integrals.

1. Introduction

In the classical theory of hypergeometric functions, a prominent role is played by the Euler integral formula

$$ {}_2F_1(s;t;u) = \frac{\Gamma(t)}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 x^{s_1-1}(1-x)^{t-s_1-1}(1-ux)^{-s_2}\,dx, $$

which yields an analytic continuation of the Gauss hypergeometric series ${}_2F_1$ from the unit disk $|u|<1$ to the larger domain $|\arg(1-u)|<\pi$. However, this Euler integral is not symmetric in s_1 and s_2, even though the function ${}_2F_1$ enjoys such symmetry. Following Erdélyi [Erd37], we can introduce another variable of integration and obtain the symmetric formula

$$ \begin{aligned} {}_2F_1(s;t;u) = G(s,t) \int_0^1 \int_0^1 & x^{s_1-1}y^{s_2-1}(1-x)^{t-s_1-1} \\ & \times (1-y)^{t-s_2-1}(1-uxy)^{-t}\,dx\wedge dy, \end{aligned} $$

$$ \text{where } G(s,t) = \frac{\Gamma(t)^2}{\Gamma(s_1)\Gamma(s_2)\Gamma(t-s_1)\Gamma(t-s_2)}. \tag{1.1} $$

After making the substitutions $z = x/(1-x)$, $w = y/(1-y)$, and $c = 1-u$, we find that the double integral in (1.1) takes the simple form

$$ \int_0^\infty \int_0^\infty \frac{z^{s_1}w^{s_2}}{(1+z+w+czw)^t}\frac{dz\wedge dw}{zw}, \tag{1.2} $$

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which restricted to $t=-1$ is a twofold Mellin transform of $1/f$, where $f(z,w)=1+z+w+czw$. In this paper, we introduce a generalization of the Mellin transform of a rational function $1/f$, which we call an Euler–Mellin integral. The general form of an Euler–Mellin integral is given in Definition 2.1.

Euler–Mellin integrals are closely related to A-hypergeometric Euler-type integrals, as studied in [GKZ90; SST00]. The most notable difference between these previously studied functions and the Euler–Mellin integrals we introduce here is the domain of integration. We integrate over explicit, simply connected, but noncompact sets, whereas previous authors used compact yet rather elusive cycles. We show that the simple connectivity of our domain of integration allows us to handle the multivaluedness of the integrand; however, to achieve convergence, the noncompactness initially restricts the values of the parameters (s,t); see Theorem 2.3. In Theorem 2.5, we remove the restrictions on (s,t) through an explicit meromorphic continuation of an Euler–Mellin integral. This yields a meromorphic function whose singular locus is contained in certain families of hyperplanes. Taking these into account, we obtain a function that is entire in the parameters (s,t).

A further generalization of the results of Section 2 is achieved by considering the coamoeba of the polynomial f, as defined in (3.1). In practice, we rotate the domain of integration of the Euler–Mellin integral to $\operatorname{Arg}^{-1}(\theta)=\{z\in(\mathbb{C}_*)^n \mid \operatorname{Arg}(z)=\theta\}$ for appropriate choices of θ (see Section 3). In the previous example, this means replacing the integral (1.2) by

$$\int_{\operatorname{Arg}^{-1}(\theta)} \frac{z^{s_1}w^{s_2}}{(c_1+c_2z+c_3w+c_4zw)^t}\frac{dz\wedge dw}{zw}.$$

The A-hypergeometric approach considers a polynomial f with general coefficients on a fixed set of monomials, which are identified with a matrix A. We show in Theorem 4.2 that for a generic choice of coefficients of f, the corresponding Euler–Mellin integral with parameters (s,t) satisfies an A-hypergeometric system $H_A(\beta)$ of differential equations with parameter $\beta=-(t,s)$ (see Definition 4.1). In particular, the meromorphic continuations of Euler–Mellin integrals obtained through Sections 2–3 provide a family of A-hypergeometric functions that are entire in β.

A key problem in the study of the A-hypergeometric system is to describe the variation with the parameter β of its solution space of germs of analytic functions at a nonsingular point. To begin this study, we must first find solutions of $H_A(\beta)$ that vary nicely with β. Saito, Sturmfels, and Takayama [SST00] presented an algorithm to compute such a basis for arbitrary β, called *canonical series solutions;* however, because this algorithm uses Gröbner degeneration, the solutions it produces are not well suited to the variation of β. For generic (nonresonant) β, Gelfand, Kapranov, and Zelevinsky [GKZ90] computed a basis of Euler-type integral solutions. These integrals are also unsuitable for understanding parametric behavior, as their domains of integration are not explicit, and, at nonresonant β, they do not span the solution space of $H_A(\beta)$.

In contrast, since our meromorphic continuations of Euler–Mellin integrals are entire in β, they provide a new tool for describing the parametric variation of A-hypergeometric solutions by Theorem 4.2. It is thus natural to ask whether these "extended" Euler–Mellin integrals arising from different connected components of the complement of the corresponding coamoeba are linearly independent, and if so, in which cases they span the solution space. The final sections of this article address this question from different viewpoints.

Most notably, in Theorem 6.4, we relate Euler–Mellin integrals to Mellin–Barnes integrals. These are another class of A-hypergeometric integrals previously considered in the literature [Nil09; Beu11a] (see Definition 6.1); in particular, Mellin–Barnes integrals are used by Beukers to compute elements in the local monodromy group of an A-hypergeometric system. As a corollary, there are at least as many linearly independent extended Euler–Mellin integrals as Mellin–Barnes integrals, providing examples in which extended Euler–Mellin integrals are linearly independent at generic β.

Outline

In Section 2, we introduce Euler–Mellin integrals, show their convergence, and perform their meromorphic continuation, which is our main result. In Section 3, we employ coamoebas to extend the results of the previous section to include more general domains of integration. Euler–Mellin integrals are shown to be A-hypergeometric functions in Section 4, and in Section 5, we show that they provide a basis of solutions to A-hypergeometric systems in the case of curves. In Section 6, we relate Euler–Mellin integrals to Mellin–Barnes integrals, obtaining further insight into the linear independence of both sets of integrals. Finally, Section 7 contains an example to illustrate the behavior of Euler–Mellin integrals at a rank-jumping parameter of an A-hypergeometric system.

2. Convergence and Meromorphic Continuation of Euler–Mellin Integrals

This section contains our main result, Theorem 2.5, which provides an explicit presentation of a meromorphic continuation of the Euler–Mellin integral of a polynomial in several variables.

DEFINITION 2.1. Given a polynomial $f = \sum_{\alpha \in \operatorname{supp}(f)} c_\alpha z^\alpha$, the *Euler–Mellin integral* is a natural generalization of the Mellin transform of the rational function $1/f$ of several variables given by

$$M_f(s,t) := \int_{\mathbb{R}^n_+} \frac{z^s}{f(z)^t} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n} = \int_{\mathbb{R}^n} \frac{e^{\langle s,x\rangle}}{f(e^x)^t} dx_1 \wedge \cdots \wedge dx_n, \qquad (2.1)$$

where $\mathbb{R}^n_+ := (0,\infty)^n$ denotes the positive orthant in $\mathbb{R}^n$. Here we employ the multi-index notation for variables $z_1, \ldots, z_n$ and polynomials $f_1, \ldots, f_m$; that is, for $s \in \mathbb{C}^n$ and $t \in \mathbb{C}^m$, we write $z^s := z_1^{s_1} \cdots z_n^{s_n}$ and $f(z)^t := f_1(z)^{t_1} \cdots f_m(z)^{t_m}$.

Whenever there is no risk of confusion, we use the notation $f(z) := f(z)^{(1,\dots,1)} = \prod_{i=1}^m f_i$.

In order for such an integral to converge, restrictions must a priori be placed on both the exponent vector (s,t) and the polynomial f; it is not enough to demand only that each f_i is nonvanishing on $\mathbb{R}_+^n$. We next provide such a domain of convergence for the Euler–Mellin integral (2.1), generalizing [NP13, Theorem 1].

DEFINITION 2.2. If Γ is a face of the Newton polytope Δ_f of f, then the *truncated polynomial with support* Γ is given by

$$f_\Gamma := \sum_{\alpha \in \Gamma \cap \operatorname{supp}(f)} c_\alpha z^\alpha.$$

The polynomial f is said to be *completely nonvanishing* on a set X if for each face Γ of Δ_f (including Δ_f itself), the truncated polynomial f_Γ has no zeros on X. In particular, the polynomial f itself does not vanish on X.

For a vector $\tau \in \mathbb{R}_+^m$, we denote by $\tau\Delta_f$ the weighted Minkowski sum $\sum_{i=1}^m \tau_i \Delta_{f_i}$ of the Newton polytopes of the f_i with respect to τ. Note that with this notation, the Newton polytope of f satisfies $\Delta_f = (1,\dots,1)\Delta_f$.

THEOREM 2.3. *If each of the polynomials $f_1,\dots,f_m$ is completely nonvanishing on the positive orthant $\mathbb{R}_+^n$ (as in Definition 2.2), then the Euler–Mellin integral $M_f(s,t)$ of (2.1) converges and defines an analytic function in the tube domain*

$$\{(s,t) \in \mathbb{C}^{n+m} \mid \tau := \operatorname{Re} t \in \mathbb{R}_+^m, \sigma := \operatorname{Re} s \in \operatorname{int}(\tau\Delta_f)\}. \tag{2.2}$$

Proof. It suffices to prove that for any (s,t) with all $\tau_i > 0$ and $\sigma \in \operatorname{int}(\tau\Delta_f)$, there exist positive constants c and k such that

$$|f(e^x)^t e^{-\langle s,x\rangle}| = |f(e^x)^t| e^{-\langle \sigma,x\rangle} \geq c e^{k|x|} \quad \text{for all } x \in \mathbb{R}^n.$$

In fact, it is enough to show that this inequality holds outside some ball $B(0)$ in $\mathbb{R}^n$.

Since $\sigma \in \operatorname{int}(\tau\Delta_f)$, we can expand it as a sum $\sigma = \sigma_1 + \dots + \sigma_m$ of m vectors such that $\sigma_i/\tau_i \in \operatorname{int}(\Delta_{f_i})$. It is shown in the proof of [NP13, Theorem 1] that for each $\sigma_i \in \operatorname{int}(\Delta_{f_i})$, there are positive constants c_i and k_i such that for x outside some ball $B_i(0)$,

$$|f_i(e^x)| e^{-\langle \sigma_i, x\rangle} \geq c_i e^{k_i|x|}.$$

Note that it is essential in [NP13, Theorem 1] that f_i is completely nonvanishing on the positive orthant. Thus, for x outside of $B(0) = \bigcup_{i=0}^m B_i(0)$, we have

$$|f(e^x)^t| e^{-\langle \sigma,x\rangle} = \prod_{i=1}^m (|f_i(e^x)| e^{-\langle \sigma_i/\tau_i, x\rangle})^{\tau_i} \geq \prod_{i=1}^m c_i^{\tau_i} e^{\tau_i k_i |x|} = c e^{k|x|}, \tag{2.3}$$

where $c = c_1^{\tau_1} \cdots c_m^{\tau_m}$ and $k = \tau_1 k_1 + \dots + \tau_m k_m$ are the desired positive constants. □

EXAMPLE 2.4. By a classical integral representation of the Gauss hypergeometric function ${}_2F_1$,

$$\int_0^\infty \frac{z^s}{(1+z)^{t_1}(c+z)^{t_2}}\frac{dz}{z} = \frac{\Gamma(t_1+t_2-s)\Gamma(s)}{\Gamma(t_1+t_2)}\,{}_2F_1(t_2, t_1+t_2-s; t_1+t_2; 1-c) \tag{2.4}$$

for $\mathrm{Re}(t_1+t_2) > \mathrm{Re}(t_1+t_2-s) > 0$ and $|\arg(c)| < \pi$, where arg denotes the principal branch of the argument mapping. Note that $|\arg(c)| < \pi$ if and only if $f(z) = (1+z)(c+z)$ is completely nonvanishing on $\mathbb{R}_+$. Since $\Delta_{f_1} = \Delta_{f_2} = [0, 1]$, the condition that $\sigma \in \mathrm{int}(\tau\Delta_f)$ is the same as $0 < \mathrm{Re}(s) < \mathrm{Re}(t_1+t_2)$. We also note that the right-hand side of (2.4) is analytic in this domain. Further, since $\mathrm{Re}(t_1) > 0$ and $\mathrm{Re}(t_2) > 0$, the convergence domain given in Theorem 2.3 is not optimal; however, being full-dimensional, it is large enough for our goal of meromorphic continuation.

As the right-hand side of (2.4) is a meromorphic function in s and t, it provides a meromorphic extension of the corresponding Euler–Mellin integral. On this right side, we have the regularized ${}_2F_1$ as one factor, and thus the polar locus of the meromorphic extension is contained in two families of hyperplanes given by the polar loci of the gamma functions. Our main result shows that this kind of meromorphic continuation is possible for all Euler–Mellin integrals.

To obtain the strongest form of this result, we choose a specific presentation for $\tau\Delta_f$. To begin, each Newton polytope Δ_{f_i} can be written uniquely as the intersection of a finite number of halfspaces

$$\Delta_{f_i} = \bigcap_{j=1}^{N_i} \{\sigma \in \mathbb{R}^n \mid \langle \mu_j^i, \sigma\rangle \geq \nu_j^i\}, \tag{2.5}$$

where the ν_j^i are integer vectors, and the μ_j^i are primitive vectors. In particular, each μ_j^i has coordinates that are relatively prime.

Fixing an order, let $\{\mu_1, \dots, \mu_N\}$ be equal to the set $\{\mu_j^i \mid 1 \leq i \leq m, \leq j \leq N_i\}$, where we assume that $\mu_i \neq \mu_j$ for all $i \neq j$. We now extend the definitions of ν_j^i from (2.5) to each μ_k; namely, for each k, let $\nu_k := (\nu_k^1, \dots, \nu_k^m)$ with

$$\nu_k^i := \min\{\langle \mu_k, \alpha\rangle \mid \alpha \in \Delta_{f_i}\},$$

and set $|\nu_k| := \nu_k^1 + \cdots + \nu_k^m$. By definition of the ν_k, we have $\mathrm{int}(\tau\Delta_f) = \sum_{i=1}^m \tau_i \,\mathrm{int}(\Delta_{f_i})$ and

$$\tau\Delta_f = \bigcap_{k=1}^{N} \{\sigma \in \mathbb{R}^n \mid \langle \mu_k, \sigma\rangle \geq \langle \nu_k, \tau\rangle\}. \tag{2.6}$$

We are now prepared to state our main result, which provides a meromorphic continuation of (2.1), generalizing [NP13, Theorem 2]. In Section 3, we obtain a stronger form of the result by relaxing the condition that the f_i be completely nonvanishing on $\mathbb{R}_+^n$.

Theorem 2.5. *If the polynomials* $f_1, \ldots, f_m$ *are completely nonvanishing on the positive orthant* $\mathbb{R}^n_+$ (*as in Definition* 2.2) *and the Newton polytope* $\Delta_{f_1 \cdots f_m}$ *is of full dimension* n, *then the Euler–Mellin integral* $M_f(s,t)$ *admits a meromorphic continuation of the form*

$$M_f(s,t) = \Phi_f(s,t) \prod_{k=1}^{N} \Gamma(\langle \mu_k, s\rangle - \langle \nu_k, t\rangle), \tag{2.7}$$

where $\Phi_f(s,t)$ *is an entire function, and* μ_k, ν_k *are given by* (2.6). *We call* $\Phi_f(s,t)$ *an* extended Euler–Mellin integral.

Proof. By Theorem 2.3, the original Euler–Mellin integral $M_f(s,t)$ of (2.1) converges on

$$\{(s,t) \in \mathbb{C}^{n+m} \mid \tau := \mathrm{Re}(t) \in \mathbb{R}^m_+,\\ \sigma := \mathrm{Re}(s) \text{ such that } \langle \mu_k, \sigma\rangle > \langle \nu_k, \tau\rangle \text{ for all } 1 \le k \le N\},$$

which is a domain since Δ_f is of full dimension. Our goal is to expand the convergence domain of the integral (2.1) at the cost of multiplication by factors corresponding to the poles of the gamma functions appearing in (2.7). We do this iteratively, integrating by parts in the direction of a vector μ_k at each step. This expands the domain of convergence in the opposite direction of μ_k by a distance d_k, which we determine explicitly.

To begin, we set the notation for the first iteration in one direction. Fix k between 1 and N, and let Γ be the face of Δ_{f_i} corresponding to μ_k and ν_k. For $\alpha \in \mathrm{supp}(f)$, consider the integers

$$d_k^\alpha := \langle \mu_k, \alpha\rangle - |\nu_k|.$$

Since $\alpha \in \Delta_f$, it follows that $d_k^\alpha \ge 0$. In particular, since there is a decomposition $\alpha = \sum_i \alpha_i$ with $\alpha_i \in \Delta_{f_i}$, we see that $d_k^\alpha = 0$ if and only if $\langle \mu_k, \alpha_i\rangle = \nu_k^i$ for all i.

For a fixed i, the polynomial $(f_i)_\Gamma$ has the homogeneity $(f_i)_\Gamma(\lambda^{\mu_k} z) = \lambda^{\nu_k^i}(f_i)_\Gamma(z)$, where λ is any nonzero complex number, and $\lambda^{\mu_k} z = (\lambda^{\mu_k^1} z_1, \lambda^{\mu_k^2} z_2, \ldots, \lambda^{\mu_k^n} z_n)$. Hence, the coefficients of the scaled polynomial $\lambda^{-\nu_k^i}(f_i)_\Gamma(\lambda^{\mu_k} z)$ are independent of k and the λ. In particular, we have that the Newton polytope of

$$f_i'(z) := \frac{d}{d\lambda}(\lambda^{-\nu_k^i} f_i(\lambda^{\mu_k} z))\Big|_{\lambda=1}$$

is disjoint from Γ. This fact allows us to extend the domain of convergence of (2.1) over the hyperplane defined by $\langle \mu_k, \sigma\rangle = \langle \nu_k, \tau\rangle$ as follows. Since $M_f(s,t)$ is independent of λ, we have

$$0 = \frac{d}{d\lambda}\int_{\mathbb{R}^n_+} \frac{(\lambda^{\mu_k} z)^s}{f(\lambda^{\mu_k} z)^t}\frac{dz}{z} = \frac{d}{d\lambda}\left[\lambda^{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle}\int_{\mathbb{R}^n_+} \frac{z^s}{\lambda^{-\langle \nu_k, t\rangle} f(\lambda^{\mu_k} z)^t}\frac{dz}{z}\right].$$

Thus, differentiating (2.1) with respect to λ and setting $\lambda = 1$ yields the identity

$$M_f(s,t) = \frac{1}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle}\int_{\mathbb{R}^n_+} \frac{z^s g_k(z)}{f(z)^{t+1}}\frac{dz}{z}, \tag{2.8}$$

where g_k is the polynomial

$$g_k = \sum_{i=1}^{m} t_i \cdot f_1 \cdots f_i' \cdots f_m.$$

Note that $\operatorname{supp}(g_k)$ is contained in $\operatorname{supp}(f)$; moreover, since Γ is the face of Δ_f corresponding to μ_k and $\operatorname{supp}(f_i')$ is disjoint from $\Delta_{f_i} \cap \Gamma$, we see that $\operatorname{supp}(g_k)$ is disjoint from Γ. In other words, $d_k^\alpha > 0$ for each $\alpha \in \operatorname{supp}(g_k)$.

Rewrite (2.8) as the sum

$$M_f(s,t) = \sum_{\alpha \in \operatorname{supp}(g_k)} \frac{h_\alpha(t)}{\langle \mu_k, s\rangle - \langle \nu_k, t\rangle} \int_{\mathbb{R}_+^n} \frac{z^{s+\alpha}}{f(z)^{t+1}} \frac{dz}{z} \tag{2.9}$$

for some linear polynomials $h_\alpha(t)$, noting that each term of (2.9) is a translation of the original Euler–Mellin integral. By Theorem 2.3, the term corresponding to α converges on the domain given by $\tau + 1 > 0$ and

$$\langle \mu_j, \sigma + \alpha\rangle > \langle \nu_j, \tau + 1\rangle \quad \text{for } j = 1, \ldots, N,$$

where the latter is equivalent to

$$\langle \mu_j, \sigma\rangle > \langle \nu_j, \tau + 1\rangle - \langle \mu_j, \alpha\rangle = \langle \nu_j, \tau\rangle - d_j^\alpha \quad \text{for } j = 1, \ldots, N.$$

The sum (2.9) converges on the intersection of these domains, which is given by

$$\begin{aligned} &\tau + 1 > 0, \\ &\langle \mu_j, \sigma\rangle > \langle \nu_j, \tau\rangle \quad \text{if } j \neq k, \quad \text{and} \\ &\langle \mu_k, \sigma\rangle > \langle \nu_k, \tau\rangle - d_k, \end{aligned}$$

where $d_k := \min\{d_k^\alpha \mid \alpha \in \operatorname{supp}(g_k)\}$. Since d_k is by definition strictly greater than 0, (2.9) has a strictly larger domain of convergence than (2.1); we say that it has been extended by the "distance" d_k in the direction determined by μ_k.

Before iterating this procedure, we set some notation. Let G_k be the semigroup generated by the integers $\{d_k^\alpha \mid \alpha \in \operatorname{supp}(f) \text{ and } 1 \le k \le N\} \subseteq \mathbb{N}$. Let $\beta = (\alpha_1, \ldots, \alpha_q)$ be an ordered q-tuple with $\alpha_i \in \operatorname{supp}(f)$ for each i. We sometimes write β as an exponent of z, viewing $\beta = \alpha_1 + \cdots + \alpha_q$. Similarly, set $d_k^\beta := d_k^{\alpha_1} + \cdots + d_k^{\alpha_q} \in G_k$.

Now after q iterations, let $\mu_{j(i)}$ denote the direction of the extension in the ith iteration. Let $d_{j(i)}^{\beta_i} := d_{j(i)}^{\alpha_1} + \cdots + d_{j(i)}^{\alpha_{i-1}} \in G_{j(i)}$ be the sum of the distances of the first $i-1$ components of β in the direction $\mu_{j(i)}$. Then there is a rational function of the type

$$L_\beta(s,t) = \prod_{i=1}^{q} \frac{h_{\beta_i}(t)}{\langle \mu_{j(i)}, s\rangle - \langle \nu_{j(i)}, t\rangle + d_{j(i)}^{\beta_i}}, \tag{2.10}$$

where $h_\beta(t) := (h_{\beta_1}(t), \ldots, h_{\beta_q}(t))$ is an ordered q-tuple of linear polynomials such that M_f can be expressed as a finite sum of translations of the original Euler–Mellin integral:

$$M_f(s,t) = \sum_\beta L_\beta(s,t) \int_{\mathbb{R}_+^n} \frac{z^{s+\beta}}{f(z)^{t+q}} \frac{dz}{z}. \tag{2.11}$$

Fixing k, we next expand the domain of convergence of (2.11) in the direction determined by μ_k. This is achieved through simultaneous expansion of the domains of convergence of all terms, arguing as above. This yields the expression

$$M_f(s,t) = \sum_\beta L_\beta(s,t) \sum_{\alpha\in\operatorname{supp}(g_k)} \frac{h_{(\beta,\alpha)_{q+1}}(t)}{\langle\mu_k, s\rangle - \langle\nu_k, t\rangle + d_k^\beta} \int_{\mathbb{R}_+^n} \frac{z^{s+\beta+\alpha}}{f(z)^{t+q+1}} \frac{dz}{z}$$

$$= \sum_{\beta'} L_{\beta'}(s,t) \int_{\mathbb{R}_+^n} \frac{z^{s+\beta'}}{f(z)^{t+q'}} \frac{dz}{z}, \tag{2.12}$$

where $\beta' = (\beta, \alpha)$, $q' = q+1$, and the resulting rational function $L_{\beta'}(s,t)$ is given by

$$L_{\beta'}(s,t) = L_\beta(s,t) \frac{h_{\beta'_{q'}}(t)}{\langle\mu_k, s\rangle - \langle\nu_k, t\rangle + d_k^\beta}.$$

Since the convergence domain of each term in (2.11) is extended by the distance d_k in the direction determined by μ_k, the convergence domain of the sum is similarly extended. In addition, since $d_k^\alpha > 0$, we have that $d_k^{\beta+\alpha} > d_k^\beta$; therefore, the products $L_\beta(s,t)$ will never repeat factors in their denominators. As (2.12) is in the same form as (2.11), we may iterate this procedure to extend the domain of convergence.

Finally, note that after q iterations that have extended the domain of convergence of $M_f(s,t)$ in the direction determined by μ_j for q_j of the q steps, we obtain a meromorphic function on the tube domain given by $(s,t) \in \mathbb{C}^{n+m}$ such that $\tau + \sum_{j=1}^N q_j = \tau + q > 0$ and

$$\langle\mu_j, \sigma\rangle > \langle\nu_j, \tau\rangle - q_j d_j \quad \text{for } j = 1, \dots, N.$$

Continuing this process, $M_f(s,t)$ can be extended to a meromorphic function on $\mathbb{C}^{n+m}$ as in (2.7). We note that because the denominator of the products of the rational functions $L_\beta(s,t)$ never has repeated terms, all poles of the extended Euler–Mellin integral are simple. It now follows from the removable singularities theorem that $\Phi_f(s,t)$ in (2.7) is an entire function, as desired. □

The entire function $\Phi_f(s,t)$ is of great interest to the study of A-hypergeometric functions. The gamma functions appearing in (2.7) may introduce some unnecessary zeros in the meromorphic continuation of the Euler–Mellin integral, which hinder A-hypergeometric applications.

REMARK 2.6. In the proof of Theorem 2.5, we see that the linear form $\langle\mu_k, \sigma\rangle - \langle\nu_k, \tau\rangle - d$ appears in the denominator of some rational function L_β if and only if $d \in G_k$. Hence, if $G_k \neq \mathbb{N}$, then our meromorphic continuation has introduced unnecessary zeros into the entire function $\Phi_f(s,t)$.

REMARK 2.7. If $m = 1$, then $h_{\beta_i}(t) = k_{\beta_i}(t+i)$ for some constant k_{β_i}, where h_{β_i} is as in (2.10). Therefore, each L_β is divisible by $(t)_{i+1} = t(t+1)\cdots(t+i)$, which can thus be factored outside the sum (2.11). In particular, $\tilde{\Phi}_f(s,t) := \Gamma(t)\Phi_f(s,t)$ is an entire function.

We conclude this section with examples to illustrate Theorem 2.5 and our recent remarks.

EXAMPLE 2.8. Consider the case of $m+1$ linear functions of one variable,

$$M_f(s,t) = \int_0^\infty \frac{z^s}{(1+z)^{t_0}(c_1+z)^{t_1}\cdots(c_m+z)^{t_m}}\frac{dz}{z}. \tag{2.13}$$

Note that we have reindexed t for this example. When $m=0$, (2.13) is the beta function. Here $\Phi_f(s,t) = 1/\Gamma(t)$, or with the notation of Remark 2.7, $\tilde{\Phi}_f(s, t) = 1$. When $m=1$, we showed in Example 2.4 that

$$\Phi_f(s,t) = \frac{1}{\Gamma(t_0+t_1)}{}_2F_1(t_1, t_0+t_1-s; t_0+t_1; 1-c_1).$$

This equality is obtained by the change of variables $w = z/(1+z)$ and application of the generalized binomial theorem. By similar calculations for $m=2$,

$$\Phi_f(s,t) = \frac{1}{\Gamma(t_0+t_1+t_2)}F_1(t_0+t_1+t_2-s, t_1, t_2; t_0+t_1+t_2; 1-c_1, 1-c_2),$$

where F_1 denotes the first Appell series. For arbitrary m and $|c_i|<1$,

$$\Phi_f(s,t) = \frac{1}{\Gamma(t_0+|t|)}\sum_{k\in\mathbb{N}^m}\frac{(t_0+|t|-s)_{|k|}}{(t_0+|t|)_{|k|}}\frac{(t)_k}{k!}(1-c)^k,$$

where $t=(t_1,\ldots,t_m)$, $|t| = t_1+\cdots+t_m$, and $(t)_k = (t_1)_{k_1}\cdots(t_m)_{k_m}$.

EXAMPLE 2.9. Finally, consider the case of one linear function of n variables,

$$M_f(s,t) = \int_{\mathbb{R}_+^n}\frac{z_1^{s_1}\cdots z_n^{s_n}}{(1+z_1+\cdots+z_n)^t}\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n}.$$

We claim that

$$M_f(s,t) = \frac{\Gamma(s_1)\cdots\Gamma(s_n)\Gamma(t-s_1-\cdots-s_n)}{\Gamma(t)},$$

and hence $\Phi_f(s,t) = 1/\Gamma(t)$. This is clear when $n=1$ because we again have the beta function. For $n>1$, we can argue by induction, making the change of variables given by $w_n = z_n$ and $w_i = z_i/(1+z_n)$ for $i\neq n$. To generalize this example to an arbitrary simplex, consider the Euler–Mellin integral

$$M_f(s,t) = \int_{\mathbb{R}_+^n}\frac{z_1^{s_1}\cdots z_n^{s_n}}{(1+z^{T_1}+\cdots+z^{T_n})^t}\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n},$$

where the exponent vectors T_i are the columns of an invertible matrix T. By the change of variables $w_i = z^{T_i}$ we find that

$$M_f(s,t) = \frac{\Gamma((T^{-1}s)_1)\cdots\Gamma((T^{-1}s)_n)\Gamma(t-|T^{-1}s|)}{|\det(T)|\Gamma(t)}.$$

3. Relation to Coamoebas

For Theorems 2.3 and 2.5 to hold, each $f_i(z)$ must be completely nonvanishing on the positive orthant. This is a strong restriction that many polynomials do not fulfill. However, the goal of this section is to modify this hypothesis by considering the coamoeba of $f(z)$.

The amoeba $\mathcal{A}_f$ and the coamoeba $\mathcal{A}'_f$ of a polynomial f are defined to be the images of the zero set $Z_f = \{z \in (\mathbb{C}_*)^n \mid f(z) = 0\}$ under the real and imaginary parts of the coordinate-wise complex logarithm mapping, Log and Arg, respectively. More precisely, if $\mathrm{Log}(z) = (\log|z_1|, \dots, \log|z_n|)$ and $\mathrm{Arg}(z) = (\arg(z_1), \dots, \arg(z_n))$, then the amoeba and coamoeba of f are, respectively,

$$\mathcal{A}_f := \mathrm{Log}(Z_f) \quad \text{and} \quad \mathcal{A}'_f := \mathrm{Arg}(Z_f). \tag{3.1}$$

The amoeba $\mathcal{A}_f$ lies in $\mathbb{R}^n$; however, since the argument mapping is multivalued, the coamoeba $\mathcal{A}'_f$ can be viewed either in the n-dimensional torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ or as a multiply periodic subset of $\mathbb{R}^n$. Amoebas were introduced by Gelfand, Kapranov, and Zelevinsky [GKZ94], whereas the term coamoeba was first used by the third author in 2004 at a conference at Johns Hopkins University.

PROPOSITION 3.1. *For $\theta \in \mathbb{T}^n$, a polynomial $f(z)$ is completely nonvanishing on the set* $\mathrm{Arg}^{-1}(\theta)$ *if and only if $\theta \notin \overline{\mathcal{A}'_f}$.*

Proof. The claim is equivalent to the statement

$$\overline{\mathcal{A}'_f} = \bigcup_{\Gamma} \mathcal{A}'_{f_\Gamma},$$

where f_Γ is the truncated polynomial with support Γ. This has been proven by Johansson [Joh10] and independently by Nisse and Sottile [NS11]. □

By Proposition 3.1, when polynomials $f_1, \dots, f_m$ are such that the closure of the coamoeba of $f(z) = \prod_{i=1}^m f_i(z)$ is a proper subset of $\mathbb{T}^n$, there is a $\theta \notin \overline{\mathcal{A}'_f}$ for which the Euler–Mellin integral with respect to θ is well defined:

$$M_f^\theta(s,t) := \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{f(z)^t} \frac{dz}{z}. \tag{3.2}$$

Note that after fixing the matrix A, and hence the set of monomials of the polynomials f_i, any choice of coefficients for the f_i with positive real part ensures that $0 \in \mathbb{T}^n$. In particular, this means that there is always a choice of coefficients for the f_i so that $\overline{\mathcal{A}'_f}$ is a proper subset of $\mathbb{T}^n$. For another discussion on the components of $\mathbb{T}^n \setminus \overline{\mathcal{A}'_f}$, see the end of Section 4.

As (3.2) differs from our earlier definition of the Euler–Mellin integral in (2.1) only by a change of variables, it is immediate that θ-analogues of Theorems 2.3 and 2.5 hold. In addition, a slight perturbation of θ does not impact the value of (3.2).

THEOREM 3.2. *The Euler–Mellin integral M_f^θ of (3.2) is a locally constant function in θ. Thus, it depends only on the choice of connected component Θ of the* *complement of* $\overline{\mathcal{A}'_f}$, *and we thus write* $M_f^{\Theta} := M_f^{\theta}$. *Accordingly, there is an extended Euler–Mellin integral* $\Phi_f^{\Theta} := \Phi_f^{\theta}$ *given by a meromorphic continuation of* M_f^{Θ}.

Proof. First, consider the case $n = 1$ and suppose that θ_1 and θ_2 lie in the same connected component of the complement of $\overline{\mathcal{A}'_f}$; in fact, assume that the interval $[\theta_1, \theta_2] \subseteq \mathbb{T}^n \setminus \overline{\mathcal{A}'_f}$. In other words, $f(z)$ has no zeros with arguments in this interval, and hence $z^{s-1}/f(z)^t$ is analytic in the corresponding domain. Connecting the two rays $\mathrm{Arg}^{-1}(\theta_1)$ and $\mathrm{Arg}^{-1}(\theta_2)$ with the circle section of radius r yields a closed curve, and the integral of $z^{s-1}/f(z)^t$ over this (oriented) curve is zero by residue calculus. By the proof of Theorem 2.3, the integral over the circle section tends to 0 as $r \to \infty$, and so the two Euler–Mellin integrals $M_f^{\theta_1}$ and $M_f^{\theta_2}$ are equal.

In arbitrary dimensions, we obtain the desired equality by considering one variable at a time while the remaining variables are fixed. □

EXAMPLE 3.3. Revisiting the polynomial $f(z_1, z_2) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2$ from the Introduction, we see that if we choose $\theta = (\arg(c_1/c_2), \arg(c_1/c_3))$, then

$$\Phi_f^{\Theta}(s_1, s_2, t) = \frac{c_1^{s_1+s_2-t} c_2^{-s_1} c_3^{-s_2}}{\Gamma(t)^2} {}_2F_1\left(s_1, s_2; t; 1 - \frac{c_1 c_4}{c_2 c_3}\right),$$

where Θ is the component of the complement of $\overline{\mathcal{A}'_f}$ containing θ. By Remark 2.7, we may ignore one of the $\Gamma(t)$ in the denominator, and ${}_2F_1/\Gamma(t)$ is the regularized Gauss hypergeometric function.

4. Integral Representations of A-Hypergeometric Functions

We now fix a connected component Θ of the complement of $\overline{\mathcal{A}'_f}$ and study the entire function $\Phi_f(s,t) = \Phi_f^{\Theta}(s,t)$ from (3.2). In particular, we consider its dependence on the coefficients $c_i = \{c_{i,\alpha}\}$ of the polynomials f_i, where $f(z) = \prod_{i=1}^m f_i$ and $f_i = \sum_{j=1}^{r_i} c_{ij} z^{\alpha_{ij}}$ (so $r_i = |\operatorname{supp}(f_i)|$). In order to emphasize this dependence, we write $\Phi_f(s,t,c)$ rather than $\Phi_f(s,t)$. Generalizing [NP13, Section 6], we show that the extended Euler–Mellin integral $\Phi_f(s,t,c)$ is an A-hypergeometric function in the sense of Gelfand, Kapranov, and Zelevinsky. More precisely, Theorem 4.2 states that $c \mapsto \Phi_f(s,t,c)$ satisfies the A-hypergeometric system of partial differential equations, where the exponents α_{ij} of the f_i provide a matrix A via the Cayley trick,

$$A = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ \alpha_{11} & \cdots & \alpha_{1r_1} & \alpha_{21} & \cdots & \alpha_{2r_2} & & \alpha_{m_1} & \cdots & \alpha_{mr_m} \end{bmatrix} \in \mathbb{Z}^{(m+n)\times r}, \tag{4.1}$$

where $r := \sum_{i=1}^{m} r_i$, and the desired homogeneity parameter is $\beta = -(t,s)$.

We now recall the definition of an A-hypergeometric system. For a vector $v \in \mathbb{Z}^r$, denote by u_+ and u_- the unique vectors in $\mathbb{N}^r$ with disjoint support such that $u = u_+ - u_-$.

Definition 4.1. Let $A = (a_{ij}) \in \mathbb{Z}^{(m+n)\times r}$ be a matrix. Define the differential operators $\Box_u$ and E_i to be

$$\Box_u := \left(\frac{\partial}{\partial c}\right)^{u_+} - \left(\frac{\partial}{\partial c}\right)^{u_-} \quad \text{and} \quad E_i := \sum_{j=1}^{r} a_{ij}\frac{\partial}{\partial c_j}.$$

The *A-hypergeometric system* $H_A(\beta)$ at $\beta \in \mathbb{C}^{m+n}$ is given by

$$\begin{aligned} \Box_u F(c) &= 0 \quad \text{for } u \in \mathbb{Z}^r \text{ with } Au = 0 \\ \text{and} \quad (E_i - \beta_i)F(c) &= 0 \quad \text{for } 1 \le i \le m+n. \end{aligned}$$

A local multivalued analytic function F that solves this system is called an *A-hypergeometric function* with homogeneity parameter β. Such solutions of $H_A(\beta)$ form a $\mathbb{C}$-vector space.

The ideal I_A cuts out an affine variety $X_A \subseteq \mathbb{C}^r$, which has an action of an algebraic torus $(\mathbb{C}_*)^{m+n}$. To understand the role of the Euler operators $E_i - \beta_i$, note that a germ of an analytic function at a nonsingular point $c \in \mathbb{C}^r$ that is annihilated by $c_1\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial c_2} + \cdots + c_r\frac{\partial}{\partial c_r} - \beta_0$ is homogeneous, in the usual sense, of degree β_0. In general, the Euler operators in $H_A(\beta)$ force solutions to have weighted homogeneities. From this point of view, it becomes natural to fix A and view β as a parameter of $H_A(\beta)$.

We now consider the behavior of the entire function $(s,t) \mapsto \Phi_f(s,t,c)$, as described in Theorem 2.5, when c is viewed as a variable. Let $\Sigma_A \subseteq \mathbb{C}^r$ denote the singular locus of all A-hypergeometric functions, which is the hypersurface defined by the principal A-determinant (also known as the full A-discriminant) [GKZ94].

Theorem 4.2. *Let $c \in \mathbb{C}^r \setminus \Sigma_A$, and let Θ be a connected component of $\mathbb{R}^n \setminus \overline{\mathcal{A}'_f}$, where f is the polynomial $f(z) = \prod_{i=1}^{m} f_i$ with $f_i = \sum_{j=1}^{r_i} c_{ij} z^{\alpha_{ij}}$. Then, for any $\theta \in \Theta$, the analytic germ $\Phi_f^{\Theta}(s,t,c)$ has a (multivalued) analytic continuation to $\mathbb{C}^{m+n} \times (\mathbb{C}^r \setminus \Sigma_A)$ that is everywhere A-hypergeometric (in the variables c) with homogeneity parameter $\beta = -(t,s)$.*

Proof. Let us first consider the case $\tau := \operatorname{Re} t > 0$ and $\sigma := \operatorname{Re} s \in \operatorname{int}(\tau\Delta_f)$, where we have

$$\Phi_f^{\Theta}(s,t,c) = \frac{1}{\prod_k \Gamma(\langle \mu_k, s\rangle - \langle \nu_k, t\rangle)} \int_{\operatorname{Arg}^{-1}(\theta)} \frac{z^s}{f(z)^t}\frac{dz}{z}. \tag{4.2}$$

Fix a representative $\theta \in \Theta$. Since θ is disjoint from $\overline{\mathcal{A}'_f}$ for polynomials f with coefficients c near the original ones, say in a small ball $B(c)$, the integral in (4.2) does indeed define an analytic germ $\Phi_f = \Phi_f^{\theta}(s,t,c)$. By Theorem 2.5, Φ_f can be extended to an entire function with respect to the variables s and t. In other words, Φ_f has been analytically extended to the infinite cylinder $\mathbb{C}^{m+n} \times B(c)$.

To see that Φ_f is an A-hypergeometric function with homogeneity parameter β as given, we fix s and t under the above condition, noting that the product of gamma functions in Φ_f is simply a nonzero constant. Thus, it is enough to show that the integral itself is A-hypergeometric at β. This is accomplished through the argument of [SST00, Theorem 5.4.2], which applies since differentiation and integration may be interchanged because Euler–Mellin integrals are uniformly convergent by the bound in (2.3). See also [GKZ90, Remark 2.8(b)].

Having established that Φ_f is an A-hypergeometric function in the product domain given by (s,t,c) in $(\mathbb{R}_+ \operatorname{int}(\tau\Delta_f)+i\mathbb{R}^n)\times(\mathbb{R}^m_+ \times i\mathbb{R}^m)\times B(c)$, it follows from the uniqueness of analytic continuation that its extension to the cylinder $\mathbb{C}^{m+n}\times B(c)$ will remain A-hypergeometric. Now, for each fixed (s,t), there is a (typically multivalued) analytic continuation of $c \mapsto \Phi_f = \Phi^\theta_f(s,t,c)$ from $B(c)$ to all of $\mathbb{C}^r \setminus \Sigma_A$. As these continuations still depend analytically on s and t, we have now achieved the desired analytic continuation to the full product domain $\mathbb{C}^{m+n}\times(\mathbb{C}^r \setminus \Sigma_A)$. The uniqueness of analytic continuation again guarantees that Φ_f will everywhere satisfy the A-hypergeometric system with the homogeneity parameter β, as desired. □

A parameter β is *nonresonant* if $\beta + \mathbb{Z}^{n+m}$ does not meet any facet of the cone $\mathbb{R}_{\geq 0}A$ spanned by A. When β is nonresonant, the dimension of the solution space of $H_A(\beta)$ is equal to $\operatorname{vol}(A)$, which is $(m+n)!$ times the Euclidean volume of the convex hull of A and the origin. In general, $\operatorname{vol}(A)$ is a lower bound for this dimension.

We turn now to the question of constructing a basis of solutions of $H_A(\beta)$ via extended Euler–Mellin integrals arising from different connected components of the complement of the coamoeba of f. Before this question can be answered fully, we must gain a better knowledge of the geometry of coamoebas. Indeed, in order to construct a basis of solutions of $H_A(\beta)$ consisting of extended Euler–Mellin integrals, one must first find a coamoeba with the correct number of connected components of its complement. In the context of this article, it may come as no surprise that the conjectured maximal number of connected components of a (hypersurface) coamoeba is the normalized volume $\operatorname{vol}(A)$ of the appropriate Newton polytope. The paper [N08] discusses this issue; see also the examples in [For12, Section 2.5].

5. Bases of Solutions at Nonresonant Parameters in Low Dimension

We show in this section that if $n = 1$ and β is nonresonant, then the extended Euler–Mellin integrals given by the components of $\mathbb{T}^1 \setminus \overline{\mathcal{A}'_f}$ form a basis of solutions of $H_A(\beta)$. We then illustrate how similar methods can be used to show the linear independence of the extended Euler–Mellin integrals for the Gauss A-hypergeometric system, for which $n = 2$, at nonresonant parameters.

PROPOSITION 5.1. *If $\beta = -(t, s)$ is nonresonant and $n = 1$, then the extended Euler–Mellin integrals $\Phi_f^\Theta(s, t, c)$, where Θ ranges over the components of $\mathbb{T}^1 \setminus \overline{\mathcal{A}'_f}$, form a basis of solutions of the A-hypergeometric system $H_A(\beta)$.*

Proof. For a generic choice of coefficients c, f has distinct roots $r_1, r_2, \dots, r_{\mathrm{vol}(A)}$ with distinct arguments $0 \le \theta_1 < \theta_2 < \cdots < \theta_{\mathrm{vol}(A)} < 2\pi$. That is, $\overline{\mathcal{A}'_f}$ will have $\mathrm{vol}(A)$-many components in its complement in $\mathbb{T}^1$. For convenience, we use the convention that $\theta_{\mathrm{vol}(A)+1} := \theta_1$. Now fix $0 < \varepsilon \ll 1$ so that the circles $B_\varepsilon(r_i) := \{z \in \mathbb{C} \mid |z - r_i| = \varepsilon\}$, viewed as 1-chains oriented counterclockwise, have disjoint supports for $1 \le i \le \mathrm{vol}(A)$. By [GKZ90], the integrals

$$\int_{B_\varepsilon(r_i)} \frac{z^s}{f(z)^t} \frac{dz}{z} \quad \text{for } 1 \le i \le \mathrm{vol}(A)$$

are linearly independent, forming a basis for the solution space of $H_A(\beta)$. By the convergence of $M_f^{\theta_i}(s, t, c)$ from Theorems 2.3 and 2.5 we see that as non-compact chains, $\mathrm{Arg}^{-1}(\theta_i) - \mathrm{Arg}^{-1}(\theta_{i+1})$ and $B_\varepsilon(r_i)$ are homologous, providing the second statement. For a nongeneric choice of coefficients c for f, which still lie away from Σ_A, a similar argument implies linear independence of the Euler–Mellin integrals given by the distinct components of the complement of $\overline{\mathcal{A}'_f}$. □

EXAMPLE 5.2. In Example 3.3, it was shown that if $f(z) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2$ and θ is near $(\arg(c_1/c_2), \arg(c_1/c_3))$, then

$$\Phi_f^\theta(s, t, c) = \frac{c_1^{s_1+s_2-1} c_2^{-s_1} c_3^{-s_2}}{\Gamma(t)^2} {}_2F_1\left(s_1, s_2; t; 1 - \frac{c_1 c_4}{c_2 c_3}\right).$$

By Theorem 4.2 this is a solution of $H_A(\beta)$ when

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \beta = -(t, s).$$

Now consider points c of the form $(1, i, i, c_4)$, where c_4 is near 1, and define the polynomials

$$f_\rho := 1 + e^{(\pi i/2)\rho} z_1 + e^{(\pi i/2)\rho} z_2 + c_4 z_1 z_2 \quad \text{for } 0 \le \rho \le 1$$
$$\text{and} \quad g_\rho := 1 + e^{(\pi i/2)(2-\rho)} z_1 + e^{(\pi i/2)(2-\rho)} z_2 + c_4 z_1 z_2 \quad \text{for } 0 \le \rho \le 1.$$

As shown in Figure 1, the complement of the coamoeba for $f_1 = g_1$ has two connected components, one containing $(0, 0)$ and another containing (π, π). These yield two solutions of $H_A(\beta)$ at $c = (1, i, i, c_4)$ by Theorem 4.2, namely, $\Phi_{f_1}^{(0,0)}(s, t, c)$ and $\Phi_{f_1}^{(\pi,\pi)}(s, t, c)$. In addition, $(0, 0) \notin \overline{\mathcal{A}'_{f_\rho}}$ and $(\pi, \pi) \notin \overline{\mathcal{A}'_{g_\rho}}$ for all ρ, so we let $\Phi_{f_\rho}^{(0,0)}(s, t, c)$ and $\Phi_{g_\rho}^{(\pi,\pi)}(s, t, c)$ denote the entire functions corresponding to f_ρ and g_ρ, respectively.

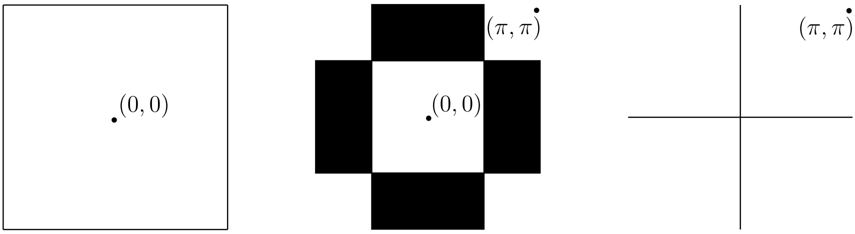

Figure 1 The coamoebas of the polynomials f_0, $f_1 = g_1$, and g_0, respectively, shown inside the fundamental domain $[-\pi,\pi]\times[-\pi,\pi]$ of $\mathbb{T}^2$ in $\mathbb{R}^2$, have been colored in black

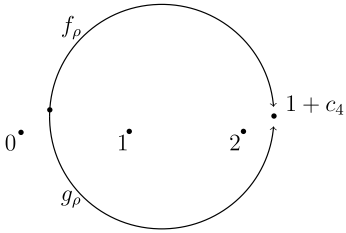

Figure 2 The loop L in the $(1 + c_4)$-complex plane is given by the reverse of the arrow labeled f_ρ, followed by the arrow labeled g_ρ

Let L be the loop in the coefficient space given by first following the coefficients in the reverse of f_ρ and then those in g_ρ. Since $f_1 = g_1$ and $\Phi_{g_0}^{(\pi,\pi)}(s,t,c) = e^{(s_1+s_2)\pi i}\,\Phi_{f_0}^{(0,0)}(s,t,c)$, we can explicitly perform an analytic continuation of $\Phi_{f_1}^{(0,0)}(s,t,c)$ along the loop L; see Figure 2. When the monodromy of $H_A(\beta)$ is irreducible, it follows that $\Phi_{f_1}^{(0,0)}(s,t,c)$ and $\Phi_{f_1}^{(\pi,\pi)}(s,t,c)$ form a basis for the solution space of $H_A(\beta)$ of analytic germs at $(1,i,i,c_4)$. Since the monodromy irreducibility of $H_A(\beta)$ is equivalent to the nonresonance of β [Beu11b; SW10], the conclusion of Proposition 5.1 also holds in this case.

6. Mellin–Barnes Integrals and Lopsided Coamoebas

In this section, we continue our investigation of the linear independence of extended Euler–Mellin integrals, obtaining partial results in arbitrary dimensions. To do this, we employ another class of integral representations of A-hypergeometric functions, known as *Mellin–Barnes integrals* [Nil09; Beu11a]. The main result of this section is Theorem 6.4, which identifies the set of Mellin–Barnes integral solutions of $H_A(\beta)$ with a certain subset of its set of extended Euler–Mellin integral solutions. This yields the linear independence of certain collections of extended Euler–Mellin integrals at totally nonresonant parameters β, as stated in Corollary 6.5. We say that $\beta \in \mathbb{C}^{m+n}$ is *totally nonresonant* for A if the shifted lattice

$\beta + \mathbb{Z}^{m+n}$ has empty intersection with any hyperplane spanned by any $m+n-1$ linearly independent columns of A.

The definition of a Mellin–Barnes integral requires the input of a *Gale dual* of A, which is an integer $r \times (r-m-n)$-matrix B with relatively prime maximal minors such that $AB = 0$. Typically in the sequel, we will not require that the condition on maximal minors holds; in this case, the matrix B is called a *dual matrix* of A. Connections between coamoebas and Gale duals are explored in [NP13; FJ12].

DEFINITION 6.1. Fix a Gale dual B of A, and let γ be such that $A\gamma = \beta$. Then for $c \in \mathbb{C}^r$, the *Mellin–Barnes integral* has the form

$$L(c) = L(c_1, \dots, c_r) = \int_{(i\mathbb{R})^m} \prod_{i=1}^{r} \Gamma(-\gamma_i - \langle b_i, w\rangle) c_i^{\gamma_i + \langle b_i, w\rangle}\, dw_1 \wedge \cdots \wedge dw_m. \tag{6.1}$$

Given $\theta \in \mathbb{T}^n$ and $c \in (\mathbb{C}_*)^r$, we write

$$L^\theta(c) := L(c_1 e^{i\langle \alpha_1, \theta\rangle}, \dots, c_r e^{i\langle \alpha_r, \theta\rangle}),$$

viewed as the germ of an analytic function at c.

Related to Mellin–Barnes integrals are two objects arising from a dual matrix B. These are the zonotope

$$\mathcal{Z}_B := \left\{ \frac{\pi}{2} \sum_{i=1}^{r} \mu_i b_i \;\middle|\; |\mu_i| < 1 \right\},$$

where b_i denotes the ith row of B, and the sublattice $\mathbb{Z}[B]$ of $\mathbb{Z}^{r-m-n}$ generated by $b_1, \dots, b_r$. The following result on Mellin–Barnes integrals summarizes Corollary 4.2, Theorem 3.1, and Proposition 4.3 of [Beu11a].

THEOREM 6.2 [Beu11a]. *Consider $c, c_1, \dots, c_k \in (\mathbb{C}_*)^r$.*

(1) *If* $\operatorname{Arg}(c)B \in \operatorname{int}(\mathcal{Z}_B)$, *then the integral $L(c)$ converges absolutely.*
(2) *If* $\operatorname{Arg}(c)B \in \operatorname{int}(\mathcal{Z}_B)$ *and $\gamma_i < 0$ for each i, then $L(c)$ is a solution of $H_A(\beta)$.*
(3) *If β is totally nonresonant for A and the $(r-m-n)$-tuples* $\operatorname{Arg}(c_1)B, \dots, \operatorname{Arg}(c_k)B$ *are distinct elements of the set* $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$, *then the Mellin–Barnes integrals $L(c_1), \dots, L(c_k)$ are linearly independent.*

By choosing $c_1, \dots, c_k$ as in Theorem 6.2.3, we obtain a set of linearly independent solutions to $H_A(\beta)$ that are in bijective correspondence with $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$, provided that β is sufficiently generic.

The set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$ is closely related to a certain subset of the set of connected components of the complement of the coamoeba associated to A. This relationship can be made precise through the notion of a lopsided coamoeba. Consider the polynomial

$$F(c, z) = \sum_{\alpha \in A} c_\alpha z^\alpha,$$

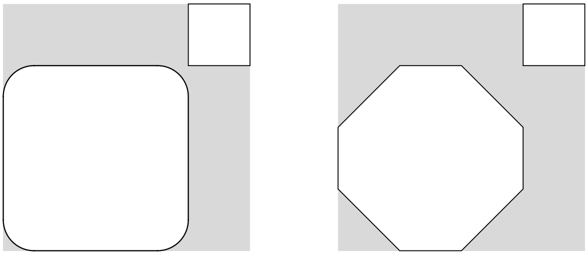

Figure 3 The coamoeba (left) and the lopsided coamoeba (right) of $f(z_1, z_2) = 1 + z_1 + z_2 + iz_1z_2$, both colored in grey

where the coefficients c are also viewed as variables. The corresponding variety has a coamoeba $\mathcal{A}'_F$, which is contained in $\mathbb{T}^{n+r}$. Given $f(z) = \sum_{\alpha \in A} c_\alpha z^\alpha$ with fixed coefficients c, the *lopsided coamoeba of f*, denoted by $\mathcal{L}\mathcal{A}'_f$, is by definition the intersection of $\mathcal{A}'_F$ with the sub-$\mathbb{T}^n$-torus of $\mathbb{T}^{n+r}$ obtained by fixing $\text{Arg}(c)$ as prescribed by f. The lopsided coamoeba $\mathcal{L}\mathcal{A}'_f$ is viewed as a subset of $\mathbb{T}^n$.

The name "lopsided coamoeba" might be misleading; the lopsided coamoeba is not per se a coamoeba, but it can be viewed as a crude approximation of one. Figure 3 provides a comparison between these objects. The following properties of lopsided coamoebas, as summarized from [FJ12, Theorem 4.1 and Propositions 4.4 and 4.5] and [For12, Theorem 2.3.10], will be used to relate Mellin–Barnes and Euler–Mellin integrals.

THEOREM 6.3 [FJ12; For12].

(1) *There is a natural inclusion $\mathcal{A}'_f \subseteq \mathcal{L}\mathcal{A}'_f$. In particular, each component of $\mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$ is contained in a component of $\mathbb{T}^n \setminus \overline{\mathcal{A}'_f}$. While this map on components is injective, it is in general not surjective.*

(2) *The lopsided coamoeba is equipped with an* order map v. *That is, there is a surjective map from the set of connected components of $\mathbb{T}^n \setminus \overline{\mathcal{A}'_f}$ to the set* $\text{int}(\mathcal{Z}_B) \cap (\text{Arg}(c)B + 2\pi\mathbb{Z}[B])$. *The fiber over each point consists of g_A many connected components, where g_A is the greatest common divisor of the maximal minors of A.*

(3) *If the polynomial f contains the constant monomial with coefficient c_0 and* $\text{Arg}(c_0)$ *is equal to zero, then for a connected component Θ of $\mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$, the value $v(\Theta)$ is*

$$v(\theta) = \arg_\pi(c_1 e^{i\langle \alpha_1, \theta\rangle}, \ldots, c_r e^{i\langle \alpha_r, \theta\rangle})B \quad \textit{some } \theta \in \Theta,$$

where $\arg_\pi$ *denotes the principal branch of the argument map.*

We now show that the order map for the set of components of $\mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$ lifts to a bijection between the set of Mellin–Barnes integrals corresponding to points in $\text{int}(\mathcal{Z}_B) \cap (\text{Arg}(c)B + 2\pi\mathbb{Z}[B])$ and the set of Euler–Mellin integrals arising from the components of $\mathbb{T}^n \setminus \mathcal{L}\mathcal{A}'_f$.

Theorem 6.4. *For all $\theta \in \mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$ and (s,t) in the tube domain* (2.2), *the Mellin–Barnes integral $L^\theta(c)$ and Euler–Mellin integral $M_f^\theta(c)$ satisfy the relation*

$$g_B L^\theta(c) = 2\pi i e^{-i\langle s,\theta\rangle}\Gamma(t) g_A M_f^\theta(c),$$

where g_A and g_B respectively denote the greatest common divisors of the maximal minors of the matrices A and B.

Proof. Since the order map ν from Theorem 6.3 sends each point in $\mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$ to a point in the set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$, the Mellin–Barnes integral $L^\theta(c)$ is convergent by Theorem 6.2.

By meromorphic extension, it is enough to give the proof in the case where the A-hypergeometric homogeneity parameter β is such that the integral expression in (2.1) converges. We may also assume that A is of the form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & A_{\mathrm{I}} & A_{\mathrm{II}} \end{bmatrix},$$

where A_{I} is a nonsingular $n \times n$-matrix; we will use the same decomposition for $c = (c_0, c_{\mathrm{I}}, c_{\mathrm{II}})$. For simplicity of notation, we will take β to be of the form $\beta = -(t, A_{\mathrm{I}}s)$. Let B denote the dual matrix of A of the form

$$B = \begin{bmatrix} -a_0 \\ A_{\mathrm{I}}^{-1}A_{\mathrm{II}} \\ -I_m \end{bmatrix} D,$$

where a_0 is chosen so that each column sum of B is zero, and D is an integer diagonal matrix chosen so that B is an integer matrix. It will be later useful that

$$\frac{g_B}{g_A} = \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|}. \tag{6.2}$$

To see this, assume that $g_A = 1$. Following [Nil09, Proposition 4.2], this implies that A can be extended to an $r \times r$ unimodular matrix

$$\widetilde{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & A_{\mathrm{I}} & A_{\mathrm{II}} \\ * & * & * \end{bmatrix} \quad \text{with inverse} \quad \widetilde{A}^{-1} = \begin{bmatrix} * & \widetilde{B} \end{bmatrix} = \begin{bmatrix} * & \tilde{b}_0 \\ * & \widetilde{B}_1 \\ * & \widetilde{B}_2 \end{bmatrix}.$$

It follows that $\widetilde{B}$ is a Gale dual of A, and by the Schur complement formula, $|A_{\mathrm{I}}| = |\widetilde{B}_2|$. Since $B = \widetilde{B}T$ for some affine transformation T, equality (6.2) thus holds.

Note that it is enough to give the proof for this particular choice of dual matrix B. Write $x_i = c^{B_i}$, where B_i denotes the ith column of B, and hence $\operatorname{Arg}(x) = \operatorname{Arg}(c)B$. Then the Euler–Mellin integral is

$$M_f^\theta(c) = \int_{\operatorname{Arg}^{-1}(\theta)} z^{A_{\mathrm{I}}s}/(c_0 + c_1 z^{\alpha_1} + \cdots + c_n z^{\alpha_n} + c_{1+n} z^{\alpha_{1+n}} + \cdots + c_{m+n} z^{\alpha_{m+n}})^t \, \frac{dz}{z}$$

$$= \frac{c_0^{|s|-t}}{c_{\mathrm{I}}{}^s} \int z^{A_{\mathrm{I}} s} / (1 + z^{\alpha_1} + \cdots + z^{\alpha_n} + x_1^{1/d_1} z^{\alpha_{1+n}} + \cdots + x_m^{1/d_m} z^{\alpha_{m+n}})^t \, \frac{dz}{z}, \tag{6.3}$$

where the integration in the second integral takes place over the domain given by the fiber of Arg over the point $\theta + \mathrm{Arg}(c_I) A_I^{-1}$. Let us denote by $M_f^\theta(x)$ the function given by the integral in (6.3). Note that $\theta \in \mathbb{T}^n \setminus \overline{\mathcal{L}\mathcal{A}'_f}$ is equivalent to the convergence of the integral

$$\int x^w M_f^\theta(x) \, \frac{dx}{x},$$

where the integration takes place over the domain given by the fiber of Arg over the point $\mathrm{Arg}(x) = \mathrm{Arg}(c)B$, and w is chosen to fulfill the requirements of Theorem 2.3. However, this integral is precisely the Mellin transform with respect to x of $M_f^\theta(x)$ with variables w. Consequently, after making the change of variables $x_i \mapsto x_i^{d_i}$, we find that

$$\begin{aligned}
&\{\mathcal{M} M_f^\theta(x)(w)\} \\
&\quad = \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|} \int \frac{z^{s - A_{\mathrm{I}}^{-1} A_2 D w} x^{Dw}}{(1 + z_1 + \cdots + z_n + x_1 + \cdots + x_m)^t} \, \frac{dz \wedge dx}{zx} \\
&\quad = \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|} \frac{\Gamma(s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} D w) \Gamma(Dw) \Gamma(t - |Dw| - |s| + |A_{\mathrm{I}}^{-1} A_{\mathrm{II}} D w|)}{\Gamma(t)},
\end{aligned}$$

by Example 2.9. For γ in (6.1), write $\gamma = (\gamma_0, \gamma_{\mathrm{I}}, \gamma_{\mathrm{II}})$. Assuming that $s_j > 0$ for all j, that $t > |s|$ (note that this is in accordance with our previous assumptions on β), and that $-1 \gg \gamma_{\mathrm{II}} > 0$, we set $\gamma_{\mathrm{I}} = -s - A_{\mathrm{I}}^{-1} A_{\mathrm{II}} \gamma_{\mathrm{II}}$ and $\gamma_0 = |s| - t + \langle b_0, \gamma_{\mathrm{II}} \rangle$. It follows that $\gamma_k < 0$ for all k. With this notation,

$$\{\mathcal{M} M_f^\theta(x)(w)\} = \frac{|\det(D)|}{|\det(A_{\mathrm{I}})|} \frac{\prod_{i=1}^r \Gamma(-\gamma_i - \langle b_i, w - \gamma_{\mathrm{II}} \rangle)}{\Gamma(t)}.$$

Furthermore, with a_i denoting the $(i+1)$th column of A,

$$\sum_{i=0}^{r-1} \gamma_i a_i = A\gamma = \begin{bmatrix} -t \\ -A_{\mathrm{I}} s \end{bmatrix}.$$

Turning to the Mellin–Barnes integral, we find that

$$\begin{aligned}
L^\theta(c) &= \int_{(i\mathbb{R})^m} \prod_{i=1}^r \Gamma(-\gamma_i - \langle b_i, w \rangle) c_i^{\gamma_i + \langle b_i, w \rangle} \, dw \\
&= \int_{\gamma_{\mathrm{II}} + (i\mathbb{R})^m} \prod_{i=1}^r \Gamma(-\gamma_i - \langle b_i, w - \gamma_{\mathrm{II}} \rangle) c_i^{\gamma_i + \langle b_i, w - \gamma_{\mathrm{II}} \rangle} \, dw \\
&= \frac{c_0^{|s|-t}}{e^{i \langle A_{\mathrm{I}} s, \theta \rangle} c_{\mathrm{I}}^s} \int_{\gamma_{\mathrm{II}} + (i\mathbb{R})^m} \left(\prod_{i=1}^r \Gamma(-\gamma_i - \langle b_i, w - \gamma_{\mathrm{II}} \rangle) \right) \frac{dw}{x^w}.
\end{aligned}$$

The bounds in the proof of Theorem 2.3 imply that we can apply the Mellin inversion formula, which yields the equality

$$|\det(D)|L^{\theta}(c) = 2\pi i e^{-i\langle A_{\mathrm{I}}s,\theta\rangle}\Gamma(t)|\det(A_{\mathrm{I}})|M_f^{\theta}(c).$$

Applying (6.2) thus completes the proof. □

COROLLARY 6.5. *If β is totally nonresonant for A, then when viewed as analytic germs at some $c \in \mathbb{C}^r \setminus \Sigma_A$, the extended Euler–Mellin integrals $\Phi_f^{\Theta}(s,t,c)$, where Θ ranges over the components of $\mathbb{T}^n \setminus \overline{\mathcal{LA}'_f}$, are linearly independent solutions of the A-hypergeometric system $H_A(\beta)$.*

Proof. Let $\theta_1,\dots,\theta_k$ be representatives for the components of $\mathbb{T}^n \setminus \overline{\mathcal{LA}'_f}$. If the indicated set of extended Euler–Mellin integrals is linearly dependent, then there exist constants $\ell_1,\dots,\ell_k$ providing a vanishing linear combination of $M_f^{\theta_1}(c)$, $\dots$, $M_f^{\theta_k}(c)$ such that

$$g_B\sum_{j=1}^{k}\ell_j e^{i\langle s,\theta_j\rangle}L^{\theta_j}(c) = 2\pi i\Gamma(t)g_A\sum_{j=1}^{k}\ell_j M_f^{\theta_j}(c) = 0.$$

It then follows from Theorem 6.2.3 that $\ell_1 = \cdots = \ell_k = 0$. □

If A is a circuit, then when β is totally nonresonant, there always exists a Mellin–Barnes, and hence an extended Euler–Mellin, basis of integral representations for solutions of the system $H_A(\beta)$ [FJ12]. However, it is noted in [Beu11a] that for general A, it is not always possible to construct a basis for the solution space of $H_A(\beta)$ by considering only Mellin–Barnes integrals of the form (6.1). By Theorem 6.4, this also holds for extended Euler–Mellin integrals arising only from the set of components of $\mathbb{T}^n \setminus \overline{\mathcal{LA}'_f}$; however, $\mathbb{T}^n \setminus \overline{\mathcal{A}'_f}$ has in general more connected components than $\mathbb{T}^n \setminus \overline{\mathcal{LA}'_f}$. In many cases, it is possible to construct a basis of Euler–Mellin integral solutions even though Mellin–Barnes integrals do not suffice, as illustrated in the following example.

EXAMPLE 6.6. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 6 \end{bmatrix}.$$

By [NP10], since the coamoeba of the A-discriminant covers $\mathbb{T}^4$, the maximal number of points in the set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(c)B + 2\pi\mathbb{Z}[B])$ is five. Hence, there is no basis of solutions of $H_A(\beta)$ represented by Mellin–Barnes integrals. However, for a generic choice of coefficients c, the coamoeba of $f(z) = c_0 + c_1z^2 + c_2z^3 + c_3z^6$ has six components in its complement. Thus, by Proposition 5.1, at each nonresonant β, this set of components provides a basis of solutions of $H_A(\beta)$ represented by extended Euler–Mellin integrals.

7. An A-Hypergeometric Rank-Jumping Example

We conclude with an example first studied in [ST98], where it was shown that some parameters β admit a higher-dimensional solution space for $H_A(\beta)$ than the expected dimension of $\mathrm{vol}(A)$. We illustrate how extended Euler–Mellin integrals capture these extra solutions at nongeneric parameters β, offering a new tool to understand how these special functions arise.

Consider the system $H_A(\beta)$ given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

and the unique parameter $\beta = (1,2)$ for which the dimension of the solution space of $H_A(\beta)$ is one larger than expected. For this A, the Euler–Mellin integral is

$$M_f^{\Theta}(s,t,c) = \int_{\mathrm{Arg}^{-1}(\theta)} \frac{z^s}{(c_1 + c_2 z + c_3 z^3 + c_4 z^4)^t} \frac{dz}{z} \tag{7.1}$$

for the polynomial $f(z) = c_1 + c_2 z + c_3 z^3 + c_4 z^4$ and $\theta \in \Theta$ for a fixed connected component Θ of $\mathbb{T}^2 \setminus \overline{\mathcal{A}'_f}$. In order to calculate the corresponding Φ_f^{Θ}, we first expand (7.1) five times in different directions, so that it converges for $(s,t) = (-2,-1)$. Upon expansion, $M_f^{\Theta}(s,t,c)$ is equal to

$$\frac{(t)_2}{s} \int \frac{z^s h_1(z)}{f(z)^{t+2}} \frac{dz}{z} + \frac{(t)_3}{s} \int \frac{z^s h_2(z)}{f(z)^{t+3}} \frac{dz}{z} + \frac{(t)_4}{s} \int \frac{z^s h_3(z)}{f(z)^{t+4}} \frac{dz}{z} + \frac{(t)_5}{s} \int \frac{z^s h_4(z)}{f(z)^{t+5}} \frac{dz}{z}, \tag{7.2}$$

where all integrals are taken over $\mathrm{Arg}^{-1}(\theta)$, and $(t)_n = \Gamma(t+n)/\Gamma(t)$ is the Pochhammer symbol. This shows that when $(s,t) = (-2,-1)$, the entire function Φ_f^{Θ} falls into the situation noted in Remark 2.7, and we thus ignore the factor $(t+1)$ in (7.2). To be explicit,

$$h_1(z) = \frac{3c_2c_3z^4}{s+1} + \frac{3c_2c_3z^4}{s+3} + \frac{4c_2c_4z^5}{s+1} + \frac{4c_2c_4z^5}{s+4},$$

$$\begin{aligned} h_2(z) = {} & \frac{36c_1c_3^2z^6}{(s+3)(4t-s+2)} + \frac{48c_1c_3c_4z^7}{(s+3)(4t-s+1)} \\ & + \frac{48c_1c_3c_4z^7}{(s+4)(4t-s+1)} + \frac{64c_1c_4^2z^8}{(s+4)(4t-s)} \\ & + \frac{c_2^3z^3}{(s+1)(s+2)} + \frac{3c_2^2c_3z^5}{(s+1)(s+2)} \\ & + \frac{4c_2^2c_4z^6}{(s+1)(s+2)} + \frac{27c_2c_3^2z^7}{(s+3)(4t-s+2)} \\ & + \frac{36c_2c_3c_4z^8}{(s+3)(4t-s+1)} + \frac{36c_2c_3c_4z^8}{(s+4)(4t-s+1)} \end{aligned}$$

$$+\frac{48c_2c_4^2z^9}{(s+4)(4t-s)}+\frac{9c_3^3z^9}{(s+3)(4t-s+2)},$$

$$h_3(z)=\frac{48c_1c_3^2c_4z^{10}}{(s+3)(4t-s+1)(4t-s+2)}+\frac{48c_1c_3^2c_4z^{10}}{(s+4)(4t-s+1)(4t-s+12)}$$
$$+\frac{64c_1c_3c_4^2z^{11}}{(s+4)(4t-s+1)^2}+\frac{36c_2c_3^2c_4z^{11}}{(s+3)(4t-s+1)(-s+4t+2)}$$
$$+\frac{36c_2c_3^2c_4z^{11}}{(s+4)(4t-s+1)(4t-s+2)}+\frac{48c_2c_3c_4^2z^{12}}{(s+4)(4t-s)(4t-s+1)}$$
$$+\frac{12c_3^3c_4z^{13}}{(s+3)(4t-s+1)(4t-s+2)}+\frac{12c_3^3c_4z^{13}}{(s+4)(4t-s+1)(4t-s+2)},$$

and

$$h_4(z)=\frac{64c_1c_3^2c_4^2z^{14}}{(s+4)(4t-s)(4t-s+1)(4t-s+2)}$$
$$+\frac{48c_2c_3^2c_4^2z^{15}}{(s+4)(4t-s)(4t-s+1)(4t-s+2)}$$
$$+\frac{16c_3^3c_4^2z^{17}}{s(s+4)(4t-s)(4t-s+1)(4t-s+2)}.$$

Each term in (7.2) corresponds to a translation of the original integral (7.1) and converges at $(s,t)=(-2,-1)$. In addition, the lack of a degree 2 term in f means that no term of any $h_i(t)$ has both $(s+2)$ and $(4t-s+2)$ as factors in its denominator. Thus, there are entire functions Φ_1, Φ_2, and Φ_3 in s and t such that

$$\Phi_f^\Theta=(4t-s+2)\Phi_1+(s+2)\Phi_2+(s+2)(4t-s+2)\Phi_3.$$

From this expression we see that since $\Phi^\Theta(-2,-1,c)=0$ independently of c and Θ, we also obtain two functions Φ_1 and Φ_2 that are also solutions of $H_A(\beta)$. Explicit calculation reveals that

$$\Phi_1^\Theta(-2,-1,c)=2\frac{c_2^2}{c_1}\quad\text{and}\quad\Phi_2^\Theta(-2,-1,c)=2\frac{c_3^2}{c_4}$$

for any choice of Θ. These span the Laurent series solutions of the system $H_A(1, 2)$, which has dimension two only at this parameter [CDD99]. The vanishing of Φ^Θ at $\beta=(1,2)$, together with the appearance of Φ_1 and Φ_2, illustrates the first direct relationship between the computation of the local cohomology of the commutative ring $\mathbb{C}[\partial_c]/\langle\Box_u \mid Au=0\rangle$ with respect to $\langle\partial_c\rangle$ and the Laurent polynomial solutions of $H_A(1,2)$.

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