Michigan Math. J. 63 (2014), 27–40

Generalized Maximum Principles and the Characterization of Linear Weingarten Hypersurfaces in Space Forms

Cícero P. Aquino, Henrique F. de Lima, & Marco Antonio L. Velásquez

1. Introduction and Statement of the Main Results

Many authors have approached the problem of characterizing hypersurfaces immersed with constant mean curvature or with constant scalar curvature in a Riemannian space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c. For instance, in the seminal work [8], Cheng and Yau introduced a new self-adjoint differential operator $\square$ acting on smooth functions defined on Riemannian manifolds. As a by-product of such an approach, they were able to classify closed hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and nonnegative sectional curvature immersed in $\mathbb{Q}_c^{n+1}$. Later on, Li [12] extended the results of Cheng and Yau [8] in terms of the squared norm of the second fundamental form of the hypersurface M^n. Shu [19] applied the generalized maximum principle of Omori [16] and Yau [21] to prove that a complete hypersurface M^n in the hyperbolic space $\mathbb{H}^{n+1}$ with constant normalized scalar curvature and nonnegative sectional curvature must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$ of $\mathbb{H}^{n+1}$. Brasil Jr., Colares, and Palmas [4] used the generalized maximum principle of Omori–Yau to characterize complete hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}$. By applying a weak Omori–Yau maximum principle stated by Pigola, Rigoli, and Setti [17], Alías and García-Martínez [2] studied the behavior of the scalar curvature R of a complete hypersurface immersed with constant mean curvature into a real space form $\mathbb{Q}_c^{n+1}$, deriving a sharp estimate for the infimum of R. More recently, Alías, García-Martínez, and Rigoli [3] obtained another suitable weak maximum principle for complete hypersurfaces with constant scalar curvature in $\mathbb{Q}_c^{n+1}$ and gave some applications in order to estimate the norm of the traceless part of its second fundamental form. In particular, they extended the main theorem of [4] for the context of $\mathbb{Q}_c^{n+1}$.

Li [13] studied the rigidity of compact hypersurfaces with nonnegative sectional curvature immersed into a unit sphere with scalar curvature proportional to

Received December 17, 2012. Revision received November 27, 2013.

The second author is partially supported by CNPq, Brazil. The second and third authors are partially supported by CAPES/CNPq, Brazil, grant Casadinho/Procad 552.464/2011-2.

mean curvature. Next, Li et al. [14] extended the results of [8] and [13] by considering *linear Weingarten* hypersurfaces immersed in the unit sphere $\mathbb{S}^{n+1}$, that is, hypersurfaces of $\mathbb{S}^{n+1}$ whose mean curvature H and normalized scalar curvature R satisfy $R = aH + b$ for some $a, b \in \mathbb{R}$. In this setting, they showed that if M^n is a compact linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$ such that $R = aH + b$ with $(n-1)a^2 + 4n(b-1) \geq 0$, then M^n is either totally umbilical or isometric to a Clifford torus $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r)$, where $1 \leq k \leq n-1$.

Thereafter, Shu [20] obtained some rigidity theorems concerning linear Weingarten hypersurfaces with two distinct principal curvatures immersed in $\mathbb{Q}_c^{n+1}$. More recently, the second author [9] used Hopf's strong maximum principle in order to obtain a suitable characterization of complete linear Weingarten hypersurfaces immersed in $\mathbb{H}^{n+1}$. Under the assumption that the mean curvature attains its maximum and supposing an appropriated restriction on the norm of the second fundamental form, he proved that such a hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$ of $\mathbb{H}^{n+1}$.

Here, our purpose is to establish characterization theorems concerning complete linear Weingarten hypersurfaces immersed in a Riemannian space form $\mathbb{Q}_c^{n+1}$. First, assuming an appropriated restriction on the norm of the traceless part Φ of the second fundamental form of M^n, we apply the generalized maximum principle of Omori–Yau jointly with Hopf's strong maximum principle in order to obtain the following result.

THEOREM 1.1. *Let M^n be a complete linear Weingarten hypersurface immersed in a Riemannian space form $\mathbb{Q}_c^{n+1}$, $n \geq 3$, such that $R = aH + b$ with $a \leq 0$ and $(n-1)a^2 + 4n(b-c) \geq 0$. Suppose that H is bounded on M^n and that $R \geq \alpha$ for some positive constant α when $c = 0$ or $c = -1$ and that $R > \frac{n-2}{n}$ when $c = 1$. If*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)},$$

then

i. *either $|\Phi| \equiv 0$ and M^n is totally umbilical, or*
ii. *$\sup_M |\Phi|^2 = n(n-1)R^2/((n-2)(nR-(n-2)c))$. In addition, if $b > c$ and $|\Phi|(p) = \sup_M |\Phi|$ at some point $p \in M^n$, then $|\Phi|^2 \equiv n(n-1)R^2/((n-2)(nR-(n-2)c))$, and M^n is isometric to a*
   (a) *Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$ when $c = 1$,*
   (b) *circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, when $c = 0$,*
   (c) *hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$ when $c = -1$,*
   where $r = \sqrt{(n-2)/(nR)}$.

Afterwards, we use an extension of a generalized maximum principle at the infinity of Yau [22] stated by Caminha [6] to derive another characterization result. More precisely, we get the following:

THEOREM 1.2. *Let M^n be a complete linear Weingarten hypersurface immersed in a Riemannian space form $\mathbb{Q}_c^{n+1}$, $n \geq 3$, such that $R = aH + b$ with $(n-1)a^2 + 4n(b-c) > 0$. Suppose that H is bounded on M^n and that $R > 0$ when $c = 0$ or $c = -1$ and that $R > \frac{n-2}{n}$ when $c = 1$. If ∇H has integrable norm on M^n and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)},$$

then

i. *either $|\Phi| \equiv 0$ and M^n is totally umbilical, or*
ii. $|\Phi|^2 \equiv n(n-1)R^2/((n-2)(nR-(n-2)c))$, *and M^n is isometric to a*
   (a) *Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$ when $c = 1$,*
   (b) *circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$ when $c = 0$,*
   (c) *hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$ when $c = -1$,*
   where $r = \sqrt{(n-2)/(nR)}$.

The proofs of Theorems 1.1 and 1.2 are given in Section 4.

2. Preliminaries

In this section we will introduce some basic facts and notation that will appear along the paper. In what follows, we will suppose that all considered hypersurfaces are orientable and connected.

Let M^n be an n-dimensional hypersurface in a real space form $\mathbb{Q}_c^{n+1}$. We choose a local orthonormal frame $\{e_A\}$ in $\mathbb{Q}_c^{n+1}$, with dual coframe $\{\omega_A\}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n, and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1, \qquad 1 \leq i, j, k, \ldots \leq n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{Q}_c^{n+1}$, we have that the structure equations of $\mathbb{Q}_c^{n+1}$ are given by

$$d\omega_A = \sum_i \omega_{Ai} \wedge \omega_i + \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Next, we restrict all the tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n, so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$, and by Cartan's lemma [7] we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.4}$$

This gives the second fundamental form of M^n, $B = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$. Furthermore, the mean curvature H of M^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l. \tag{2.6}$$

Using the structure equations, we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.7}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

The Ricci curvature and the normalized scalar curvature of M^n are given, respectively, by

$$R_{ij} = (n-1)c\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{2.8}$$

and

$$R = \frac{1}{n(n-1)} \sum_i R_{ii}. \tag{2.9}$$

From (2.8) and (2.9) we obtain

$$|B|^2 = n^2H^2 - n(n-1)(R-c), \tag{2.10}$$

where $|B|^2 = \sum_{i,j} h_{ij}^2$ is the square of the length of the second fundamental form B of M^n.

Set $\Phi_{ij} = h_{ij} - H\delta_{ij}$. We will also consider the following symmetric tensor:

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i\omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is easy to check that Φ is traceless, and from (2.10) we get

$$|\Phi|^2 = |B|^2 - nH^2 = n(n-1)H^2 - n(n-1)(R-c). \tag{2.11}$$

The components h_{ijk} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{jk}\omega_{ki}. \tag{2.12}$$

The Codazzi equation and the Ricci identity are, respectively, given by

$$h_{ijk} = h_{ikj} \tag{2.13}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.14}$$

where h_{ijk} and h_{ijkl} denote the first and second covariant derivatives of h_{ij}.

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. From equations (2.13) and (2.14) we obtain that

$$\Delta h_{ij} = \sum_k h_{kkij} + \sum_{k,l} h_{kl}R_{lijk} + \sum_{k,l} h_{li}R_{lkjk}. \tag{2.15}$$

Since $\Delta|B|^2 = 2(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2)$, from (2.15) we get

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_{i,i,k} h_{ij}h_{kkij} + \sum_{i,j,k,l} h_{ij}h_{lk}R_{lijk} + \sum_{i,j,k,l} h_{ij}h_{il}R_{lkjk}. \tag{2.16}$$

Consequently, taking a (local) orthonormal frame $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.16) we obtain the following Simons-type formula:

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + \sum_i \lambda_i(nH)_{,ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.17}$$

Now, let $\phi = \sum_{i,j}\phi_{ij}\omega_i\omega_j$ be the symmetric tensor on M^n defined by

$$\phi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng and Yau [8], we introduce the operator $\square$ associated to ϕ and acting on any smooth function f by

$$\square f = \sum_{i,j}\phi_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.18}$$

Now, setting $f = nH$ in (2.18) and taking a local frame field $\{e_1, \dots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.10) we obtain:

$$\begin{aligned}\square(nH) &= nH\Delta(nH) - \sum_i \lambda_i(nH)_{,ii}\\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i(nH)_{,ii}\\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta|B|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{,ii}.\end{aligned}$$

Consequently, taking into account equation (2.17), we get

$$\square(nH) = \frac{n(n-1)}{2}\Delta R + |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.19}$$

REMARK 2.1. Concerning the previous computation of $\square(nH)$, we also would like to suggest the readers to see Corollary 3.3 (case $r = 1$) in [5].

To close this section, we will quote three auxiliary lemmas. The first one is a classic algebraic result of Okumura [15], completed with the equality case proved by Alencar and do Carmo [1].

LEMMA 2.2. *Let $\mu_1, \dots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$ with $\beta \geq 0$. Then,*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3, \tag{2.20}$$

and the equality holds if and only if at least $n-1$ of the numbers μ_i are equal.

The second one is the well-known generalized maximum principle of Omori [16].

LEMMA 2.3. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below, and $f : M^n \to \mathbb{R}$ be a smooth function that is bounded from above on M^n. Then, there exists a sequence of points $\{p_k\}_{k\geq 1}$ in M^n satisfying the following properties*:

$$\lim_{k\to\infty} f(p_k) = \sup f, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0, \quad \textit{and} \quad \limsup_{k\to\infty}(\Delta f(p_k)) \leq 0.$$

Yau [22] established the following version of Stokes' theorem on an n-dimensional, complete noncompact Riemannian manifold M^n: *if $\omega \in \Omega^{n-1}(M)$ is an $(n-1)$-differential form on M^n, then there exists a sequence B_i of domains on M^n such that $B_i \subset B_{i+1}$, $M^n = \bigcup_{i\geq 1} B_i$, and*

$$\lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

Suppose that M^n is oriented by the volume element dM. If $\omega = \iota_X dM$ is the contraction of dM in the direction of a smooth vector field X on M^n, then Caminha [6] obtained a suitable consequence of Yau's result, stated in Lemma 2.4. We denote by $\mathcal{L}^1(M)$ and $\operatorname{div}_M X$ the space of Lebesgue-integrable functions and the divergence of a smooth vector field X on M^n, respectively.

LEMMA 2.4 (Proposition 2.1 of [6]). *Let X be a smooth vector field on the n-dimensional complete oriented Riemannian manifold M^n such that* $\operatorname{div}_M X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\operatorname{div}_M X = 0$.

REMARK 2.5. As was observed by the referee, Lemma 2.4 is also a consequence of the theorem of Karp [10].

3. Some Auxiliary Results

Along this section, we will establish some auxiliary results, which we will use to prove Theorems 1.1 and 1.2. In the first one, we will reason as in the proof of Lemma 2.1 of [14].

PROPOSITION 3.1. *Let M^n be a linear Weingarten hypersurface in a Riemannian space form $\mathbb{Q}_c^{n+1}$ such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n(b-c) \geq 0. \tag{3.1}$$

Then

$$|\nabla B|^2 \geq n^2 |\nabla H|^2. \tag{3.2}$$

Moreover, if inequality (3.1) *is strict and the equality holds in* (3.2) *on M^n, then H is constant on M^n.*

Proof. Since we are supposing that $R = aH + b$, from equation (2.10) we get

$$2\sum_{i,j} h_{ij}h_{ijk} = (2n^2H - n(n-1)a)H_{,k}.$$

Thus,

$$4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = (2n^2H - n(n-1)a)^2|\nabla H|^2.$$

Consequently, using Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned} 4|B|^2|\nabla B|^2 &= 4\left(\sum_{i,j} h_{ij}^2\right)\left(\sum_{i,j,k} h_{ijk}^2\right) \\ &\geq 4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \\ &= (2n^2H - n(n-1)a)^2|\nabla H|^2. \end{aligned} \tag{3.3}$$

On the other hand, since $R = aH + b$, from equation (2.10) we easily see that

$$(2n^2H - n(n-1)a)^2 = n^2(n-1)((n-1)a^2 + 4n(b-c)) + 4n^2|B|^2.$$

Consequently, from (3.3) we have

$$|B|^2|\nabla B|^2 \geq n^2|B|^2|\nabla H|^2.$$

Therefore, we obtain that either $|B| = 0$ and $|\nabla B|^2 = n^2|\nabla H|^2$ or $|\nabla B|^2 \geq n^2|\nabla H|^2$. Moreover, if $(n-1)a^2 + 4n(b-c) > 0$, from the previous identity we get that $(2n^2H - n(n-1)a)^2 > 4n^2|B|^2$. Now, let us assume in addition that the equality holds in (3.2) on M^n. In this case, we wish to show that H is constant on M^n. Suppose, by contradiction, that this does not occur. Consequently, there exists a point $p \in M^n$ such that $|\nabla H(p)| > 0$. So, one deduces from (3.3) that $4|B(p)|^2|\nabla B(p)|^2 > 4n^2|B(p)|^2|\nabla H(p)|^2$, and since $|\nabla B(p)|^2 = n^2 \times |\nabla H(p)|^2 > 0$, we arrive at a contradiction. Hence, in this case, we conclude that H must be constant on M^n. □

In what follows, we will consider the Cheng–Yau modified operator

$$L = \square - \frac{n-1}{2}a\Delta. \tag{3.4}$$

In our next result, we extend the generalized maximum principle of Omori to the Cheng–Yau modified operator.

PROPOSITION 3.2. *Let M^n be a complete linear Weingarten hypersurface immersed in a Riemannian space form $\mathbb{Q}_c^{n+1}$ such that $R = aH + b$ with $a \leq 0$ and $(n-1)a^2 + 4n(b-c) \geq 0$. If H is bounded on M^n, then there exists a sequence of points $\{p_k\}_{k\geq 1}$ in M^n satisfying the following properties*:

$$\lim_{k\to\infty} nH(p_k) = n\sup H, \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0, \quad \textit{and}$$
$$\limsup_{k\to\infty}(L(nH)(p_k)) \leq 0.$$

Proof. Let us choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i \delta_{ij}$. From (3.4) we have that

$$L(nH) = \sum_i \left(nH - \frac{n-1}{2}a - \lambda_i \right)(nH)_{ii}. \tag{3.5}$$

On the other hand, we observe that if H vanishes identically on M^n, then the proposition is obvious. So, let us suppose that H is not identically zero. By changing the orientation of M^n if necessary, we may assume that $\sup H > 0$. Thus, for all $i = 1, \ldots, n$, from (2.10) by a straightforward computation we get

$$\begin{aligned}
(\lambda_i)^2 &\le |B|^2 \\
&= n^2H^2 - n(n-1)(aH + b - c) \\
&= \left(nH - \frac{n-1}{2}a \right)^2 - \frac{n-1}{4}((n-1)a^2 + 4n(b-c)) \\
&\le \left(nH - \frac{n-1}{2}a \right)^2,
\end{aligned}$$

where we have used our assumption that $(n-1)a^2 + 4n(b-c) \ge 0$ to obtain the last inequality. Consequently, for all $i = 1, \ldots, n$, we have

$$|\lambda_i| \le \left| nH - \frac{n-1}{2}a \right|. \tag{3.6}$$

Thus, from (2.7) and (3.6) we obtain

$$R_{ijij} = c + \lambda_i \lambda_j \ge c - \left(nH - \frac{n-1}{2}a \right)^2. \tag{3.7}$$

Hence, since we are supposing that H is bounded on M^n, it follows from (3.7) that the sectional curvatures of M^n are bounded from below. Therefore, we may apply Lemma 2.3 to the function nH, obtaining a sequence of points $\{p_k\}_{k\ge1}$ in M^n such that

$$\begin{gathered}
\lim_{k\to\infty} nH(p_k) = n \sup H, \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0, \\
\limsup_{k\to\infty}((nH)_{ii}(p_k)) \le 0.
\end{gathered} \tag{3.8}$$

However, since H is bounded, taking subsequences if necessary, we can arrive at a sequence $\{p_k\}_{k\ge1}$ in M^n that satisfies (3.8) and such that $H(p_k) \ge 0$. Thus, taking into account that $a \le 0$, from (3.6) we get

$$\begin{aligned}
0 \le nH(p_k) - \frac{n-1}{2}a - |\lambda_i(p_k)| &\le nH(p_k) - \frac{n-1}{2}a - \lambda_i(p_k) \\
\le nH(p_k) - \frac{n-1}{2}a + |\lambda_i(p_k)| &\le 2nH(p_k) - (n-1)a.
\end{aligned} \tag{3.9}$$

Consequently, using once more that H is bounded on M^n, from (3.9) we infer that $nH(p_k) - \frac{n-1}{2}a - \lambda_i(p_k)$ is nonnegative and bounded on M^n. Therefore, from

(3.5), (3.8), and (3.9) we obtain that

$$\limsup_{k\to\infty}(L(nH)(p_k)) \le \sum_i \limsup_{k\to\infty}\left[\left(nH - \frac{n-1}{2}a - \lambda_i\right)(p_k)(nH)_{ii}(p_k)\right] \le 0. \qquad \square$$

In the last result of this section, we establish a sufficient criteria of ellipticity for the Cheng–Yau modified operator.

PROPOSITION 3.3. *Let M^n be a linear Weingarten hypersurface immersed in a Riemannian space form $\mathbb{Q}_c^{n+1}$ such that $R = aH + b$ with $b > c$. Then L is elliptic.*

Proof. Since $R = aH + b$ with $b > c$, from equation (2.10) we easily see that H cannot vanish on M^n, and, by choosing the appropriate Gauss mapping, we may assume that $H > 0$ on M^n.

Let us consider the case that $a = 0$. Since $R = b > c$, choosing a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.10) we have that $\sum_{i<j}\lambda_i\lambda_j > 0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j > \lambda_i^2$$

for every $i = 1, \ldots, n$, and hence we have that $nH - \lambda_i > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From equation (2.10) we get that

$$a = -\frac{1}{n(n-1)H}(|B|^2 - n^2H^2 + n(n-1)(b-c)).$$

Consequently, for every $i = 1, \ldots, n$, by a straightforward algebraic computation we verify that

$$\begin{aligned} nH - \lambda_i - \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH}(|B|^2 - n^2H^2 + n(n-1)(b-c)) \\ &= \frac{1}{2nH}\left(\sum_{j\neq i}\lambda_j^2 + \left(\sum_{j\neq i}\lambda_j\right)^2 + n(n-1)(b-c)\right). \end{aligned}$$

Therefore, since $b > c$, we also conclude in this case that L is elliptic. $\square$

4. Proofs of Theorems 1.1 and 1.2

Now, we are in position to prove Theorem 1.1.

Proof. Let us choose a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$. Since $R = aH + b$, from (2.19) and (3.4) we have that

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}R_{ijij}(\lambda_i - \lambda_j)^2. \tag{4.1}$$

Thus, since from (2.7) we have that $R_{ijij} = \lambda_i\lambda_j + c$, from (4.1) we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nc(|B|^2 - nH^2) - |B|^4 + nH\sum_i \lambda_i^3. \quad (4.2)$$

Moreover, we have $\Phi_{ij} = \mu_i\delta_{ij}$, and by a straightforward computation we verify that

$$\sum_i \mu_i = 0, \quad \sum_i \mu_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \lambda_i^3 - 3H|\Phi|^2 - nH^3. \quad (4.3)$$

Thus, using Gauss equation (2.7) jointly with (4.3) into (4.2), we get

$$L(nH) = |\nabla B|^2 - n^2|\nabla H|^2 + nH\sum_i \mu_i^3 + |\Phi|^2(-|\Phi|^2 + nH^2 + nc). \quad (4.4)$$

By applying Lemma 2.2 and Proposition 3.1, from (4.4) we have

$$L(nH) \geq |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nH^2 + nc\right). \quad (4.5)$$

Furthermore, from (2.11) we obtain

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R - c). \quad (4.6)$$

Thus, from (4.5) and (4.6) we get

$$L(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|), \quad (4.7)$$

where $Q_R(x)$ is the function introduced by Alías, García-Martínez, and Rigoli [3] and given by

$$Q_R(x) = -(n-2)x^2 - (n-2)x\sqrt{x^2 + n(n-1)(R-c)} + n(n-1)R.$$

Since we are supposing that $R > 0$, we have that $Q_R(0) = n(n-1)R > 0$ and the function $Q_R(x)$ is strictly decreasing for $x \geq 0$, with $Q_R(x^*) = 0$ at

$$x^* = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}} > 0.$$

Consequently, from our restriction on the norm of Φ we obtain that

$$L(nH) \geq \frac{1}{n-1}|\Phi|^2 Q_R(|\Phi|) \geq 0. \quad (4.8)$$

On the other hand, since H is supposed to be bounded on M^n, by Proposition 3.2 it is possible to obtain a sequence of points $\{p_k\}_{k\geq 1}$ in M^n such that

$$\lim_{k\to\infty} H(p_k) = \sup H > 0 \quad \text{and} \quad \limsup_{k\to\infty}(L(nH)(p_k)) \leq 0. \quad (4.9)$$

Thus, since from (4.6) we have that

$$|\Phi|^2 = n(n-1)(H^2 - aH - b + c), \quad (4.10)$$

our assumption that $a \le 0$, jointly with (4.9), gives

$$\lim_{k\to\infty} |\Phi(p_k)| = \sup|\Phi|. \tag{4.11}$$

Consequently, from (4.8) and (4.11) we have that

$$0 \ge \limsup_{k\to\infty}(L(nH)(p_k)) \ge \frac{1}{n-1}\sup|\Phi|^2 Q_{\inf R}(\sup|\Phi|) \ge 0,$$

and, hence, we conclude that $\sup|\Phi|^2 Q_{\inf R}(\sup|\Phi|) = 0$. Therefore, we have that either $|\Phi| \equiv 0$ and M^n is totally umbilical or $\sup|\Phi| = x^*$. Moreover, if $|\Phi|(p) = \sup|\Phi|$ at some point $p \in M^n$ and since we are assuming that $a \le 0$, equation (4.10) implies that H attains its maximum on M^n. Thus, since Proposition 3.3 guarantees that L is elliptic when $b > c$, from inequality (4.8) we can apply Hopf's strong maximum principle to conclude that H is constant on M^n. Consequently, $|\Phi| = \sup|\Phi|$ on M^n, and, since the equality holds in (2.20) of Lemma 2.2, we conclude that M^n must be an isoparametric hypersurface with two distinct principal curvatures one of which is simple. Hence, by the classical results on isoparametric hypersurfaces of real space forms [7; 11; 18], and since we are supposing that $R > 0$, we conclude that either $|\Phi| = 0$ and M^n is totally umbilical or $|\Phi|^2 = n(n-1)R^2/((n-2)(nR-(n-2)c))$ and M^n is isometric to a

(a) Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$ with $0 < r < 1$ if $c = 1$,
(b) circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$ with $r > 0$ if $c = 0$,
(c) hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r)$ with $r > 0$ if $c = -1$.

When $c = 1$, for a given radius $0 < r < 1$, it is a standard fact that the product embedding $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{S}^{n+1}$ has constant principal curvatures given by

$$k_1 = \frac{r}{\sqrt{1-r^2}}, \qquad k_2 = \cdots = k_n = -\frac{\sqrt{1-r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2-(n-1)}{nr\sqrt{1-r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1-r^2)}.$$

When $c = 0$, for a given radius $r > 0$, $\mathbb{R} \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{R}^{n+1}$ has constant principal curvatures given by

$$k_1 = 0, \qquad k_2 = \cdots = k_n = \frac{1}{r}.$$

In this case,

$$H = \frac{n-1}{nr} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2}.$$

Finally, when $c = -1$, for a given radius $r > 0$, $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \hookrightarrow \mathbb{H}^{n+1}$ has constant principal curvatures given by

$$k_1 = \frac{r}{\sqrt{1+r^2}}, \qquad k_2 = \cdots = k_n = \frac{\sqrt{1+r^2}}{r}.$$

Thus, in this case,

$$H = \frac{nr^2 + (n-1)}{nr\sqrt{1+r^2}} \quad \text{and} \quad |\Phi|^2 = \frac{n-1}{nr^2(1+r^2)}.$$

Therefore, in order to finish our proof, from equation (2.11) by algebraic computations we verify that in all these previously described situations we must have $r = \sqrt{(n-2)/(nR)}$. □

We close our paper by presenting the proof of Theorem 1.2.

Proof. First, we observe that from (2.18) and (3.4) it is not difficult to verify that

$$L(nH) = \mathrm{div}_M(P(\nabla H)), \tag{4.12}$$

where

$$P = \left(n^2 H + \frac{n(n-1)}{2}a\right)I - nB, \tag{4.13}$$

and I denotes the identity in the algebra of smooth vector fields on M^n.

Moreover, since H is supposed to be bounded on M^n, from equation (2.10) we have that B is also bounded on M^n. Consequently, from (4.13) we see that there exists a positive constant C such that $|P| \leq C$. Thus, since we are also assuming that $|\nabla H| \in \mathcal{L}^1(M)$, we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \leq C|\nabla H| \in \mathcal{L}^1(M). \tag{4.14}$$

Thus, because of (4.8), (4.12), and (4.14), we can apply Lemma 2.4 to obtain that $L(nH) = 0$ on M^n. Consequently, taking into account that all the inequalities obtained are, in fact, equalities, from equation (4.1) we have that

$$|\nabla B|^2 = n^2 |\nabla H|^2.$$

Hence, since we are assuming that $(n-1)a^2 + 4n(b+1) > 0$, by applying Proposition 3.1 we get that H is constant on M^n. Therefore, since the equality holds in (2.20) of Lemma 2.2, we conclude that M^n is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple. At this point, the proof follows the same steps of the proof of Theorem 1.1. □

Acknowledgments. The authors would like to thank the referee for giving some valuable suggestions that improved the paper.

References

[1] H. Alencar and M. do Carmo, *Hypersurfaces with constant mean curvature in spheres,* Proc. Amer. Math. Soc. 120 (1994), 1223–1229.

[2] L. J. Alías and S. C. García-Martínez, *On the scalar curvature of constant mean curvature hypersurfaces in space forms,* J. Math. Anal. Appl. 363 (2010), 579–587.

[3] L. J. Alías, S. C. García-Martínez, and M. Rigoli, *A maximum principle for hypersurfaces with constant scalar curvature and applications,* Ann. Glob. Anal. Geom. 41 (2012), 307–320.

[4] A. Brasil Jr., A. G. Colares, and O. Palmas, *Complete hypersurfaces with constant scalar curvature in spheres,* Monatsh. Math. 161 (2010), 369–380.

[5] A. Caminha, *On hypersurfaces into Riemannian spaces of constant sectional curvature,* Kodai Math. J. 29 (2006), 185–210.

[6] ———, *The geometry of closed conformal vector fields on Riemannian spaces,* Bull. Braz. Math. Soc. 42 (2011), 277–300.

[7] É. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante,* Ann. Mat. Pura Appl. 17 (1938), 177–191.

[8] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature,* Math. Ann. 225 (1977), 195–204.

[9] H. F. de Lima, *Complete linear Weingarten hypersurfaces immersed in the hyperbolic space,* J. Math. Soc. Japan 66 (2014), 415–423.

[10] L. Karp, *On Stokes' theorem for noncompact manifolds,* Proc. Amer. Math. Soc. 82 (1981), 487–490.

[11] T. Levi-Civita, *Famiglia di superfici isoparametriche nell'ordinario spazio Euclideo,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. 26 (1937), 355–362.

[12] H. Li, *Hypersurfaces with constant scalar curvature in space forms,* Math. Ann. 305 (1996), 665–672.

[13] ———, *Global rigidity theorems of hypersurfaces,* Ark. Math. 35 (1997), 327–351.

[14] H. Li, Y. J. Suh, and G. Wei, *Linear Weingarten hypersurfaces in a unit sphere,* Bull. Korean Math. Soc. 46 (2009), 321–329.

[15] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor,* Amer. J. Math. 96 (1974), 207–213.

[16] H. Omori, *Isometric immersions of Riemannian manifolds,* J. Math. Soc. Japan 19 (1967), 205–214.

[17] S. Pigola, M. Rigoli, and A. G. Setti, *Maximum principles on Riemannian manifolds and applications,* Mem. American Math. Soc., 822, 2005.

[18] B. Segre, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. 27 (1938), 203–207.

[19] S. Shu, *Complete hypersurfaces with constant scalar curvature in a hyperbolic space,* Balkan J. Geom. Appl. 12 (2007), 107–115.

[20] ———, *Linear Weingarten hypersurfaces in a real space form,* Glasgow Math. J. 52 (2010), 635–648.

[21] S. T. Yau, *Harmonic functions on complete Riemannian manifolds,* Comm. Pure Appl. Math. 28 (1975), 201–228.

[22] ———, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry,* Indiana Univ. Math. J. 25 (1976), 659–670.

C. P. Aquino
Departamento de Matemática
Universidade Federal do Piauí
64.049-550 Teresina, Piauí
Brazil

cicero.aquino@ufpi.edu.br

H. F. de Lima
Departamento de Matemática
Universidade Federal de Campina Grande
58.429-970 Campina Grande, Paraíba
Brazil

henrique@dme.ufcg.edu.br

M. A. L. Velásquez
Departamento de Matemática
Universidade Federal de Campina Grande
58.429-970 Campina Grande, Paraíba
Brazil

marco.velasquez@pq.cnpq.br