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The Exactness of a General Skoda Complex

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Abstract. We show that a Skoda complex with a general plurisubharmonic weight function is exact if its ‘degree’ is sufficiently large. This answers a question of Lazarsfeld and implies that not every integrally closed ideal is equal to a multiplier ideal even if we allow general plurisubharmonic weights for the multiplier ideal, extending the result of Lazarsfeld and Lee [LL].

1. Introduction

In complex algebraic geometry, a singular weight function of the form $1/|f|^2 =: e^{-\phi}$, where f is a holomorphic function, plays an important role. It is natural to consider more generally a singular weight $e^{-\phi}$, where ϕ is a plurisubharmonic (psh) function. Given $e^{-\phi}$, there are two fundamental ways to define an ideal sheaf of local holomorphic function germs, say u: collecting those with $|u|^2 e^{-\phi}$ locally bounded above, on the one hand, and collecting those with the local integral $\int_\Omega |u|^2 e^{-\phi}$ finite, on the other hand. The former gives an *integrally closed* ideal and the latter gives a *multiplier* ideal.

A multiplier ideal is always an integrally closed ideal, but the converse had been unknown in a general dimension until [LL] showed the existence of an integrally closed ideal that is not a multiplier ideal of a psh function with analytic singularities (e.g. of the form $\log |f|^2$ for f holomorphic). In this paper, we extend this result to the full generality of multiplier ideals of arbitrary psh functions. The proof of [LL] used the exactness of a *Skoda complex* – a Koszul-type complex of sheaves which involves multiplier ideals in a natural way (see Definition 2.2). For the special case of a psh function with analytic singularities, the exactness is a rather elementary consequence of a local vanishing theorem [L, (9.4.4), (9.6.36)]. However, the general case of the exactness of a Skoda complex cannot be equally shown from the vanishing theorem because of the difficult openness conjecture (3.1) which is not known beyond dimension 2 (see [FJ]). Instead of vanishing, we use the L^2 methods of [Sk72] to a Skoda complex setting and prove the following theorem.

Theorem 1.1. *Let X be a complex manifold, and let L and M be line bundles on X. Let $e^{-\psi}$ be a singular hermitian metric with psh weight for the line bundle M. Let $g_1, \dots, g_p \in H^0(X, L)$. Then there exists an integer $q \geq p$ such that*

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the qth Skoda complex associated with $g_1, \dots, g_p$ is exact. More precisely, one can take $q = \lfloor \frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4} \rfloor$.

Theorem 1.1 can be considered as a generalized version of the Skoda-type division theorem [Sk72] in that we divide not only at the right end of the Skoda complex but also at all the other intermediate terms of the Skoda complex. Indeed, we separate the division statement as Proposition 4.9 which follows from the proof of (1.1) (see Remark 4.8).

The value of q in (1.1) comes from $\lfloor \frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4} \rfloor = \max_{0 \le m \le p} m(p - m + 1) + 1$. We believe that the optimal value of q, which can be used in the statement, will be $q = p$ as is also indicated by (3.4). However, the present value of q is sufficient to establish the following generalization of the main result of [LL].

COROLLARY 1.2. *There exist a complex algebraic variety X and an integrally closed ideal sheaf $\mathfrak{b}$ on it such that $\mathfrak{b}$ cannot be written as a multiplier ideal sheaf $\mathcal{J}(\phi)$ even if we allow ϕ to be a general plurisubharmonic function.*

Note that this is a generalization of the main result of [LL] since a priori the class of all analytic multiplier ideal sheaves associated with a psh function might be strictly larger than the class of all algebraic multiplier ideal sheaves. (We also remark here that the note at the end of [LL] was made when we had not realized yet that the proof of implication (3.4) depended on (3.1).)

This paper is organized as follows. In Section 1, we motivate and give the definition of a Skoda complex. In Section 2, we discuss the openness conjecture of [DK] and show that it implies the exactness of a Skoda complex in full generality. In Section 3, we prove our main theorem (1.1), the exactness of a Skoda complex, not assuming the openness conjecture, of course. In Section 4, we follow [LL] to derive the existence of an integrally closed ideal that is not a multiplier ideal (1.2).

REMARK 1.3. Apart from the use for local syzygy in [LL], a Skoda complex was originally used to prove the Skoda-type division theorem in an algebraic way (using cohomology vanishing) in [EL]. On the other hand, the original analytic way of [Sk72] to prove the Skoda-type division theorem (not via cohomology vanishing) does not involve the use of a Skoda complex. But it is interesting to note that [H67] had used a Koszul complex (together with his L^2 methods for $\overline{\partial}$) for a prototype result toward Skoda division. Later it was replaced by the more refined L^2 methods of [Sk72].

2. Definition of a Skoda Complex

Let X be a complex manifold and L be a line bundle on X. Let $g_1, \dots, g_p \in H^0(X, L)$ be holomorphic sections. Let M be another line bundle. Let $\mathcal{A}(M)$ denote either the set of all holomorphic sections of M or the set of all complex-valued measurable sections of M. Given $u \in \mathcal{A}(M)$, we can ask whether there exist $h_1, \dots, h_p \in \mathcal{A}(M - L)$ such that $u = h_1 g_1 + \dots + h_p g_p$. Such a division problem is concerned with the surjectivity of the multiplication map $P : \mathcal{A}(M - L)^{\oplus p} \to \mathcal{A}(M)$ given by $(v_1, \ldots, v_p) \mapsto v_1 g_1 + \cdots + v_p g_p$ from the direct sum of p copies of $\mathcal{A}(M-L)$ on the left. In some approaches to the division problem, one needs to extend the single map P to a Koszul-type complex to the left:

$$\begin{aligned} 0 &\to \mathcal{A}(M - pL)^{\oplus\binom{p}{p}} \to \cdots \to \mathcal{A}(M-2L)^{\oplus\binom{p}{2}} \to \mathcal{A}(M-L)^{\oplus p} \\ &\to \mathcal{A}(M) \to 0, \end{aligned}$$

where we use the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \le i_1 < \cdots < i_m \le p\}$ for $\mathcal{A}(M - mL)^{\oplus\binom{p}{m}} =: \mathcal{B}_m$, and the map $P : \mathcal{B}_m \to \mathcal{B}_{m-1}$ in the complex is the usual Koszul map

$$P(e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{k=1}^{m} (-1)^{k-1} g_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_m}. \tag{1}$$

The map P is defined by (1) and the linearity in $\mathcal{A}$. In effect, we have introduced e_i in order to define the map P. Now it is also natural to consider the weight $\phi = \log |g|^2$ (defining $|g|^2 := |g_1|^2 + \cdots + |g_p|^2$ throughout this paper) and (for $1 \le m \le p$) the subset $\mathcal{A}(M - mL, e^{-(p-m)\phi} e^{-\psi}) \subset \mathcal{A}(M - mL)$ of sections that are square-integrable with respect to $e^{-(p-m)\phi} e^{-\psi}\, dV$, where dV is a volume form and $e^{-\psi}$ is an (auxiliary) weight for the line bundle $(M - pL)$.

PROPOSITION 2.1. *The restriction of the Koszul map P to $\mathcal{A}(M - mL, e^{-(p-m)\phi} \times e^{-\psi})^{\oplus\binom{p}{m}}$ has its image contained in $\mathcal{A}(M - (m-1)L, e^{-(p-m+1)\phi} e^{-\psi})^{\oplus\binom{p}{m-1}}$.*

Proof. Let $u \in \mathcal{A}(M - mL, e^{-(p-m)\phi} e^{-\psi})^{\oplus\binom{p}{m}}$ and write it as $u = \sum_J u_J e_J$, where the index J denotes $(j_1, \ldots, j_m)$ with $1 \le j_1 < \cdots < j_m \le p$. We know that for each index J, $\int_X |u_J|^2 e^{-(p-m)\phi} e^{-\psi}\, dV < \infty$. Consider $P(u)$ and its Ith component where the index I denotes $(i_1, \ldots, i_{m-1})$ with $1 \le i_1 < \cdots < i_{m-1} \le p$. Call the Ith component σ. Then

$$\sigma = \sum_{t \notin I, 1 \le t \le p} g_t u_{I \cup t},$$

where the index $I \cup t$ of $u_{I \cup t}$ denotes the rearrangement in the right order, $|I \cup t|$ being m. Now the conclusion follows from Cauchy–Schwarz:

$$\left| \sum_{t \notin I, 1 \le t \le p} g_t u_{I \cup t} \right|^2 e^{-(p-(m-1))\phi} e^{-\psi} \le |g|^2 \sum_J |u_J|^2 e^{-(p-(m-1))\phi} e^{-\psi}. \qquad \square$$

Consequently, we have the following Koszul-type complex:

$$\begin{aligned} 0 &\to \mathcal{A}(M - pL, e^{-\psi})^{\oplus\binom{p}{p}} \to \mathcal{A}(M - (p-1)L, e^{-(\psi+\phi)})^{\oplus\binom{p}{p-1}} \to \cdots \\ &\to \mathcal{A}(M - L, e^{-(\psi + (p-1)\phi)})^{\oplus p} \to \mathcal{A}(M, e^{-(\psi + p\phi)}) \to 0. \end{aligned}$$

Now if we restrict our attention to holomorphic sections and the corresponding sheaves, we get a complex of coherent sheaves as follows.

DEFINITION 2.2 [EL; L, (9.6.36)]. Let X be a complex manifold, and let L and M be line bundles on X. Let $(M, e^{-\psi})$ be a singular hermitian metric. Let V be the vector space spanned by holomorphic sections $g_1, \dots, g_p \in H^0(X, L)$. Let $q \geq p$. The qth *Skoda complex* (Skod_q) is the complex of Koszul maps (which we will construct in what follows):

$$\begin{aligned} 0 &\to \Lambda^p V \otimes \mathcal{J}((q-p)\phi + \psi) \otimes \mathcal{O}((q-p)L + M) \to \cdots \\ &\to \Lambda^m V \otimes \mathcal{J}((q-m)\phi + \psi) \otimes \mathcal{O}((q-m)L + M) \to \cdots \\ &\to \Lambda^1 V \otimes \mathcal{J}((q-1)\phi + \psi) \otimes \mathcal{O}((q-1)L + M) \\ &\to \mathcal{J}(q\phi + \psi) \otimes \mathcal{O}(qL + M) \to 0, \end{aligned} \tag{2}$$

where we recall that $\phi = \log |g|^2 = \log(|g_1|^2 + \cdots + |g_p|^2)$.

CONSTRUCTION OF (2). Let $f : Y \to X$ be a log-resolution of the ideal $\mathfrak{a} \subset \mathcal{O}_X$ generated by $g_1, \dots, g_p$ by Hironaka's theorem. Later in (3.4), we will use the fact that f is given by composition of blow-ups along smooth subvarieties. Let F be the exceptional divisor on Y such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. Consider the Koszul complex defined by pullbacks of generators of $\mathfrak{a}$ where V is the vector space spanned by the pullbacks:

$$0 \to \Lambda^p V \otimes \mathcal{O}_Y(pF) \to \cdots \to \Lambda^2 V \otimes \mathcal{O}_Y(2F) \to V \otimes \mathcal{O}_Y(F) \to \mathcal{O}_Y \to 0.$$

Then twist through by a coherent sheaf $O_Y(K_{Y/X} - qF) \otimes \mathcal{J}(f^*\psi)$, and it stays exact since [EP, p. 7, footnote 2] says that the Koszul complex is locally split and its syzygies are locally free, so twisting by any coherent sheaf preserves exactness. We get our Skoda complex by taking the pushforward of this exact sequence under f since (for $0 \leq m \leq p$)

$$f_*(\mathcal{O}_Y(K_{Y/X} - (q-m)F) \otimes \mathcal{J}(Y, f^*\psi)) = \mathcal{J}(\Omega, (q-m)\phi + \psi).$$

This is from the change of variables formula [L, (9.3.43)] and the fact that $\mathcal{O}_Y(-(q-m)F) \otimes \mathcal{J}(f^*\psi) = \mathcal{J}(f^*((q-m)\phi + \psi))$, which in turn comes from comparing the two sides using holomorphic function germs satisfying the local integrability conditions of the multiplier ideal sheaves.

When ϕ has analytic singularities, the complex (Skod_q) is exact for all $q \geq p$ by Theorem 9.6.36 [L]. In the general case, we will prove that it is exact for sufficiently large $q \geq p$.

3. Openness Conjecture for Plurisubharmonic Functions

Suppose that a plurisubharmonic function $e^{-\phi}$ is given on a complex manifold X. Let $0 \leq c < d$ be real numbers. If $(e^{-\phi})^d$ is L^1, then $(e^{-\phi})^c$ is L^1 as well since $(e^{-\phi})^c \leq (e^{-\phi})^d$. So (fixing any compact set $K \subset X$) the set $T := \{c \geq 0 \mid e^{-2c\phi}$ is L^1 on a neighborhood of $K\}$ is an interval. We call $\sup T$ the *singularity exponent* of ϕ and write $c_K(\phi) := \sup T$. Is the interval T open at the right end? The openness conjecture of [DK] says so, that is, $c_K(\phi) \notin T$. On the other hand, the following statement in terms of multiplier ideal sheaves is also natural to consider.

CONJECTURE 3.1. *Let α and β be plurisubharmonic functions on a complex manifold. Let $\mathcal{J}_+(\alpha,\beta)$ be the maximal element of $\{\mathcal{J}(\alpha+t\beta) \mid t>0\}$ as $t\to 0$. Then $\mathcal{J}_+(\alpha,\beta)=\mathcal{J}(\alpha)$.*

The special case $\beta=\alpha$ gives the conjecture $\mathcal{J}_+(\alpha)=\mathcal{J}(\alpha)$ (where $\mathcal{J}_+(\alpha):=\mathcal{J}_+(\alpha,\alpha)$), which was considered in [D, (15.2.2)] and [DEL]. This special case of (3.1) implies the openness conjecture of [DK] though the converse is not known. (Boucksom informed the author that it might be shown that the special case $\alpha=\beta$ of (3.1) implies the general cases using the methods of [BFJ].)

PROPOSITION 3.2. *Conjecture* (3.1) *implies the openness conjecture of* [DK].

Proof. Let $c=c_K(\phi)$ be the singularity exponent. Take $\alpha=c\phi$ and $\beta=\phi$. Suppose that c belongs to the interval T. Then $e^{-2c\phi}$ is L^1, so $\mathcal{J}(c\phi)$ is trivial whereas for any $t>0$, we have that $\mathcal{J}(c\phi+t\phi)$ is nontrivial. This contradicts (3.1). □

PROPOSITION 3.3. *Conjecture* (3.1) *is true if α has analytic singularities* (*see* [D, *Definition 1.10*]).

Proof. The special case $\alpha=0$ is a result of Skoda ([Sk72], see [D, Lemma (5.6)]), which we will use. We need to show that $\mathcal{J}(\alpha)\subset\mathcal{J}(\alpha+\delta\beta)$ for some $\delta>0$. Let f be a holomorphic function germ which belongs to $\mathcal{J}(\alpha)$. Then $\exp(\log|f|^2-\alpha)$ is locally integrable. Since α has analytic singularities, we can use a log-resolution of α and $\operatorname{div}(f)$ to have $\epsilon>0$ such that $\exp((1+\epsilon)(\log|f|^2-\alpha))$ is still integrable. Now choose p such that $\frac{1}{1+\epsilon}+\frac{1}{p}=1$. Then we can choose $\delta>0$ such that $(e^{-\beta})^{\delta p}$ is integrable from the special case $\alpha=0$ of Skoda: [D, (5.6) **a**] says that the multiplier ideal sheaf of $\delta p\beta$ is trivial when the Lelong number of $\delta p\beta$ is less than 1. Then it gives the finiteness of the first factor on the right in the following Hölder inequality:

$$\int_\Omega |f|^2 e^{-(\alpha+\delta\beta)}\,dV \le \left(\int_\Omega e^{-\delta p\beta}\,dV\right)^{1/p}\left(\int_\Omega |f|^{2(1+\epsilon)}e^{-(1+\epsilon)\alpha}\,dV\right)^{1/(1+\epsilon)} < \infty.$$

□

On the other hand, the special case of β having analytic singularities does not seem to make (3.1) easier, as in the above way of using Hölder inequality.

Now we show that, in fact, deep Conjecture 3.1 implies the exactness of a Skoda complex via vanishing. Note that this is completely independent of our main result, Theorem 1.1, where the exactness of a Skoda complex is proved without assuming (3.1). It seems that the methods in the proof of (3.4) might be useful elsewhere as well.

PROPOSITION 3.4. *If Conjecture 3.1 is true, then the qth Skoda complex* (2) *is exact for any $q\ge p$.*

For this, we will use Demailly's version of the Nadel vanishing theorem [D, (5.11)] for a *weakly pseudoconvex* Kähler manifold: a weakly pseudoconvex manifold is a complex manifold that possesses a smooth plurisubharmonic exhaustion function ϕ. For example, a compact complex manifold is weakly pseudoconvex when $\phi = 0$. Also, Stein manifolds are weakly pseudoconvex.

Proof of Proposition 3.4. Going back to the construction of (2.2), first note that the log-resolution Y is weakly pseudoconvex since it has the pullback under f of a smooth plurisubharmonic exhaustion function on Ω as its own such exhaustion function.

We need to show the vanishing of the higher direct images: $R^i f_*(O_Y(K_{Y/\Omega} \otimes \mathcal{J}(Y, f^*((q-m)\phi + \psi))$ for $i \geq 1$. Using the projection formula and the fact that Ω is Stein, it suffices to show the vanishing of $H^i(Y, O_Y(K_{Y/\Omega} + f^*L) \otimes \mathcal{J}(f^*((q-m)\phi + \psi)))$ for a sufficiently positive line bundle L on Ω, $i \geq 1$ and $0 \leq j \leq p$. Taking $L - K_\Omega$ positive enough, it suffices to show that

$$H^i(Y, O_Y(K_Y + f^*L) \otimes \mathcal{J}(f^*((q-m)\phi + \psi))) = 0. \tag{3}$$

For this, we take $F = f^*(L)$. To apply the above-mentioned Nadel vanishing theorem [D, (5.11)], we need to construct a singular metric h of $f^*(L)$ such that its curvature current is strictly positive and the multiplier ideal sheaf $\mathcal{J}(Y, h)$ is the same as $\mathcal{J}(f^*((q-m)\phi + \psi))$. This will be shown possible by giving h as a product of a smooth metric of a positive line bundle (which is f^*L minus E, sum of small multiples of exceptional divisors to be specified in what follows), the singular metric precisely given by E for the $\mathbf{Q}$-line bundle $\mathcal{O}(E)$, and $f^*((q-m)\phi + \psi)$ (a plurisubharmonic function, which can be seen as a singular metric of $\mathcal{O}_Y$). We use the following lemma.

LEMMA 3.5 [Vo, Proposition 3.24]. *Let $f_1 : Y_1 \to Y_0$ be the blow-up of Y_0 along a complex submanifold Z_0 of codimension k_0, and let E_1 be the exceptional divisor of f_1 in Y_1. Let A_0 be any ample line bundle on Y_0. Then there is a large enough integer $a > 1$ such that the $\mathbf{Q}$-line bundle $f^*A_0 - \frac{1}{a}E_1$ is positive.*

Now let us suppose that the log-resolution f is composed of smooth blow-ups $f_M \circ f_{M-1} \circ \cdots \circ f_1$. Set $Y_0 := \Omega$ and $A_0 := L$. By abuse of notation, E_m denotes all the proper transforms of the exceptional divisor E_m in Y_m up to Y_M. We choose large enough integers $a_1, \ldots, a_M$ as follows.

- Integer a_1 is chosen to be large enough to satisfy that $f_1^*L - (1/a_1)E_1$ is positive by (3.5).
- Integer a_2 is chosen to be large enough so that

$$f_2^* f_1^* L = f_2^*\Bigg(\underbrace{\left(f_1^*L - \frac{1}{a_1}E_1\right)}_{positive} + \frac{1}{a_1}E_1\Bigg)$$

$$= \underbrace{f_2^*\left(f_1^*L - \frac{1}{a_1}E_1\right) - \frac{1}{a_2}E_2}_{positive} + \frac{1}{a_2}E_2 + f_2^*\left(\frac{1}{a_1}E_1\right).$$

- Integer a_3 is chosen to be large enough so that

$$f_3^* f_2^* f_1^* L = \underbrace{f_3^*(\cdots) - \frac{1}{a_3}E_3}_{\text{positive}} + \frac{1}{a_3}E_3 + f_3^*\left(\frac{1}{a_2}E_2 + f_2^*\left(\frac{1}{a_1}E_1\right)\right),$$

where the last three terms are rewritten as $(1/a_3)E_3 + (1/a_2)f_3^*E_2 + (1/a_1) \times f_3^* f_2^* E_1$.

- Similarly, for a_m $(m \geq 4)$,

$$\begin{aligned} f_4^* f_3^* f_2^* f_1^* L &= \text{(a positive line bundle)} \\ &\qquad + \frac{1}{a_4}E_4 + \frac{1}{a_3}f_4^*E_3 + \frac{1}{a_2}f_4^*f_3^*E_2 + \frac{1}{a_1}f_4^*f_3^*f_2^*E_1, \\ &\vdots \\ f^*L &= f_M^* \cdots f_1^* L \\ &= \text{(positive)} + \frac{1}{a_M}E_M + \frac{1}{a_{M-1}}f_M^*E_{M-1} + \cdots \\ &\qquad + \frac{1}{a_1}f_M^*f_{M-1}^* \cdots f_2^*E_1. \end{aligned}$$

It is now clear that we can take E to be

$$E := \frac{1}{a_M}E_M + \frac{1}{a_{M-1}}f_M^*E_{M-1} + \cdots + \frac{1}{a_1}f_M^*f_{M-1}^* \cdots f_2^*E_1$$

for large enough integers $a_1, \ldots, a_M$ so that $\mathcal{J}(\phi_E + f^*((q-m)\phi + \psi)) = \mathcal{J}(f^*((q-m)\phi + \psi))$ (ϕ_E is the weight function associated with the **Q**-divisor E) according to Conjecture 3.1. This proves (3.4). $\square$

4. Proof of the Main Theorem

4.1. Algebraic Preliminaries

Let A be a commutative ring and M be the dual of free module $M' := A^{\oplus p}$ of rank p. We view an element of $\bigwedge^k M$ as an alternating function on $(v_1, \ldots, v_k)$, where $v_i \in A^{\oplus p}$. Let $\varepsilon_1, \ldots, \varepsilon_p$ be the basis of M' and let $e_1, \ldots, e_p$ be the dual basis of M. Let $h \in M'$. Let $i(h)$ be the contraction by h, that is, (for each $m \geq 1$) the map $i(h) : \bigwedge^m M \to \bigwedge^{m-1} M$ determined by

$$(i(h)(\eta))(v_1, \ldots, v_{m-1}) = \eta(h, v_1, \ldots, v_{m-1})$$

for every m-form $\eta \in \bigwedge^m M$. Then the following is well known.

PROPOSITION 4.1. *For every $l, n \geq 1$ and $\phi \in \bigwedge^l M$, $\psi \in \bigwedge^n M$, we have*

$$i(h)(\phi \wedge \psi) = (i(h)\phi) \wedge \psi + (-1)^l \phi \wedge (i(h)\psi).$$

Now, taking $h = g_1\varepsilon_1 + \cdots + g_p\varepsilon_p$, it is easy to see that the map $i(h) : \bigwedge^m M \to \bigwedge^{m-1} M$ is our Koszul map P of (1). Also, we let $P^\vee$ denote the map $e(\psi) : \bigwedge^{m-1} M \to \bigwedge^m M$ given by taking a wedge with $(1/|g|^2)\psi$, where $\psi = \overline{g_1}e_1 +$

$\cdots + \overline{g_p} e_p$. That is, for each $u \in \bigwedge^{m-1} M$, we have

$$P^\vee(u) = \frac{1}{|g|^2} u \wedge \psi = \frac{1}{|g|^2} \left(\sum_I u_I e_I \right) \wedge (\overline{g_1} e_1 + \cdots + \overline{g_p} e_p)$$
$$= \frac{1}{|g|^2} \sum_J \sum_{k=1}^m (-1)^{m-k} \overline{g_{j_k}} u_{j_1 \cdots \widehat{j_k} \cdots j_m}. \tag{4}$$

In the first summation, we take $J = (j_1, \ldots, j_m)$ where $1 \le j_1 < \cdots < j_m \le p$. The last equality comes from the following argument: $e_I \wedge e_\ell$ is not zero for ℓ such that $\{i_1, \ldots, i_{m-1}, \ell\}$ has m elements. Rewriting the set $\{i_1, \ldots, i_{m-1}, \ell\}$ in the increasing order as $\{j_1, \ldots, j_m\}$, where $1 \le j_1 < \cdots < j_m \le p$, we note that

$$e_{i_1} \wedge \cdots \wedge e_{i_{m-1}} \wedge e_\ell = (-1)^{m-k} e_{j_1} \wedge \cdots \wedge e_{j_m}$$

when $\ell = j_k$ for some $1 \le k \le m$.

As a consequence of (4.1), we have (taking $l = m - 1$).

COROLLARY 4.2. 1. *$P(P^\vee u) = P^\vee(Pu) + (-1)^{m-1} u$ for all $u \in \bigwedge^{m-1} M$;*
2. *$P(P^\vee u) = (-1)^{m-1} u$ if $Pu = 0$,*
since the map $i(h)\psi : \bigwedge^m M \to \bigwedge^m M$ is the multiplication by 1.

Now we turn to define our Hilbert spaces and their double complex. For $i \ge 0$ (though we actually need $i = 0, 1, 2$ only) and $0 \le m \le p$, let $\widetilde{\mathcal{H}_i^m}$ be the Hilbert space completion of the smooth $((q-m)L + M)$-valued $(0, i)$ forms that are square-integrable with respect to $e^{-(q-m)\phi} e^{-\psi}$. Let $\mathcal{H}_i^m$ be the direct sum of $\binom{p}{m}$ copies of $\widetilde{\mathcal{H}_i^m}$, for which we use the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \le i_1 < \cdots < i_m \le p\}$. Each $\mathcal{H}_i^m$ is given the inner product of the direct sum Hilbert space. Let $T : \mathcal{H}_0^m \to \mathcal{H}_1^m$ and $S : \mathcal{H}_1^m \to \mathcal{H}_2^m$ be the direct sum of $\bar\partial$ operators.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{H}_0^2 & \xrightarrow{P} & \mathcal{H}_0^1 & \xrightarrow{P} & \mathcal{H}_0^0 & \longrightarrow & 0 \\
{\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & \\
\cdots & \longrightarrow & \mathcal{H}_1^2 & \xrightarrow{P} & \mathcal{H}_1^1 & \xrightarrow{P} & \mathcal{H}_1^0 & \longrightarrow & 0 \\
{\scriptstyle S}\downarrow & & {\scriptstyle S}\downarrow & & {\scriptstyle S}\downarrow & & {\scriptstyle S}\downarrow & & \\
\cdots & \longrightarrow & \mathcal{H}_2^2 & \xrightarrow{P} & \mathcal{H}_2^1 & \xrightarrow{P} & \mathcal{H}_2^0 & \longrightarrow & 0 \\
{\scriptstyle \bar\partial}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & {\scriptstyle \bar\partial}\downarrow & & \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & 0
\end{array} \tag{5}$$

Let each P be the Koszul map of (1). First we compute $P^* u$ using the fact that $(P^* u, v) = (u, Pv)$ for all v.

Throughout the rest of this paper, the index I denotes $(i_1, \ldots, i_{m-1})$, where $1 \le i_1 < \cdots < i_{m-1} \le p$, and the index J denotes $(j_1, \ldots, j_m)$, where $1 \le j_1 < \cdots < j_m \le p$.

PROPOSITION 4.3. *If $Pu = 0$, then $\|P^*u\|^2 = \|u\|^2$.*

Proof. Let $u = \sum_I u_I e_I$ and $v = \sum_J v_J e_J$. Then

$$P(v) = \sum_J v_J P(e_J) = \sum_J v_J \sum_{k=1}^{m} (-1)^{k-1} g_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_m}.$$

From the condition that $(P^*u, v) = (u, Pv)$ for all $v \in \mathcal{H}_0^m$, we can determine P^*u. Namely, the coefficient for v_J in the summation (u, Pv) must be the coefficient for e_J in P^*u. Also, we take into account the fact that (P^*u, v) is in $\mathcal{H}_0^m$ and (u, Pv) is in $\mathcal{H}_0^{m-1}$ with one less power of $1/|g|^2$ in the weight for the inner product. Thus we have

$$P^*u = \frac{1}{|g|^2} \sum_J x_{j_1 \cdots j_m} e_{j_1} \wedge \cdots \wedge e_{j_m}, \tag{6}$$

where $x_{j_1 \cdots j_m} = \sum_{k=1}^m u_{j_1 \cdots \widehat{j_k} \cdots j_m} (-1)^{k-1} \overline{g_{j_k}}$. From (4), we have $P^*u = (-1)^{m-1} P^\vee u$. Then $\|P^*u\|^2 = (P^*u, P^*u) = (u, P(P^*u)) = (u, (-1)^{m-1} \times P(P^\vee u)) = (u, u)$ by (4.2). $\square$

4.2. *Some Analytic Preliminaries*

We recall the following fundamental lemmas in the methods of L^2 estimates for $\overline{\partial}$ as in [Sk72].

LEMMA 4.4 ([Sk72, Proposition 1], see also [Va, (3.2)]). *Let $\mathcal{E}_0$, $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ be Hilbert spaces. Let $P : \mathcal{F}_0 \to \mathcal{E}_0$ be a bounded operator. Let $T : \mathcal{F}_0 \to \mathcal{F}_1$ and $S : \mathcal{F}_1 \to \mathcal{F}_2$ be unbounded, closely defined operators such that $S \circ T = 0$. Let $\mathcal{G} \subset \mathcal{E}_0$ be a closed subspace such that $P(\operatorname{Ker} T) \subset \mathcal{G}$. We have $P(\operatorname{Ker} T) = \mathcal{G}$ if and only if there exists a constant $C > 0$ such that*

$$\|P^*u + T^*\beta\|^2 + \|S\beta\|^2 \geq C\|u\|^2$$

for all $u \in \mathcal{G}$ and all $\beta \in \operatorname{Dom} T^* \cap \operatorname{Dom} S \subset \mathcal{F}_1$. Moreover, in this case, for every $u \in \mathcal{G}$, there exists $v \in \operatorname{Ker} T$ such that $Pv = u$ and $\|v\| \leq (1/\sqrt{C})\|u\|$.

All the norms in the above are taken in the respective Hilbert spaces.

Another important ingredient of the L^2 estimates for $\overline{\partial}$ is the following Bochner–Kodaira inequality, also known as the *basic estimate*. Let $\Omega \subset \mathbf{C}^n$ be a Stein bounded open subset, and let L be a line bundle on Ω. Let $e^{-\psi}$ be a singular hermitian metric of L. Let $\mathcal{F}_i$ be the Hilbert space of L-valued $(0, i)$ forms that are square integrable with respect to $e^{-\psi}$. Let $T : \mathcal{F}_0 \to \mathcal{F}_1$ and $S : \mathcal{F}_1 \to \mathcal{F}_2$ be the $\overline{\partial}$ operators.

LEMMA 4.5 (Bochner-Kodaira [H65]). *For all $\beta \in \operatorname{Dom} T^* \cap \operatorname{Dom} S$, we have*

$$\|T^*\beta\|^2 + \|S\beta\|^2 \geq \int_\Omega \left(\sqrt{-1}\partial\overline{\partial}\psi\right)(\beta, \beta) e^{-\psi}.$$

Here we define $(\sqrt{-1}\partial\overline{\partial}\psi)(\beta,\beta)$ to be (for $\beta=\beta^1\,d\overline{z}_1+\cdots+\beta^n\,d\overline{z}_n$)

$$\big(\sqrt{-1}\partial\overline{\partial}\psi\big)(\beta,\beta):=\sum_{1\le p,q\le n}\frac{\partial^2\psi}{\partial z_p\partial\overline{z}_q}\beta^p\overline{\beta^q}$$

and regard $\int_\Omega(\sqrt{-1}\partial\overline{\partial}\psi)(\beta,\beta)e^{-\psi}$ as the norm of β with respect to $\sqrt{-1}\partial\overline{\partial}\psi$.

REMARK 4.6. In the use of Hörmander's L^2 estimates for $\overline{\partial}$ with these lemmas, we need the standard procedure of regularizing the plurisubharmonic weight ψ by a sequence of smooth plurisubharmonic functions $(\psi_\nu)_{\nu\ge1}$. For simplicity in notations, here and in the next section, we adopt the convention that each ψ means the νth regularized ψ_ν so that we can take $\sqrt{-1}\partial\overline{\partial}\psi_\nu$ and so on. The resulting holomorphic function u_ν at each step comes with a uniform bound that is independent of ν, so we can take the limit u as $\nu\to\infty$ in the usual way.

4.3. *Proof of* (1.1)

In the qth Skoda complex (2), let S_m denote the sheaf $\Lambda^m V\otimes\mathcal{J}((q-m)\phi+\psi)\otimes\mathcal{O}((q-m)L+M)$ $(0\le m\le p)$. We want to show the exactness in the middle of $S_{m+1}\xrightarrow{P}S_m\xrightarrow{P}S_{m-1}$ for every $m\ge0$ (defining $S_{-1}:=0$). Since the exactness of a complex is a local property, it is sufficient to show (see (4.9) for a stronger statement with estimates)

$$\operatorname{Im}P|_\Omega=\operatorname{Ker}P|_\Omega\quad\text{in}H^0(\Omega,S_m),\tag{7}$$

where $\Omega\subset X$ is a Stein open subset.

To apply the functional analysis lemma (4.4) to this, we consider the corresponding Hilbert spaces on Ω in (5) and the Koszul maps $P_{m-1}:\mathcal{H}_0^{m-1}\to\mathcal{H}_0^{m-2}$, $P_m:\mathcal{H}_0^m\to\mathcal{H}_0^{m-1}$. In the setting of (4.4), we take $P:=P_m$, $\mathcal{E}_0:=\mathcal{H}_0^{m-1}$, $\mathcal{F}_i:=\mathcal{H}_i^m$ $(i=0,1,2)$ and take $\mathcal{G}:=\operatorname{Ker}P_{m-1}$ in $\mathcal{H}_0^{m-1}$. It suffices to show that $P_m(\operatorname{Ker}T)=\mathcal{G}$. Now consider

$$\begin{aligned}&\|P^*u+T^*\beta\|^2+\|S\beta\|^2\\&\qquad=\|T^*\beta\|^2+\|S\beta\|^2+\|P^*u\|^2+2\operatorname{Re}(P^*u,T^*\beta).\end{aligned}\tag{8}$$

First, note that $\|P^*u\|^2=\|u\|^2$ from (4.3). Our plan toward having $C\|u\|^2$ as in (4.4) is to divide $2\operatorname{Re}(P^*u,T^*\beta)$ into a u part and a β part. Then the β part being less than $\|T^*\beta\|^2+\|S\beta\|^2$ will finish the proof. More precisely, now apply the Bochner-Kodaira inequality (4.5) to get

$$\begin{aligned}\|T^*\beta\|^2+\|S\beta\|^2\ge\int_\Omega\sum_J&\big((q-m)\sqrt{-1}\partial\overline{\partial}\log|g|^2+\sqrt{-1}\partial\overline{\partial}\psi\big)\\&\times(\beta_J,\beta_J)e^{-\phi_1}\,dV,\end{aligned}$$

where dV is the Lebesgue volume form and $\phi_1:=(q-m)\log|g|^2+\psi$ is the weight of the Hilbert spaces $\mathcal{H}_i^m$. Here we have $(q-m)$ times the norm of β with respect to $\sqrt{-1}\partial\overline{\partial}\log|g|^2$. This will be canceled out by another multiple of the same norm of β coming out of $2\operatorname{Re}(P^*u,T^*\beta)$. More precisely, we will show that $2\operatorname{Re}(P^*u, T^*\beta) \geq -\frac{1}{B}\|u\|^2 - T_1$, where B is a constant $B > 1$ which we fix throughout, and (see (10))

$$T_1 := B\int_\Omega |g|^{-2} \sum_{i_1<\cdots<i_{m-1}} |e^\phi T_{i_1\cdots i_{m-1}}|^2 e^{-\phi_1}\, d\lambda$$

is the second term of the RHS of (10). Now our main inequality to show is

$$T_1 \leq (m(p-(m-1))+1)\int_\Omega \sum_J \big(\sqrt{-1}\partial\overline{\partial}\log|g|^2\big)(\beta_J, \beta_J) e^{-\phi_1}\, dV.$$

Then we can take

$$q = \max_{0\leq m\leq p} m(p-m+1)+1$$

(which gives q in (1.1)) so that (Skod_q) is exact, where we apply (4.4) with $C = 1 - \frac{1}{B}$.

Now we begin the main computations following the previous outline. In order to consider $2\operatorname{Re}(P^*u, T^*\beta)$, we write arbitrary $u \in \mathcal{H}_0^{m-1}$, $\beta \in \mathcal{H}_1^m$ as

$$u = \sum_{i_1<\cdots<i_{m-1}} u_{i_1\cdots i_{m-1}} e_{i_1}\wedge\cdots\wedge e_{i_{m-1}},$$

$$\beta = \sum_{j_1<\cdots<j_m} (\beta^1_{j_1\cdots j_m}\, d\bar{z}_1 + \cdots + \beta^n_{j_1\cdots j_m}\, d\bar{z}_n) e_{j_1}\wedge\cdots\wedge e_{j_m}.$$

We use (6) to obtain (letting $\phi := \log|g|^2$)

$$2\operatorname{Re}(P^*u, T^*\beta) = 2\operatorname{Re}(\overline{\partial}(P^*u), \beta) = 2\operatorname{Re}\int_U \sum_{j_1<\cdots<j_m} \mathcal{S}_J e^{-\phi_1}\, dV,$$

where

$$\begin{aligned}
\mathcal{S}_J := {} & u_{j_2\cdots j_m}\left(\overline{\frac{\partial}{\partial z_1}(g_{j_1}e^{-\phi})\beta^1_{j_1\cdots j_m} + \cdots + \frac{\partial}{\partial z_n}(g_{j_1}e^{-\phi})\beta^n_{j_1\cdots j_m}}\right)\\
& + u_{j_1 j_3\cdots j_m}\left(\overline{\frac{\partial}{\partial z_1}(g_{j_2}e^{-\phi})\beta^1_{j_1\cdots j_m} + \cdots + \frac{\partial}{\partial z_n}(g_{j_2}e^{-\phi})\beta^n_{j_1\cdots j_m}}\right)\\
& + \cdots\\
& + u_{j_1\cdots j_{m-1}}\left(\overline{\frac{\partial}{\partial z_1}(g_{j_m}e^{-\phi})\beta^1_{j_1\cdots j_m} + \cdots + \frac{\partial}{\partial z_n}(g_{j_m}e^{-\phi})\beta^n_{j_1\cdots j_m}}\right).
\end{aligned}$$

Then we can rewrite this sum over J as the sum over I:

$$2\operatorname{Re}(\overline{\partial}(P^*u), \beta) = \int_\Omega \sum_{i_1<\cdots<i_{m-1}} u_{i_1\cdots i_{m-1}} \overline{T_{i_1\cdots i_{m-1}}} e^{-\phi_1}\, dV,$$

where (understanding that the index $I \cup t$ of $\beta_{I\cup t}$ denotes the rearrangement in the right order as far as $|I \cup t| = m$) we define

$$\overline{T_{i_1\cdots i_{m-1}}} := \sum_{t\notin I, 1\leq t\leq p}\left(\overline{\frac{\partial}{\partial z_1}(g_t e^{-\phi})\beta^1_{I\cup t} + \cdots + \frac{\partial}{\partial z_n}(g_t e^{-\phi})\beta^n_{I\cup t}}\right). \tag{9}$$

Remembering that $|u|^2 := \sum_{i_1<\cdots<i_{m-1}} |u_{i_1\cdots i_{m-1}}|^2$, we have

$$\begin{aligned}2\,\mathrm{Re}(\overline{\partial}(P^*u), \beta) \geq -\frac{1}{B}\int_\Omega |g|^2|u|^2 e^{-2\phi-\phi_1}\,dV \\ - B\int_\Omega |g|^{-2}\sum_{i_1<\cdots<i_{m-1}} |e^{\phi}T_{i_1\cdots i_{m-1}}|^2 e^{-\phi_1}\,dV,\end{aligned} \tag{10}$$

where we have used the fact that for any complex numbers X, Y: $|X|^2+|Y|^2+2\times \mathrm{Re}(XY) \geq 0$, and also $\frac{1}{B}|X|^2 + B|Y|^2 + 2\,\mathrm{Re}(XY) \geq 0$ for any $B \geq 1$. Then we consider

$$\begin{aligned}|e^{\phi}T_{i_1\cdots i_{m-1}}|^2 &= \left| e^{\phi}\sum_{t\notin I, 1\leq t\leq p} \overline{\left(\frac{\partial}{\partial z_1}(g_t e^{-\phi})\beta^1_{I\cup t} + \cdots + \frac{\partial}{\partial z_n}(g_t e^{-\phi})\beta^n_{I\cup t}\right)}\right|^2 \\ &= \left|\sum_{t\notin I, 1\leq t\leq p}\left(e^{\phi}\frac{\partial}{\partial z_1}(g_t e^{-\phi})\beta^1_{I\cup t} + \cdots + e^{\phi}\frac{\partial}{\partial z_n}(g_t e^{-\phi})\beta^n_{I\cup t}\right)\right|^2 \\ &= |g|^{-4}\left|\sum_{t\notin I, 1\leq t\leq p}\sum_{k=1}^{n}\sum_{s=1}^{p}\overline{g_s}\left(g_s\frac{\partial g_t}{\partial z_k} - g_t\frac{\partial g_s}{\partial z_k}\right)\beta^k_{I\cup t}\right|^2\end{aligned}$$

noting that (for each t)

$$e^{\phi}\frac{\partial}{\partial z_k}(g_t e^{-\phi}) = |g|^{-2}\sum_{s=1}^{p}\overline{g_s}\left(g_s\frac{\partial g_t}{\partial z_k} - g_t\frac{\partial g_s}{\partial z_k}\right). \tag{11}$$

Finally, we have the following inequalities:

$$\begin{aligned}&B|g|^{-2}\sum_{i_1<\cdots<i_{m-1}}|e^{\phi}T_{i_1\cdots i_{m-1}}|^2 \\ &= B|g|^{-2}\sum_{i_1<\cdots<i_{m-1}}|g|^{-4}\left|\sum_{t\notin I, 1\leq t\leq p}\sum_{k=1}^{n}\sum_{s=1}^{p}\overline{g_s}\left(g_s\frac{\partial g_t}{\partial z_k} - g_t\frac{\partial g_s}{\partial z_k}\right)\beta^k_{I\cup t}\right|^2 \\ &= B|g|^{-6}\sum_{i_1<\cdots<i_{m-1}}\left|\sum_{s=1}^{p}\overline{g_s}\sum_{\substack{t\notin I\\ 1\leq t\leq p}}\sum_{k=1}^{n}\left(g_s\frac{\partial g_t}{\partial z_k} - g_t\frac{\partial g_s}{\partial z_k}\right)\beta^k_{I\cup t}\right|^2 &(12)\\ &\leq B|g|^{-6}\sum_{i_1<\cdots<i_{m-1}}|g|^2\left|\sum_{s=1}^{p}\sum_{\substack{t\notin I\\ 1\leq t\leq p}}[[s,t]]_{I\cup t}\right|^2 &(13)\\ &\leq (p-(m-1))B|g|^{-4}\sum_{i_1<\cdots<i_{m-1}}\sum_{\substack{t\notin I\\ 1\leq t\leq p}}\left|\sum_{s=1}^{p}[[s,t]]_{I\cup t}\right|^2 &(14)\\ &\leq m(p-(m-1))B|g|^{-4}\sum_{J}\sum_{1\leq q<r\leq p}|[[q,r]]_J|^2 &(15)\\ &\leq (m(p-(m-1))+1)\sum_J\left(\sqrt{-1}\partial\overline{\partial}\log|g|^2\right)(\beta_J,\beta_J). &(16)\end{aligned}$$

Here we have defined (from (13) on)

$$[[r,s]]_J := \sum_{k=1}^{n}\left(g_r\frac{\partial g_s}{\partial z_k} - g_s\frac{\partial g_r}{\partial z_k}\right)\beta_J^k$$

for any $1 \le r, s \le p$ and the index J with $|J| = m$. The implications (12) → (13) and (13) → (14) are given by Cauchy–Schwarz whereas (14) → (15) follows from an elementary counting argument. Also, we have used the following:

$$\begin{aligned}\left(\sqrt{-1}\partial\overline{\partial}\log|g|^2\right)(\beta_J,\beta_J) &= \sum_{1\le p,q\le n}\frac{\partial^2}{\partial z_p\partial\overline{z}_q}(\log|g|^2)\beta_J^p\overline{\beta_J^q}\\ &= |g|^{-4}\sum_{1\le q<r\le p}\left|\sum_{k=1}^{n}\left(g_q\frac{\partial g_r}{\partial z_k} - g_r\frac{\partial g_q}{\partial z_k}\right)\beta_J^k\right|^2.\end{aligned}$$

This completes the proof of (1.1).

REMARK 4.7. The value of q in Theorem 1.1 can be improved if we have a better inequality between (12) and $\sum_J(\sqrt{-1}\partial\overline{\partial}\log|g|^2)(\beta_J,\beta_J)$ in (16). For example, suppose that $p = 2$, then (1.1) says that $q = \lfloor\frac{1}{4}p^2 + \frac{1}{2}p + \frac{5}{4}\rfloor = 3$ works. However, an easy computation shows that (12) is less than two times $\sum_J(\sqrt{-1}\partial\overline{\partial}\log|g|^2)(\beta_J,\beta_J)$ in this case. Therefore (the optimal value) $q = 2$ also works in this case in (1.1).

REMARK 4.8. We point out that the above method of proof yields the following proposition, which is equality (7) together with L^2 estimates. This generalizes the original Skoda division theorem in the setting of a Skoda complex. Indeed, it is completely standard in Hörmander's L^2 estimates that the use of a functional analysis lemma similar to (4.4) automatically produces a solution of $\overline{\partial}$ together with L^2 estimates. The only difference between the following proposition and (7) is that we now use the last sentence of (4.4).

We use the setting and notation around (5) of Section 4 in the following proposition.

PROPOSITION 4.9. *Let $X = \Omega$ be a Stein manifold. Let p and q be integers as in (1.1). For every m such that $1 \le m \le p$, let u be an element of the direct sum space $\mathcal{H}_0^{m-1}$ such that each component of u is holomorphic. If the norm of u in $\mathcal{H}_0^{m-1}$ is finite, then there exists $v \in \mathcal{H}_0^m$ such that $Pv = u$ and each component of v is holomorphic. Moreover, there exists a constant $C > 0$ such that $\|v\| \le C\|u\|$, where C is independent of u.*

It is also not hard to formulate a version for a projective manifold X, similar to the Skoda division theorem for such X.

5. Applications to the Local Syzygy

As we discussed in Introduction, [LL] used the exactness of a Skoda complex to show that not every integrally closed ideal is a multiplier ideal. Recall the setting from Introduction: Let $\Omega \subset X$ be a connected open subset of a complex manifold X. Let $e^{-\phi}$ be a singular weight on Ω where ϕ is a plurisubharmonic function on Ω. From $e^{-\phi}$, there are two fundamental ways to define an ideal sheaf of local holomorphic function germs u: collecting those with $|u|^2 e^{-\phi}$ bounded above locally, on the one hand; and collecting those with the integral $\int_\Omega |u|^2 e^{-\phi}$ finite, on the other hand. Let us denote the former by $\mathcal{I}(\phi)$ and the latter by $\mathcal{J}(\phi)$. If ϕ has analytic singularities and is of the form $\phi = \log(|f_1|^2 + \cdots + |f_p|^2)$, then $\mathcal{I}(\phi)$ is the integral closure of the ideal generated by $f_1, \ldots, f_p$. It is important to distinguish $\mathcal{I}(\phi)$ from $\mathcal{J}(\phi)$ clearly, so let us call $\mathcal{I}(\phi)$ the *sublevel* ideal sheaf of ϕ (whereas $\mathcal{J}(\phi)$ is well known as the *multiplier* ideal sheaf of ϕ). The following is essentially contained in [D].

PROPOSITION 5.1. *A sublevel ideal sheaf $\mathcal{I}(\phi)$ is integrally closed.*

Proof. Suppose that a local holomorphic function f satisfies an equation

$$f^k + a_1 f^{k-1} + \cdots + a_{k-1} s + f_k = 0,$$

where $a_i \in \mathcal{I}(\phi)^i$. We have the following elementary bound [D97, Ch. II, Lemma 4.10] for the roots of a monic polynomial

$$|f| \le 2 \max_{1 \le i \le k} |a_i|^{1/i}$$

locally. Therefore $|f|^2 e^{-\phi}$ is locally bounded above. □

Now we apply our main theorem in the local setting as in [LL]. We only need a slight modification of the statements from [LL] due to the fact that our q in (1.1) is not optimal as in the algebraic case of (1.1). Let X be a smooth complex algebraic variety of dimension n, and let $(\mathcal{O}, \mathfrak{m})$ be the local ring of a point $x \in X$. Let $h_1, \ldots, h_p \in \mathfrak{m}$ be any collection of nonzero elements generating an ideal $\mathfrak{a} \subset \mathcal{O}$. Our main theorem (1.1) implies the following version of Theorem B in [LL], for which we just note that our exact Skoda complex sits in-between the two Koszul complexes in the statement.

THEOREM 5.2 (see Theorem B [LL]). *Let $\mathcal{J}(\psi)$ be the multiplier ideal sheaf of a plurisubharmonic function ψ which is defined in a neighborhood of x. There exists an integer $p' \ge p$ such that for every $0 \le r \le p$, the natural map*

$$H_r(K_\bullet(h_1, \ldots, h_p) \otimes \mathfrak{a}^{p'-r} \mathcal{J}(\psi)) \to H_r(K_\bullet(h_1, \ldots, h_p) \otimes \mathcal{J}(\psi))$$

vanishes.

The original Theorem B was stronger when ϕ is algebraic, which means that we could actually take $p' = p$ in that case. Now, using the isomorphism between H_r and Tor_r and taking $p = n$, we have the following.

Corollary 5.3 (see Corollary C [LL]). *There exists an integer $n' \geq n$ such that the natural maps*

$$\mathrm{Tor}_r(\mathfrak{m}^{n'-r}\mathcal{J}(\psi), \mathbf{C}) \to \mathrm{Tor}_r(\mathcal{J}(\psi), \mathbf{C})$$

vanish for all $0 \leq r \leq n$.

Corollary 5.4 (see Theorem A [LL]). *Let $\mathcal{J} = \mathcal{J}(\psi) \subset \mathcal{O}$ be* (*the germ at x of*) *any multiplier ideal. Then there exists an integer $n' \geq n$ with the property*: *For $p \geq 1$, no minimal pth syzygy of $\mathcal{J}$ vanishes modulo $\mathfrak{m}^{n'+1-p}$.*

(5.3) implies (5.4). Take n' from (5.3). Suppose that a minimal pth syzygy of $\mathcal{J}$ vanishes modulo $\mathfrak{m}^{n'+1-p}$. That is, given a minimal free resolution, a linear combination of the columns of u_p, namely $u_p(e)$ for some $e \in R_p = \mathcal{O}^{b_p}$, satisfies $u_p(e) \in \mathfrak{m}^a R_{p-1} = \mathfrak{m}^a \cdot \mathcal{O}^{b_{p-1}}$, where $a = n' + 1 - p \geq 2$. Then Proposition 1.1 [LL] says that e represents a class lying in the image of $\mathrm{Tor}_p(\mathfrak{m}^{a-1}\mathcal{I}, \mathbf{C}) \to \mathrm{Tor}_p(\mathcal{I}, \mathbf{C})$. This contradicts (5.3).

Finally, [LL, Example 2.2] says that there exists an integrally closed ideal $\mathcal{I}$ supported at a point with a first syzygy vanishing to arbitrary order a at the origin. It cannot be a multiplier ideal due to (5.4). Hence Corollary 1.2 is proved since it is sufficient to examine at the local ring of a point.

Note added. The referee kindly informed the author of recently published papers [J1] and [J2] (formerly arXiv:1102.3950, arXiv:1105.4474) which contain independently obtained results similar to ours in this paper (formerly arXiv:1007.0551): division in the Koszul complex setting (1.1), (4.9). We note that [J1, Corollary 5.7] gives an improved lower bound of q from (1.1) and (4.9). Its method seems useful toward obtaining the optimal value of q. The author is grateful to the referee for informing him of [J1] and [J2].

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